$$\begin{cases} \dot x = \dfrac{1}{\varepsilon}\left((y-x) f(x,a) - x g(x)\right) \\ \dot y = h_x - xy \\ \dot a = \dfrac{1}{\delta}\left(H(y) - a\right) \end{cases}$$

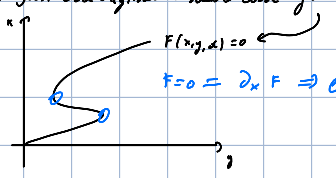
Where
$$f(x,a) = a + b\frac{x^n}{K^n + x^n}$$
$$g(x) = a' + b'\frac{K^{m'}}{x^{m'} + K^{m'}}$$
$$H(y) = \bar a + \Delta a \tanh(K(y - x_c))$$

In the limit of $\varepsilon \ll \delta \ll 1$: $\begin{cases} x \to \text{fast} & (\frac{1}{\varepsilon} \text{ timescale}) \\ a \to \text{slow} & \\ y \to \text{slow} & (t \text{ timescale}) \end{cases} \Rightarrow \dfrac{dy}{d\tau} = \varepsilon \dfrac{dy}{dt} \simeq 0$

1) rescale to fast time scale $\tau \equiv \varepsilon t$

$$\Rightarrow \begin{cases} x' = (y-x) f(x,a) - x g(x) \equiv F(x,y,a) \\ y' = 0 \\ a' = 0 \end{cases}$$

So the equilibria on this manifold satisfy $(y-x) f(x,a) = x g(x) \Rightarrow$ Just the original bistable curve $y = x\left(1 + \dfrac{f(x,a)}{g(x)}\right)$



Stability determined by $\dfrac{\partial F}{\partial x}$: $\Rightarrow (y-x)\dfrac{\partial f}{\partial x} - f(x,a) - g(x,a) - x\dfrac{\partial g}{\partial x} = 0$

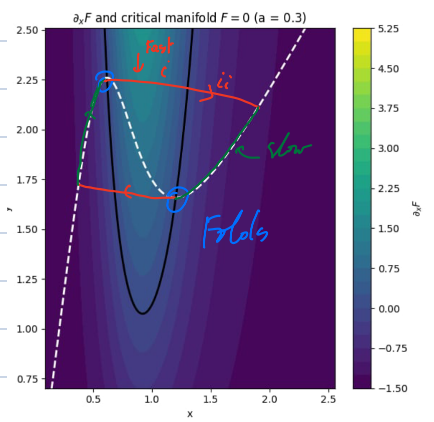
$\Rightarrow$ 0 points and particular

Two points satisfy $\dfrac{\partial F}{\partial x} = 0 = F$: $\begin{cases} F_+ \\ F_- \end{cases}$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(a + b\dfrac{x^n}{K^n + x^n}\right) = b\dfrac{(K^n + x^n) n x^{n-1} - x^n n x^{n-1}}{(K^n + x^n)^2} = b n x^{n-1} \dfrac{K^n + x^n - x^n}{(K^n + x^n)^2} = \dfrac{b n x^{n-1} K^n}{(K^n + x^n)^2} \\ \dfrac{\partial g}{\partial x} = \dfrac{\partial}{\partial x}\left(a' + b'\dfrac{K^{m'}}{x^{m'} + K^{m'}}\right) = b' K^{m'}\left(-\dfrac{m' x^{m'-1}}{(x^{m'} + K^{m'})^2}\right) = -\dfrac{m' b' K^{m'} x^{m'-1}}{(x^{m'} + K^{m'})^2} \end{cases}$$

Move slightly around fold $\partial_x F = 0$; $F(x+\eta, y, a) = \overset{0}{\overset{\parallel}{F(x,y,a)}} + \partial_x F(x,y,a)\eta + \mathcal{O}(\eta^2)$ around $F=0$

* $\partial_x F > 0$: Repulsive for fast subsystem $\Rightarrow$ Rapid move away from these points, fast dynamics dominate $\equiv$ Jump
* $\partial_x F < 0$: Attractive for fast subsystem $\Rightarrow$ System goes back to $F=0$ solution, fast dynamics "frozen" $\equiv$ Creep along nullcline



* Fast :

i) Repulsive from "Starting branch"
{ Too close to old $\dot x$ dynamic

ii) Attractive to "New branch"
{ Not close to new $\dot x$ dynamics

Slow : $\dot x \simeq 0$ and attractive

2) Compute slow dynamics on attracting branches : choose $x^* = x^*(a,y)$ on the Twia !
↳ only relevant here since $\dot x \simeq 0$

$$\Rightarrow \begin{cases} \dot y = h_x - x^* y \\ \dot a = \dfrac{1}{\delta}\left(H(y) - a\right) \end{cases}$$

$$(y-x) f(x,a) - x g(x) \equiv F(x,y,a)$$

3) Check drift at folds $D = \dfrac{\partial F}{\partial y}\dot y + \dfrac{\partial F}{\partial a}\dot a$

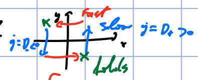
* Tiny perturbation
$$\begin{cases} x_f \to x_f + \delta_x \\ y_f \to y_f + \delta_y \end{cases}$$

$$\begin{cases} (y-x) f(x,a) - x g(x) \equiv F(x,y,a) \\ \Rightarrow \dfrac{\partial F}{\partial y} = f(x,a) \\ \Rightarrow \dfrac{\partial F}{\partial a} = (y-x)\dfrac{\partial f}{\partial a} = (y-x)\dfrac{\partial a}{\partial a} = y - x \end{cases}$$

$$= f(x,a)\dot y + (y-x)\dot a$$

$$\begin{cases} \dot y = h_x - xy \\ \dot\alpha = \frac{1}{\delta}\left(H(y) - \alpha\right) \end{cases}$$

$$= f(x,\alpha)(h_x - xy) + \frac{1}{\delta}\left(H(y) - \alpha\right)$$

* $D_- < 0 < D_+ \Rightarrow$ Relaxation cycle 

Opposite signs essential but not enough. $D_\pm$ are defined through eq conditions on x

* Same sign $\Rightarrow$ No limit cycle

The red directions are set!