Classical Electrodynamics

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1 Maxwell equations

Let's start from the basics:

$$\begin{aligned} \nabla \cdot \mathbf{D} = \rho \qquad & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \cdot \mathbf{B} = 0 \qquad & \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \end{aligned} \tag{1}$$

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-1}\,\mathrm{N} \cdot \mathrm{A}^{-2} \\ \epsilon_0 &= 8.854 \times 10^{-12}\,\mathrm{F} \cdot \mathrm{m}^{-1} \\ c &= 2.998\,\mathrm{m} \cdot \mathrm{s}^{-1} \\ v &= \frac{c}{n} \end{aligned} \tag{2}$$

$$\begin{aligned} D_i = D_i(\mathbf{E}) &= \sum_j \epsilon_{ij} \mathbf{E}_j + \mathrm{O}(E^2) \\ H_i = H_i(\mathbf{B}) &= \sum_j \mu_{ij} \mathbf{B}_j + \mathrm{O}(B^2) \end{aligned} \tag{3}$$

$$W = \frac{1}{2}(\epsilon E^2 + \mu B^2) \tag{4}$$

1.1 Useful properties

$$\begin{aligned} A \cdot (B \times C) &= (A \times B) \cdot C \\ A \times (B \times C) &= B(A \cdot C) - C(A \cdot B) \end{aligned} \tag{5}$$

Every field $\mathbf{A}$ can be decomposed in this way

$$\mathbf{A} \coloneqq \mathbf{A}_l + \mathbf{A}_t \qquad \textbf{such that} \qquad \begin{aligned} \nabla \times \mathbf{A}_l = 0 \\ \nabla \cdot \mathbf{A}_t = 0 \end{aligned}$$

1.2 Useful theorems

$$\begin{aligned} \int_V d^3\mathbf{x}\ \nabla \cdot \mathbf{A} &= \int_S d\mathbf{s} \cdot \mathbf{A} \qquad \textbf{Gauss} \\ \int_S d\mathbf{s} \cdot (\nabla \times \mathbf{A}) &= \oint_C d\mathbf{l} \cdot \mathbf{A} \qquad \textbf{Stokes} \end{aligned} \tag{6}$$

1.3 Maxwell equations

Using the two homogenous Maxwell equations, we define the potentials $\mathbf{A}$ and ϕ; using the two inhomogenous, selecting the Lorenz gauge ($\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi = 0$), we kinda obtain *wave equations* (if in the vacuum, we obtain proper wave equations)

$$\begin{aligned} \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J} \end{aligned} \tag{7}$$

With the Coulomb gauge, we obtain the Poisson equation. We can solve eq. 7 by the means of the *Green function* $\mathrm{G}(x, x', t, t')$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\ \mathrm{G}(x, x', t, t') = -4\pi\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') \tag{8}$$

It is easy to find the fourier transform of $\mathrm{G}(x, x', t, t')$

$$g(\mathbf{k}, \omega) = \frac{1}{k^2 - \frac{\omega^2}{c^2}} \tag{9}$$

$$\mathrm{G}(x, x', t, t') = \int d^3\mathbf{k} d\omega g(\mathbf{k}, \omega) e^{\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') - \omega(t - t')} \tag{10}$$

After some calculations (omitted), this is the retarded Green function

$$\mathrm{G}(\mathbf{x}-\mathbf{x}', t-t') = \mathrm{G}(\mathbf{R}, \tau) = \frac{t - t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c}}{|\mathbf{x}-\mathbf{x}'|} \tag{11}$$

And thus the potentials and the fields

$$\begin{aligned} \Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})}{\frac{|\mathbf{x}-\mathbf{x}'|}{c}} \\ \mathbf{A}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi} \int d^3\mathbf{x}' \frac{J(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})}{\frac{|\mathbf{x}-\mathbf{x}'|}{c}} \end{aligned} \tag{12}$$

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{x}' \frac{1}{R} \left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial}{\partial t'}\mathbf{J} \right]_{rit} \\ \mathbf{B}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi} \int d^3\mathbf{x}' \frac{1}{R} \left[\nabla' \times \mathbf{J} \right] \end{aligned} \tag{13}$$

We can separate eq. 14 into a static and a time dependent term, obtaining the *Jefimenko equations* (omitted)

1.4 Continuity equation

In general, when a quantity f is conserved, the *continuity equation yields*:

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot \mathbf{J}_f = \left(\frac{\partial \rho_f}{\partial t} \right)_S \tag{14}$$

From Maxwell equation, we obtain the *Poynting theorem* (conservation of energy)

$$\frac{\partial u}{\partial t} + \nabla \cdot S = -\mathbf{J} \cdot \mathbf{E} \tag{15}$$

We can do the same thing for the conservation of the moment.

1.5 Plane waves and wave propagation

Considering a plane wave propagating in the z direction, or $\mathbf{E} = \mathbf{E}(z, t)$. Then the Maxwell equations can be written as

$$\textit{omitted} \tag{16}$$

blabla

$$\begin{aligned} Z &= \sqrt{\frac{\mu}{\epsilon}} \\ v &= \frac{1}{\epsilon Z} = \frac{Z}{\mu} = \frac{1}{\sqrt{\epsilon\mu}} \\ n &= \sqrt{\frac{\epsilon}{\epsilon_0}\frac{\mu}{\mu_0}} \end{aligned} \tag{17}$$

One more time, from this set of equations, we can obtain *wave equations.* (this part may be omitted)

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial z} &= -\frac{1}{v}\frac{\partial}{\partial t}(Z\mathbf{H} \times \hat{z}) \\ \frac{\partial (Z\mathbf{H} \times \hat{z})}{\partial z} &= -\frac{1}{v}\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{18}$$

Which has solution (with $\mathbf{F}$,$\mathbf{G}$ arbitrary functions)

$$\begin{aligned} \mathbf{E}(z, t) &= \mathbf{F}(z - vt) + \mathbf{G}(z + vt) \\ \mathbf{B}(z, t) &= \frac{1}{Z}\hat{z} \left[\mathbf{F}(z - vt) - \mathbf{G}(z + vt) \right] \end{aligned} \tag{19}$$

Let's now see what happes in a lossy medium. We have

$$\begin{aligned} \mathbf{J}_{cond} &= \sigma\mathbf{E} \\ \mathbf{J}_{disp} &= -i\omega\mathbf{D} = -i\omega\epsilon_d\mathbf{E} \\ \mathbf{J} = \mathbf{J}_{cond} + \mathbf{J}_{disp} &= -i\omega\epsilon_c\mathbf{E} \end{aligned} \tag{20}$$

With the same logic as before, but this time with different starting equation, we still obtain wave equations for the fields

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{0+}e^{ik_c z} + \mathbf{E}_{0-}e^{-ik_c z} \\ \mathbf{H} &= \frac{1}{z_c}\hat{k} \times \mathbf{E} \end{aligned} \tag{21}$$

$$\begin{aligned} k_c &= \beta + i\frac{\alpha}{2} = \omega\sqrt{\mu\epsilon_d}(1 + i\tau)^{\frac{1}{2}} = \frac{\omega\mu}{Z_c} \\ \tau &:= \frac{\mathrm{Im}(\epsilon_c)}{\mathrm{Re}(\epsilon_c)} = \frac{\frac{\sigma_s}{\omega} + \mathrm{Im}(\epsilon_d)}{\mathrm{Re}(\epsilon_d)} \end{aligned} \tag{22}$$

Where β is the *wave number* and α is the *absorption coefficient*. The following relations hold:

$$\begin{aligned} \alpha &= \beta \frac{\mathrm{Im}(\frac{\epsilon}{\epsilon_o})}{\mathrm{Re}\frac{\epsilon}{\epsilon_0}} \\ \beta &= k_0\sqrt{\mathrm{Re}\frac{\epsilon}{\epsilon_0}} \end{aligned} \tag{23}$$

$$k = \omega\sqrt{\mu\epsilon} \tag{24}$$

If we take a wave propagatin in a generic direction $\hat{x}$, we can reduce it to the precedent (propagating along $\hat{z}$) case by the means of a rotation. Then the wave can be **TE** ($E\perp(x,y)$)or **TM** ($H\perp(x,y)$) [pg. 27]

We have planes of constant amplitude ($\alpha \cdot \mathbf{x}$ = const), and planes of constant phase ($\beta \cdot \mathbf{x}$ = const); in a uniform wave they are the same ($\hat{\alpha} = \hat{\beta} = \hat{k}$).

If a medium has $\epsilon < 0$ and $\mu < 0$, then $k < 0$, and we have the following effects:

- Negative Doppler effect
- Inverted Cerenkov effect
- Inverted diffraction (the diffraction angle is inverted)

The ... has the following expression

$$W = \frac{1}{2}\left[\frac{\partial}{\partial\omega}(\epsilon\omega)E^2 + \frac{\partial}{\partial\omega}(\mu\omega)B^2\right] \tag{25}$$

$\Longrightarrow$ where $\epsilon\mu < 0$ the wave can't propagate (consequences about refraction and reflection).

Let's now see what happens at the interface between two media, considering an interface at $z = 0$, and a plane wave normally incident; we can study *dynamic properties* and *cinematic properties.*

Cinematic properties can be determined from the boundary conditions.

$$\begin{aligned} \omega^i = \omega^t &= \omega^r \\ \mathbf{k}_i \cdot \mathbf{x}|_{z=0} = \mathbf{k}_t \cdot \mathbf{x}|_{z=0} &= \mathbf{k}_r \cdot \mathbf{x}|_{z=0} \\ \frac{\sin i}{\sin t} = \frac{k_t}{k} &= \frac{n'}{n} \end{aligned} \tag{26}$$

For the dynamic properties, we recall the following laws, derived from the Maxwell equations

$$\begin{aligned} (\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{n} &= \sigma \\ (\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{n} &= 0 \\ \hat{n} \times (\mathbf{E}_2 - \mathbf{E}_1) &= 0 \\ \hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{J} \end{aligned} \tag{27}$$

Considering TE ($\mathbf{E} = E\hat{y}$), we find the *Fresnel formulas*

1.6 Frequency Dispersion Characteristics of Dielectrics, Conductors, and Plasmas

$$\mathrm{D} = \epsilon_0 \mathrm{E} + \mathrm{P} \tag{28}$$

$$\begin{cases} \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \omega_p^2 \sum_j \frac{\frac{f_j}{Z}}{w_j^2 - \gamma_j\omega - \omega^2} \\ \epsilon_p = \frac{Ze^2N}{\epsilon_0 m} \end{cases} \tag{29}$$

$$\begin{gathered} \omega_j \approx 10^{15}\,\mathrm{Hz};\ \gamma_j \approx 10^{11 \div 12}\mathrm{Hz};\ w_p \approx 10^{16}\,\mathrm{Hz} \\ \omega_j >> \gamma_j \implies \epsilon(\omega) \text{ in generale è reale} \end{gathered} \tag{30}$$

$$\begin{aligned} \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \omega_p^2 \sum_j \frac{\frac{f_j}{Z}(\omega_j^2 - \omega^2)}{(\omega_y^2 - \omega^2)^2 + \gamma_j^2\omega^2} + \\ i\omega_p^2 \sum_j \frac{\frac{f_j}{Z}\gamma_j\omega}{(\omega_y^2 - \omega^2)^2 + \gamma_j^2\omega^2} \end{aligned} \tag{31}$$

If the charateristic frequancy $\omega_0 = 0$, we are modeling a conductor (there are free electrons).

$$\begin{aligned} \frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2 \frac{f_0}{Z}}{\omega(\omega + i\gamma_0)} + \sum_{j\neq 0} \frac{\omega_p^2 \frac{f_j}{Z}}{\omega_j^2 - i\gamma_j\omega - \omega^2} \\ \xrightarrow[\omega\to 0]{} \epsilon_{bound} + i\frac{\omega_p^2 \frac{f_0}{Z}}{\omega(\gamma_0 - i\omega)} \end{aligned} \tag{32}$$

$$\omega^2 = k^2c^2 + \omega_p^2 \tag{33}$$

Using the aproximation of small ω, we can develop the *Drude model*

$$\sigma_{Drude} = \frac{Nf_0e^2}{m\gamma_0\epsilon_0} \tag{34}$$

If σ_{Drude} is small, but not too small, it also has a complex component

1.7 Ionosphere propagation

$$\left.\begin{aligned} \omega_j = 0 \quad \forall\ j \\ \gamma_0 = 0 \end{aligned}\right\} \qquad \epsilon(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\epsilon_o \tag{35}$$

In this case, the magnetic field is not negligible ($B \sim 0.1 \div 1\mathrm{G}$). Considering, B in the $\hat{z}$ direction, and a circularly polarized radiation propagating along the direction $\hat{z}$, we find

$$\begin{aligned} x_0 &= \frac{e\left(\frac{E_o}{n}\right)}{\omega(\omega \mp \omega_B)} \\ \frac{\epsilon(\omega)}{\epsilon_0} &= 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)} \end{aligned} \tag{36}$$

So, if we launch a radiation from earth to the ionosphere, we see that, separating the polarization into a clockwise ad a counterclockwise component, one of the two can propagate, while the other one cannot. this is an example of *birefrengence.*

1.8 Group velocity

What happens to a wave packet in a medium? Let's consider a multimode wave

$$E(z,t) = \int_{-\infty}^{\infty} \mathrm{d}k A(k) e^{ikz - i\omega t} \tag{37}$$

A typical packet has different k, but centered around a k_0 and with a small spread. so we can expand, and by ignoring the orders ≥ 2 we can write

$$E(z,t) = e^{i(V_g k_0 - \omega_0 t)} \int \mathrm{d}k A(k) e^{ik(z - V_g t)} \tag{38}$$

So we have a packet that is rigidly translated at a velocity $v_g \implies$ energy is rigidly transferred at grup velocity.

We can also find v_g by noting that $\frac{\mathrm{d}k}{\mathrm{d}k} = \frac{\mathrm{d}\omega n(\omega)c^{-1}}{\mathrm{d}k}$ In conclusion:

$$f_g = \frac{1}{n + \omega\frac{\mathrm{d}n}{\mathrm{d}\omega}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} v_f = \frac{c}{n} = \frac{\omega}{k} \tag{39}$$

Considering also higher orders, we find that v_g can be higher than c, and also negative. So how do we interpret v_gy? It's not the speed of the information transfer; we transfer information with a discontinuity in the progation, so information always travels at speed c. v_g looses meaning in regions with anomal dispersion. E.g. when there is absorption, the packet is highly distorted. We have to be careful when the approximation no longer yields

1.9 Arrival of a signal after propagation through a dispersive medium

$$u(z,t) = \int_{-\infty}^{\infty} \mathrm{d}\omega \left[\frac{2}{1+n(\omega)}\right] A(\omega) e^{i\left(\frac{\omega n(\omega)}{c} z - \omega t\right)} \tag{40}$$

(not really important)

1.10 Causality in the connection between D and E

We saw that

$$\mathbf{D}(\mathbf{x},\omega) = \epsilon(\omega)\mathbf{E}(\mathbf{x},\omega) \tag{41}$$

With some Fourier transform we can find

$$\begin{aligned} \mathbf{D}(\mathbf{x},t) &= \epsilon_0 \mathbf{E}(\mathbf{x},t) + \epsilon_0 \int \mathrm{d}\tau G(\tau)\mathbf{E}(\mathbf{x}, t-\tau) \\ G(\tau) &:= \frac{1}{2\pi}\int \mathrm{d}\omega \left[\frac{\epsilon(\omega)}{\epsilon_0} - 1\right] e^{-i\omega\tau} \\ \tau &:= t - t' \end{aligned} \tag{42}$$

This equation is not local in time. A couple observations:

- If the medium is not dispersive ($\epsilon(\omega) = const$), then the equation is local in time
- By calculation, we can find that $G(\tau) = 0$ if $\tau < 0$ (causality is conserved).

Using the previous model for $\epsilon(\omega)$, we have

$$\mathrm{G}(\tau) = \omega_p^2 e^{-\gamma_0 \frac{\tau}{2}} \sin(\nu_0 t)\frac{1}{\nu_0} \qquad \nu_0 := \omega_o^2 - \frac{\gamma_0^2}{4} \tag{43}$$

Reality condition (to be satisfied by both E and ϵ)

$$f(x) = f^*(x^*) \tag{44}$$

So far we didn't consider a spatial non-locality; in that, more general, case, we would have $\epsilon = \epsilon(k, w)$ ...

$$\frac{numerator}{\beta} \tag{45}$$

There is a relation between the real and the imaginary part of ϵ. Measuring one, the other can be obtained

2 Special relativity

2.1 Introduction

$$ds = \frac{d\tau}{\gamma} \tag{46}$$

Trasformazioni delle velocità, dove $\mathbf{u}$ è la velocità di traslazione fra i due sistemi, e $\mathbf{v}$ è la velocità della particella nel primo sistema

$$\begin{aligned} v_{\parallel} &= \frac{v'_{\parallel} + u}{1 + \frac{\mathbf{v}' \cdot \mathbf{u}}{c^2}} \\ \mathbf{v}_{\perp} &= \frac{\mathbf{v}'_{\perp}}{\gamma(1 + \frac{\mathbf{v}' \cdot \mathbf{u}}{c^2})} \end{aligned} \tag{47}$$

$$v'_{\parallel} = \frac{v_{\parallel} - u}{1 - \frac{\mathbf{v}' \cdot \mathbf{u}}{c^2}} \tag{48}$$

Supponendo $\mathbf{u} = u\hat{\mathbf{x}}$

$$a_x = \tag{49}$$

e:

$$a_{\perp} = \frac{a'_{\perp} + []}{denominator} \tag{50}$$

[Lasciamo perdere!]

Questo quadrivettore velocità è invariante

$$u^{\mu} := \frac{dx^{\mu}}{d\tau} = \begin{pmatrix} c\gamma \\ \mathbf{v}\gamma \end{pmatrix} \tag{51}$$

Vediamo ora il quadrivettore accelerazione:

$$a^{\mu} := \frac{du^{\mu}}{d\tau} = \gamma \begin{pmatrix} c\frac{d\gamma}{dt} \\ \frac{d\gamma}{dt}\mathbf{v} + \gamma\mathbf{a} \end{pmatrix} = \begin{pmatrix} c\gamma^4 \dot{\beta} \cdot \beta \\ \gamma^4 \dot{\beta} \cdot \beta\mathbf{v} + \gamma\mathbf{a} \end{pmatrix} \tag{52}$$

$$a^2 = -\gamma^6 \left[a^2 - \frac{(\mathbf{v} \times \mathbf{a})^2}{c^2} \right] \tag{53}$$

$$\mathcal{L} = -\frac{mc^2}{\gamma} \tag{54}$$

$$\mathbf{p} = \frac{d\mathcal{L}}{d\mathbf{v}} = m\gamma\mathbf{v} \tag{55}$$

$$H = \mathbf{p} \cdot \mathbf{v} - \mathcal{L} = m\gamma c^2 = \epsilon \qquad \text{(Hamiltonian)}$$

Introduciamo il quadrivettore momento:

$$p^{\mu} = mv^{\mu} = m \begin{pmatrix} \gamma c \\ \gamma\mathbf{u} \end{pmatrix} = \begin{pmatrix} \frac{\epsilon}{c} \\ \mathbf{p} \end{pmatrix} \tag{56}$$

$$p^2 = mc^2 \tag{57}$$

Consideriamo ora un'onda piana, abbiamo **invarianza della fase**, poichè la fase è un conteggio di creste

$$\phi = k \cdot \mathbf{x} - \omega t = k' \cdot \mathbf{x}' - \omega' t' \tag{58}$$

Da qui, sostituendo x'^{μ} usando il boost di Lorentz, ricavo l'ultimo quadrivettore:

$$k^{\mu} = \begin{pmatrix} \frac{\omega}{c} \\ \mathbf{k} \end{pmatrix} \tag{59}$$

Queste formule contengono l'effetto Doppler e la legge di aberrazione:

$$\omega' = \gamma\omega(1 - \beta\cos\theta) \tan\theta' = \frac{\sin\theta}{\gamma\cos\theta - \beta} \tag{60}$$

$$\frac{dp^{\mu}}{d\tau} = F^{\mu} \tag{61}$$

2.2 Covarianza dell'elettrodinamica

$$\frac{d}{d\tau} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \frac{q}{c}\mathbf{u} \cdot \mathbf{E} \\ \frac{q}{c}(u_0\mathbf{E} + \mathbf{u} \times \mathbf{B}) \end{pmatrix} \tag{62}$$

Voglio che il membro di destra sia un quadrivett, per cui introduco:

$$J^{\mu} := \begin{pmatrix} \rho c \\ \rho\frac{dx}{dt} \end{pmatrix} \tag{63}$$

$$\partial^{\mu} J_{\mu} = \frac{\partial\rho}{\partial t} + \nabla \cdot \mathbf{J} \tag{64}$$

$$\partial^{\mu} A_{\mu} \qquad \text{gauge di Lorenz}$$

$$\Box A^{\mu} = 4\pi J^{\mu} \tag{65}$$

Da cui:

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu \tag{66}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{67}$$

$$F^{\mu\nu} = (\mathbf{E}, \mathbf{B}) \tag{68}$$

$$F_{\mu\nu} = (-\mathbf{E}, \mathbf{B}) \tag{69}$$

$$F^{*\mu\nu} = (\mathbf{B}, \mathbf{E}) \tag{70}$$

$$F^*_{\mu\nu} = (-\mathbf{B}, -\mathbf{E}) \tag{71}$$

Riscriviamo le eq. di Maxwell

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{72}$$

$$\partial_\mu F^{*\mu\nu} = 0 \tag{73}$$

$$\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} = 0$$

(forma alternativa per la seconda)

Posso riscrivere le eq. del moto in forma covariante

$$\frac{dp^\mu}{d\tau} = m\frac{du^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} u_\nu \tag{74}$$

2.3 Leggi di trasformazione dei campi

$$stranote \tag{75}$$

Vediamo alcuni invarianti

$$\mathbf{E}^2 - \mathbf{B}^2 = cost \tag{76}$$

$$\mathbf{E} \cdot \mathbf{B} = cost \tag{77}$$

2.4 Lagrangiana e Hamiltoniana di particella

Un po' di formule a caso

$$\mathcal{L}_{free} = -\frac{mc^2}{\gamma} \tag{78}$$

$$\mathcal{L}\gamma = cost \tag{79}$$

$$\frac{d}{dt}\frac{d\mathcal{L}}{d\mathbf{v}} = \frac{d\mathcal{L}}{d\mathbf{x}} \tag{80}$$

$$\frac{d\mathcal{L}_{free}}{d\mathbf{x}} = 0 \tag{81}$$

$$\tag{82}$$

2.5 Soluzione all'eq. delle onde in forma covariante

Risolviamo l'equazione 65 a pagina 5, supponendo $J^\mu = J^\mu(x)$, utilizzando una funzione di Green:

$$\Box_x D(x - x') = \delta^{(4)}(x - x') \tag{83}$$

$$z := x - x' \tag{84}$$

Passando ad uno spazio di Fourier si ha

$$D(k) = \frac{1}{k \cdot k} \tag{85}$$

$$D(z) = -\frac{1}{(2\pi)^4} \int dk D(k) e^{-ik\cdot x} \tag{86}$$

Risolvendo, si hanno due soluzioni:

$$D_{ritardata} = \frac{1}{2\pi}\Theta(x_0 - x_0')\delta[(x - x')^2] \tag{87}$$

$$D_{anticipata} = \frac{1}{2\pi}\Theta(x_0' - x_0)\delta[(x - x')^2] \tag{88}$$

3 Moving charges

Un po' di notazione

x^μ	osservatore
r^μ	carica in moto
R	distanza fra osservatore e carica
$\hat{\mathbf{n}}$	versore dalla carica all'osservatore

Posso scrivere il quadrivettore delle sorgenti per una carica in moto come:

$$J^\mu = qc \int d\tau u^\mu(\tau) \delta^{(4)}(x - r(\tau)) \tag{89}$$

$$u^\mu := \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix} \qquad r(t) := \begin{pmatrix} ct \\ r(t) \end{pmatrix} \tag{90}$$

3.1 Lienerd-Wichert

Partiamo trovando i potenziali

$$A^\mu(x) = \frac{4\pi}{c} \int d^4x' D_r(x - x') J^\mu(x') \tag{91}$$

Sostituendo l'eq 89 a pagina 6, si ottiene

$$A^\mu(x) = 2q \int d\tau u^\mu(\tau) \Theta(x_0 - r_0(\tau)) \delta([x - r(\tau)]^2) \tag{92}$$

Considering the properties of the delta, and that $\delta([x-r(\tau)])$ implies that only the points on the trajectory that lie on the backward light cone starting from x^μ can contribute to the potential (and also that $x_0 - r_0(\tau_0) = |\mathbf{x} - \mathbf{r}(\tau)|$), we have:

$$A^\mu(x) = \left.\frac{qu^\mu(\tau)}{u^\nu(\tau)(x-r(\tau))_\nu}\right|_{\tau=\tau_0} \tag{93}$$

$$\Phi(\mathbf{x},t) = \left.\frac{q}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right|_{\tau=\tau_0}$$
$$A(\mathbf{x},t) = \left.\frac{q\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right|_{\tau=\tau_0} \tag{94}$$

$$\text{con } \tau_0 \text{ definito da } (x-r(\tau_0))^2 = 0 \tag{95}$$

Tramite derivazione di 92, si trova il tensore del campo EM

$$F^{\mu\nu} = \frac{e}{u\cdot(x-r)}\frac{d}{d\tau}\left.\left[\frac{(x-r)^\mu u^\nu - (x-r)^\nu u^\nu}{u\cdot(x-r)}\right]\right|_{\tau=\tau_0} \tag{96}$$

E di conseguenza i campi

$$\mathbf{E} = q\left.\frac{\mathbf{n}-\beta}{\gamma^2(1-\mathbf{n}\cdot\beta)^3R^2}\right|_{\tau=\tau_0} + \frac{q}{c}\left.\frac{\mathbf{n}\times(\mathbf{n}-\beta)\times\dot{\beta}}{(1-\mathbf{n}\cdot\beta)^3R}\right|_{\tau=\tau_0} \tag{97}$$

$$\mathbf{E} = \text{c. velocita}'(\propto \frac{1}{r^2}) + \text{c. accelerazione}(\propto \frac{1}{r})$$
$$\mathbf{B} = [\mathbf{n}]_{rit}\times\mathbf{E} \tag{98}$$

Sappiamo che

$$\frac{dP}{d\Omega} = R^2\ \mathbf{S}\cdot\mathbf{n} \tag{99}$$

Considerando solo il campo di radiazione, e mettendoci nel caso non relativistico, otteniamo le *formule di Larmor* non relativistica

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\dot{v}^2\sin^2\theta \tag{100}$$

$$P = \frac{2}{3}\frac{q^2}{c^2}|\dot{v}|^2 \tag{101}$$

E le equivalenti relativistiche

$$P = \frac{2}{3}\frac{q^2}{c^2}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = \frac{2}{3}\frac{q^2}{c}\gamma^6[\dot{\beta}^2 - (\beta\times\dot{\beta})^2] \tag{102}$$

$$\frac{dP}{d\Omega}(t') = \frac{q^2}{4\pi c}\frac{\left|\mathbf{n}\times(\mathbf{n}-\beta\times\dot{\beta})\right|^2}{(1-\mathbf{n}\cdot\beta)^5} \tag{103}$$

4 Notazione

X_μ	covariante
X^μ	controvariante

5 M.U.

$$1\text{ eV} \approx 1.6\cdot 10^{-19}\text{J} \tag{104}$$

5.1 Gaussian CGS

We set the unit for q such that

$$F = \frac{q_1 q_2}{d^2} \tag{105}$$

Thus the unit of charge is called *esu*, or *statcoulomb*

$$esu = \sqrt{\text{dyne}\cdot\text{cm}^2} = \sqrt{\text{g}\cdot\text{cm}^3}/\text{s} \tag{106}$$

Which results in

$$\frac{F}{L} = \frac{2}{c^2}\frac{I_1 I_2}{d} \tag{107}$$

We can convert a charge q from CGS to SI and viceversa by noting that the Coulomb force is the same in every system (for a more thourough explanation, see `http://www.rpi.edu/dept/phys/Courses/PHYS4210/S10/NotesOnUnits.pdf`)

$$q_C = \frac{q_{esu}}{10\cdot c_{SI}} \tag{108}$$

Where q_C and q_{esu} are the "number" of the respective measure unit contained in the charge q (that wasn't veryclear, take a look at the link before)