Classical Electrodynamics

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1 Maxwell equations

Let's start from the basics:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{D} \end{aligned} \tag{1}$$

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-1}\,\mathrm{N} \cdot \mathrm{A}^{-2} \\ \epsilon_0 &= 8.854 \times 10^{-12}\,\mathrm{F} \cdot \mathrm{m}^{-1} \\ c &= 2.998\,\mathrm{m} \cdot \mathrm{s}^{-1} \end{aligned} \tag{2}$$

$$\begin{aligned} D_i = D_i(\mathbf{E}) &= \sum_j \epsilon_{ij} \mathbf{E}_j + \mathrm{O}(E^2) \\ H_i = H_i(\mathbf{B}) &= \sum_j \mu_{ij} \mathbf{B}_j + \mathrm{O}(B^2) \end{aligned} \tag{3}$$

1.1 Useful properties

$$\begin{aligned} A \cdot (B \times C) &= (A \times B) \cdot C \\ A \times (B \times C) &= B(A \cdot C) - C(A \cdot B) \end{aligned} \tag{4}$$

Every field $\mathbf{A}$ can be decomposed in this way

$$\mathbf{A} \coloneqq \mathbf{A}_l + \mathbf{A}_t \qquad \textbf{such that} \qquad \begin{aligned} \nabla \times \mathbf{A}_l &= 0 \\ \nabla \cdot \mathbf{A}_t &= 0 \end{aligned}$$

1.2 Useful theorems

$$\begin{aligned} \int_V d^3\mathbf{x}\ \nabla \cdot \mathbf{A} &= \int_S d\mathbf{s} \cdot \mathbf{A} \qquad \textbf{Gauss} \\ \int_S d\mathbf{s} \cdot (\nabla \times \mathbf{A}) &= \oint_C d\mathbf{l} \cdot \mathbf{A} \qquad \textbf{Stokes} \end{aligned} \tag{5}$$

1.3 Maxwell equations

Using the two homogenous Maxwell equations, we define the potentials $\mathbf{A}$ and ϕ; using the two inhomogenous, selecting the Lorenz gauge ($\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi = 0$), we kinda obtain *wave equations* (if in the vacuum, we obtain proper wave equations)

$$\begin{aligned} \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J} \end{aligned} \tag{6}$$

With the Coulomb gauge, we obtain the Poisson equation. We can solve eq. 6 by the means of the *Green function* $\mathrm{G}(x, x', t, t')$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\ \mathrm{G}(x, x', t, t') = -4\pi\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') \tag{7}$$

It is easy to find the fourier transform of $\mathrm{G}(x, x', t, t')$

$$g(\mathbf{k}, \omega) = \frac{1}{k^2 - \frac{\omega^2}{c^2}} \tag{8}$$

$$\mathrm{G}(x, x', t, t') = \int d^3\mathbf{k} d\omega g(\mathbf{k}, \omega) e^{\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - \omega(t-t')} \tag{9}$$

After some calculations (omitted), this is the retarded Green function

$$\mathrm{G}(\mathbf{x} - \mathbf{x}', t - t') = \mathrm{G}(\mathbf{R}, \tau) = \frac{t - t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c}}{|\mathbf{x} - \mathbf{x}'|} \tag{10}$$

And thus the potentials and the fields

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}', t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{\frac{|\mathbf{x}-\mathbf{x}'|}{c}}$$
$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\frac{J(\mathbf{x}', t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{\frac{|\mathbf{x}-\mathbf{x}'|}{c}} \tag{11}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{x}'\frac{1}{R}\left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial}{\partial t'}\mathbf{J}\right]_{rit}$$
$$\mathbf{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\frac{1}{R}\left[\nabla'\times\mathbf{J}\right] \tag{12}$$

We can separate eq. 12 into a static and a time dependent term, obtaining the *Jefimenko equations* (omitted)

1.4 idk lol

$$\vec{A}(\mathbf{x},t) = \int d^3\mathbf{x}'dt'\vec{J}(\mathbf{x}',t)\frac{\delta(t-t'-\frac{|\mathbf{x}'-\mathbf{x}|}{c})}{|\mathbf{x}'-\mathbf{x}|}$$

(α costanti se S.I.)

$$\phi(\mathbf{x},t) = \int d^3\mathbf{x}'dt'\rho(\mathbf{x}',t)\frac{\delta(t-t'-\frac{|\mathbf{x}'-\mathbf{x}|}{c})}{|\mathbf{x}'-\mathbf{x}|}$$

(α altre costanti se S.I.)

1.5 Waves in dielectrics

$$\mathrm{D} = \epsilon_0\mathrm{E} + \mathrm{P} \tag{13}$$

$$\begin{cases} \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \omega_p^2\sum_j \frac{\frac{f_j}{Z}}{w_j^2 - \gamma_j\omega - \omega^2} \\ \epsilon_p = \frac{Ze^2N}{\epsilon_0 m} \end{cases} \tag{14}$$

$$\omega^2 = k^2c^2 + \omega_p^2 \tag{15}$$

$$\sigma_{Drude} = \frac{Nf_0e^2}{m\gamma_0\epsilon_0} \tag{16}$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{Per i plasmi} \tag{17}$$

$$u(x,t) = \int_{-\infty}^{\infty} d\omega\left[\frac{2}{1+n(\omega)}\right]A(\omega)e^{i(k\cdot x-\omega t)}$$

(onda piana incidente in mezzo con $n(\omega)$)

2 Special relativity

2.1 Introduction

$$ds = \frac{d\tau}{\gamma} \tag{18}$$

Trasformazioni delle velocità, dove $\mathbf{u}$ è la velocità di traslazione fra i due sistemi, e $\mathbf{v}$ è la velocità della particella nel primo sistema

$$v_\parallel = \frac{v'_\parallel + u}{1+\frac{\mathbf{v}'\cdot\mathbf{u}}{c^2}}$$
$$\mathbf{v}_\perp = \frac{\mathbf{v}'_\perp}{\gamma(1+\frac{\mathbf{v}'\cdot\mathbf{u}}{c^2})} \tag{19}$$

$$v'_\parallel = \frac{v_\parallel - u}{1-\frac{\mathbf{v}'\cdot\mathbf{u}}{c^2}} \tag{20}$$

Supponendo $\mathbf{u} = u\hat{\mathbf{x}}$

$$a_x = \tag{21}$$

e:

$$a_\perp = \frac{a'_\perp + []}{denominator} \tag{22}$$

[Lasciamo perdere!]

Questo quadrivettore velocità è invariante

$$u^\mu := \frac{dx^\mu}{d\tau} = \begin{pmatrix} c\gamma \\ \mathbf{v}\gamma \end{pmatrix} \tag{23}$$

Vediamo ora il quadrivettore accelerazione:

$$a^\mu := \frac{du^\mu}{d\tau} = \gamma\begin{pmatrix} c\frac{d\gamma}{dt} \\ \frac{d\gamma}{dt}\mathbf{v} + \gamma\mathbf{a} \end{pmatrix} = \begin{pmatrix} c\gamma^4\dot{\beta}\cdot\beta \\ \gamma^4\dot{\beta}\cdot\beta\mathbf{v} + \gamma\mathbf{a} \end{pmatrix} \tag{24}$$

$$a^2 = -\gamma^6\left[a^2 - \frac{(\mathbf{v}\times\mathbf{a})^2}{c^2}\right] \tag{25}$$

$$\mathcal{L} = -\frac{mc^2}{\gamma} \tag{26}$$

$$\mathbf{p} = \frac{d\mathcal{L}}{d\mathbf{v}} = m\gamma\mathbf{v} \tag{27}$$

$$H = \mathbf{p}\cdot\mathbf{v} - \mathcal{L} = m\gamma c^2 = \epsilon \quad \text{(Hamiltonian)}$$

Introduciamo il quadrivettore momento:

$$p^\mu = mv^\mu = m\begin{pmatrix} \gamma c \\ \gamma\mathbf{u} \end{pmatrix} = \begin{pmatrix} \frac{\epsilon}{c} \\ \mathbf{p} \end{pmatrix} \tag{28}$$

$$p^2 = mc^2 \tag{29}$$

Consideriamo ora un'onda piana, abbiamo **invarianza della fase**, poichè la fase è un conteggio di creste

$$\phi = k \cdot \mathbf{x} - \omega t = k' \cdot \mathbf{x}' - \omega' t' \tag{30}$$

Da qui, sostituendo x'^μ usando il boost di Lorentz, ricavo l'ultimo quadrivettore:

$$k^\mu = \begin{pmatrix} \frac{\omega}{c} \\ \mathbf{k} \end{pmatrix} \tag{31}$$

Queste formule contengono l'effetto Doppler e la legge di aberrazione:

$$\omega' = \gamma\omega(1 - \beta\cos\theta) \tan\theta' = \frac{\sin\theta}{\gamma\cos\theta - \beta} \tag{32}$$

$$\frac{dp^\mu}{d\tau} = F^\mu \tag{33}$$

2.2 Covarianza dell'elettrodinamica

$$\frac{d}{d\tau}\begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \frac{q}{c}\mathbf{u} \cdot \mathbf{E} \\ \frac{q}{c}(u_0\mathbf{E} + \mathbf{u} \times \mathbf{B}) \end{pmatrix} \tag{34}$$

Voglio che il membro di destra sia un quadrivett, per cui introduco:

$$J^\mu := \begin{pmatrix} \rho c \\ \rho\frac{dx}{dt} \end{pmatrix} \tag{35}$$

$$\partial^\mu J_\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \tag{36}$$

$$\partial^\mu A_\mu \qquad \text{gauge di Lorenz}$$

$$\Box A^\mu = 4\pi J^\mu \tag{37}$$

Da cui:

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu \tag{38}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{39}$$

$$F^{\mu\nu} = (\mathbf{E}, \mathbf{B}) \tag{40}$$

$$F_{\mu\nu} = (-\mathbf{E}, \mathbf{B}) \tag{41}$$

$$F^{*\mu\nu} = (\mathbf{B}, \mathbf{E}) \tag{42}$$

$$F^*_{\mu\nu} = (-\mathbf{B}, -\mathbf{E}) \tag{43}$$

Riscriviamo le eq. di Maxwell

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{44}$$

$$\partial_\mu F^{*\mu\nu} = 0 \tag{45}$$

$$\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} = 0$$

(forma alternativa per la seconda)

Posso riscrivere le eq. del moto in forma covariante

$$\frac{dp^\mu}{d\tau} = m\frac{du^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu \tag{46}$$

2.3 Leggi di trasformazione dei campi

$$stranote \tag{47}$$

Vediamo alcuni invarianti

$$\mathbf{E}^2 - \mathbf{B}^2 = cost \tag{48}$$

$$\mathbf{E} \cdot \mathbf{B} = cost \tag{49}$$

2.4 Lagrangiana e Hamiltoniana di particella

Un po' di formule a caso

$$\mathcal{L}_{free} = -\frac{mc^2}{\gamma} \tag{50}$$

$$\mathcal{L}\gamma = cost \tag{51}$$

$$\frac{d}{dt}\frac{d\mathcal{L}}{d\mathbf{v}} = \frac{d\mathcal{L}}{d\mathbf{x}} \tag{52}$$

$$\frac{d\mathcal{L}_{free}}{d\mathbf{x}} = 0 \tag{53}$$

$$\tag{54}$$

2.5 Soluzione all'eq. delle onde in forma covariante

Risolviamo l'equazione 37 a pagina 3, supponendo $J^\mu = J^\mu(x)$, utilizzando una funzione di Green:

$$\Box_x D(x - x') = \delta^{(4)}(x - x') \tag{55}$$

$$z := x - x' \tag{56}$$

Passando ad uno spazio di Fourier si ha

$$D(k) = \frac{1}{k \cdot k} \tag{57}$$

$$D(z) = -\frac{1}{(2\pi)^4} \int dk D(k) e^{-ik \cdot x} \tag{58}$$

Risolvendo, si hanno due soluzioni:

$$D_{ritardata} = \frac{1}{2\pi} \Theta(x_0 - x'_0) \delta[(x - x')^2] \tag{59}$$

$$D_{anticipata} = \frac{1}{2\pi} \Theta(x'_0 - x_0) \delta[(x - x')^2] \tag{60}$$

3 Moving charges

Un po' di notazione

x^μ	osservatore
r^μ	carica in moto
R	distanza fra osservatore e carica
$\hat{\mathbf{n}}$	versore dalla carica all'osservatore

Posso scrivere il quadrivettore delle sorgenti per una carica in moto come:

$$J^\mu = qc \int d\tau u^\mu(\tau) \delta^{(4)}(x - r(\tau)) \tag{61}$$

$$u^\mu := \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix} \qquad r(t) := \begin{pmatrix} ct \\ r(t) \end{pmatrix} \tag{62}$$

3.1 Lienerd-Wichert

Partiamo trovando i potenziali

$$A^\mu(x) = \frac{4\pi}{c} \int d^4x' D_r(x - x') J^\mu(x') \tag{63}$$

Sostituendo l'eq 61 a pagina 4, si ottiene

$$A^\mu(x) = 2q \int d\tau u^\mu(\tau) \Theta(x_0 - r_0(\tau)) \delta([x - r(\tau)]^2) \tag{64}$$

Considering the properties of the delta, and that $\delta([x - r(\tau)])$ implies that only the points on the trajectory that lie on the backward light cone starting from x^μ can contribute to the potential (and also that $x_0 - r_0(\tau_0) = |\mathbf{x} - \mathbf{r}(\tau)|$), we have:

$$A^\mu(x) = \left. \frac{q u^\mu(\tau)}{u^\nu(\tau)(x - r(\tau))_\nu} \right|_{\tau=\tau_0} \tag{65}$$

$$\Phi(\mathbf{x}, t) = \left. \frac{q}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})R} \right|_{\tau=\tau_0}$$

$$A(\mathbf{x}, t) = \left. \frac{q\boldsymbol{\beta}}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})R} \right|_{\tau=\tau_0} \tag{66}$$

$$\text{con } \tau_0 \text{ definito da } (x - r(\tau_0))^2 = 0 \tag{67}$$

Tramite derivazione di 64, si trova il tensore del campo EM

$$F^{\mu\nu} = \frac{e}{u \cdot (x - r)} \frac{d}{d\tau} \left[\frac{(x - r)^\mu u^\nu - (x - r)^\nu u^\nu}{u \cdot (x - r)} \right] \Bigg|_{\tau=\tau_0} \tag{68}$$

E di conseguenza i campi

$$\mathbf{E} = q \left. \frac{\mathbf{n} - \beta}{\gamma^2 (1 - \mathbf{n} \cdot \beta)^3 R^2} \right|_{\tau=\tau_0} + \left. \frac{q}{c} \frac{\mathbf{n} \times (\mathbf{n} - \beta) \times \dot{\beta}}{(1 - \mathbf{n} \cdot \beta)^3 R} \right|_{\tau=\tau_0} \tag{69}$$

$$\mathbf{E} = \text{c. velocita}'(\propto \frac{1}{r^2}) + \text{c. accelerazione}(\propto \frac{1}{r})$$

$$\mathbf{B} = [\mathbf{n}]_{rit} \times \mathbf{E} \tag{70}$$

Sappiamo che

$$\frac{dP}{d\Omega} = R^2 \ \mathbf{S} \cdot \mathbf{n} \tag{71}$$

Considerando solo il campo di radiazione, e mettendoci nel caso non relativistico, otteniamo le *formule di Larmor* non relativistica

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \dot{v}^2 \sin^2 \theta \tag{72}$$

$$P = \frac{2}{3} \frac{q^2}{c^2} |\dot{v}|^2 \tag{73}$$

E le equivalenti relativistiche

$$P = \frac{2}{3} \frac{q^2}{c^2} \frac{dp_\mu}{d\tau} \frac{dp^\mu}{d\tau} = \frac{2}{3} \frac{q^2}{c} \gamma^6 [\dot{\beta}^2 - (\beta \times \dot{\beta})^2] \tag{74}$$

$$\frac{dP}{d\Omega}(t') = \frac{q^2}{4\pi c} \frac{\left| \mathbf{n} \times (\mathbf{n} - \beta \times \dot{\beta}) \right|^2}{(1 - \mathbf{n} \cdot \beta)^5} \tag{75}$$

4 Notazione

X_μ	covariante
X^μ	controvariante

5 M.U.

$$1 \text{ eV} \approx 1.6 \cdot 10^{-19}\text{J} \tag{76}$$

5.1 Gaussian CGS

We set the unit for q such that

$$F = \frac{q_1 q_2}{d^2} \tag{77}$$

Thus the unit of charge is called *esu*, or *statcoulomb*

$$esu = \sqrt{\text{dyne} \cdot \text{cm}^2} = \sqrt{\text{g} \cdot \text{cm}^3}/\text{s} \tag{78}$$

Which results in

$$\frac{F}{L} = \frac{2}{c^2}\frac{I_1 I_2}{d} \tag{79}$$

We can convert a charge q from CGS to SI and viceversa by noting that the Coulomb force is the same in every system (for a more thourough explanation, see `http://www.rpi.edu/dept/phys/Courses/PHYS4210/S10/NotesOnUnits.pdf`)

$$q_C = \frac{q_{esu}}{10 \cdot c_{SI}} \tag{80}$$

Where q_C and q_{esu} are the "number" of the respective measure unit contained in the charge q (that wasn't veryclear, take a look at the link before)