Sets and functions

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Contents

Part I
Sets

1 Sets

Axiom 1. **(SF 01 01 603161)** Every set is an element.

Axiom 2. **(SF 01 01 617091)** Every element of any set is an element.

Let x, y, z denote sets. Let u, v, w denote elements.

Theorem 3. **(Russell)** If every class is a set then we have a contradiction.

Proof. Assume that every class is a set. Define $R = \{\text{set } x \mid x \notin x\}$. Then R is a set. Hence $R \in R$ iff $R \notin R$. Contradiction. $\square$

1.1 Subsets

Definition 4. A subset of x is a set y such that every element of y is an element of x.

Let $y \subseteq x$ stand for y is a subset of x. Let $y \subset x$ stand for $y \subseteq x$. Let a superset of x stand for a set y such that $x \subseteq y$. Let $y \supseteq x$ stand for y is a superset of x. Let $y \supset x$ stand for $y \supseteq x$.

Definition 5. A proper subset of x is a subset of x that is not equal to x.

Let $y \subsetneq x$ stand for x is a proper subset of x. Let a proper superset of x stand for a set y such that $x \subsetneq y$. Let $y \supsetneq x$ stand for y is a proper superset of x.

Proposition 6. **(SF 01 01 375611)** $x \subseteq x$.

Proposition 7. **(SF 01 01 726162)** If $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$.

1.2 Set extensionality

Axiom 8. **(Set extensionality)** If $x \subseteq y$ and $y \subseteq x$ then $x = y$.

1.3 Separation

Axiom 9. **(Separation)** Let C be a class and x be a set. Assume that every element of C is contained in x. Then C is a set.

1.4 Set existence

Axiom 10. **(Set existence)** There exists a set.

1.5 The empty set

Definition 11. x is empty iff x has no elements.

Let x is nonempty stand for x is not empty.

Lemma 12. There exists an empty set.

Proof. Define $C = \{\text{element } u \mid \text{contradiction}\}$. Take a set x (by set existence). Then every element of C is contained in x. Hence C is a set (by separation). C has no element. Hence the thesis. □

Lemma 13. If x and y are empty then $x = y$.

Proof. Assume that x and y are empty. Then every element of x is an element of y and every element of y is an element of x. Hence $x \subseteq y$ and $y \subseteq x$. Thus $x = y$. □

Definition 14. $\emptyset$ is the empty set.

Let $\{\}$ stand for $\emptyset$. Let the empty set stand for $\emptyset$.

Proposition 15. **(SF 01 01 656396)** $\emptyset$ is a subset of every set.

Proof. Let x be a set. Then every element of $\emptyset$ is an element of x. Indeed $\emptyset$ has no element. Hence $\emptyset \subseteq x$. □

1.6 Pairing

Axiom 16. **(Pairing)** There exists a set z such that $z = \{\text{element } w \mid w = u \text{ or } w = v\}$.

Definition 17. $\{u, v\}$ is the set z such that $z = \{\text{element } w \mid w = u \text{ or } w = v\}$.

Let the unordered pair of u and v stand for $\{u, v\}$.

Lemma 18. There exists a set z such that $z = \{\text{element } w \mid w = u\}$.

Proof. Take $z = \{u, u\}$. Then $z = \{\text{element } w \mid w = u\}$. □

Definition 19. $\{u\}$ is the set z such that $z = \{\text{element } w \mid w = u\}$.

Let the singleton set of u stand for $\{u\}$.

Definition 20. A singleton set is a set x such that $x = \{u\}$ for some element u.

1.7 Set-systems

Definition 21. A system of sets is a set X such that every element of X is a set.

Let X, Y, Z denote systems of sets.

Definition 22. A system of nonempty sets is a system of sets X such that every element of X is nonempty.

Proposition 23. **(SF 01 01 261697)** $\{x\}$ is a system of sets.

Proposition 24. **(SF 01 01 176500)** $\{x, y\}$ is a system of sets.

Definition 25. A system of subsets of x is a set X such that every element of X is a subset of x.

Proposition 26. **(SF 01 01 366869)** Every system of subsets of x is a system of sets.

1.8 Intersections

Lemma 27. Let x be a nonempty system of sets. Then there exists a set z such that $z = \{$element $u \mid u$ is an element of every element of $x\}$.

Proof. Take an element y of x. Then y is a set. (1) Define $z = \{$element $u \mid u$ is an element of every element of $x\}$. Every element of z is contained in y. Hence z is a set. Then we have the thesis (by 1). $\square$

Definition 28. Let x be a nonempty system of sets. $\bigcap x$ is the set z such that $z = \{$element $u \mid u$ is an element of every element of $x\}$.

Let the intersection over x stand for $\bigcap x$.

Lemma 29. Let x, y be sets. Then there exists a set z such that $z = \{$element $u \mid u \in x$ and $u \in y\}$.

Proof. Take $z = \bigcap\{x, y\}$. Then $z = \{$element $u \mid u$ is an element of every element of $\{x, y\}\}$. Hence $z = \{$element $u \mid u \in x$ and $u \in y\}$. $\square$

Definition 30. $x \cap y$ is the set z such that $z = \{$element $u \mid u \in x$ and $u \in y\}$.

Let the intersection of x and y stand for $x \cap y$.

Proposition 31. **(SF 01 01 220491)** $\bigcap\{x, y\} = x \cap y$.

Proof. Let us show that $\bigcap\{x, y\} \subseteq x \cap y$. Let $u \in \bigcap\{x, y\}$. Then u is an element of every element of $\{x, y\}$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$. End.

Let us show that $x \cap y \subseteq \bigcap\{x, y\}$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence u is an element of every element of $\{x, y\}$. Thus $u \in \bigcap\{x, y\}$. End. $\square$

Corollary 32. **(SF 01 01 485484)** $\bigcap\{x\} = x$.

Proof. $\bigcap\{x\} = \bigcap\{x, x\} = x \cap x = x$. □

Proposition 33. **(SF 01 01 517087)** Let x be a nonempty system of sets. Then $y \subseteq \bigcap x$ iff y is a subset of every element of x.

Proof. Case $y \subseteq \bigcap x$. Let z be an element of x. Let $u \in y$. Then $u \in \bigcap x$. Hence $u \in z$. End.

Case y is a subset of every element of x. Let $u \in y$. Then $u \in z$ for all sets z such that $z \in x$. Hence $u \in \bigcap x$. End. □

Definition 34. x and y are disjoint iff $x \cap y = \emptyset$.

Proposition 35. **(SF 01 01 300845)** If x and y are disjoint then y and x are disjoint.

Proof. Assume that x and y are disjoint. Then $x \cap y$ is empty. Hence there is no element u such that $u \in x$ and $u \in y$. Thus $y \cap x$ is empty. Therefore y and x are disjoint. □

1.9 Unions

Axiom 36. **(Union)** Let x be a system of sets. Then there exists a set z such that $z = \{\text{element } u \mid u \text{ is an element of some element of } x\}$.

Definition 37. Let x be a system of sets. $\bigcup x$ is the set z such that $z = \{\text{element } u \mid u \text{ is an element of some element of } x\}$.

Let the union over x stand for $\bigcup x$.

Lemma 38. Let x, y be sets. Then there exists a set z such that $z = \{\text{element } u \mid u \in x \text{ or } u \in y\}$.

Proof. Take $z = \bigcup\{x, y\}$. Then $z = \{\text{element } u \mid u \text{ is an element of some element of } \{x, y\}\}$. Hence $z = \{\text{element } u \mid u \in x \text{ or } u \in y\}$. □

Definition 39. $x \cup y$ is the set z such that $z = \{\text{element } w \mid w \in x \text{ or } w \in y\}$.

Let the union of x and y stand for $x \cup y$.

Proposition 40. **(SF 01 01 519005)** $\bigcup\{x, y\} = x \cup y$.

Proof. Let us show that $\bigcup\{x, y\} \subseteq x \cup y$. Let $u \in \bigcup\{x, y\}$. Then u is an element of some element of $\{x, y\}$. Hence $u \in x$ or $u \in y$. Thus $u \in x \cup y$. End.

Let us show that $x \cup y \subseteq \bigcup\{x, y\}$. Let $u \in x \cup y$. Then $u \in x$ or $u \in y$. Hence u is an element of some element of $\{x, y\}$. Thus $u \in \bigcup\{x, y\}$. End. □

Corollary 41. (SF 01 01 820534) $\bigcup\{x\} = x$.

Proof. Hence $\bigcup\{x\} = \bigcup\{x, x\} = x \cup x = x$. □

Proposition 42. (SF 01 01 251673) Let x be a system of sets. Then $\bigcup x \subseteq y$ iff every element of x is a subset of y.

Proof. Case $\bigcup x \subseteq y$. Let z be an element of x. Let $u \in z$. Then u is an element of some element of x. Hence $u \in \bigcup x$. Thus $u \in y$. End.

Case every element of x is a subset of y. Let $u \in \bigcup x$. Take a set z such that $z \in x$ and $u \in z$. Then z is a subset of y. Hence $u \in y$. End. □

Proposition 43. (SF 01 01 675114) $\bigcup\emptyset = \emptyset$.

Proof. $\emptyset$ has no elements. Hence there is no $x \in \emptyset$ that has an element. Thus $\bigcup\emptyset$ is empty. Therefore $\bigcup\emptyset = \emptyset$. □

1.10 Complements

Lemma 44. Let x, y be sets. There exists a set z such that $z = \{\text{element } w \mid w \in x \text{ and } w \notin y\}$.

Proof. Define $z = \{\text{element } w \mid w \in x \text{ and } w \notin y\}$. Then every element of z is contained in x. Hence z is a set (by separation). □

Definition 45. $x \setminus y$ is the set such that $x \setminus y = \{\text{element } w \mid w \in x \text{ and } w \notin y\}$.

Let the complement of y in x stand for $x \setminus y$.

1.11 Computation laws

Proposition 46. (SF 01 01 830899)

$$x \cup y = y \cup x.$$

Proof. Let us show that $x \cup y \subseteq y \cup x$. Let $u \in x \cup y$. Then $u \in x$ or $u \in y$. Hence $u \in y$ or $u \in x$. Thus $u \in y \cup x$. End.

Let us show that $y \cup x \subseteq x \cup y$. Let $u \in y \cup x$. Then $u \in y$ or $u \in x$. Hence $u \in x$ or $u \in y$. Thus $u \in x \cup y$. End. □

Proposition 47. (SF 01 01 728823)

$$x \cap y = y \cap x.$$

Proof. Let us show that $x \cap y \subseteq y \cap x$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \in y$ and $u \in x$. Thus $u \in y \cap x$. End.

Let us show that $y \cap x \subseteq x \cap y$. Let $u \in y \cap x$. Then $u \in y$ and $u \in x$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$. End. □

Proposition 48. (SF 01 01 665069)

$$((x \cup y) \cup z) = x \cup (y \cup z).$$

Proof. Let us show that $((x \cup y) \cup z) \subseteq x \cup (y \cup z)$. Let $u \in (x \cup y) \cup z$. Then $u \in x \cup y$ or $u \in z$. Hence $u \in x$ or $u \in y$ or $u \in z$. Thus $u \in x$ or $u \in (y \cup z)$. Therefore $u \in x \cup (y \cup z)$. End.

Let us show that $x \cup (y \cup z) \subseteq (x \cup y) \cup z$. Let $u \in x \cup (y \cup z)$. Then $u \in x$ or $u \in y \cup z$. Hence $u \in x$ or $u \in y$ or $u \in z$. Thus $u \in x \cup y$ or $u \in z$. Therefore $u \in (x \cup y) \cup z$. End. □

Proposition 49. (SF 01 01 368359)

$$((x \cap y) \cap z) = x \cap (y \cap z).$$

Proof. Let us show that $((x \cap y) \cap z) \subseteq x \cap (y \cap z)$. Let $u \in (x \cap y) \cap z$. Then $u \in x \cap y$ and $u \in z$. Hence $u \in x$ and $u \in y$ and $u \in z$. Thus $u \in x$ and $u \in (y \cap z)$. Therefore$u \in x \cap (y \cap z)$. End.

Let us show that $x \cap (y \cap z) \subseteq (x \cap y) \cap z$. Let $u \in x \cap (y \cap z)$. Then $u \in x$ and $u \in y \cap z$. Hence $u \in x$ and $u \in y$ and $u \in z$. Thus $u \in x \cap y$ and $u \in z$. Therefore$u \in (x \cap y) \cap z$. End. □

Proposition 50. (SF 01 01 106755)

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z).$$

Proof. Let us show that $x \cap (y \cup z) \subseteq (x \cap y) \cup (x \cap z)$. Let $u \in x \cap (y \cup z)$. Then $u \in x$ and $u \in y \cup z$. Hence $u \in x$ and ($u \in y$ or $u \in z$). Thus ($u \in x$ and $u \in y$) or ($u \in x$ and $u \in z$). Therefore $u \in x \cap y$ or $u \in x \cap z$. Hence $u \in (x \cap y) \cup (x \cap z)$. End.

Let us show that $((x \cap y) \cup (x \cap z)) \subseteq x \cap (y \cup z)$. Let $u \in (x \cap y) \cup (x \cap z)$. Then $u \in x \cap y$ or $u \in x \cap z$. Hence ($u \in x$ and $u \in y$) or ($u \in x$ and $u \in z$). Thus $u \in x$ and ($u \in y$ or $u \in z$). Therefore $u \in x$ and $u \in y \cup z$. Hence$u \in x \cap (y \cup z)$. End. □

Proposition 51. (SF 01 01 836290)

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z).$$

Proof. Let us show that $x \cup (y \cap z) \subseteq (x \cup y) \cap (x \cup z)$. Let $u \in x \cup (y \cap z)$. Then $u \in x$ or $u \in y \cap z$. Hence $u \in x$ or ($u \in y$ and $u \in z$). Thus ($u \in x$ or $u \in y$) and ($u \in \text{x}$ or $u \in z$). Therefore $u \in x \cup y$ and $u \in x \cup z$. Hence $u \in (x \cup y) \cap (x \cup z)$. End.

Let us show that $((x \cup y) \cap (x \cup z)) \subseteq x \cup (y \cap z)$. Let $u \in (x \cup y) \cap (x \cup z)$. Then $u \in x \cup y$ and $u \in x \cup z$. Hence ($u \in x$ or $u \in y$) and ($u \in x$ or $u \in z$). Thus $u \in x$ or ($u \in y$ and $u \in z$). Therefore $u \in x$ or $u \in y \cap z$. Hence $u \in x \cup (y \cap z)$. End. □

Proposition 52. (SF 01 01 496190)

$$x \cup x = x.$$

Proof. $x \cup x = \{\text{element } u \mid u \in x \text{ or } u \in x\}$. Hence $x \cup x = \{\text{element } u \mid u \in x\}$. Thus $x \cup x = x$. □

Proposition 53. (SF 01 01 783425)

$$x \cap x = x.$$

Proof. $x \cap x = \{\text{element } u \mid u \in x \text{ and } u \in x\}$. Hence $x \cap x = \{\text{element } u \mid u \in x\}$. Thus $x \cap x = x$. □

Proposition 54. (SF 01 01 339365)

$$x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z).$$

Proof. Let us show that $x \setminus (y \cap z) \subseteq (x \setminus y) \cup (x \setminus z)$. Let $u \in x \setminus (y \cap z)$. Then $u \in x$ and $u \notin y \cap z$. Hence it is wrong that ($u \in y$ and $u \in z$). Thus $u \notin y$ or $u \notin z$. Therefore $u \in x$ and ($u \notin y$ or $u \notin z$). Then ($u \in x$ and $u \notin y$) or ($u \in x$ and $u \notin z$). Hence $u \in x \setminus y$ or $u \in x \setminus z$. Thus $u \in (x \setminus y) \cup (x \setminus z)$. End.

Let us show that $((x \setminus y) \cup (x \setminus z)) \subseteq x \setminus (y \cap z)$. Let $u \in (x \setminus y) \cup (x \setminus z)$. Then $u \in x \setminus y$ or $u \in x \setminus z$. Hence ($u \in x$ and $u \notin y$) or ($u \in x$ and $u \notin z$). Thus $u \in x$ and ($u \notin y$ or $u \notin z$). Therefore $u \in x$ and not ($u \in y$ and $u \in z$). Then $u \in x$ and not $u \in y \cap z$. Hence $u \in x \setminus (y \cap z)$. End. □

Proposition 55. (SF 01 01 403962)

$$x \setminus (y \cup z) = (x \setminus y) \cap (x \setminus z).$$

Proof. Let us show that $x \setminus (y \cup z) \subseteq (x \setminus y) \cap (x \setminus z)$. Let $u \in x \setminus (y \cup z)$. Then $u \in x$ and $u \notin y \cup z$. Hence it is wrong that ($u \in y$ or $u \in z$). Thus $u \notin y$ and $u \notin z$. Therefore $u \in x$ and ($u \notin y$ and $u \notin z$). Then ($u \in x$ and $u \notin y$) and ($u \in x$ and $u \notin z$). Hence $u \in x \setminus y$ and $u \in x \setminus z$. Thus $u \in (x \setminus y) \cap (x \setminus z)$. End.

Let us show that $((x \setminus y) \cap (x \setminus z)) \subseteq x \setminus (y \cup z)$. Let $u \in (x \setminus y) \cap (x \setminus z)$. Then $u \in x \setminus y$ and $u \in x \setminus z$. Hence ($u \in x$ and $u \notin y$) and ($u \in x$ and $u \notin z$). Thus $u \in x$ and ($u \notin y$ and $u \notin z$). Therefore $u \in x$ and not ($u \in y$ or $u \in z$). Then $u \in x$ and not $u \in y \cup z$. Hence $u \in x \setminus (y \cup z)$. End. □

Proposition 56. (SF 01 01 628970)

$$x \subseteq x \cup y.$$

Proof. Let $u \in x$. Then $u \in x$ or $u \in y$. Hence $u \in x \cup y$. ☐

Proposition 57. (SF 01 01 368515)

$$x \cap y \subseteq x.$$

Proof. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \in x$. ☐

Proposition 58. (SF 01 01 591527)

$$x \subseteq y \iff x \cup y = y.$$

Proof. Case $x \subseteq y$.

Let us show that $x \cup y \subseteq y$. Let $u \in x \cup y$. Then $u \in x$ or $u \in y$. If $u \in x$ then $u \in y$. Hence $u \in y$. End.

Let us show that $y \subseteq x \cup y$. Let $u \in y$. Then $u \in x$ or $u \in y$. Hence $u \in x \cup y$. End. End.

Case $x \cup y = y$. Let $u \in x$. Then $u \in x$ or $u \in y$. Hence $u \in x \cup y = y$. End. ☐

Proposition 59. (SF 01 01 681535)

$$x \subseteq y \iff x \cap y = x.$$

Proof. Case $x \subseteq y$.

Let us show that $x \cap y \subseteq x$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \in x$. End.

Let us show that $x \subseteq x \cap y$. Let $u \in x$. Then $u \in y$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$. End. End.

Case $x \cap y = x$. Let $u \in x$. Then $u \in x \cap y$. Hence $u \in x$ and $u \in y$. Thus $u \in y$. End. ☐

Proposition 60. (SF 01 01 402739)

$$x \setminus x = \emptyset.$$

Proof. $x \setminus x$ has no elements. Indeed $x \setminus x = \{\text{element } u \mid u \in x \text{ and } u \notin x\}$. Hence the thesis. ☐

Proposition 61. (SF 01 01 661163)

$$x \setminus \emptyset = x.$$

Proof. $x \setminus \emptyset = \{\text{element } u \mid u \in x \text{ and } u \notin \emptyset\}$. No element is an element of $\emptyset$. Hence $x \setminus \emptyset = \{\text{element } u \mid u \in x\}$. Then we have the thesis. □

Proposition 62. (SF 01 01 408438)

$$x \setminus (x \setminus y) = x \cap y.$$

Proof. Let us show that $x \setminus (x \setminus y) \subseteq x \cap y$. Let $u \in x \setminus (x \setminus y)$. Then $u \in x$ and $u \notin x \setminus y$. Hence $u \notin x$ or $u \in y$. Thus $u \in y$. Therefore $u \in x \cap y$. End.

Let us show that $x \cap y \subseteq x \setminus (x \setminus y)$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \notin x$ or $u \in y$. Thus $u \notin x \setminus y$. Therefore $u \in x \setminus (x \setminus y)$. End. □

Proposition 63. (SF 01 01 185130)

$$y \subseteq x \iff x \setminus (x \setminus y) = y.$$

Proof. Case $y \subseteq x$. Obvious.

Case $x \setminus (x \setminus y) = y$. Then every element of y is an element of $x \setminus (x \setminus y)$. Thus every element of y is an element of x. Then we have the thesis. End. □

Proposition 64. (SF 01 01 878796)

$$x \cap (y \setminus z) = (x \cap y) \setminus (x \cap z).$$

Proof. Let us show that $x \cap (y \setminus z) \subseteq (x \cap y) \setminus (x \cap z)$. Let $u \in x \cap (y \setminus z)$. Then $u \in x$ and $u \in y \setminus z$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$ and $u \notin z$. Therefore $u \notin x \cap z$. Then we have $u \in (x \cap y) \setminus (x \cap z)$. End.

Let us show that $((x \cap y) \setminus (x \cap z)) \subseteq x \cap (y \setminus z)$. Let $u \in (x \cap y) \setminus (x \cap z)$. Then $u \in x$ and $u \in y$. $u \notin x \cap z$. Hence $u \notin z$. Thus $u \in y \setminus z$. Therefore $u \in x \cap (y \setminus z)$. End. □

2 The powerset

Axiom 65. There exists a set z such that $z = \{\text{set } y \mid y \subseteq x\}$.

Definition 66. $\mathcal{P}(x)$ is the set z such that $z = \{\text{set } y \mid y \subseteq x\}$.

Let the powerset of x stand for $\mathcal{P}(x)$.

Proposition 67. (SF 01 02 481481) $\emptyset$ and x are elements of $\mathcal{P}(x)$.

Proof. We have $\emptyset, x \subseteq x$. Hence the thesis. □

Corollary 68. (SF 01 02 671341) $\mathcal{P}(x)$ is nonempty.

Proposition 69. **(SF 01 02 833606)** $\mathcal{P}(x)$ is a system of subsets of x.

Proposition 70. **(SF 01 02 706547)** $\bigcup \mathcal{P}(x) = x$.

Proof. Every element of $\mathcal{P}(x)$ is a subset of x. Hence $\bigcup \mathcal{P}(x) \subseteq x$.

We have $x \in \mathcal{P}(x)$. Hence every element of x is an element of some element of $\mathcal{P}(x)$. Thus every element of x belongs to $\bigcup \mathcal{P}(x)$. Therefore $x \subseteq \bigcup \mathcal{P}(x)$.

Then we have the thesis. ☐

Proposition 71. **(SF 01 02 818609)** $\bigcap \mathcal{P}(x) = \emptyset$.

Proof. We have $\emptyset \in \mathcal{P}(x)$. Hence every element of $\bigcap \mathcal{P}(x)$ is an element of $\emptyset$. Thus $\bigcap \mathcal{P}(x)$ is empty. Therefore $\bigcap \mathcal{P}(x) = \emptyset$. ☐

3 The axiom of regularity

Axiom 72. **(Regularity)** Every nonempty set x that contains some set contains some set y such that x and y are disjoint.

Proposition 73. **(SF 01 03 877283)** No set x is an element of x.

Proof. Assume the contrary. Take a set x such that $x \in x$. We can take an element y of $\{x\}$ such that $\{x\}$ and y are disjoint (by regularity). Indeed $\{x\}$ contains some set. Then $y = x$. Hence $\{x\}$ and x are disjoint. Contradiction. Indeed $x \in \{x\}$ and $x \in x$. ☐

Corollary 74. **(SF 01 03 722484)** There is no set that contains every set.

Proof. Assume the contrary. Take a set V that contains every set. Then V is an element of V. Contradiction. ☐

Proposition 75. **(SF 01 03 512352)** There exist no sets x, y such that $x \in y$ and $y \in x$.

Proof. Assume the contrary. Take sets x, y such that $x \in y$ and $y \in x$. Consider an element z of $\{x, y\}$ such that $\{x, y\}$ and z are disjoint (by regularity). Indeed $\{x, y\}$ contains some set. We have $z = x$ or $z = y$.

Case $z = x$. Then x and $\{x, y\}$ are disjoint. Hence $y \notin x$. Contradiction. End.

Case $z = y$. Then y and $\{x, y\}$ are disjoint. Hence $x \notin y$. Contradiction. End. ☐

4 The symmetric difference

4.1 Definition

Definition 76. $x \triangle y = (x \cup y) \setminus (x \cap y)$.

Let the symmetric difference of x and y stand for $x \triangle y$.

Lemma 77. $x \triangle y$ is a set.

Proof. x and y are sets. Hence $x \cup y$ and $x \cap y$ are sets. Thus $(x \cup y) \setminus (x \cap y)$ is a set. Therefore $x \triangle y$ is a set. □

Proposition 78. **(SF 01 04 470605)** $x \triangle y = (x \setminus y) \cup (y \setminus x)$.

Proof. Let us show that $x \triangle y \subseteq (x \setminus y) \cup (y \setminus x)$. Let $u \in x \triangle y$. Then $u \in x \cup y$ and $u \notin x \cap y$. Hence ($u \in x$ or $u \in y$) and not ($u \in x$ and $u \in y$). Thus ($u \in x$ or $u \in y$) and ($u \notin x$ or $u \notin y$). Therefore if $u \in x$ then $u \notin y$. If $u \in y$ then $u \notin x$. Then we have ($u \in x$ and $u \notin y$) or ($u \in y$ and $u \notin x$). Hence $u \in x \setminus y$ or $u \in y \setminus x$. Thus $u \in (x \setminus y) \cup (y \setminus x)$. End.

Let us show that $((x \setminus y) \cup (y \setminus x)) \subseteq x \triangle y$. Let $u \in (x \setminus y) \cup (y \setminus x)$. Then ($u \in x$ and $u \notin y$) or ($u \in y$ and $u \notin x$). If $u \in x$ and $u \notin y$ then $u \in x \cup y$ and $u \notin x \cap y$. If $u \in y$ and $u \notin x$ then $u \in x \cup y$ and $u \notin x \cap y$. Hence $u \in x \cup y$ and $u \notin x \cap y$. Thus $u \in (x \cup y) \setminus (x \cap y) = x \triangle y$. End. □

4.2 Computation laws

Proposition 79. **(SF 01 04 688675)**

$$x \triangle y = y \triangle x.$$

Proof. $x \triangle y = (x \cup y) \setminus (x \cap y) = (y \cup x) \setminus (y \cap x) = y \triangle x$. □

Proposition 80. **(SF 01 04 606646)**

$$((x \triangle y) \triangle z) = x \triangle (y \triangle z).$$

Proof. Take $A = (((x \setminus y) \cup (y \setminus x)) \setminus z) \cup (z \setminus ((x \setminus y) \cup (y \setminus x)))$.

Take $B = (x \setminus ((y \setminus z) \cup (z \setminus y))) \cup (((y \setminus z) \cup (z \setminus y)) \setminus x)$.

We have $x \triangle y = (x \setminus y) \cup (y \setminus x)$ and $y \triangle z = (y \setminus z) \cup (z \setminus y)$. Hence $(x \triangle y) \triangle z = A$ and $x \triangle (y \triangle z) = B$.

Let us show that (A) $A \subseteq B$. Let $u \in A$.

(A 1) Case $u \in ((x \setminus y) \cup (y \setminus x)) \setminus z$. Then $u \notin z$.

(A 1a) Case $u \in x \setminus y$. Then $u \notin y \setminus z$ and $u \notin z \setminus y$. $u \in x$. Hence $u \in x \setminus ((y \setminus z) \cup (z \setminus y))$. Thus $u \in B$. End.

(A 1b) Case $u \in y \setminus x$. Then $u \in y \setminus z$. Hence $u \in (y \setminus z) \cup (z \setminus y)$. $u \notin x$. Thus $u \in ((y \setminus z) \cup (z \setminus y)) \setminus x$. Therefore $u \in B$. End. End.

(A 2) Case $u \in z \setminus ((x \setminus y) \cup (y \setminus x))$. Then $u \in z$. $u \notin x \setminus y$ and $u \notin y \setminus x$. Hence not ($u \in x \setminus y$ or $u \in y \setminus x$). Thus not (($u \in x$ and $u \notin y$) or ($u \in y$ and $u \notin x$)). Therefore ($u \notin x$ or $u \in y$) and ($u \notin y$ or $u \in x$).

(A 2a) Case $u \in x$. Then $u \in y$. Hence $u \notin (y \setminus z) \cup (z \setminus y)$. Thus $u \in x \setminus ((y \setminus z) \cup (z \setminus y))$. Therefore $u \in B$. End.

(A 2b) Case $u \notin x$. Then $u \notin y$. Hence $u \in z \setminus y$. Thus $u \in (y \setminus z) \cup (z \setminus y)$. Therefore $u \in ((y \setminus z) \cup (z \setminus y)) \setminus x$. Then we have $u \in B$. End. End. End.

Let us show that (B) $B \subseteq A$. Let $u \in B$.

(B 1) Case $u \in x \setminus ((y \setminus z) \cup (z \setminus y))$. Then $u \in x$. $u \notin y \setminus z$ and $u \notin z \setminus y$. Hence not ($u \in y \setminus z$ or $u \in z \setminus y$). Thus not (($u \in y$ and $u \notin z$) or ($u \in z$ and $u \notin y$)). Therefore ($u \notin y$ or $u \in z$) and ($u \notin z$ or $u \in y$).

(B 1a) Case $u \in y$. Then $u \in z$. $u \notin x \setminus y$ and $u \notin y \setminus x$. Hence $u \notin (x \setminus y) \cup (y \setminus x)$. Thus $u \in z \setminus ((x \setminus y) \cup (y \setminus x))$. Therefore $u \in A$. End.

(B 1b) Case $u \notin y$. Then $u \notin z$. $u \in x \setminus y$. Hence $u \in (x \setminus y) \cup (y \setminus x)$. Thus $u \in ((x \setminus y) \cup (y \setminus x)) \setminus z$. Therefore $u \in A$. End. End.

(B 2) Case $u \in ((y \setminus z) \cup (z \setminus y)) \setminus x$. Then $u \notin x$.

(B 2a) Case $u \in y \setminus z$. Then $u \in y \setminus x$. Hence $u \in (x \setminus y) \cup (y \setminus x)$. Thus $u \in ((x \setminus y) \cup (y \setminus x)) \setminus z$. Therefore $u \in A$. End.

(B 2b) Case $u \in z \setminus y$. Then $u \in z$. $u \notin x \setminus y$ and $u \notin y \setminus x$. Hence $u \notin (x \setminus y) \cup (y \setminus x)$. Thus $u \in z \setminus ((x \setminus y) \cup (y \setminus x))$. Therefore $u \in A$. End. End. End. □

Proposition 81. (SF 01 04 751668)

$$x \cap (y \triangle z) = (x \cap y) \triangle (x \cap z).$$

Proof. $x \cap (y \triangle z) = x \cap ((y \setminus z) \cup (z \setminus y)) = (x \cap (y \setminus z)) \cup (x \cap (z \setminus y))$.

$x \cap (y \setminus z) = (x \cap y) \setminus (x \cap z)$. $x \cap (z \setminus y) = (x \cap z) \setminus (x \cap y)$.

Hence $x \cap (y \triangle z) = ((x \cap y) \setminus (x \cap z)) \cup ((x \cap z) \setminus (x \cap y)) = (x \cap y) \triangle (x \cap z)$. □

Proposition 82. (SF 01 04 420961)

$$x \subseteq y \iff x \triangle y = y \setminus x.$$

Proof. Case $x \subseteq y$. Then $x \cup y = y$ and $x \cap y = x$. Hence the thesis. End.

Case $x \triangle y = y \setminus x$. Let $u \in x$. Then $u \notin y \setminus x$. Hence $u \notin x \triangle y$. Thus $u \notin x \cup y$ or $u \in x \cap y$. Indeed $x \triangle y = (x \cup y) \setminus (x \cap y)$. If $u \notin x \cup y$ then we have a contradiction. Therefore $u \in x \cap y$. Then we have the thesis. End. □

Proposition 83. **(SF 01 04 241267)**

$$x \triangle y = x \triangle z \iff y = z.$$

Proof. Case $x \triangle y = x \triangle z$.

Let us show that $y \subseteq z$. Let $u \in y$.

Case $u \in x$. Then $u \notin x \triangle y$. Hence $u \notin x \triangle z$. Therefore $u \in x \cap z$. Indeed $x \triangle z = (x \cup z) \setminus (x \cap z)$. Hence $u \in z$. End.

Case $u \notin x$. Then $u \in x \triangle y$. Indeed $u \in x \cup y$ and $u \notin x \cap y$. Hence $u \in x \triangle z$. Thus $u \in x \cup z$ and $u \notin x \cap z$. Therefore $u \in x$ or $u \in z$. Then we have the thesis. End. End.

Let us show that $z \subseteq y$. Let $u \in z$.

Case $u \in x$. Then $u \notin x \triangle z$. Hence $u \notin x \triangle y$. Therefore $u \in x \cap y$. Indeed $u \notin x \cup y$ or $u \in x \cap y$. Hence $u \in y$. End.

Case $u \notin x$. Then $u \in x \triangle z$. Indeed $u \in x \cup z$ and $u \notin x \cap z$. Hence $u \in x \triangle y$. Thus $u \in x \cup y$ and $u \notin x \cap y$. Therefore $u \in x$ or $u \in y$. Then we have the thesis. End. End. End. □

Proposition 84. **(SF 01 04 496712)**

$$x \triangle x = \emptyset.$$

Proof. $x \triangle x = (x \cup x) \setminus (x \cap x) = x \setminus x = \emptyset$. □

Proposition 85. **(SF 01 04 182395)**

$$x \triangle \emptyset = x.$$

Proof. $x \triangle \emptyset = (x \cup \emptyset) \setminus (x \cap \emptyset) = x \setminus \emptyset = x$. □

Proposition 86. **(SF 01 04 814558)**

$$x = y \iff x \triangle y = \emptyset.$$

Proof. Case $x = y$. Then $x \triangle y = (x \cup x) \setminus (x \cap x) = x \setminus x = \emptyset$. Hence the thesis. End.

Case $x \triangle y = \emptyset$. Then $(x \cup y) \setminus (x \cap y)$ is empty. Hence every element of $x \cup y$ is an element of $x \cap y$. Thus for all elements u if $u \in x$ or $u \in y$ then $u \in x$ and $u \in y$. Therefore every element of x is an element of y. Every element of y is an element of x. Then we have the thesis. End. □

5 Ordered pairs and Cartesian products

Let u', v', w' denote elements. Let x', y', z' denote sets.

5.1 Ordered pairs

Note that Naproche provides an built-in function symbol $(\cdot, \cdot)$, i.e. for any two objects a, b there is an object (a, b).

Axiom 87. $(u, v) = \{\{u\}, \{u, v\}\}$.

Proposition 88. **(SF 01 05 366682)** Let u, v be elements. Then (u, v) is an element.

Proof. $\{u\}$ and $\{u, v\}$ are elements. Hence $(u, v) = \{\{u\}, \{u, v\}\}$. Thus (u, v) is an element. □

Proposition 89. **(SF 01 05 270653)** If $(u, v) = (u', v')$ then $u = u'$ and $v = v'$.

Proof. Assume $(u, v) = (u', v')$. (1) Then $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$. Hence $(\{u\} = \{u'\}$ or $\{u\} = \{u', v'\})$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. Thus $(\{u\} = \{u'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}))$ or $(\{u\} = \{u', v'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}))$.

Case $\{u\} = \{u'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. We have $\{u\} = \{u'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = u' = v$. Hence $\{\{u\}, \{u, u\}\} = \{\{u\}, \{u, v'\}\}$ (by 1). Thus $\{\{u\}\} = \{\{u\}, \{u, v'\}\}$. Therefore $\{u\} = \{u, v'\}$. Consequently $v' = u = v$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End.

Case $\{u\} = \{u', v'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. We have $\{u\} = \{u', v'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = v = u'$. Hence $v = v'$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End. □

5.2 Cartesian products

Lemma 90. There exists a set z such that $z = \{(u, v) \mid u \in x \text{ and } v \in y\}$.

Proof. (1) Define $z = \{(u, v) \mid u \in x \text{ and } v \in y\}$. Take $z' = \mathcal{P}(\mathcal{P}(x \cup y))$. Then z' is a set.

Let us show that every element of z is contained in z'. Let $w \in z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and $v \in y$. Hence $\{u\}$ and $\{u, v\}$ are subsets of $x \cup y$. Thus $\{u\}$ and $\{u, v\}$ are elements of $\mathcal{P}(x \cup y)$. Therefore $w = \{\{u\}, \{u, v\}\} \subseteq \mathcal{P}(x \cup y)$. Consequently $w \in \mathcal{P}(\mathcal{P}(x \cup y)) = z'$. End.

Hence z is a set (by separation). Therefore the thesis (by 1). $\square$

Definition 91. $x \times y$ is the set z such that $z = \{(u, v) \mid u \in x \text{ and } v \in y\}$.

Let the Cartesian product of x and y stand for $x \times y$.

Proposition 92. **(SF 01 05 773790)** $(u, v) \in x \times y$ iff $u \in x$ and $v \in y$.

Proof. Case $(u, v) \in x \times y$. Assume $(u, v) \in x \times y$. We can take $u' \in x$ and $v' \in y$ such that $(u, v) = (u', v')$. Then $u = u'$ and $v = v'$. Hence $u \in x$ and $v \in y$. End.

Case $u \in x$ and $v \in y$. u and v are elements. Hence (u, v) is an element. Therefore $(u, v) \in x \times y$. Indeed $x \times y = \{(u', v') \mid u' \in x \text{ and } v' \in y\}$. End. $\square$

Proposition 93. **(SF 01 05 279635)** $x \times y$ is empty iff x is empty or y is empty.

Proof. Case $x \times y$ is empty. Assume that x and y are nonempty. Thus we can take an element u of x and an element v of y. Then (u, v) is an element of $x \times y$. Contradiction. End.

Case x is empty or y is empty. Assume that $x \times y$ is nonempty. Then we can take an element z of $x \times y$. Then $z = (u, v)$ for some $u \in x$ and some $v \in y$. Hence x and y are nonempty. Contradiction. End. $\square$

Proposition 94. **(SF 01 05 784919)** $\{u\} \times \{v\} = \{(u, v)\}$.

Proof. Let us show that $\{u\} \times \{v\} \subseteq \{(u, v)\}$. Let $w \in \{u\} \times \{v\}$. Take $a \in \{u\}$ and $b \in \{v\}$ such that $w = (a, b)$. We have $a = u$ and $b = v$. Hence $w = (u, v)$. Thus $w \in \{(u, v)\}$. End.

Let us show that $\{(u, v)\} \subseteq \{u\} \times \{v\}$. Let $w \in \{(u, v)\}$. Then $w = (u, v)$. We have $u \in \{u\}$ and $v \in \{v\}$. Hence $w \in \{u\} \times \{v\}$. End. $\square$

5.3 Computation laws

Proposition 95. **(SF 01 05 197314)**

$$x \subseteq y \implies x \times z \subseteq y \times z.$$

Proof. Assume $x \subseteq y$. Let $w \in x \times z$. Take $u \in x$ and $v \in z$ such that $w = (u, v)$. Then $u \in y$. Hence $(u, v) \in y \times z$. $\square$

Proposition 96. **(SF 01 05 238807)** Assume that x and x' are nonempty.

$$(x \times x') \subseteq (y \times y') \iff (x \subseteq y \text{ and } x' \subseteq y').$$

Proof. Case $(x \times x') \subseteq (y \times y')$. Let us show that for all $u \in x$ and all $v \in x'$ we have $u \in y$ and $v \in y'$. Let $u \in x$ and $v \in x'$. Then $(u, v) \in x \times x'$. Hence $(u, v) \in y \times y'$. Thus $u \in y$ and $v \in y'$. End. End.

Case $x \subseteq y$ and $x' \subseteq y'$. Let $w \in x \times x'$. Take $u \in x$ and $v \in x'$ such that $w = (u, v)$. Then $u \in y$ and $v \in y'$. Hence $(u, v) \in y \times y'$. End. ☐

Proposition 97. **(SF 01 05 138531)**

$$((x \cup y) \times z) = (x \times z) \cup (y \times z).$$

Proof. Let us show that $((x \cup y) \times z) \subseteq (x \times z) \cup (y \times z)$. Let $w \in (x \cup y) \times z$. Take $u \in x \cup y$ and $v \in z$ such that $w = (u, v)$. Then $u \in x$ or $u \in y$. If $u \in x$ then $w \in x \times z$ and if $u \in y$ then $w \in y \times z$. Hence $w \in x \times z$ or $w \in y \times z$. Thus $w \in (x \times z) \cup (y \times z)$. End.

Let us show that $((x \times z) \cup (y \times z)) \subseteq (x \cup y) \times z$. Let $w \in (x \times z) \cup (y \times z)$. Then $w \in x \times z$ or $w \in y \times z$. Take elements u, v such that $w = (u, v)$. Then $(u \in x$ or $u \in y)$ and $v \in z$. Hence $u \in x \cup y$. Thus $w \in (x \cup y) \times z$. End. ☐

Proposition 98. **(SF 01 05 575129)**

$$x \times (y \cup z) = (x \times y) \cup (x \times z).$$

Proof. Let us show that $x \times (y \cup z) \subseteq (x \times y) \cup (x \times z)$. Let $w \in x \times (y \cup z)$. Take $u \in x$ and $v \in y \cup z$ such that $w = (u, v)$. Then $v \in y$ or $v \in z$. Hence $w \in x \times y$ or $w \in x \times z$. Indeed if $v \in y$ then $w \in x \times y$ and if $v \in z$ then $w \in x \times z$. Thus $w \in (x \times y) \cup (x \times z)$. End.

Let us show that $((x \times y) \cup (x \times z)) \subseteq x \times (y \cup z)$. Let $w \in (x \times y) \cup (x \times z)$. Then $w \in x \times y$ or $w \in x \times z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and $(v \in y$ or $v \in z)$. Hence $w \in x \times (y \cup z)$. End. ☐

Proposition 99. **(SF 01 05 811990)**

$$((x \cap y) \times z) = (x \times z) \cap (y \times z).$$

Proof. Let us show that $((x \cap y) \times z) \subseteq (x \times z) \cap (y \times z)$. Let $w \in (x \cap y) \times z$. Take $u \in x \cap y$ and $v \in z$ such that $w = (u, v)$. Then $u \in x$ and $u \in y$. Hence $w \in x \times z$ and $w \in y \times z$. Thus $w \in (x \times z) \cap (y \times z)$. End.

Let us show that $((x \times z) \cap (y \times z)) \subseteq (x \cap y) \times z$. Let $w \in (x \times z) \cap (y \times z)$. Then $w \in x \times z$ and $w \in y \times z$. Take elements u, v such that $w = (u, v)$. Then $(u \in x$ and $u \in y)$ and $v \in z$. Hence $u \in x \cap y$. Thus $w \in (x \cap y) \times z$. End. ☐

Proposition 100. **(SF 01 05 427022)**

$$x \times (y \cap z) = (x \times y) \cap (x \times z).$$

Proof. Let us show that $x \times (y \cap z) \subseteq (x \times y) \cap (x \times z)$. Let $w \in x \times (y \cap z)$. Take $u \in x$ and $v \in y \cap z$ such that $w = (u, v)$. Then $v \in y$ and $v \in z$. Hence $w \in x \times y$ and $w \in x \times z$. Thus $w \in (x \times y) \cap (x \times z)$. End.

Let us show that $((x \times y) \cap (x \times z)) \subseteq x \times (y \cap z)$. Let $w \in (x \times y) \cap (x \times z)$. Then $w \in x \times y$ and $w \in x \times z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and ($v \in y$ and $v \in z$). Hence $w \in x \times (y \cap z)$. End. □

Proposition 101. **(SF 01 05 517847)**

$$((x \setminus y) \times z) = (x \times z) \setminus (y \times z).$$

Proof. Let us show that $((x \setminus y) \times z) \subseteq (x \times z) \setminus (y \times z)$. Let $w \in (x \setminus y) \times z$. Take $u \in x \setminus y$ and $v \in z$ such that $w = (u, v)$. Then $u \in x$ and $u \notin y$. Hence $w \in x \times z$ and $w \notin y \times z$. Thus $w \in (x \times z) \setminus (y \times z)$. End.

Let us show that $((x \times z) \setminus (y \times z)) \subseteq (x \setminus y) \times z$. Let $w \in (x \times z) \setminus (y \times z)$. Then $w \in x \times z$ and $w \notin y \times z$. Take $u \in x$ and $v \in z$ such that $w = (u, v)$. Then $u \notin y$. Indeed if $u \in y$ then $w \in y \times z$. Hence $u \in x \setminus y$. Thus $w \in (x \setminus y) \times z$. End. □

Proposition 102. **(SF 01 05 773842)**

$$x \times (y \setminus z) = (x \times y) \setminus (x \times z).$$

Proof. Let us show that $x \times (y \setminus z) \subseteq (x \times y) \setminus (x \times z)$. Let $w \in x \times (y \setminus z)$. Take $u \in x$ and $v \in y \setminus z$ such that $w = (u, v)$. Then $v \in y$ and $v \notin z$. Hence $w \in x \times y$ and $w \notin x \times z$. Thus $w \in (x \times y) \setminus (x \times z)$. End.

Let us show that $((x \times y) \setminus (x \times z)) \subseteq x \times (y \setminus z)$. Let $w \in (x \times y) \setminus (x \times z)$. Then $w \in x \times y$ and $w \notin x \times z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and ($v \in y$ and $v \notin z$). Hence $w \in x \times (y \setminus z)$. End. □

Proposition 103. **(SF 01 05 472623)** Assume that x and x' are nonempty or y and y' are nonempty.

$$(x \times x') = (y \times y') \iff (x = y \text{ and } x' = y').$$

Proof. Case $x \times x' = y \times y'$. Then x and x' are nonempty iff y and y' are nonempty.

Let us show that for all $u \in x$ and all $v \in x'$ we have $u \in y$ and $v \in y'$. Let $u \in x$ and $v \in x'$. Then $(u, v) \in x \times x'$. Hence we can take $w \in y \times y'$ such that $w = (u, v)$. Thus $u \in y$ and $v \in y'$. End.

Therefore $x \subseteq y$ and $x' \subseteq y'$. Indeed x and x' are nonempty.

Let us show that for all $u \in y$ and all $v \in y'$ we have $u \in x$ and $v \in x'$. Let $u \in y$ and $v \in y'$. Then $(u, v) \in y \times y'$. Hence we can take $w \in x \times x'$ such that $w = (u, v)$. Thus $(u, v) \in x \times x'$. End.

Therefore $y \subseteq x$ and $y' \subseteq x'$. Indeed y and y' are nonempty. End.

Case $x = y$ and $x' = y'$. Trivial. □

Proposition 104. (SF 01 05 261950)

$$((x \times y) \cap (x' \times y')) = (x \cap x') \times (y \cap y').$$

Proof. Let us show that $((x \times y) \cap (x' \times y')) \subseteq (x \cap x') \times (y \cap y')$. Let $w \in (x \times y) \cap (x' \times y')$. Then $w \in x \times y$ and $w \in x' \times y'$. Take elements u, v such that $w = (u, v)$. Then $u \in x, x'$ and $v \in y, y'$. Hence $u \in x \cap x'$ and $v \in y \cap y'$. Thus $w \in (x \cap x') \times (y \cap y')$. End.

Let us show that $(x \cap x') \times (y \cap y') \subseteq (x \times y) \cap (x' \times y')$. Let $w \in (x \cap x') \times (y \cap y')$. Take elements u, v such that $w = (u, v)$. Then $u \in x \cap x'$ and $v \in y \cap y'$. Hence $u \in x, x'$ and $v \in y, y'$. Thus $w \in x \times y$ and $w \in x' \times y'$. Therefore $w \in (x \times y) \cap (x' \times y')$. End. □

Proposition 105. (SF 01 05 687547)

$$((x \times y) \cup (x' \times y')) \subseteq (x \cup x') \times (y \cup y').$$

Proof. Let $w \in (x \times y) \cup (x' \times y')$. Then $w \in x \times y$ or $w \in x' \times y'$. Take elements u, v such that $w = (u, v)$. Then ($u \in x$ or $u \in x'$) and ($v \in y$ or $v \in y'$). Hence $u \in x \cup x'$ and $v \in y \cup y'$. Thus $w \in (x \cup x') \times (y \cup y')$. □

Proposition 106. (SF 01 05 247770)

$$((x \times y) \setminus (x' \times y')) = (x \times (y \setminus y')) \cup ((x \setminus x') \times y).$$

Proof. Let us show that $((x \times y) \setminus (x' \times y')) \subseteq (x \times (y \setminus y')) \cup ((x \setminus x') \times y)$. Let $w \in (x \times y) \setminus (x' \times y')$. Then $w \in x \times y$ and $w \notin x' \times y'$. Take $u \in x$ and $v \in y$ such that $w = (u, v)$. Then it is wrong that $u \in x'$ and $v \in y'$. Hence $u \notin x'$ or $v \notin y'$. Thus $u \in x \setminus x'$ or $v \in y \setminus y'$. Therefore $w \in x \times (y \setminus y')$ or $w \in (x \setminus x') \times y$. Hence we have $w \in (x \times (y \setminus y')) \cup ((x \setminus x') \times y)$. End.

Let us show that $(x \times (y \setminus y')) \cup ((x \setminus x') \times y) \subseteq (x \times y) \setminus (x' \times y')$. Let $w \in (x \times (y \setminus y')) \cup ((x \setminus x') \times y)$. Then $w \in (x \times (y \setminus y'))$ or $w \in ((x \setminus x') \times y)$. Take elements u, v such that $w = (u, v)$. Then ($u \in x$ and $v \in y \setminus y'$) or ($u \in x \setminus x'$ and $v \in y$).

Case $u \in x$ and $v \in y \setminus y'$. Then $u \in x$ and $v \in y$. Hence $w \in x \times y$. We have $v \notin y'$. Thus $w \notin x' \times y'$. Therefore $w \in (x \times y) \setminus (x' \times y')$. End.

Case $u \in x \setminus x'$ and $v \in y$. Then $u \in x$ and $v \in y$. Hence $w \in x \times y$. We have $u \notin x'$. Thus $w \notin x' \times y'$. Therefore $w \in (x \times y) \setminus (x' \times y')$. End. End. □

6 The axiom of infinity

Axiom 107. **(infinity)** There exists a system of sets ω such that $\emptyset \in \omega$ and for all $x \in \omega$ we have $x \cup \{x\} \in \omega$.

Part II
Functions

7 Functions

7.1 Function axioms

Let u, v, w denote elements. Let x, y, z denote sets. Let f, g, h denote functions.
Let the domain of f stand for $\mathrm{dom}(f)$. Let the value of f at u stand for $f(u)$. Let f_u stand for $f(u)$.

Definition 108. A value of f is an object v such that $v = f(u)$ for some $u \in \mathrm{dom}(f)$.

Definition 109. A fixed point of f is an element u of the domain of f such that $f(u) = u$.

Note that the following two axioms are already hard-coded into Naproche.

Axiom 110. (Function extensionality) Let f, g be functions. If $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(u) = g(u)$ for all $u \in \mathrm{dom}(f)$ then $f = g$.

Axiom 111. (SF 02 01 459591) The domain of any function is a set.

Axiom 112. (SF 02 01 303112) Every value of f is an element.

Important note: The current version of Naproche[1] allows to define functions *manually* whose values are not elements. Hence such a manual definition will introduce an inconsistency to the theory. Fortunately the ATP is not able to deduce the existence of a function which contradicts this axiom. So as long as you do not define a non-element-valued function yourself, there will not be any consistency issues (assuming that our is consistent at all).

Axiom 113. (Replacement) Let f be a function. There exists a set y such that $y = \{f(u) \mid u \in \mathrm{dom}(f)\}$.

7.2 The range

Definition 114. Let f be a function. $\mathrm{range}(f)$ is the set y such that $y = \{f(u) \mid u \in \mathrm{dom}(f)\}$.

Let the range of f stand for $\mathrm{range}(f)$.

Proposition 115. (SF 02 01 324423) v is a value of f iff $v \in \mathrm{range}(f)$.

[1] Isabelle/Naproche 2021

Proof. Case v is a value of f. Take $u \in \mathrm{dom}(f)$ such that $v = f(u)$. v is an element. Hence $v \in \mathrm{range}(f)$. End.

Case $v \in \mathrm{range}(f)$. Then $v = f(u)$ for some $u \in \mathrm{dom}(f)$. Hence v is a value of f. End. $\square$

7.3 Functions between sets

Definition 116. A function of x is a function f such that $\mathrm{dom}(f) = x$.

Definition 117. A function to y is a function f such that $f(u) \in y$ for all $u \in \mathrm{dom}(f)$.

Let a function from x to y stand for a function f of x such that f is a function to y. Let $f : x \to y$ stand for f is a function from x to y.

Proposition 118. **(SF 02 01 694542)** Let f be a function from x to y. Then $\mathrm{range}(f) \subseteq y$.

Proof. Let $v \in \mathrm{range}(f)$. Take $u \in x$ such that $v = f(u)$. Then $v \in y$. $\square$

Definition 119. A function onto y is a function f such that $y = \mathrm{range}(f)$.

Definition 120. A function from x onto y is a function f of x such that f is a function onto y.

Let $f : x \twoheadrightarrow y$ stand for f is a function from x onto y.

Proposition 121. **(SF 02 01 677451)** f is a function onto $\mathrm{range}(f)$.

Proposition 122. **(SF 02 01 495468)** Let f be a function onto y. Then f is a function to y.

Proof. Let $u \in \mathrm{dom}(f)$. Then $f(u) \in \mathrm{range}(f)$. Hence $f(u) \in y$. $\square$

Definition 123. A function on x is a function from x to x.

Definition 124. f is one to one iff for all $u, v \in \mathrm{dom}(f)$ if $f(u) = f(v)$ then $u = v$.

Definition 125. A function into y is an one to one function to y.

Definition 126. A function from x into y is a function f of x such that f is a function into y.

Let $f : x \hookrightarrow y$ stand for f is a function from x into y.

Definition 127. A bijection between x and y is a one to one function f from x onto y.

Let a bijection from x to y stand for a bijection between x and y.

Proposition 128. **(SF 02 01 717927)** Let f be a function from x into y. Then f is a bijection between x and $\mathrm{range}(f)$.

Proof. f is one to one and f is a function from x onto range(f). Hence f is a bijection between x and range(f). □

Definition 129. A permutation of x is a bijection between x and x.

7.4 The identity function

Lemma 130. There is a function ι of x such that $\iota(u) = u$ for all $u \in x$.

Proof. Define $\iota(u) = u$ for $u \in x$. □

Definition 131. id$x$ is the function of x such that id$x(u) = u$ for all $u \in x$.

Let the identity function on x stand for id$x$.

Proposition 132. **(SF 02 01 848243)** id$x$ is a permutation of x.

Proof. (1) id$x$ is a function of x.

(2) id$x$ is a function onto x. Proof. Let $v \in x$. Then $v = \mathrm{id}x(v)$. Hence $v \in \mathrm{range}(\mathrm{id}x)$. Qed.

(3) id$x$ is a function into x. Proof. Let $v, v' \in x$. Assume $\mathrm{id}x(v) = \mathrm{id}x(v')$. Then $v = v'$. Qed. □

7.5 Constant functions

Lemma 133. Let x be a set and v be an element. There is a function c of x such that $c(u) = v$ for all $u \in x$.

Proof. Define $c(u) = v$ for $u \in x$. □

Definition 134. $\mathrm{const}_{x,v}$ is the function of x such that $\mathrm{const}_{x,v}(u) = v$ for all $u \in x$.

Let the constant function on x with value v stand for $\mathrm{const}_{x,v}$.

Proposition 135. **(SF 02 01 180417)** Assume $v \in y$. Then $\mathrm{const}_{x,v}$ is a function from x to y.

Proof. We have $\mathrm{dom}(\mathrm{const}_{x,v}) = x$ and $\mathrm{const}_{x,v}(u) = v$ for all $u \in x$. Hence $\mathrm{const}_{x,v}(u)$ is an element of y for all $u \in x$. Thus $\mathrm{range}(\mathrm{const}_{x,v}) \subseteq y$. Therefore $\mathrm{const}_{x,v}$ is a function from x to y. □

Definition 136. Let f be a function. f is constant iff there exists an object v such that $f(u) = v$ for all $u \in \mathrm{dom}(f)$.

Proposition 137. **(SF 02 01 359618)** $\mathrm{const}_{x,v}$ is constant.

Proof. We have $\mathrm{const}_{x,v}(u) = v$ for all $u \in x$. Hence the thesis. □

7.6 Composition

Lemma 138. Assume $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$. Then there is a function h such that $\mathrm{dom}(h) = \mathrm{dom}(f)$ and $h(u) = g(f(u))$ for all $u \in \mathrm{dom}(h)$.

Proof. Define $h(u) = g(f(u))$ for $u \in \mathrm{dom}(f)$. □

Definition 139. Assume $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$. $g \circ f$ is the function h such that $\mathrm{dom}(h) = \mathrm{dom}(f)$ and $h(u) = g(f(u))$ for all $u \in \mathrm{dom}(h)$.

Let the composition of g and f stand for $g \circ f$.

Proposition 140. **(SF 02 01 289732)** Let f be a function from x to y and g be a function from y to z. Then $g \circ f$ is a function from x to z.

Proof. (1) $g \circ f$ is a function of x. Indeed $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = x$.

(2) $\mathrm{range}(g \circ f) \subseteq z$. Proof. Let $w \in \mathrm{range}(g \circ f)$. Take $u \in x$ such that $(g \circ f)(u) = w$. Then $w = g(f(u))$. We have $f(u) \in y$. Hence $w \in z$. Qed. □

Proposition 141. **(SF 02 01 718601)** Let f be a function from x to y. Then $f \circ \mathrm{id}x = f = \mathrm{id}y \circ f$.

Proof. x is the domain of $f \circ \mathrm{id}x$ and the domain of f and the domain of $\mathrm{id}y \circ f$. $(f \circ \mathrm{id}x)(u) = f(\mathrm{id}x(u)) = f(u) = \mathrm{id}y(f(u)) = (\mathrm{id}y \circ f)(u)$ for all $u \in x$. Hence the thesis (by function extensionality). □

Proposition 142. **(SF 02 01 558108)** Let f be a function from x to y and v be an element. Then $\mathrm{const}_{y,v} \circ f = \mathrm{const}_{x,v}$.

Proof. We have $\mathrm{dom}(\mathrm{const}_{y,v} \circ f) = \mathrm{dom}(f) = x = \mathrm{dom}(\mathrm{const}_{x,v})$. $(\mathrm{const}_{y,v} \circ f)(u) = \mathrm{const}_{y,v}(f(u)) = v = \mathrm{const}_{x,v}(u)$ for all $u \in x$. Hence the thesis (by function extensionality). □

Proposition 143. **(SF 02 01 795869)** Let f be a function from y to z and $v \in y$. Then $f \circ \mathrm{const}_{x,v} = \mathrm{const}_{x,f(v)}$.

Proof. We have $\mathrm{dom}(f \circ \mathrm{const}_{x,v}) = \mathrm{dom}(\mathrm{const}_{x,v}) = x = \mathrm{dom}(\mathrm{const}_{x,f(v)})$. $(f \circ \mathrm{const}_{x,v})(u) = f(\mathrm{const}_{x,v}(u)) = f(v) = \mathrm{const}_{x,f(v)}(u)$ for all $u \in x$. Hence the thesis (by function extensionality). □

Proposition 144. **(SF 02 01 205975)** Let f be a function from x onto y and g be a function from y onto z. Then $g \circ f$ is a function from x onto z.

Proof. $g \circ f$ is a function of x.

Let us show that $g \circ f$ is a function onto z. Let $w \in z$. Take $v \in y$ such that $w = g(v)$. Take $u \in x$ such that $v = f(u)$. Then $w = g(f(u)) = (g \circ f)(u)$. End. □

Proposition 145. **(SF 02 01 784576)** Let f be a function from x into y and g be a function from y into z. Then $g \circ f$ is a function from x into z.

Proof. $g \circ f$ is a function of x.

Let us show that $g \circ f$ is one to one. Let $u, u' \in x$. Assume $(g \circ f)(u) = (g \circ f)(u')$. Then $g(f(u)) = g(f(u'))$. Hence $f(u) = f(u')$. Indeed $f(u), f(u') \in y$. Thus $u = u'$. End. □

Corollary 146. **(SF 02 01 627406)** Let f be a bijection between x and y and g be a bijection between y and z. Then $g \circ f$ is a bijection between x and z.

Proof. $g \circ f$ is a function from x onto z and a function into z. Hence the thesis. □

Proposition 147. **(SF 02 01 517102)** Let w be a set. Let $f : w \to x$ and $g : x \to y$ and $h : y \to z$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. $\mathrm{dom}(h \circ (g \circ f)) = \mathrm{dom}(g \circ f) = \mathrm{dom}(f) = w$. $\mathrm{dom}((h \circ g) \circ f) = \mathrm{dom}(f) = w$. Hence $\mathrm{dom}(h \circ (g \circ f)) = \mathrm{dom}((h \circ g) \circ f)$.

Let us show that $(h \circ (g \circ f))(u) = ((h \circ g) \circ f)(u)$ for all $u \in w$. Let $u \in w$. Then

$$\begin{gathered}(h \circ (g \circ f))(u) \\ = h((g \circ f)(u)) \\ = h(g(f(u))) \\ = (h \circ g)(f(u)) \\ = ((h \circ g) \circ f)(u).\end{gathered}$$

End.

Thus $h \circ (g \circ f) = (h \circ g) \circ f$ (by function extensionality). □

7.7 Restriction

Lemma 148. Let $a \subseteq \mathrm{dom}(f)$. Then there is a function h of a such that $h(u) = f(u)$ for all $u \in a$.

Proof. Define $h(u) = f(u)$ for $u \in a$. □

Definition 149. Let $a \subseteq \mathrm{dom}(f)$. $f \upharpoonright a$ is the function h of a such that $h(u) = f(u)$ for all $u \in a$.

Let the restriction of f to a stand for $f \upharpoonright a$.

Proposition 150. **(SF 02 01 589280)** Let f be a function from x to

y and $a \subseteq x$. Then $f \restriction a$ is a function from a to y.

Proof. We have $\mathrm{dom}(f \restriction a) = a$. Then $(f \restriction a)(u) = f(u) \in y$ for all $u \in a$. Hence $f \restriction a$ is a function from a to y. □

Proposition 151. **(SF 02 01 795968)** Let $a \subseteq x$. Then $\mathrm{id}x \restriction a = \mathrm{id}a$.

Proof. We have $\mathrm{dom}(\mathrm{id}x \restriction a) = a = \mathrm{dom}(\mathrm{id}a)$. $(\mathrm{id}x \restriction a)(u) = \mathrm{id}x(u) = u = \mathrm{id}a(u)$ for all $u \in a$. Hence the thesis (by function extensionality). □

Proposition 152. **(SF 02 01 575265)** Let v be an element and $a \subseteq x$. Then $\mathrm{const}_{x,v} \restriction a = \mathrm{const}_{a,v}$.

Proof. We have $\mathrm{dom}(\mathrm{const}_{x,v} \restriction a) = a = \mathrm{dom}(\mathrm{const}_{a,v})$. $(\mathrm{const}_{x,v} \restriction a)(u) = \mathrm{const}_{x,v}(u) = v = \mathrm{const}_{a,v}(u)$ for all $u \in a$. Hence the thesis (by function extensionality). □

Proposition 153. **(SF 02 01 507691)** Let f be an one to one function from x to y and $a \subseteq x$. Then $f \restriction a$ is one to one.

Proof. Let $u, u' \in a$. Assume $(f \restriction a)(u) = (f \restriction a)(u')$. Then $f(u) = f(u')$. Hence $u = u'$. □

8 Image and preimage

8.1 The image

Lemma 154. Let f be a function. There exists a set y such that $y = \{f(u) \mid u \in \mathrm{dom}(f) \cap z\}$.

Proof. Take $y = \mathrm{range}(f \restriction (\mathrm{dom}(f) \cap z))$. Then $y = \{(f \restriction (\mathrm{dom}(f) \cap z))(u) \mid u \in \mathrm{dom}(f) \cap z\}$. Hence $y = \{f(u) \mid u \in \mathrm{dom}(f) \cap z\}$. □

Definition 155. Let f be a function. $f[z]$ is the set y such that $y = \{f(u) \mid u \in \mathrm{dom}(f) \cap z\}$.

Let the image of z under f stand for $f[z]$. Let the direct image of z under f stand for $f[z]$.

Proposition 156. **(SF 02 02 549225)** Let f be a function from x to y and $a \subseteq x$. Then $f[a] = \{f(u) \mid u \in a\}$.

Proof. $f[a] = \{f(u) \mid u \in \mathrm{dom}(f) \cap a\}$. $\mathrm{dom}(f) \cap a = x \cap a = a$. Hence the thesis. □

Corollary 157. **(SF 02 02 516307)** Let f be a function from x to y. Then $f[x] = \mathrm{range}(f)$.

Proof. We have $f[x] = \{f(u) \mid u \in x\}$. Hence $f[x] = \mathrm{range}(f)$. □

Corollary 158. **(SF 02 0 216993)** Let f be a function from x to y and $a \subseteq x$. Then $f[a] = \mathrm{range}(f \restriction a)$.

Proof. We have $f[a] = \{f(u) \mid u \in a\}$. Hence $f[a] = \mathrm{range}(f \restriction a)$. □

Proposition 159. **(SF 02 02 560324)** Let $a \subseteq x$. Then $\mathrm{id}x[a] = a$.

Proof. $\mathrm{id}x[a] = \{\mathrm{id}x(u) \mid u \in a\}$. We have $\mathrm{id}x(u) = u$ for all $u \in a$. Hence $\mathrm{id}x[a] = \{\text{element } u \mid u \in a\}$. Thus $\mathrm{id}x[a] = a$. □

Proposition 160. **(SF 02 02 196036)** Let $a \subseteq x$ and v be an element. Assume that a is nonempty. Then $\mathrm{const}_{x,v}[a] = \{v\}$.

Proof. Let us show that $\mathrm{const}_{x,v}[a] \subseteq \{v\}$. Let $w \in \mathrm{const}_{x,v}[a]$. Take $u \in a$ such that $w = \mathrm{const}_{x,v}(u)$. Then $w = v$. Hence $w \in \{v\}$. End.

Let us show that $\{v\} \subseteq \mathrm{const}_{x,v}[a]$. Let $w \in \{v\}$. Then $w = v$. Take $u \in a$. Then $\mathrm{const}_{x,v}(u) = v = w$. Hence $w \in \mathrm{const}_{x,v}[a]$. End. □

Proposition 161. **(SF 02 01 257685)** Let f be a function from x into y and $a \subseteq x$. Then $f \restriction a$ is a bijection between a and $f[a]$.

Proof. (1) $f \restriction a$ is a function of a.

(2) $f \restriction a$ is one to one.

(3) $\mathrm{range}(f \restriction a) = f[a]$. Proof. Let us show that $\mathrm{range}(f \restriction a) \subseteq f[a]$. Let $v \in \mathrm{range}(f \restriction a)$. Take $u \in a$ such that $v = (f \restriction a)(u)$. Then $v = f(u)$. Hence $v \in f[a]$. End.

Let us show that $f[a] \subseteq \mathrm{range}(f \restriction a)$. Let $v \in f[a]$. Take $u \in a$ such that $v = f(u)$. Then $v = (f \restriction a)(u)$. Hence $v \in \mathrm{range}(f \restriction a)$. End. Qed.

Thus $f \restriction a$ is an one to one function from a onto $f[a]$. Therefore $f \restriction a$ is a bijection between a and $f[a]$. □

8.2 The preimage

Lemma 162. Let f be a function. There exists a set y such that $y = \{u \in \mathrm{dom}(f) \mid f(u) \in z\}$.

Proof. Case $f(u) \in z$ for all $u \in \mathrm{dom}(f)$. Obvious.

Case $f(u) \notin z$ for some $u \in \mathrm{dom}(f)$. Take $w \in \mathrm{dom}(f)$ such that $f(w) \notin z$. Define

$$g(u) = \begin{cases} u & f(u) \in z \\ w & f(u) \notin z \end{cases}$$

for $u \in \mathrm{dom}(f)$. $\mathrm{range}(g) = \{g(u) \mid u \in \mathrm{dom}(f)\}$. Hence $\mathrm{range}(g) = \{u \in \mathrm{dom}(f) \mid f(u) \in z \text{ or } u = w\}$. Take $y = \mathrm{range}(g) \setminus \{w\}$. Then $y = \{u \in \mathrm{dom}(f) \mid f(u) \in z\}$. End. □

Definition 163. Let f be a function. $f^{-}[z]$ is the set y such that $y = \{u \in \mathrm{dom}(f) \mid f(u) \in z\}$.

Let the preimage of z under f stand for $f^{-}[z]$. Let the inverse image of z under f stand for $f^{-}[z]$.

Proposition 164. **(SF 02 02 317629)** Let $b \subseteq y$. Then $\mathrm{id}y^{-}[b] = b$.

Proof. $\mathrm{id}y^{-}[b] = \{u \in y \mid \mathrm{id}y(u) \in b\}$. $\mathrm{id}y(u) = u$ for all $u \in y$. Hence $\mathrm{id}y^{-}[b] = \{u \in y \mid u \in b\}$. Thus $\mathrm{id}y^{-}[b] = b$. □

Proposition 165. **(SF 02 02 732231)** Let v be an element and z be a set that contains v. Then $\mathrm{const}^{-}_{x,v}[z] = x$.

Proof. $\mathrm{const}^{-}_{x,v}[z] = \{u \in x \mid \mathrm{const}_{x,v}(u) \in z\}$. $\mathrm{const}_{x,v}(u) = v$ for every $u \in x$. Hence $\mathrm{const}^{-}_{x,v}[z] = \{u \in x \mid v \in z\}$. We have $v \in z$. Thus $\mathrm{const}^{-}_{x,v}[z] = x$. □

Proposition 166. **(SF 02 02 483725)** Let v be an element and z be a set that does not contain v. Then $\mathrm{const}^{-}_{x,v}[z] = \emptyset$.

Proof. $\mathrm{const}^{-}_{x,v}[z] = \{u \in x \mid \mathrm{const}_{x,v}(u) \in z\}$. $\mathrm{const}_{x,v}(u) = v$ for every $u \in x$. Hence $\mathrm{const}^{-}_{x,v}[z] = \{u \in x \mid v \in z\}$. It is wrong that $v \in z$. Thus $\mathrm{const}^{-}_{x,v}[z] = \emptyset$. □

8.3 Computation rules

Proposition 167. **(SF 02 02 206888)** Let f be a function from x to y and $a \subseteq x$ and $u \in x$. Then $u \in a \implies f(u) \in f[a]$.

Proof. Assume $u \in a$. We have $f[a] = \{f(u') \mid u' \in a\}$. Hence $f(u) \in f[a]$. □

Proposition 168. **(SF 02 02 451910)** Let f be a function from x to y and $b \subseteq y$ and $u \in x$. Then $f(u) \in b \iff u \in f^{-}[b]$.

Proof. We have $f^{-}[b] = \{u' \in x \mid f(u') \in b\}$. Hence $u \in f^{-}[b]$ iff $u \in x$ and $f(u) \in b$. Then we have the thesis. □

Proposition 169. **(SF 02 02 186101)** Let f be a function from x to y. Then $f[x] \subseteq y$.

Proof. $f[x] = f[\mathrm{dom}(f)] = \mathrm{range}(f) \subseteq y$. □

Proposition 170. **(SF 02 02 104059)** Let f be a function from x to y. Then $f^{-}[y] = x$.

Proof. We have $f^{-}[y] = \{u \in x \mid f(u) \in y\}$. $f(u)$ is an element of y for all $u \in x$. Hence the thesis. □

Proposition 171. **(SF 02 02 481295)** Let f be a function from x to y. Then $f[f^{-}[y]] = f[x]$.

Proof. Let us show that $f[f^{-}[y]] \subseteq f[x]$. Let $v \in f[f^{-}[y]]$. Take $u \in f^{-}[y] \cap x$ such that $v = f(u)$. Then $u \in x$. Hence $v \in f[x]$. End.

Let us show that $f[x] \subseteq f[f^{-}[y]]$. Let $v \in f[x]$. Take $u \in x$ such that $v = f(u)$. We have $v \in y$. Hence $u \in f^{-}[y]$. Thus $f(u) \in f[f^{-}[y]]$. Indeed $f^{-}[y] \subseteq x$. Therefore $v \in f[f^{-}[y]]$. End. □

Proposition 172. **(SF 02 02 253830)** Let f be a function from x to y. Then $f^{-}[f[x]] = x$.

Proof. $f^{-}[f[x]] = \{u \in x \mid f(u) \in f[x]\}$. For all $u \in x$ we have $f(u) \in f[x]$. Hence every element of $f^{-}[f[x]]$ is contained in x and every element of x is contained in $f^{-}[f[x]]$. Thus $f^{-}[f[x]] = x$. □

Proposition 173. **(SF 02 02 163978)** Let f be a function from x to y and $b \subseteq y$. Then $f[f^{-}[b]] = b \cap f[x]$.

Proof. Let us show that $f[f^{-}[b]] \subseteq b \cap f[x]$. Let $v \in f[f^{-}[b]]$. Take $u \in f^{-}[b]$ such that $v = f(u)$. Then $f(u) \in b \cap f[x]$. Hence we have $v \in b \cap f[x]$. End.

Let us show that $b \cap f[x] \subseteq f[f^{-}[b]]$. Let $v \in b \cap f[x]$. Take $u \in x$ such that $v = f(u)$. Then $u \in f^{-}[b]$. Hence $f(u) \in f[f^{-}[b]]$. End. □

Corollary 174. **(SF 02 02 422873)** Let f be a function from x to y and $b \subseteq y$. Then $f[f^{-}[b]] \subseteq b$.

Proof. We have $f[f^{-}[b]] = b \cap f[x] \subseteq b$. Hence $f[f^{-}[b]] \subseteq b$. □

Proposition 175. **(SF 02 02 171121)** Let f be a function from x to y and $a \subseteq x$. Then $f^{-}[f[a]] \supseteq a$.

Proof. Let $u \in a$. Then $f(u) \in f[a]$. Hence $u \in f^{-}[f[a]]$. Indeed $f[a] \subseteq y$. □

Proposition 176. **(SF 02 02 693086)** Let f be a function from x to y and $a \subseteq x$. Then $f[a] = \emptyset \iff a = \emptyset$.

Proof. Case $f[a] = \emptyset$. Then there is no $u \in a$ such that $f(u) \in f[a]$. For all $u \in a$ we have $f(u) \in f[a]$. Hence a is empty. End.

Case $a = \emptyset$. For all $v \in f[a]$ we have $v = f(u)$ for some $u \in a$. There is no $u \in a$. Hence $f[a]$ is empty. End. □

Proposition 177. **(SF 02 02 464503)** Let f be a function from x to y and $b \subseteq y$. Then $f^{-}[b] = \emptyset \iff b \subseteq y \setminus f[x]$.

Proof. Case $f^{-}[b] = \emptyset$. Let $v \in b$. Then $v \in y$.

There is no $u \in x$ such that $v = f(u)$.
Proof. Assume the contrary. Take $u \in x$ such that $v = f(u)$. Then $u \in f^{-}[b]$. Contradiction. Qed.

Hence $v \notin f[x]$. Therefore $v \in y \setminus f[x]$. End.

Case $b \subseteq y \setminus f[x]$. Then no element of b is an element of $f[x]$. Assume that $f^{-}[b]$ is nonempty. Take $u \in f^{-}[b]$. Then $f(u) \in b$ and $f(u) \in f[x]$. Contradiction. End. ☐

Proposition 178. (SF 02 02 474184) Let f be a function from x to y and $a \subseteq x$ and $b \subseteq y$. Then $f[a] \cap b = \emptyset \iff a \cap f^{-}[b] = \emptyset$.

Proof. Case $f[a] \cap b = \emptyset$. Assume that $a \cap f^{-}[b]$ is nonempty. Take $u \in a \cap f^{-}[b]$. Then $f(u) \in f[a]$ and $f(u) \in b$. Hence $f(u) \in f[a] \cap b$. Contradiction. End.

Case $a \cap f^{-}[b] = \emptyset$. Assume that $f[a] \cap b$ is nonempty. Take $v \in f[a] \cap b$. Consider a $u \in a$ such that $v = f(u)$. Then $u \in f^{-}[b]$. Indeed $v \in b$. Hence $u \in a \cap f^{-}[b]$. Contradiction. End. ☐

Proposition 179. (SF 02 02 522811) Let f be a function from x to y and g be a function from y to z and $a \subseteq x$. Then $(g \circ f)[a] = g[f[a]]$.

Proof. $((g \circ f)[a]) = \{g(f(u)) \mid u \in a\}$. We have $g[f[a]] = \{g(v) \mid v \in f[a]\}$ and $f[a] = \{f(u) \mid u \in a\}$. Thus $g[f[a]] = \{g(f(u)) \mid u \in a\}$. Therefore $(g \circ f)[a] = g[f[a]]$. ☐

Proposition 180. (SF 02 02 819065) Let f be a function from x to y and g be a function from y to z and $c \subseteq z$. Then $(g \circ f)^{-}[z] = f^{-}[g^{-}[z]]$.

Proof. $((g \circ f)^{-}[z]) = \{u \in x \mid g(f(u)) \in z\}$. We have $g^{-}[z] = \{v \in y \mid g(v) \in z\}$ and $f^{-}[g^{-}[z]] = \{u \in x \mid f(u) \in g^{-}[z]\}$. Hence $f^{-}[g^{-}[z]] = \{u \in x \mid g(f(u)) \in z\}$. Thus $(g \circ f)^{-}[z] = f^{-}[g^{-}[z]]$. ☐

Proposition 181. (SF 02 02 889945) Let f be a function from x to y and $a, a' \subseteq x$. Then $a \subseteq a' \implies f[a] \subseteq f[a']$.

Proof. Assume $a \subseteq a'$. Let $v \in f[a]$. Take $u \in a$ such that $f(u) = v$. Then $u \in a'$. Hence $v = f(u) \in f[a']$. ☐

Proposition 182. (SF 02 02 514409) Let f be a function from x to y and $b, b' \subseteq y$. Then $b \subseteq b' \implies f^{-}[b] \subseteq f^{-}[b']$.

Proof. Assume $b \subseteq b'$. Let $u \in f^{-}[b]$. Then $f(u) \in b$. Hence $f(u) \in b'$. Thus $u \in f^{-}[b']$. ☐

Proposition 183. (SF 02 02 319894) Let f be a function from x to y and $a, a' \subseteq x$. Then $f[a \cup a'] = f[a] \cup f[a']$.

Proof. Let us show that $f[a \cup a'] \subseteq f[a] \cup f[a']$. Let $v \in f[a \cup a']$. Take $u \in a \cup a'$ such that $v = f(u)$. Then $u \in a$ or $u \in a'$. Hence $f(u) \in f[a]$ or $f(u) \in f[a']$. Thus $v = f(u) \in f[a] \cup f[a']$. End.

Let us show that $f[a] \cup f[a'] \subseteq f[a \cup a']$. Let $v \in f[a] \cup f[a']$.

Case $v \in f[a]$. Take $u \in a$ such that $v = f(u)$. Then $u \in a \cup a'$. Hence $v \in f[a \cup a']$. End.

Case $v \in f[a']$. Take $u \in a'$ such that $v = f(u)$. Then $u \in a \cup a'$. Hence $v \in f[a \cup a']$. End. End. □

Proposition 184. **(SF 02 02 357044)** Let f be a function from x to y and $b, b' \subseteq y$. Then $f^{-}[b \cup b'] = f^{-}[b] \cup f^{-}[b']$.

Proof. Let us show that $f^{-}[b \cup b'] \subseteq f^{-}[b] \cup f^{-}[b']$. Let $u \in f^{-}[b \cup b']$. Then $f(u) \in b \cup b'$. Hence $f(u) \in b$ or $f(u) \in b'$. If $f(u) \in b$ then $u \in f^{-}[b]$. If $f(u) \in b'$ then $u \in f^{-}[b']$. Thus $u \in f^{-}[b] \cup f^{-}[b']$. End.

Let us show that $f^{-}[b] \cup f^{-}[b'] \subseteq f^{-}[b \cup b']$. Let $u \in f^{-}[b] \cup f^{-}[b']$. Then $u \in f^{-}[b]$ or $u \in f^{-}[b']$. If $u \in f^{-}[b]$ then $f(u) \in b$. If $u \in f^{-}[b']$ then $f(u) \in b'$. Hence $f(u) \in b \cup b'$. Thus $u \in f^{-}[b \cup b']$. End. □

Proposition 185. **(SF 02 02 512404)** Let f be a function from x to y and $a, a' \subseteq x$. Then $f[a \cap a'] \subseteq f[a] \cap f[a']$.

Proof. Let $v \in f[a \cap a']$. Take $u \in a \cap a'$ such that $v = f(u)$. Then $u \in a$ and $u \in a'$. Hence $f(u) \in f[a]$ and $f(u) \in f[a']$. Thus $v \in f[a] \cap f[a]$. □

Proposition 186. **(SF 02 02 266480)** Let f be a function from x to y and $b, b' \subseteq y$. Then $f^{-}[b \cap b'] = f^{-}[b] \cap f^{-}[b']$.

Proof. Let us show that $f^{-}[b \cap b'] \subseteq f^{-}[b] \cap f^{-}[b']$. Let $u \in f^{-}[b \cap b']$. Then $f(u) \in b \cap b'$. Hence $f(u) \in b$ and $f(u) \in b'$. Thus $u \in f^{-}[b]$ and $u \in f^{-}[b']$. Therefore $u \in f^{-}[b] \cap f^{-}[b']$. End.

Let us show that $f^{-}[b] \cap f^{-}[b'] \subseteq f^{-}[b \cap b']$. Let $u \in f^{-}[b] \cap f^{-}[b']$. Then $u \in f^{-}[b]$ and $u \in f^{-}[b']$. Hence $f(u) \in b$ and $f(u) \in b'$. Thus $f(u) \in b \cap b'$. Therefore $u \in f^{-}[b \cap b']$. End. □

Proposition 187. **(SF 02 02 560446)** Let f be a function from x to y and $a, a' \subseteq x$. Then $f[a \setminus a'] \supseteq f[a] \setminus f[a']$.

Proof. Let $v \in f[a] \setminus f[a']$. Then $v \in f[a]$ and $v \notin f[a']$. Take $u \in a$ such that $v = f(u)$. If $u \in a'$ then $v \in f[a']$. Hence $u \notin a'$. Thus $u \in a \setminus a'$. Therefore $v = f(u) \in f[a \setminus a']$. □

Proposition 188. **(SF 02 02 523450)** Let f be a function from x to y and $b, b' \subseteq y$. Then $f^{-}[b \setminus b'] = f^{-}[b] \setminus f^{-}[b']$.

Proof. Let us show that $f^{-}[b \setminus b'] \subseteq f^{-}[b] \setminus f^{-}[b']$. Let $u \in f^{-}[b \setminus b']$. Then $f(u) \in b \setminus b'$. Hence $f(u) \in b$ and $f(u) \notin b'$. Thus $u \in f^{-}[b]$ and $u \notin f^{-}[b']$. Therefore $u \in f^{-}[b] \setminus f^{-}[b']$. End.

Let us show that $f^{-}[b] \setminus f^{-}[b'] \subseteq f^{-}[b \setminus b']$. Let $u \in f^{-}[b] \setminus f^{-}[b']$. Then $u \in f^{-}[b]$ and $u \notin f^{-}[b']$. Hence $f(u) \in b$ and $f(u) \notin b'$. Thus $f(u) \in b \setminus b'$. Therefore $u \in f^{-}[b \setminus b']$. End. $\square$

9 Invertible functions

9.1 Definitions and basic properties

Definition 189. An inverse of f is a function g from range(f) to dom(f) such that

$$f(u) = v \iff g(v) = u$$

for all $u \in \mathrm{dom}(f)$ and all $v \in \mathrm{dom}(g)$.

Definition 190. f is invertible iff f has an inverse.

Lemma 191. Let g, g' be inverses of f. Then $g = g'$.

Proof. We have $\mathrm{dom}(g) = \mathrm{range}(f) = \mathrm{dom}(g')$.

Let us show that $g(v) = g'(v)$ for all $v \in \mathrm{range}(f)$. Let $v \in \mathrm{range}(f)$. Take $u = g'(v)$. Then $g(v) = u$ iff $f(u) = v$. We have $f(u) = v$ iff $g'(v) = u$. Thus $g(v) = g'(v)$. End. $\square$

Definition 192. Let f be invertible. f^{-1} is the inverse of f.

Let f is involutory stand for f is the inverse of f. Let f is selfinverse stand for f is the inverse of f.

Proposition 193. **(SF 02 03 587168)** Let f be a function from x onto y and g be a function from y onto x. Then g is the inverse of f iff $g \circ f = \mathrm{id}x$ and $f \circ g = \mathrm{id}y$.

Proof. Case g is the inverse of f. We have $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = x = \mathrm{dom}(\mathrm{id}x)$. For all $u \in x$ we have $(g \circ f)(u) = g(f(u)) = u$. Hence $g \circ f = \mathrm{id}x$.

We have $\mathrm{dom}(f \circ g) = \mathrm{dom}(g) = y = \mathrm{dom}(\mathrm{id}y)$. For all $v \in y$ we have $(f \circ g)(v) = f(g(v)) = v$. Hence $f \circ g = \mathrm{id}y$. End.

Case $g \circ f = \mathrm{id}x$ and $f \circ g = \mathrm{id}y$. Then $\mathrm{dom}(g) = y = \mathrm{range}(f)$ and $\mathrm{range}(g) = x = \mathrm{dom}(f)$. Let $u \in \mathrm{dom}(f)$ and $v \in \mathrm{dom}(g)$. If $f(u) = v$ then $g(v) = g(f(u)) = (g \circ f)(u) = \mathrm{id}x(u) = u$. If $g(v) = u$ then $f(u) = f(g(v)) = (f \circ g)(v) = \mathrm{id}y(v) = v$. Hence $f(u) = v$ iff $g(v) = u$. End. $\square$

Proposition 194. **(SF 02 03 196251)** Let f be an invertible function from x onto y. Then f^{-1} is an invertible function from y onto x such that $(f^{-1})^{-1} = f$.

Proof. f^{-1} is a function from y to x. Indeed $\mathrm{range}(f) = y$ and $\mathrm{dom}(f) = x$. Hence f^{-1} is a function from y onto x. f^{-1} is the inverse of f. Thus $f \circ f^{-1} = \mathrm{id}y$ and $f^{-1} \circ f = \mathrm{id}x$. Therefore f is the inverse of f^{-1} (by SF 02 03 587168). □

Proposition 195. **(SF 02 03 601485)** Let f be an invertible function from x onto y. Then $f \circ f^{-1} = \mathrm{id}y$ and $f^{-1} \circ f = \mathrm{id}x$.

Proof. f^{-1} is a function from y onto x (by SF 02 03 196251). f^{-1} is the inverse of f. Hence the thesis (by SF 02 03 587168). □

Proposition 196. **(SF 02 03 173329)** Let f be an invertible function from x onto y. Then $(f^{-1}(f(u)) = u$ for all $u \in x)$ and $(f(f^{-1}(v)) = v$ for all $v \in y)$.

Proof. Let us show that $f^{-1}(f(u)) = u$ for all $u \in x$. Let $u \in x$. Then $f^{-1}(f(u)) = (f^{-1} \circ f)(u) = \mathrm{id}x(u) = u$. End.

Let us show that $f(f^{-1}(v)) = v$ for all $v \in y$. Let $v \in y$. Then $f(f^{-1}(v)) = (f \circ f^{-1})(v) = \mathrm{id}y(v) = v$. End. □

Proposition 197. **(SF 02 03 430030)** Let f be an invertible function from x onto y and g be an invertible function from y onto z. Then $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. f^{-1} is a function from y onto x. g^{-1} is a function from z onto y. Take $h = f^{-1} \circ g^{-1}$. Then h is a function from z onto x (by SF 02 01 205975). Hence h is a function from z to x.

Let us show that $((g \circ f) \circ h) = \mathrm{id}z$. We have $f \circ (f^{-1} \circ g^{-1}) = (f \circ f^{-1}) \circ g^{-1}$. $f \circ h$ is a function from z to y. Hence

$$(g \circ f) \circ h$$

$$= g \circ (f \circ h)$$
$$= g \circ (f \circ (f^{-1} \circ g^{-1}))$$
$$= g \circ ((f \circ f^{-1}) \circ g^{-1})$$
$$= g \circ (\mathrm{id}y \circ g^{-1})$$
$$= g \circ g^{-1}$$
$$= \mathrm{id}z.$$

End.

Let us show that $h \circ (g \circ f) = \mathrm{id}x$. We have $(f^{-1} \circ g^{-1}) \circ g = f^{-1} \circ (g^{-1} \circ g)$. $g \circ f$ is a function from x to z. Hence

$$\begin{aligned} & h \circ (g \circ f) \\ &= (h \circ g) \circ f \\ &= ((f^{-1} \circ g^{-1}) \circ g) \circ f \\ &= (f^{-1} \circ (g^{-1} \circ g)) \circ f \\ &= (f^{-1} \circ \mathrm{id}y) \circ f \\ &= f^{-1} \circ f \\ &= \mathrm{id}x. \end{aligned}$$

End.

Thus h is the inverse of $g \circ f$ (by SF 02 03 587168). □

Proposition 198. (SF 02 03 908585) Let f be an invertible function from x onto y and $a \subseteq x$. Then $f \restriction a$ is invertible and $(f \restriction a)^{-1} = f^{-1} \restriction f[a]$.

Proof. $f \restriction a$ is a function from a onto $f[a]$. Take $g = f^{-1} \restriction f[a]$. Then g is a function of $f[a]$.

Let us show that $a \subseteq \mathrm{range}(g)$. Let $u \in a$. Then $f(u) \in f[a]$. Hence $g(f(u)) = f^{-1}(f(u)) = u$. Thus u is a value of g. End.

Let us show that $\mathrm{range}(g) \subseteq a$. Let $u \in \mathrm{range}(g)$. Take $v \in f[a]$ such that $u = g(v)$. Take $w \in a$ such that $v = f(w)$. Then $u = (f^{-1} \restriction f[a])(v) = f^{-1}(v) = f^{-1}(f(w)) = w$. Hence $u \in a$. End.

Hence $\mathrm{range}(g) = a$. Thus g is a function onto a.

Let us show that $g((f \restriction a)(u)) = u$ for all $u \in a$. Let $u \in a$. Then $g((f \restriction a)(u)) = g(f(u)) = (f^{-1} \restriction f[a])(f(u)) = f^{-1}(f(u)) = u$. End.

Let us show that $((f \restriction a)(g(v))) = v$ for all $v \in f[a]$. Let $v \in f[a]$. Take $u \in a$ such that $v = f(u)$. We have $g(v) = g(f(u)) = (f^{-1} \restriction f[a])(f(u)) = f^{-1}(f(u)) = u$. Hence $(f \restriction a)(g(v)) = (f \restriction a)(u) = f(u) = v$. End.

Thus $g \circ (f \restriction a) = \mathrm{id}a$ and $(f \restriction a) \circ g = \mathrm{id}f[a]$. Therefore g is the inverse of $f \restriction a$. □

Proposition 199. (SF 02 03 293037) Let f be an invertible function from x onto y and $b \subseteq y$. Then $f^{-}[b] = f^{-1}[b]$.

Proof. We have $f^{-1}[b] = \{f^{-1}(v) \mid v \in b\}$ and $f^{-}[b] = \{u \in x \mid f(u) \in b\}$.

Let us show that $f^{-}[b] \subseteq f^{-1}[b]$. Let $u \in f^{-}[b]$. Take $v \in b$ such that $v = f(u)$. Then $f^{-1}(v) = f^{-1}(f(u)) = u$. Hence $u \in f^{-1}[b]$. End.

Let us show that $f^{-1}[b] \subseteq f^{-}[b]$. Let $u \in f^{-1}[b]$. Take $v \in b$ such that $u = f^{-1}(v)$. Then $f(u) = f(f^{-1}(v)) = v$. Hence $u \in f^{-}[b]$. End. □

Corollary 200. **(SF 02 03 265073)** Let f be an invertible function from x onto y and $v \in y$. Then $f^{-}[\{v\}] = \{f^{-1}(v)\}$.

Proof. $f^{-}[\{v\}] = f^{-1}[\{v\}]$. We have $f^{-1}[\{v\}] = \{f^{-1}(w) \mid w \in \{v\}\}$. Hence $f^{-1}[\{v\}] = \{f^{-1}(v)\}$. □

Proposition 201. **(SF 02 03 394829)** Let f be a function from x onto y. f is invertible iff f is one to one.

Proof. Case f is invertible. Let $u, v \in x$. Assume $f(u) = f(v)$. Then $u = f^{-1}(f(u)) = f^{-1}(f(v)) = v$. End.

Case f is one to one. Define $g(v) =$ choose $u \in x$ such that $f(u) = v$ in u for $v \in y$. g is a function from y to x. For all $v \in y$ and all $u, u' \in x$ such that $f(u) = v = f(u')$ we have $u = u'$. Hence g is a function from y onto x. For all $u \in x$ we have $g(f(u)) = u$. For all $v \in y$ we have $f(g(v)) = v$. Hence g is the inverse of f. End. □

Corollary 202. **(SF 02 03 187673)** Let f be an invertible function from x onto y. Then f^{-1} is a bijection between y and x.

Proof. f^{-1} is a function from y onto x. f^{-1} is invertible. Hence f^{-1} is one to one. Thus f^{-1} is a function from y into x. Therefore f^{-1} is a bijection between y and x. □

9.2 Involutions

Definition 203. An involution on x is a selfinverse function f on x.

Proposition 204. **(SF 02 03 305935)** $\mathrm{id}x$ is an involution on x.

Proof. $\mathrm{id}x$ is a function on x. We have $\mathrm{id}x \circ \mathrm{id}x = \mathrm{id}x$. Hence $\mathrm{id}x$ is selfinverse. □

Proposition 205. **(SF 02 03 610247)** Let f and g be involutions on x. Then $g \circ f$ is an involution on x iff $g \circ f = f \circ g$.

Proof. Case $g \circ f$ is an involution on x. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1} = f \circ g$. End.

Case $g \circ f = f \circ g$. $f \circ f$, $f \circ g$ and $f \circ g$ are functions on x. Hence

$$(g \circ f) \circ (g \circ f)$$

$$= (g \circ f) \circ (f \circ g)$$

$$= ((g \circ f) \circ f) \circ g$$

$$= (g \circ (f \circ f)) \circ g$$

$$= (g \circ \mathrm{id}x) \circ g$$

$$= g \circ g$$

$$= \mathrm{id}x.$$

Thus $g \circ f$ is selfinverse. End. □

Corollary 206. (SF 02 03 310947) Let f be an involutions on x. Then $f \circ f$ is an involution on x.

Proposition 207. (SF 02 03 280184) Let f be an involution on x. Then f is a permutation of x.

Proof. f is an invertible function from x onto x. Hence f is a bijection between x and x. Thus f is a permutation of x. □

10 Functions and the symmetric difference

Proposition 208. (SF 02 04 657921) Let f be a function from x to y and $a, a' \subseteq x$. Then

$$f[a \triangle a'] \supseteq f[a] \triangle f[a'].$$

Proof. Let $v \in f[a] \triangle f[a']$. Then $v \in f[a] \cup f[a']$ and $v \notin f[a] \cap f[a']$. We have $f[a] \cup f[a'] = f[a \cup a']$. Hence we can take $u \in a \cup a'$ such that $v = f(u)$.

Let us show that $u \notin a \cap a'$. Assume the contrary. Then $v = f(u) \in f[a \cap a']$. We have $f[a \cap a'] \subseteq f[a] \cap f[a']$. Hence $v \in f[a] \cap f[a']$. Contradiction. End.

Thus $u \in a \triangle a'$. Therefore $v \in f[a \triangle a']$. □

Proposition 209. (SF 02 04 661750) Let f be a function from x to y and $b, b' \subseteq y$. Then

$$f^{-}[b \triangle b'] \supseteq f^{-}[b] \triangle f^{-}[b'].$$

Proof. Let $u \in f^{-}[b] \triangle f^{-}[b']$. Then $u \in f^{-}[b] \cup f^{-}[b']$ and $u \notin f^{-}[b] \cap f^{-}[b']$. We have $f^{-}[b] \cup f^{-}[b'] = f^{-}[b \cup b']$. Hence we can take $v \in b \cup b'$ such that $f(u) = v$.

Let us show that $v \notin b \cap b'$. Assume the contrary. Then $v = f(u) \in b \cap b'$. Hence $u \in f^{-}[b \cap b'] = f^{-}[b] \cap f^{-}[b']$. Thus $v = f(u) \in b \cap b'$. Contradiction. End.

Therefore $v \in b \triangle b'$. Hence $u \in f^{-}[b \triangle b']$. □

11 Functions and set-systems

Definition 210. A function between systems of sets is a function f such that f is a function from X to Y for some systems of sets X, Y.

Definition 211. Let f be a function between systems of sets. f preserves subsets iff for all $x, y \in \mathrm{dom}(f)$ if $x \subseteq y$ then $f(x) \subseteq f(y)$.

Definition 212. Let f be a function between systems of sets. f preserves supersets iff for all $x, y \in \mathrm{dom}(f)$ if $x \supseteq y$ then $f(x) \supseteq f(y)$.

Lemma 213. Let f be a function between systems of sets. Then f preserves subsets iff f preserves supersets.

Proof. Case f preserves subsets. Let $x, y \in \mathrm{dom}(f)$. Assume $x \supseteq y$. Then $y \subseteq x$. Hence $f(y) \subseteq f(x)$. Thus $f(x) \supseteq f(y)$. End.

Case f preserves supersets. Let $x, y \in \mathrm{dom}(f)$. Assume $x \subseteq y$. Then $y \supseteq x$. Hence $f(y) \supseteq f(x)$. Thus $f(x) \subseteq f(y)$. End. $\square$

Theorem 214. (SF 01 05 636019) Let h be a function from $\mathcal{P}(x)$ to $\mathcal{P}(x)$ that preserves subsets. Then h has a fixed point.

Proof. (1) Define $A = \{y \subseteq x \mid y \subseteq h(y)\}$. Then A is a subset of $\mathcal{P}(x)$ (by separation). We have $\bigcup A \in \mathcal{P}(x)$.

Let us show that (2) $\bigcup A \subseteq h(\bigcup A)$. Let $u \in \bigcup A$. Take $y \in A$ such that $u \in y$. Then $u \in h(y)$. We have $y \subseteq \bigcup A$. Hence $h(y) \subseteq h(\bigcup A)$. Thus $h(y) \subseteq h(\bigcup A)$. Therefore $u \in h(\bigcup A)$. End.

Then $h(\bigcup A) \in A$ (by 1). (3) Hence $h(\bigcup A) \subseteq \bigcup A$. Indeed every element of $h(\bigcup A)$ is an element of some element of A.

Thus $h(\bigcup A) = \bigcup A$ (by 2, 3). $\square$

12 Cantor's theorem

Theorem 215. (Cantor) Let x be a set. There exists no function from x onto $\mathcal{P}(x)$.

Proof. Assume the contrary. Take a function f from x onto $\mathcal{P}(x)$. Define $N = \{u \in x \mid u \notin f(u)\}$. Then N is a subset of x (by separation). Hence $N \in \mathcal{P}(x)$. Thus we can take an element u of x such that $f(u) = N$. Then $u \in N$ iff $u \in f(u)$ iff $u \notin N$. Contradiction. $\square$

13 Equipollency

Definition 216. x and y are equipollent iff there exists a bijection between x and y.

Let x and y are equipotent stand for x and y are equipollent.

Proposition 217. **(SF 01 07 639059)** x and x are equipollent.

Proof. id$x$ is a bijection between x and x. □

Proposition 218. **(SF 01 07 467393)** If x and y are equipollent then y and x are equipollent.

Proof. Assume that x and y are equipollent. Take a bijection f between x and y. Then f^{-1} is a bijection between y and x. Hence y and x are equipollent. □

Proposition 219. **(SF 01 07 956273)** If x and y are equipollent and y and z are equipollent then x and z are equipollent.

Proof. Assume that x and y are equipollent and y and z are equipollent. Take a bijection f between x and y. Take a bijection g between y and z. Then $g \circ f$ is a bijection between x and z. Hence x and z are equipollent. □

Proposition 220. **(SF 01 07 430789)** x and $\emptyset$ are equipollent iff x is empty.

Proof. Case x and $\emptyset$ are equipollent. Take a bijection f between x and $\emptyset$. Assume that x is nonempty. Take an element u of x. Then $f(u) \in \emptyset$. Contradiction. End.

Case x is empty. Then $x = \emptyset$. Hence x and $\emptyset$ are equipollent. End. □

14 The Cantor-Schröder-Bernstein theorem

The proof of the following theorem is adopted from a formalization of set theory from 2019[2].

Theorem 221. **(Cantor Schroeder Bernstein)** Let x, y be sets. x and y are equipollent iff there exists a function from x into y and there exists a function from y into x.

Proof. Case x and y are equipollent. Take a bijection f between x and y. Then f^{-1} is a bijection between y and x. Hence f is a function from x into y and f^{-1} is a function from y into x. End.

Case there exists a function from x into y and there exists a function from

[2] `https://github.com/naproche/FLib/tree/master/SetTheory2019`

y into x. Take a function f from x into y. Take a function g from y into x. We have $y \setminus f[a] \subseteq y$ for any $a \in \mathcal{P}(x)$.

(1) Define $h(a) = x \setminus g[y \setminus f[a]]$ for $a \in \mathcal{P}(x)$.

h is a function from $\mathcal{P}(x)$ to $\mathcal{P}(x)$.

Let us show that h preserves subsets. Let a, b be subsets of x. Assume $a \subseteq b$. Then $f[a] \subseteq f[b]$. Hence $y \setminus f[b] \subseteq y \setminus f[a]$. Thus $g[y \setminus f[b]] \subseteq g[y \setminus f[a]]$. Therefore $x \setminus g[y \setminus f[a]] \subseteq x \setminus g[y \setminus f[b]]$. Consequently $h[a] \subseteq h[b]$. End.

Hence we can take a fixed point c of h.

(2) Define $F(u) = f(u)$ for $u \in c$.

We have $c = h(c)$ iff $x \setminus c = g[y \setminus f[c]]$. g^{-1} is a bijection between range(g) and y. Thus $x \setminus c = g[y \setminus f[c]] \subseteq \text{range}(g)$.

(3) Define $G(u) = g^{-1}(u)$ for $u \in x \setminus c$.

F is a bijection between c and range(F). G is a bijection between $x \setminus c$ and range(G).

Define

$$H(u) = \begin{cases} F(u) & u \in c \\ G(u) & u \notin c \end{cases}$$

for $u \in x$.

Let us show that H is a function to y. Let v be a value of H. Take $u \in x$ such that $H(u) = v$. If $u \in c$ then $v = H(u) = F(u) = f(u) \in y$. If $u \notin c$ then $v = H(u) = G(u) = g^{-1}(u) \in y$. End.

Let us show that every element of y is a value of H. Let $v \in y$.

Case $v \in f[c]$. Take $u \in c$ such that $f(u) = v$. Then $F(u) = v$. End.

Case $v \notin f[c]$. Then $v \in y \setminus f[c]$. Hence $g(v) \in g[y \setminus f[c]]$. Thus $g(v) \in x \setminus h(c)$. We have $g(v) \in x \setminus c$. Therefore we can take $u \in x \setminus c$ such that $G(u) = v$. Then $v = H(u)$. End. End.

Let us show that H is one to one. Let $u, v \in \text{dom}(H)$. Assume $u \neq v$.

Case $u, v \in c$. Then $H(u) = F(u)$ and $H(v) = F(v)$. We have $F(u) \neq F(v)$. Hence $H(u) \neq H(v)$. End.

Case $u, v \notin c$. Then $H(u) = G(u)$ and $H(v) = G(v)$. We have $G(u) \neq G(v)$. Hence $H(u) \neq H(v)$. End.

Case $u \in c$ and $v \notin c$. Then $H(u) = F(u)$ and $H(v) = G(v)$. Hence $v \in g[y \setminus f[c]]$. We have $G(v) \in y \setminus F[c]$. Thus $G(v) \neq F(u)$. End.

Case $u \notin c$ and $v \in c$. Then $H(u) = G(u)$ and $H(v) = F(v)$. Hence $u \in g[y \setminus f[c]]$. We have $G(u) \in y \setminus f[c]$. Thus $G(u) \neq F(v)$. End. End.

Hence H is a bijection between x and y. End. $\square$

15 The axiom of choice

Definition 222. Let X be a system of nonempty sets. Assume that y and y' are disjoint for all $y, y' \in X$ such that $y \neq y'$. A choice set of X is a set z such that for all $y \in X$ there exists an element w such that $y \cap z = \{w\}$.

Axiom 223. (Choice) Let X be a nonempty system of nonempty sets. Assume that y and y' are disjoint for all $y, y' \in X$ such that $y \neq y'$. Then X has a choice set.

Definition 224. Let X be a system of nonempty sets. A choice function of X is a function g of X such that $g(y) \in y$ for all $y \in X$.

Proposition 225. Let X be a system of nonempty sets. Assume that y and y' are disjoint for all $y, y' \in X$ such that $y \neq y'$. X has a choice function iff X has a choice set.

Proof. Case X has a choice function. Take a choice function g of X. Define $z = \{g(y) \mid y \in X\}$. $\mathrm{range}(g)$ is a set. $g(y) \in \mathrm{range}(g)$ for each $y \in X$. Hence z is a set (by separation).

Let us show that for all $y \in X$ we have $y \cap z = \{g(y)\}$. Let $y \in X$. We have $\{g(y)\} \subseteq y \cap z$. Indeed $g(y) \in y$ and $g(y) \in z$.

$y \cap z \subseteq \{g(y)\}$.
Proof. Let $u \in y \cap z$. Then $u \in y$ and $u \in z$. Take $y' \in X$ such that $u = g(y')$. Then $y' = y$. Indeed if $y' \neq y$ then y' and y are disjoint. Qed.

Hence $y \cap z = \{g(y)\}$. End. End.

Case X has a choice set. Take a choice set z of X. Then for all $y \in X$ there exists an element w such that $y \cap z = \{w\}$. Define $g(y) =$ choose the element w such that $y \cap z = \{w\}$ in w for $y \in X$.

Let us show that $g(y) \in y$ for all $y \in X$. Let $y \in X$. Take an element w such that $y \cap z = \{w\}$. Then $g(y) = w$. We have $\{w\} \subseteq y \cap z \subseteq y$. Hence $\{w\} \subseteq y$. Thus $w \in y$. Therefore $g(y) \in y$. End.

Hence g is a choice function of X. End. $\square$