Proving Binomial Identities in Naproche

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The Language of Arithmetic

[synonym number/numbers]

Signature. A number is a mathematical object.

Let x, y, z denote numbers.

Signature. $x + y$ is a number.

Let the sum of x and y denote $x + y$.

Signature. $x * y$ is a number.

Let the product of x and y denote $x * y$.

Signature. $-x$ is a number.

Let the negative of x denote $-x$.

Signature. 0 is a number.

Signature. 1 is a number such that $1 \neq 0$.

Signature. Assume that $x \neq 0$. $\frac{1}{x}$ is a number.

The Axioms of Fields

Axiom. $(x + y) + z = x + (y + z)$.

Axiom. $x + y = y + x$.

Axiom. $x + 0 = x$.

Axiom. $x + (-x) = 0$.

Axiom. $(x * y) * z = x * (y * z)$.

Axiom. $x * y = y * x$.

Axiom. $x * 1 = x$.

Axiom. Let $x \neq 0$. Then $x * \frac{1}{x} = 1$.

Axiom. $x * (y + z) = (x * y) + (x * z)$.

Simple Consequences

Lemma. $(y * x) + (z * x) = (y + z) * x$.

Lemma. If $x + y = x + z$ then $y = z$.

Proof. Assume $x + y = x + z$. Then

$$y = ((-x) + x) + y = (-x) + (x + y) = (-x) + (x + z) = ((-x) + x) + z = z.$$

$\square$

Lemma. If $x + y = x$ then $y = 0$.

Lemma. $-(-x) = x$.

Lemma. If $x \neq 0$ and $x * y = x * z$ then $y = z$.

Lemma. If $x \neq 0$ and $x * y = 1$ then $y = \frac{1}{x}$.

Lemma. If $x \neq 0$ then $\frac{1}{\frac{1}{x}} = x$.

Lemma. $0 * x = 0$.

Lemma. If $x \neq 0$ and $y \neq 0$ then $x * y \neq 0$.

Lemma. $(-x) * y = -(x * y)$.

Proof.

$$(x * y) + (-x * y) = (x + (-x)) * y = 0 * x = 0.$$

$\square$

Lemma. $-x = -1 * x$.

Lemma. $(-x) * (-y) = x * y$.

The Binomial Identities

Let $x - y$ stand for $x + (-y)$. Let x^2 stand for $x * x$. Let 2 stand for $1 + 1$.

Lemma. $(x + y)^2 = (x^2 + ((2 * x) * y)) + y^2$.

Proof.

$$(x + y)^2 = (x^2 + (x * y)) + ((y * x) + y^2) = (x^2 + ((x * y) + (y * x))) + y^2.$$

$\square$

Lemma. $(x - y)^2 = (x^2 - ((2 * x) * y)) + y^2$.

Proof.

$$(x - y)^2 = (x^2 - (x * y)) + (-(y * x) + (-y)^2) = (x^2 - ((x * y) + (y * x))) + y^2.$$

$\square$

Lemma. $(x + y) * (x - y) = x^2 - y^2$.

Proof.

$$(x+y)*(x-y) = (x^2+(-(x*y)))+((y*x)+(-y^2)) = x^2+(((-x*y)+(y*x))+(-y^2)) = x^2-y^2.$$

□