Vera Xie
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Problem Set 7
Math 146 Spring 2023
Due: Friday, May 5, 11:59 PM

Note: All are Matlab problems that require code submissions. All answers should be submitted in a write-up.

1. (50 points)

   (a) Use Matlab to find the singular value decomposition of the matrix A given below.

   $$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

   (b) Write code to make a set of points giving a unit circle S (Hint: Use a common parametrization of a circle, given in terms of the angle θ, and then pick a list of θ values that are equally spaced. Plot S.

   (c) Apply the matrix A to S to get the resulting ellipse, and plot it.

   (d) Does your image match what you saw in the SVD? Explain.

2. (75 points) Take the almost singular linear system $A\mathbf{x} = \mathbf{b}$ given below.

   $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 8 & 12 & 2 \\ 1 & 2 & 3+10^{-15} & 4 \\ 2 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 30 \\ 66 \\ 30 \\ 7 \end{bmatrix}$$

   (a) Use Matlab's backslash to estimate a solution to this system. What do you expect the size of the relative error to be? Based on this relative error, what part of your solution can you trust?

   (b) Use Matlab to find the SVD of the matrix A. At what singular value do you think you should truncate? Explain. What will be the resulting rank r of your matrix A_r? What is the condition number of the resulting A_r (Hint: You do not need to form A_r to answer this.)

   (c) Write Matlab code to find the solution to the now over-determined problem resulting from the rank-r approximation of A.

   (d) Does your solution satisfy $A\mathbf{x} = \mathbf{b}$? Explain. If it does not satisfy the solution, what is the residual? What is the relative difference between your solution and the solution from (a). Why would we choose to "solve" this system this way?

3. (75 points)

   (a) Load the two pictures from Matlab's repository of images, titled "mandrill.mat" and "durer.mat". What is the rank of the associated matrices? Write a Matlab script to computing the truncated SVD. For both pictures, examine a range of rank values at which to truncate, r, starting with 2 and going up by powers of 2. Provide the output. (You may want to use `subplot` to help with the presentation.)

(b) Comment on the performance of the truncated SVD for each image. Explain the difference in the effectiveness of the technique for the two images for small r.

(c) State how much storage is required as a function of r (excluding the colormap) and how much storage is required for the original image.

1. (50 points)

(a) Use Matlab to find the singular value decomposition of the matrix A given below.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

(b) Write code to make a set of points giving a unit circle S (Hint: Use a common parametrization of a circle, given in terms of the angle θ, and then pick a list of θ values that are equally spaced. Plot S.

(c) Apply the matrix A to S to get the resulting ellipse, and plot it.

(d) Does your image match what you saw in the SVD? Explain.

①

a.) use MATLAB to find the SINGULAR VALUE DECOMP. of the matrix A given below

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

⟹ MATLAB's output :

$\boxed{A = USV^T}$

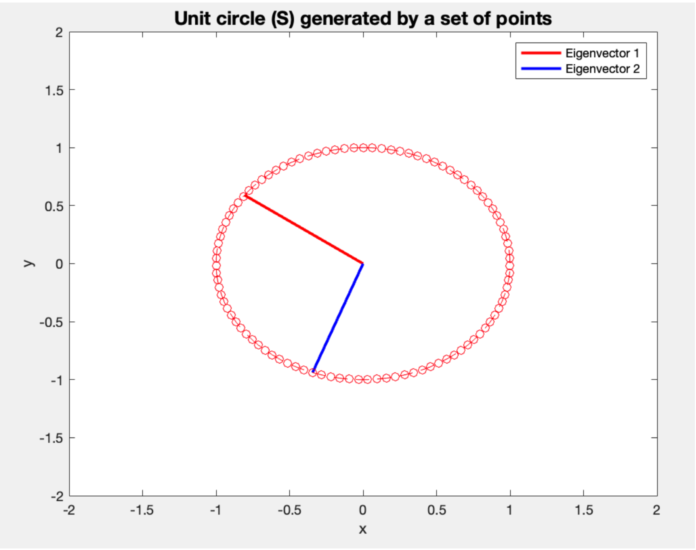
$$\boxed{\begin{aligned} U &= \begin{bmatrix} -0.3245 & -0.9459 \\ -0.9459 & 0.3245 \end{bmatrix} \\ S &= \begin{bmatrix} 6.7678 & 0 \\ 0 & 0.4433 \end{bmatrix} \\ V &= \begin{bmatrix} -0.6070 & 0.7947 \\ -0.7947 & -0.6070 \end{bmatrix} \end{aligned}}$$

b.) Write code to make a set of points giving a UNIT CIRCLE (S)

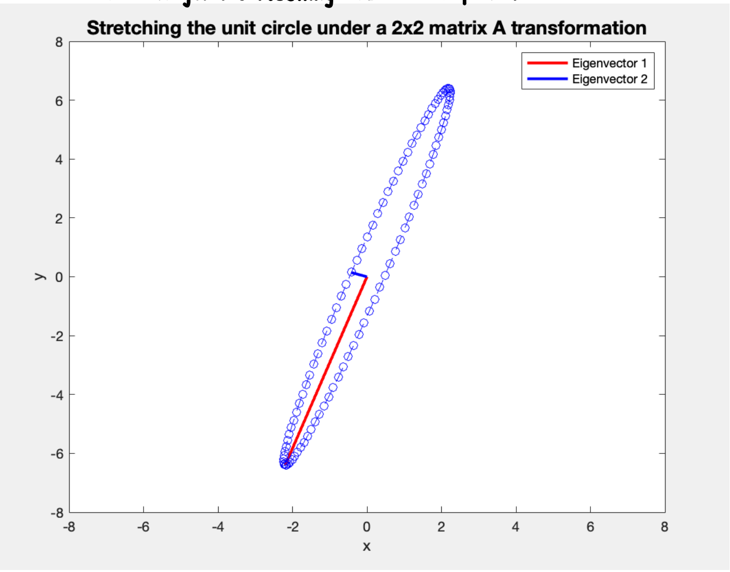
→ HINT : use a common PARAMETRIZATION of a circle, given in terms of the angle θ

Then, pick a list of θ values that are EQUALLY – SPACED

→ Plot S



c.) Apply the matrix A to S to get the resulting ELLIPSE & plot it



d.) Does your image match what you saw in the SVD? Explain

The given 2x2 matrix A acts on the right of the unit circle (S), resulting in the stretching of the unit circle that transforms the surface into a hyperellipse in R^2. The unit circle (S) stretches by the magnitude of the singular values of matrix A in the orthogonal direction of the left singular vectors of matrix A. The lengths of the hyperellipse are given by the singular values of matrix A, which are $\sigma_1 = 6.7678$ and $\sigma_2 = 0.4433$. The orientation of the principal semi-axes of the hyperellipse is given by the left singular vectors of matrix A. The left singular vectors of matrix A are given by the columns of matrix U from the singular value decomposition of matrix A, which are $\overline{u}_1 = [-0.3245; \; -0.9459]$ and $\overline{u}_2 = [-0.9459; \; 0.3245]$. Since $\sigma_1 >> \sigma_2$, the largest principal semi-axes is $\sigma_1\overline{u}_1 = [-2.1964; \; -6.4015]$. The other principal semi-axes is $\sigma_2\overline{u}_2 = [-0.4193; \; 0.1439]$. As there is a significant difference between the singular values, the condition number of matrix A is large, which is evident as K(A) = 15.2678. This causes the unit circle to undergo significant stretching under the matrix A, resulting in a hyperellipse that is elongated and narrow in shape.

2. (75 points) Take the almost singular linear system $A\mathbf{x} = \mathbf{b}$ given below.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 8 & 12 & 2 \\ 1 & 2 & 3+10^{-15} & 4 \\ 2 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 30 \\ 66 \\ 30 \\ 7 \end{bmatrix}$$

(a) Use Matlab's backslash to estimate a solution to this system. What do you expect the size of the relative error to be? Based on this relative error, what part of your solution can you trust?

(b) Use Matlab to find the SVD of the matrix A. At what singular value do you think you should truncate? Explain. What will be the resulting rank r of your matrix A_r? What is the condition number of the resulting A_r (Hint: You do not need to form A_r to answer this.)

(c) Write Matlab code to find the solution to the now over-determined problem resulting from the rank-r approximation of A.

(d) Does your solution satisfy $A\mathbf{x} = \mathbf{b}$? Explain. If it does not satisfy the solution, what is the residual? What is the relative difference between your solution and the solution from (a). Why would we choose to "solve" this system this way? Removed

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 8 & 12 & 2 \\ 1 & 2 & 3+10^{-15} & 4 \\ 2 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 30 \\ 66 \\ 30 \\ 7 \end{bmatrix}$$

a.)

As the condition number of matrix A with respect to 2-norm is $\sim 1.497 \times 10^{17}$, we lose about 17 digits of accuracy, so we can expect the size of the relative error to be $O(10)$. Since the condition number and relative error are both large, it indicates the matrix is extremely ill-conditioned. Since machine epsilon accounts for only 16 digits of accuracy, we cannot trust any part of MATLAB's backslash solution to the almost singular linear system as the condition number indicates we lose all digits of accuracy. The given matrix is close to being singular as the element in matrix A is $3 + 10^{-15}$. The addition of this extremely small perturbation is close to machine epsilon, where it accumulates round-off errors due this floating point arithmetic in the matrix.

b.)

The condition number of matrix A is defined as the ratio between the largest and smallest singular values of matrix A, which is $K(A) = \frac{\sigma_1}{\sigma_n}$. The largest and smallest singular values of matrix A are ~ 17.040608 and 0, respectively. By truncating the matrix A at the singular value 0, it removes the division by 0 when calculating the condition number, which would have resulted in a significantly large condition number where the matrix is extremely ill-conditioned. The smallest singular value of matrix A is now ~1.2762292, which effectively allows for a better condition number. Since the smallest singular value is now slightly "bigger", the ratio $\frac{\sigma_1}{\sigma_n}$ decreases, which corresponds to a smaller condition number. The condition number of the resulting matrix A_r is $K(A_r) = \frac{\sigma_1}{\sigma_r}$, where the rank (r) is 3. Thus, the condition number of A_r is

$$K(A_r) = \frac{17.040608}{1.2762292} = \sim 13.352308$$

c.)

$$\underline{x} = \begin{bmatrix} 0.999999999999995 \\ 1.999999999999999 \\ 3.000000000000000 \\ 3.999999999999997 \end{bmatrix}$$

d.)

Our solution did not satisfy $A\overline{x} = \overline{b}$ as we did not obtain the same $\overline{b}$ given on the right-hand side (i.e., $\overline{b} = [30;\ 66;\ 30;\ 7]$), instead we obtained $\overline{b} = [29.999999;\ 65.999999;\ 29.999999;\ 6.999999]$ after multiplying the given matrix A with our approximated solution $\overline{x}$. The resulting residual with respect to 2-norm is $\sim 3.79326 \times 10^{-14}$. Due to the ill-conditioning nature of the system, it is more beneficial to use truncated SVD to solve the nearly-singular linear system. By truncating the matrix to remove the smallest singular value that causes ill-conditioning, we utilize the rank-r approximation of matrix A (A_r) to obtain a more stable solution where the system is better conditioned, which is evident in the condition number of A_r. The condition number of A_r is ~ 13.352308, which is significantly less than the condition number of the original matrix A (i.e., $K(A) = 4.5208 \times 10^{16}$). Thus, it is beneficial to solve a nearly-singular or overdetermined system $A\overline{x} = \overline{b}$ using truncated SVD rather than normal SVD. Truncated SVD removes the smallest singular value that causes poor-conditioning in the system. By truncating the matrix at the r-th singular value, the rank-r approximation of matrix A becomes closer to the original matrix A, which minimizes the 2-norm between the rank-r approximation and the given matrix A.

3. (75 points)

(a) Load the two pictures from Matlab's repository of images, titled "mandrill.mat" and "durer.mat". What is the rank of the associated matrices? Write a Matlab script to computing the truncated SVD. For both pictures, examine a range of rank values at which to truncate, r, starting with 2 and going up by powers of 2. Provide the output. (You may want to use `subplot` to help with the presentation.)

(b) Comment on the performance of the truncated SVD for each image. Explain the difference in the effectiveness of the technique for the two images for small r.

(c) State how much storage is required as a function of r (excluding the colormap) and how much storage is required for the original image.

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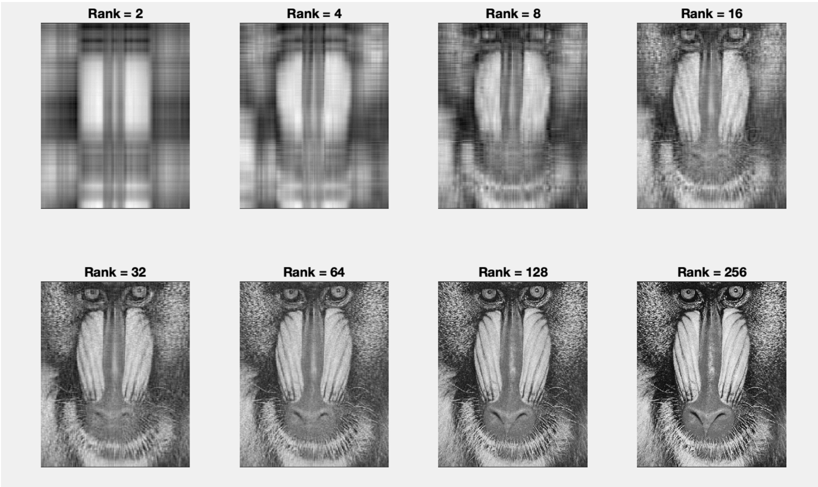
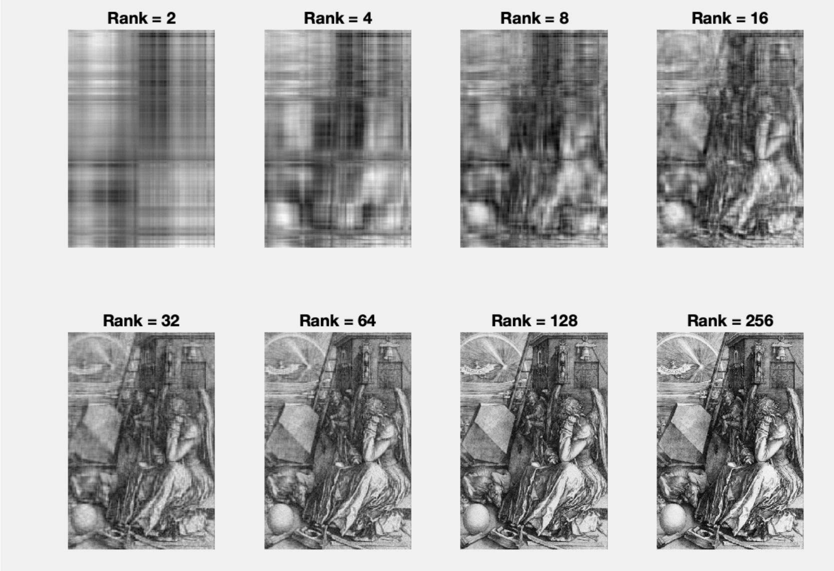
a.)

For the Mandrill image, the rank and size of the associated matrix (X) is 256 and 480 × 500, respectively. For the Durer image, the rank and size of the associated matrix (X) is 256 and 648 × 509, respectively.

⟹ MANDRILL



⟹ DURER



b.)

As rank (r) increases, the rank-r approximation of the image becomes more accurate. The truncated SVD approximation of the image keeps the most important details of the image, while removing the more subtle details, capturing more information about the image to lead to a better reconstruction of the original image. As the rank increases by powers of 2, the visual quality of the image becomes increasingly clearer, such that the image appears less pixelated and with better definition. The level of detail in an image impacts the required rank for clear display. A highly detailed image needs a higher rank, whereas a less detailed one needs a lower rank. While compressing images, it is crucial to consider the level of detail and adjust the rank to maintain quality while reducing file size. However, this comes at the cost of increasing the storage and computational cost since it requires storing more singular values and corresponding vectors. The Mandril image has a lower rank and smaller size compared to the Durer image. Thus, the Mandrill image requires less storage to approximate the image as it requires 240,000 pixels, whereas the Durer image requires 329,832 pixels. The Mandrill image has a rank of 256 and a size of 480×500, while the Durer image has a rank of 256 and a size of 648×509. This indicates the Mandrill image has fewer non-zero singular values in its SVD, resulting in a better reconstruction of the image as it is easier to approximate using truncated SVD. As a result, the performance of the truncated SVD for the Mandrill image is better than the Durer image.

c.)

Truncated SVD stores the first r singular values and their corresponding left and right singular vectors, such that we keep the first r-columns in U, the $r \times r$ submatrix of singular values in Σ, and the first r-rows in V^T. Thus, truncated SVD stores the $m \times r$ left singular vectors of U, the r singular values of the $r \times r$ diagonal matrix Σ, and the $n \times r$ right singular vectors of V. The resulting storage required for the truncated SVD approximation of an image is $r \cdot (m + n + 1) \leq mn$. If $r \leq mn / (m + n + 1)$ then the storage of the truncated SVD takes less storage otherwise, using the original uses less storage. Since the size of the associated matrix (X) of the Mandrill image is 480×500, it requires 240,000 pixels to store this image, and the compressed representation of this image using truncated SVD is $256 \cdot (480 + 500 + 1) \approx 251,136$. The original storage requirements for the Durer image is 329,832 pixels, and the compressed representation requires $256 \cdot (648 + 509 + 1) \approx 296,448$ to store this image. Depending on the shape of the original image, it will affect the storage requirement used in truncated SVD. The rank for both images is the same at $r = 256$. The shape of the Durer image resembles a rectangular shape since $m >> n$, whereas the Mandrill image appears more square since m is relatively close to n. Thus, the truncated SVD approximation of the Durer image yielded less pixels required to store the image compared to the actual original storage requirements since the difference between m and n is large.