Problem Set 4
Math 146 Spring 2023
Due Thursday, March 9, 11:59 PM

Note: Number 4(a-d) is a "pen and paper" problem for which you should show work. Numbers 1 - 3 & 4e are Matlab problems that require code submissions. All answers should be submitted in a write-up.

1. (a) (20 points) Create a function to solve the system $AX = B$, for A, X, and B, $n \times n$ matrices. The function should take as input matrices A and B and give the output X. Validate your code. Explain how you validated and provide any relevant output. Note: This only requires a slight modification to the code you created in Problem Set 3, Number 3b.

   (b) (20 points) Create a function to compute the inverse of a square $n \times n$ matrix A. The function should take as input the matrix A and give the output A^{-1}. Validate your code. Explain how you validated and provide any relevant output.

2. Included in the PS 4 Folder is a function that uses the Thomas Algorithm to solve the tridiagonal system of equations given below. Notice that the algorithm has been altered from the pseudocode provided in class in order to eliminate one of the for loops.

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x1 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \\ \vdots \\ \\ d_n \end{bmatrix} \tag{1}$$

   (a) (20 points) The function takes as input $n \times 1$ vectors **a**, **b**, **c**, and **d**. Modify the algorithm to make a new function, Thomas_2, which takes as input the tridiagonal matrix A and vector **d** instead. Include in your function errors if A is not a square tridiagonal matrix. (Hint: Use the command `isbanded`.) Validate your code by creating a sparse 1000×1000 tridiagonal matrix made up of random numbers from 0 to 50 and a similarly formed vector for **d**. Compare your solution to the solution found by using Matlab backslash, and report the relative difference between the two solutions, with respect to the max norm.

   (b) (30 BONUS points) Modify the algorithm to solve problems of the form

$$\begin{bmatrix} b_1 & c_1 & & & a_1 \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x1 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \\ \vdots \\ \\ d_n \end{bmatrix}, \tag{2}$$

   where all terms not on the tridiagonal or in the upper right corner are zeros. Show any work you did or explain how you made your modification. (You may want to explore the original algorithm a little first.) Validate your code by solving the system

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 0 & -1 \\ 2 & -2 & 3 & 0 & 0 & 0 \\ 0 & -2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -4 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x1 \\ \\ \vdots \\ \\ x_6 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \\ 9 \\ 22 \\ -34 \\ 17 \end{bmatrix}. \tag{3}$$

The exact solution is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$. Report your relative error, with respect to the 1-norm, 2-norm, and max norm.

3. (a) (32 points) Create a function to implement the Cholesky factorization pseudocode from class. Your function should take as input an $n \times n$ positive definite symmetric matrix A and give output $R^\intercal$. Validate your code by factoring the symmetric positive definite matrix $A = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$. Explain how you then verify that your solution is correct.

   (b) (18 points) Create a function that takes as input an $n \times n$ positive definite symmetric matrix A and a vector $\mathbf{b}$ and uses your Cholesky factorization from part (a) to solve the linear system $A\mathbf{x} = \mathbf{b}$. Validate your code. Explain how you validated the code and provide any relevant output.

4. You would like to fit the following data with a second order polynomial (a parabola).

x	y
-2	9
-1	5
0	3
1	4
2	8
3	12

   (a) (12 points) Let the polynomial be $p(x) = c_0 + c_1 x + c_2 x^2$. Write by hand the 6 equations given to us by this data. Also give the system of equations in matrix form.

   (b) (17 points) Find the normal equations to this system.

   (c) (22 points) Solve the normal equations by hand. What polynomial gives the Least Squares best fit polynomial for this data?

   (d) (17 points) Find the residual and its magnitude.

   (e) (22 points) Confirm your work in Matlab. Note: You can solve the normal equations using function from Problem (3b) since $A^\intercal A$ is a symmetric positive definite matrix. Plot the data and your Least Squares polynomial.