Name: Omansh Mathur

Roll no: 17123008

Branch: Maths & Computing.

Assignment – Functional Analysis

Q1) Let (X, d) be a complete metric space and $S \subseteq X$. Let $x \in \bar{S}$. Then $\exists$ a sequence $(x_n)_{n \in N} \subseteq S$ converging to x. Obviously this sequence is a cauchy sequence and since S is complete (acc to ques), it converges to some $\bar{x} \in S$. Since the limit of a sequence is unique in a metric space, we see that $x = \bar{x} \in S$. Hence we can say that S is closed.

Hence proved.

Q2 Given the map $f: x \to ||x||$ on $\mathbb{R}^n$

We can see that f is a norm.

Thus distance of $f(x)$ & $f(y) = f(x-y)$.

Also, for norms $f(x-y) = f(y-x)$.

Now, f is continuous if for all $\varepsilon > 0$

$\exists \delta > 0$ st if $|x-y| < \varepsilon$, then

$|f(x) - f(y)| < \delta$.

Here, it suffices to take $\delta = \varepsilon$.

Suppose distance from x & y $< \delta$. ie

$f(x-y) = f(y-x) < \varepsilon$. Now, norms

follow $f(x) \leq f(y) + f(x-y)$

$f(y) \leq f(x) + f(y-x)$

So $f(y) - f(x) \leq f(y-x) < \varepsilon$

and $f(x) - f(y) \leq f(x-y) < \varepsilon$

Thus $|f(x) - f(y)| < \varepsilon$

Hence proved.

Q3 The map $||x|| \to \max\{|x_1|, |x_2|, |x_3|\}$ on R^3 is banach space.

→ It is obvious that R^3 with the norm $||x|| = \max\{|x_1|, |x_2|, |x_3|\}$ is a normed space.

We need to show that it is complete.

To prove: The linear space l^∞ with norm $||x||_\infty = \sup_{1 < \xi_i < \infty} |\xi_i|$ is a banach space. (This is to show that sup norm on l^∞ is banach space)

Here $||x|| = \sup_{1 < i < 3} |\xi_i|$ $x = \{\xi_i\} \in l^\infty$

Let $\{x_m\}$ be a cauchy sequence in l^∞. Then for each $e > 0$ $\exists$ positive integer N such that

$$||x_m - x_n|| = \sup_{1 \le i \le 3} |\xi_i^{(m)} - \xi_i^{(n)}| < e$$

$$\Rightarrow |\xi_i^{(m)} - \xi_i^{(n)}| < e \quad \forall\, m, n \geq N$$

$$i = (1, 2, 3)$$

— ①

This shows that for each fixed i $(1 \le i \le 3)$ the sequence $\{\xi_i^{(m)}\}$ is cauchy seq.
Since $\mathbb{R}$ is complete, it converges in $\mathbb{R}$.

Let $\xi_i^{(m)} \to \xi_i$ as $m \to \infty$.
Using these limits, we define $x = (\xi_1, \xi_2, \xi_3)$ and show that $x \in l^\infty$ and $x_m \to x$
Letting $n \to \infty$ in ①, we get.

$$|\xi_i^{(m)} - \xi_i| \le e \quad \forall\, m \geq N \quad (i = 1, 2, 3 \ldots)$$

Since $x_m \in l^\infty$, $\exists$ real no R_m st

$$|\xi_i^{(m)}| \le R_m \;\forall\, i$$

Thus $|\xi_i| = |\xi_i - \xi_i^{(m)} + \xi_i^{(m)}|$

$$\le |\xi_i^{(m)} - \xi_i| + |\xi_i^{(m)}| \quad (\text{triangle inequality})$$

$$\le e + R_m \quad \forall \quad m \geq N$$

This shows that RHS is independent of i & true for all i. Hence $|\xi_i|$ is a bounded sequence of numbers.

This implies $x = \{\xi_i\} \in l^\infty$

Further, from (2), we get

$$\|x_m - x\| = \sup_{1 \le i \le 3} |\xi_i^{(m)} - \xi_i| \le \epsilon$$

$\therefore x_m \to x$ in l^∞ $\therefore$ l^∞ is banach space.

Hence proved that $\|x\| = \max\{|x_1|, |x_2|, |x_3|\}$ on $\mathbb{R}^3$ is banach space.

Q4 Suppose $T: X \to Y$ is bounded, $\exists\, C > 0$ st $\|Tx\| \leq C\|x\|$ $\forall x \in X$. Take any bounded subset A of X. $\exists\, M_A > 0$ st $\|x\| \leq M_A$ $\forall x \in A$. For any $x \in A$

$$\|Tx\| \leq C\|x\| \leq C M_A$$

This shows that T maps bounded set in X to bounded sets in Y.

Conversely Let $T: X \to Y$ map bounded sets in X to bounded sets in Y. This means that for any fixed $R > 0$, $\exists\, M_R > 0$ st $\|x\| \leq R \Rightarrow \|Tx\| \leq M_R$. Now, take any non-zero $y \in X$ and set

$$x = R\frac{y}{\|y\|} \Rightarrow \|x\| = R.$$

Thus $\frac{R}{\|y\|}\|Ty\| = \left\|T\left(\frac{Ry}{\|y\|}\right)\right\| = \|Tz\|$

and $\|Tz\| \leq M_R \Rightarrow \|Ty\| \leq \frac{M_R}{R}\|y\|$

Here, we crucially use the linearity of T.

Taking supremum over all y, we show that T is bounded