12 Unsteady Conduction with Robin Conditions

This example illustrates the use of Robin boundary conditions for the unsteady heat equation

$$\frac{\partial u}{\partial t} = \nabla \cdot \nu \nabla u, \text{ in } \Omega, \qquad \alpha u + \beta \nabla u \cdot \hat{\mathbf{n}} = \gamma \text{ on } \partial\Omega_R,$$

with either Dirichlet or Neumann conditions on the remainder of the domain boundary, $\partial\Omega \backslash \partial\Omega_R$. In this case, we consider the unit square, $\Omega = [0,1]^2$, with $\partial\Omega_R := [1, y]$ and homogeneous Neumann conditions, $\nabla u \cdot \hat{\mathbf{n}} = 0$, elsewhere. Without loss of generality, we consider the case with $\gamma = 0$ and initial condition $u(t = 0, x, y)$=1.

Under the stated conditions, a series solution exists of the form $\sum_k \hat{u}_k(t)\phi_k(x)$ with

$$\phi_k(x) = \cos \sqrt{\lambda_k} x, \qquad \sqrt{\lambda_k} \tan \sqrt{\lambda_k} = \frac{\alpha}{\beta}.$$

For example, for $\alpha/\beta = 2$, the first eigenvalue would be $\sqrt{\lambda_1} \approx 1.077$. After an initial transient in which high wave-number components decay rapidly, we would expect

$$||u|| \sim C e^{-\lambda_1 \nu t} \tag{20}$$

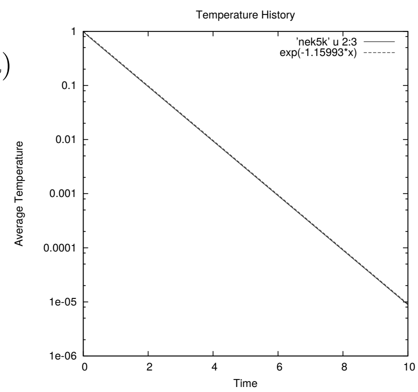
for an order-unity constant C. We can check this hypothesis with a simple run.

Setting up a Nek Run. In Nek5000, Robin boundary conditions are cast in the form of a Newton law of cooling. On $\partial\Omega_R$, we have $-\nu\nabla T \cdot \hat{\mathbf{n}} = h(T - T_\infty)$, where T (=u) is the unknown scalar field, T_∞ is the ambient temperature outside the computational domain, and $h \geq 0$ is the dimensional heat transfer coefficient governing the heat flux between $\partial\Omega_R$ and the ambient surroundings. In the conditions of our example, we would have T_∞=0, $\alpha = h$, and $\beta = \nu$. There are only two free parameters in this case and they are set in subroutine `userbc()` as `tinf` and `hc`. The following code tracks the mean temperature history in `usrchk()`.

```
    n=nx1*ny1*nz1*nelt
    tbar=glsc2(bm1,t,n)/voltm1 ! BM1=diag. mass matrix on Mesh 1
    if (nid.eq.0) write(6,1) istep,time,tbar,' tbar '
  1 format(i9,1p2e18.10,a6)
```

The results for ν=1 are plotted in the figure below along with $e^{-\lambda_1 \nu t}$.

(a)



(b)