

Chapter-11

Limits, Continuity and Differentiability

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I. SUBJECTIVE PROBLEMS

- 1) Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for real x . If $f'(0)$ exists, find its value. (1987 - 2 Marks)
- 2) Find the values of a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2} \sin x & \text{if } 0 \leq x < 4 \\ 2x \cot x + b & \text{if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ a \cos 2x - b \sin x & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases}$$
 is continuous for $0 \leq x \leq \pi$. (1989 - 2 Marks)
- 3) Draw a graph of the function $y = [x] + |1 - x|$, $-1 \leq x \leq 3$. Determine the points, if any, where this function is not differentiable. (1989 - 4 Marks)
- 4) Let $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & \text{if } x > 0 \end{cases}$
 Determine the value of a , if possible, so that the function is continuous at $x = 0$. (1990 - 4 Marks)
- 5) A function $f : R \rightarrow R$ satisfies the equation $f(x+y) = f(x)f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R . Let the function be differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R . Hence, determine $f(x)$. (1990 - 4 Marks)
- 6) Find $\lim_{x \rightarrow 0} \{\tan(\frac{\pi}{4} + x)\}^{\frac{1}{x}}$ (1993 - 2 Marks)
- 7) Let $f(x) = \begin{cases} \{1 + |\sin x|\}^{\frac{a}{|\sin x|}} & \text{if } -\frac{\pi}{6} < x < 0 \\ b & \text{if } x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & \text{if } 0 < x < \frac{\pi}{6} \end{cases}$
 Determine a and b such that $f(x)$ is continuous at $x = 0$ (1994 - 4 Marks)
- 8) Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for real x and y . If $f'(0)$ exists and equals -1 and $f(0) = 1$, find $f(2)$. (1995 - 5 Marks)
- 9) Determine the values of x for which the following function fails to be continuous or differentiable:

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ (1 - x)(2 - x), & \text{if } 1 \leq x \leq 2 \\ 3 - x, & \text{if } x > 2 \end{cases}$$

Justify your answer. (1997 - 5 Marks)

- 10) Let $f(x), x \geq 0$, be a non-negative continuous function, and let $F(x) = \int_0^x f(t) dt, x \geq 0$. If for some $c > 0, f(x) \leq cF(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$. (2001 - 5 Marks)
- 11) Let $\alpha \in R$. Prove that a function $f; R \rightarrow R$ is differentiable at α if and only if there is a function $g : R \rightarrow R$ is differentiable at continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. (1995 - 5 Marks)
- 12) Let $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ |x + a| & \text{if } x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x - 1)^2 + b & \text{if } x \geq 0 \end{cases}$, where a and b are non-negative real numbers. Determine the composite function $(g \circ f)(x)$ is continuous for all real x , determine the values of a and b . Further, for these values of a and b , is $g \circ f$ is differentiable at $x = 0$? Justify your answer. (2002 - 5 Marks)
- 13) If a function $[-2a, 2a] \rightarrow R$ is an function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivatives at $x = a$ is 0 then find the left hand derivative at $x = -a$. (2003 - 2 Marks)
- 14) $f'(0) = \lim_{n \rightarrow \infty} n f(\frac{1}{n})$ and $f(0) = 0$. Using this find $\lim_{x \rightarrow \infty} \left((n+1)^{\frac{2}{\pi}} \cos^{-1}(\frac{1}{n}) - n \right), \left| \cos^{-1} \frac{1}{n} \right| < \frac{\pi}{2}$ (2004 - 2 Marks)
- 15) If $|c| \leq \frac{1}{2}$ and $f(x)$ is a differentiable function at $x = 0$ given by

$$f(x) = \begin{cases} b \sin^{-1} \left(\frac{c+x}{2} \right) & \text{if } -\frac{1}{2} < x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{e^{\frac{ax}{2}} - 1}{x} & \text{if } 0 < x < \frac{1}{2} \end{cases}$$

Find the value of 'a' and prove that $b^2 = 4 - c^2$.
(2004 - 4 Marks)