Chapter-11- Limits,Continuity and Differentiability

EE24BTECH11051 - Prajwal

I. Subjective Problems

1) Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for real x.If $f'(0)$ exists, find its value. (1987 - 2 Marks)
2) Find the values of a and b so that the function
$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & \text{if } 0 \le x < 4 \\ 2x\cot x + b & \text{if } \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & \text{if } \frac{\pi}{2} < x \le \pi \end{cases}$$
is continuous for $0 \le x \le \pi$. (1989 - 2 Marks)
3) Draw a graph of the function $y = [x] + |1-x|, -1 \le x \le 3$. Determine the points, if any, where this function is not differentiable. (1989 - 4 Marks)
4) Let $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & \text{if } x > 0 \end{cases}$
Determine the value of a, if possible, so that the function is continuous at $x = 0$. (1990 - 4 Marks)
5) A function $f : R \to R$ satisfies the equation $f(x+y) = f(x)f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R. Let the function be differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R. Hence, determine $f(x)$. (1990 - 4 Marks)
6) Find $\lim_{x\to 0}\{\tan(\frac{\pi}{4} + x)\}^{\frac{1}{x}}$ (1993 - 2 Marks)
7) Let $f(x) = \begin{cases} \{1 + |\sin x|\}^{\frac{a}{|\sin x|}} & \text{if } -\frac{\pi}{6} < x < 0 \\ b & \text{if } x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & \text{if } 0 < x < \frac{\pi}{6} \end{cases}$
Determine a and b such that $f(x)$ is continuous at $x = 0$ (1994 - 4 Marks)
8) Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for real x and y. If $f'(0)$ exists and equals -1 and $f(0) = 1$, find $f(2)$. (1995 - 5 Marks)
9) Determine the values of x for which the following function fails to be continuous or diff-ferentiable:
$$f(x) = \begin{cases} 1-x, & \text{if } x < 1 \\ (1-x)(2-x), & \text{if } 1 \le x \le 2 \\ 3-x, & \text{if } x > 2 \end{cases}$$
Justify your answer. (1997 - 5 Marks)
10) Let $f(x), x \ge 0$, be a non-negative continuous function, and let $F(x) = \int_0^x f(t)\,dt, x \ge 0$. If for some $c > 0, f(x) \le cF(x)$ for all $x \ge 0$, then show that $f(x) = 0$ for all $x \ge 0$. (2001 - 5 Marks)
11) Let $\alpha \in R$.Prove that a function $f; R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ is differentiable at continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. (1995 - 5 Marks)
12) Let $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ |x + a| & \text{if } x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0, \end{cases}$ where a and b are non-negative real numbers. Determine the composite function $(gof)(x)$ is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof is differentiable at $x = 0$? Justify your answer. (2002 - 5 Marks)
13) If a function $[-2a, 2a] \to R$ is an function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivatives at $x = a$ is 0 then find the left hand derivative at $x = -a$. (2003 - 2 Marks)
14) $\prime f(0) = \lim_{n\to\infty} nf(\frac{1}{n})$ and $f(0) = 0$. Using this find
$\lim_{x\to\infty}\left((n+1)\frac{2}{\pi}\cos^{-1}(\frac{1}{n}) - n\right), \left|\cos^{-1}\frac{1}{n}\right| < \frac{\pi}{2}$ (2004 - 2 Marks)
15) If $|c| \le \frac{1}{2}$ and $f(x)$ is a diiferentiable function at $x = 0$ given by
$$f(x) = \begin{cases} b\sin^{-1}\left(\frac{c+x}{2}\right) & \text{if } -\frac{1}{2} < x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{e^{\frac{ax}{2}}-1}{x} & \text{if } 0 < x < \frac{1}{2} \end{cases}$$
Find the value of $'a'$ and prove that $b^2 = 4 - c^2$. (2004 - 4 Marks)