

# Chapter-11

## Limits, Continuity and Differentiability

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### I. SUBJECTIVE PROBLEMS

- 1) Let  $f(x)$  be a function satisfying the condition  $f(-x) = f(x)$  for real  $x$ . If  $f'(0)$  exists, find its value. (1987 - 2 Marks)
- 2) Find the values of  $a$  and  $b$  so that the function
 
$$f(x) = \begin{cases} x + a\sqrt{2} \sin x & \text{if } 0 \leq x < 4 \\ 2x \cot x + b & \text{if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ a \cos 2x - b \sin x & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases}$$
 is continuous for  $0 \leq x \leq \pi$ . (1989 - 2 Marks)
- 3) Draw a graph of the function  $y = [x] + |1 - x|$ ,  $-1 \leq x \leq 3$ . Determine the points, if any, where this function is not differentiable. (1989 - 4 Marks)
- 4) Let  $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & \text{if } x > 0 \end{cases}$ 
 Determine the value of  $a$ , if possible, so that the function is continuous at  $x = 0$ . (1990 - 4 Marks)
- 5) A function  $f : R \rightarrow R$  satisfies the equation  $f(x+y) = f(x)f(y)$  for all  $x, y$  in  $R$  and  $f(x) \neq 0$  for any  $x$  in  $R$ . Let the function be differentiable at  $x = 0$  and  $f'(0) = 2$ . Show that  $f'(x) = 2f(x)$  for all  $x$  in  $R$ . Hence, determine  $f(x)$ . (1990 - 4 Marks)
- 6) Find  $\lim_{x \rightarrow 0} \{\tan(\frac{\pi}{4} + x)\}^{\frac{1}{x}}$  (1993 - 2 Marks)
- 7) Let  $f(x) = \begin{cases} \{1 + |\sin x|\}^{\frac{a}{|\sin x|}} & \text{if } -\frac{\pi}{6} < x < 0 \\ b & \text{if } x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & \text{if } 0 < x < \frac{\pi}{6} \end{cases}$ 
 Determine  $a$  and  $b$  such that  $f(x)$  is continuous at  $x = 0$  (1994 - 4 Marks)
- 8) Let  $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$  for real  $x$  and  $y$ . If  $f'(0)$  exists and equals  $-1$  and  $f(0) = 1$ , find  $f(2)$ . (1995 - 5 Marks)
- 9) Determine the values of  $x$  for which the following function fails to be continuous or differentiable:

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ (1 - x)(2 - x), & \text{if } 1 \leq x \leq 2 \\ 3 - x, & \text{if } x > 2 \end{cases}$$

Justify your answer. (1997 - 5 Marks)

- 10) Let  $f(x), x \geq 0$ , be a non-negative continuous function, and let  $F(x) = \int_0^x f(t) dt, x \geq 0$ . If for some  $c > 0, f(x) \leq cF(x)$  for all  $x \geq 0$ , then show that  $f(x) = 0$  for all  $x \geq 0$ . (2001 - 5 Marks)
- 11) Let  $\alpha \in R$ . Prove that a function  $f; R \rightarrow R$  is differentiable at  $\alpha$  if and only if there is a function  $g : R \rightarrow R$  is differentiable at continuous at  $\alpha$  and satisfies  $f(x) - f(\alpha) = g(x)(x - \alpha)$  for all  $x \in R$ . (1995 - 5 Marks)
- 12) Let  $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ |x + a| & \text{if } x \geq 0 \end{cases}$  and  $g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x - 1)^2 + b & \text{if } x \geq 0 \end{cases}$  where  $a$  and  $b$  are non-negative real numbers. Determine the composite function  $(g \circ f)(x)$  is continuous for all real  $x$ , determine the values of  $a$  and  $b$ . Further, for these values of  $a$  and  $b$ , is  $g \circ f$  is differentiable at  $x = 0$ ? Justify your answer. (2002 - 5 Marks)
- 13) If a function  $[-2a, 2a] \rightarrow R$  is an function such that  $f(x) = f(2a - x)$  for  $x \in [a, 2a]$  and the left hand derivatives at  $x = a$  is 0 then find the left hand derivative at  $x = -a$ . (2003 - 2 Marks)
- 14)  $f'(0) = \lim_{n \rightarrow \infty} n f(\frac{1}{n})$  and  $f(0) = 0$ . Using this find  $\lim_{x \rightarrow \infty} \left( (n+1)^{\frac{2}{\pi}} \cos^{-1}(\frac{1}{n}) - n \right), |\cos^{-1} \frac{1}{n}| < \frac{\pi}{2}$  (2004 - 2 Marks)
- 15) If  $|c| \leq \frac{1}{2}$  and  $f(x)$  is a differentiable function at  $x = 0$  given by

$$f(x) = \begin{cases} b \sin^{-1} \left( \frac{c+x}{2} \right) & \text{if } -\frac{1}{2} < x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{e^{\frac{ax}{2}} - 1}{x} & \text{if } 0 < x < \frac{1}{2} \end{cases}$$

Find the value of 'a' and prove that  $b^2 = 4 - c^2$ .  
(2004 - 4 Marks)