Assignment 10

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Download the latex-tikz codes from

https://github.com/sachinomdubey/Matrix−theory/Assignment10

1 Problem

(Hoffman/Page123/8) : If F is a field and h is a polynomial over F of degree ≥ 1, show that the mapping $f \to f(h)$ is a one-one linear transformation of $F[x]$ into $F[x]$. Show that this transformation is an isomorphism of $F[x]$ onto $F[x]$ if and only if deg $h = 1$.

2 Solution

Here, $F[x]$ is a set of polynomials over field F, written as:

$$F[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \quad | \quad a_i \in F \right\} \tag{2.0.1}$$

Let,

$$G(f) = f(h) \tag{2.0.2}$$

Thus, $G(f)$ is clearly a function from $F[x]$ into $F[x]$. Now, we need to show that the function G is one-one linear transformation. Let us first show that G is a linear transformation:

$$\text{Let, } f, g \in F[x] \text{ and } \alpha \in F$$

$$\begin{aligned} G(\alpha f + g) &= (\alpha f + g)(h) \\ &= (\alpha f)(h) + g(h) \\ &= \alpha f(h) + g(h) \\ &= \alpha G(f) + G(g) \end{aligned} \tag{2.0.3}$$

From (2.0.3), G is a linear transformation.
For G to be one-one linear transformation, it should map a set of linearly independent polynomials in $F(x)$ to another set of linearly independent polynomials in $F(x)$. let us consider the following basis set for $F(x)$:

$$S = \{f_0, f_1, f_2, f_3, f_4, \ldots\} \tag{2.0.4}$$

Where,

$$f_i = x^i \tag{2.0.5}$$

Since, the set S forms the basis for $F(x)$, the set S is a set of linearly independent polynomials. Let us apply the transformation G to set S, then we obtain another set S' as:

$$S' = \{f_0(h), f_1(h), f_2(h), f_3(h), f_4(h), \ldots\} \tag{2.0.6}$$

Where,

$$f_i = x^i \tag{2.0.7}$$

Here, The degree of each polynomial in set S' is distinct and given by $i{\cdot}\deg(h)$. Thus, set S' is also a set of linearly independent polynomials.

Conclusion: G will maps any arbitrary set S_a of linearly independent polynomials in $F(x)$ to another set S'_a of linearly independent polynomials in $F(x)$. (Since any arbitrary set S_a can be written in terms of basis set S). Hence, <u>G is one-one linear transformation</u>.

Now, Let us prove that G is an isomorphism of $F(x)$ onto $F(x)$ if and only if $\deg(h) = 1$.
Let $\deg(h) = 1$, then h can be written as:

$$h = a + bx, \quad \text{Where, } b \neq 0 \tag{2.0.8}$$

Let us define h' such that:

$$h' = \frac{1}{b}x - \frac{a}{b} \tag{2.0.9}$$

Let G' be the linear transformation from $F(x)$ to $F(x)$ given by:

$$G'(f) = f\left(\frac{1}{b}x - \frac{a}{b}\right) \tag{2.0.10}$$

It can be shown that G' is inverse of G as follow:

$$G(G'(f)) = G\left(f\left(\frac{1}{b}x - \frac{a}{b}\right)\right) \tag{2.0.11}$$

$$= f\left(a\left(\frac{1}{a}x - \frac{b}{a}\right) + b\right) \tag{2.0.12}$$

$$= f(x) \tag{2.0.13}$$

Similarly,

$$G'(G(f)) = G'\left(f\left(ax + b\right)\right) \tag{2.0.14}$$

$$= f\left(\frac{1}{a}\left(ax + b\right) - \frac{b}{a}\right) \tag{2.0.15}$$

$$= f(x) \tag{2.0.16}$$

Thus, G' is inverse of G. Therefore, G is isomorphism and we can say:

$$\boxed{\deg(h) = 1 \implies G \text{ is isomorphism.}} \tag{2.0.17}$$

Let $\deg(h) > 1$, then

$$\deg f(h) = \deg f \cdot \deg h \tag{2.0.18}$$

$$\implies \deg f(h) \geq 1 \tag{2.0.19}$$

$$\implies G(f) = f(h) \neq x \tag{2.0.20}$$

This means the image of G does not contain polynomials of degree one. Hence G is not onto and therefore G can not be an isomorphism. Thus we can write:

$$\boxed{\deg(h) > 1 \implies G \text{ is not isomorphism.}} \tag{2.0.21}$$

From (2.0.17) and (2.0.21), We can conclude:

$$\boxed{G \text{ is isomorphism.} \iff \deg(h) = 1} \tag{2.0.22}$$