Assignment 12

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Download the latex-tikz codes from

https://github.com/sachinomdubey/Matrix−theory/Assignment12

1 Problem

(Hoffman/Page261/5) :
Let $\mathbf{T}$ be the linear operator on $\mathbf{R^8}$ which is represented in the standard basis by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.0.1}$$

1) Find the characteristic polynomial and the invariant factors.
2) Find the primary decomposition of $\mathbf{R^8}$ under $\mathbf{T}$ and the projections on the primary components. Find cyclic decompositions of each primary component as in Theorem 3.
3) Find the Jordan form of $\mathbf{A}$.
4) Find a direct-sum decomposition of $\mathbf{R^8}$ into T-cyclic subspaces as in Theorem 3.

2 Definition and Result used

Characteristic Polynomial	The characteristic polynomial of a $n \times n$ matrix $\mathbf{A}$ is given by: $\boxed{\det(\lambda\mathbf{I} - \mathbf{A})}$ Where, $\mathbf{I}$ is $n \times n$ identity matrix
Minimal Polynomial	The minimal polynomial of an $n \times n$ matrix $\mathbf{A}$ over a field F is the monic polynomial P over F of least degree such that $P(\mathbf{A}) = 0$.
Invariant factors	The invariant factors of a $n \times n$ matrix $\mathbf{A}$ are: $f_1,\ f_2,\ f_3,\ \cdots,\ f_n$ Where $f_1,\ f_2,\ \cdots,\ f_n$ are monic non-zero elements of $F[x]$ (Set of all polynomials over field F) and satisfy the following:

	• f_1 divides f_2, which in turn divides f_3, and so on, denoted as: $f_1 \mid f_2 \mid f_3 \mid \cdots \mid f_n$ • f_n is the minimal polynomial of $\mathbf{A}$ • The product $f_1 f_2 f_3 f_4 \cdots f_n = \mathrm{char}_A(x) = \det(x\mathbf{I} - \mathbf{A})$
Primary decomposition theorem	Let $\mathbf{T}$ be a linear operator on the Finite-dimensional vector space $\mathbf{V}$ over the field F. Let p be the minimal polynomial for $\mathbf{T}$, $$p = p_1^{r_1} \cdots p_k^{r_k} \quad (2.0.1)$$ where the p_i are distinct irreducible monic polynomials over F and the r_i are positive integers. Let W_i be the null space of $p_i(T)^{r_i}$, $i = 1, \cdots, k$. Then • $\mathbf{V} = \mathbf{W}_1 \oplus ... \oplus \mathbf{W}_k$; • Each $\mathbf{W}_i$ is invariant under $\mathbf{T}$ • If $\mathbf{T}_i$ is the operator induced on $\mathbf{W}_i$ by $\mathbf{T}$, then the minimal polynomial for $\mathbf{T}_i$ is $p_i^{r_i}$
Projections associated with direct decomposition of a vector space.	If $\mathbf{V} = \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_k$ then there exist k linear operators (called projections) $\mathbf{E}_1, \cdots, \mathbf{E}_k$ on $\mathbf{V}$ such that: • Each $\mathbf{E}_i$ is a projection ($\mathbf{E}_i^2 = \mathbf{E}_i$); • $\mathbf{E}_i \mathbf{E}_j = 0$, if $i \neq j$; • $\mathbf{I} = \mathbf{E_1} + \cdots + \mathbf{E_k}$ Also, for $i \in [1, k]$, $$\mathbf{E}_i(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{for } \mathbf{v} \in \mathbf{W}_i \\ 0 & \text{for } \mathbf{v} \notin \mathbf{W}_i \end{cases} \quad (2.0.2)$$
Cyclic decomposition theorem	Let T be a linear operator on a finite-dimensional vector space $\mathbf{V}$ and let $\mathbf{W}_0$ be a proper T-admissible subspace of $\mathbf{V}$. There exists non zero vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$ in $\mathbf{V}$ with respective T-annihilators $p_1, p_2, \ldots, p_r$ such that: • $\mathbf{V} = \mathbf{W}_0 \oplus \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \oplus \ldots \oplus \mathbf{Z}(\alpha_r; T)$ • p_k divides p_{k-1}, $k = 2, \ldots, r$

	Here, the T-cyclic subspace $\mathbf{Z}(\alpha_i;T)$ is defined as : • $\mathbf{Z}(\alpha_i;T) = \text{Span}\left\{\alpha_i, \mathbf{T}\alpha_i, \ldots, \mathbf{T}^{k-1}\alpha_i\right\}$ • Where k = Degree of p_i
Jordan form of a matrix	Every matrix $\mathbf{A}$ can be expressed as: $$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J} \tag{2.0.3}$$ Where $\mathbf{J}$ is an upper triangular matrix of a particular form called a Jordan matrix. Matrix $\mathbf{J}$ is of the form: $$\mathbf{J} = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix} \qquad \text{Where, } J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

3 Solution

Characteristic polynomial	
Finding the Characte- -ristic polynomial	The linear operator $\mathbf{T}$ is represented in standard basis by matrix $\mathbf{A}$ given as: $$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.1}$$ The characteristic polynomial is given by: $$\left\|\lambda\mathbf{I} - \mathbf{A}\right\| = \begin{vmatrix} \lambda-1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & \lambda-1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & \lambda-1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & \lambda-1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} \tag{3.0.2}$$

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$$= (\lambda - 1)\begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & \lambda - 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & \lambda - 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & \lambda - 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} \tag{3.0.3}$$

$$= (\lambda - 1)\lambda\begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ -1 & -1 & \lambda & 0 & 0 & 0 \\ 0 & 0 & -1 & \lambda - 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & \lambda - 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & \lambda - 1 \end{vmatrix} \tag{3.0.4}$$

$$= (\lambda - 1)^4\lambda^4 = \boxed{\lambda^8 - 4\lambda^7 + 6\lambda^6 - 4\lambda^5 + \lambda^4} \tag{3.0.5}$$

This is the required characteristic polynomial.

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Invariant factors

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<td>Finding the Invariant factors</td>
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From the obtained characteristic polynomials, Let us find the minimum polynomial $p(\lambda)$ which satisfies the condition $p(\mathbf{A}) = 0$ and has least degree. Starting from smallest degree and moving up:

Consider $\quad (\lambda - 1)\lambda = \lambda^2 - \lambda$
Verification:
$p(\mathbf{A}) = \mathbf{A}^2 - \mathbf{A}$

$$= \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & 2 & 1 & 0 & 2 \\ 0 & -2 & -2 & -2 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$p(\mathbf{A}) \neq \mathbf{0}$
Thus, $\lambda^2 - \lambda$ is not our minimal polynomial

Consider $\quad (\lambda - 1)\lambda^2 = \lambda^3 - \lambda^2$
Verification:
$p(\mathbf{A}) = \mathbf{A}^3 - \mathbf{A}^2$

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$$= \begin{pmatrix} 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 3 & 3 & 3 & 1 & 0 & 3 \\ 0 & -3 & -3 & -3 & -3 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & 2 & 1 & 0 & 2 \\ 0 & -2 & -2 & -2 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$p(\mathbf{A}) \neq \mathbf{0}$

Thus, $\lambda^3 - \lambda^2$ is not our minimal polynomial

Consider $(\lambda - 1)^2 \lambda = \lambda^3 - 2\lambda^2 + \lambda$

Verification:

$p(\mathbf{A}) = \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A}$

$$= \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$p(\mathbf{A}) \neq \mathbf{0}$

Thus, $\lambda^3 - 2\lambda^2 + \lambda$ is not our minimal polynomial

Consider $(\lambda - 1)^2 \lambda^2 = \lambda^4 - 2\lambda^3 + \lambda^2$

Verification:

$p(\mathbf{A}) = \mathbf{A}^4 - 2\mathbf{A}^3 + \mathbf{A}^2$

$$= \begin{pmatrix} 1 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 4 & 4 & 4 & 4 & 1 & 0 & 4 \\ 0 & -4 & -4 & -4 & -4 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 3 & 3 & 3 & 1 & 0 & 3 \\ 0 & -3 & -3 & -3 & -3 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & 2 & 1 & 0 & 2 \\ 0 & -2 & -2 & -2 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore p(\mathbf{A}) = \mathbf{0}$

Thus, $\boxed{\lambda^4 - 2\lambda^3 + \lambda^2}$ is the minimal polynomial

Let $f_1, f_2, f_3, \cdots, f_8$ be the invariant factors of A. Then f_8 is the minimal polynomial of $\mathbf{A}$ and so $f_8 = \lambda^4 - 2\lambda^3 + \lambda^2$. We also know that the product $f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 = \text{char}_A(x) = \det(x\mathbf{I} - \mathbf{A})$. Thus, the invariant factors are:

$$\implies \boxed{1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1 \mid (\lambda^4 - 2\lambda^3 + \lambda^2) \mid (\lambda^4 - 2\lambda^3 + \lambda^2)}$$

- Here, each f_i divides f_{i+1},
- The last factor f_8 is our minimal polynomial, and
- The product of all factors is equal to the characteristic polynomial.

Therefore, The given factors are valid invariant factors of matrix **A**

Primary decomposition of R^8 under T.

Finding the primary decomposition of $\mathbf{R^8}$ under $\mathbf{T}$

The minimal polynomial of the matrix **A** is:

$$p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda = (\lambda - 1)^2 \lambda^2 \tag{3.0.6}$$

By Primary decomposition theorem, The vector space $\mathbf{R^8}$ can be decomposed into the primary components (or subspaces) as:

$$\mathbf{R^8} = \mathbf{W}_1 \oplus \mathbf{W}_2; \tag{3.0.7}$$

Where,

$$\mathbf{W}_1 = \text{ Null space of } (\mathbf{A} - \mathbf{I})^2 \tag{3.0.8}$$

$$\mathbf{W}_2 = \text{ Null space of } (\mathbf{A})^2 \tag{3.0.9}$$

Finding primary component $\mathbf{W_1}$:

$$(\mathbf{A}-\mathbf{I})^2\mathbf{v}=0 \tag{3.0.10}$$

$$\begin{pmatrix}0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&-2\\0&0&1&0&0&0&0&2\\0&-2&-2&1&0&0&0&-2\\0&1&1&-1&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\end{pmatrix}\mathbf{v}=0 \tag{3.0.11}$$

Row reduced echelon form:

$$\begin{pmatrix}0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}\mathbf{v}=0 \tag{3.0.12}$$

$$\therefore \mathbf{W}_1=\begin{pmatrix}v_1\\0\\0\\0\\v_5\\v_6\\v_7\\0\end{pmatrix} \tag{3.0.13}$$

$$\therefore \mathbf{W}_1=\text{Span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix}\right\} \tag{3.0.14}$$

<u>Finding primary component $\mathbf{W_2}$</u>:

$$\mathbf{A}^2\mathbf{v} = 0 \tag{3.0.15}$$

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & 2 & 1 & 0 & 2 \\ 0 & -2 & -2 & -2 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{v} = 0 \tag{3.0.16}$$

Row reduced echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{v} = 0 \tag{3.0.17}$$

$$\therefore \mathbf{W}_2 = \begin{pmatrix} 0 \\ -v_3 - v_4 - v_5 - v_8 \\ v_3 \\ v_4 \\ v_5 \\ 0 \\ 0 \\ v_8 \end{pmatrix} \tag{3.0.18}$$

$$\therefore \mathbf{W}_2 = \text{Span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{3.0.19}$$

Projections on primary components $\mathbf{W}_1$ and $\mathbf{W}_2$.

Finding the projection $\mathbf{E}_1$ on the primary component $\mathbf{W}_1$

Finding the projection $\mathbf{E}_1$ that projects $\mathbf{R^8}$ on $\mathbf{W}_1$:
We know that projection $\mathbf{E}_1$ will satisfy the following:

$$\mathbf{E}_1(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{for } \mathbf{v} \in \mathbf{W}_1 \\ 0 & \text{for } \mathbf{v} \notin \mathbf{W}_1 \end{cases} \tag{3.0.20}$$

Using the above result and the equations (3.0.14) and (3.0.19), we can write:

$$\mathbf{E}_1\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \tag{3.0.21}$$

$$\mathbf{E}_1\begin{pmatrix}0&0&0&0\\-1&-1&-1&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{3.0.22}$$

By using the equation (3.0.21), we can write:

$$\begin{pmatrix}E_{11}&E_{15}&E_{16}&E_{17}\\E_{21}&E_{25}&E_{26}&E_{27}\\E_{31}&E_{35}&E_{36}&E_{37}\\E_{41}&E_{45}&E_{46}&E_{47}\\E_{51}&E_{55}&E_{56}&E_{57}\\E_{61}&E_{65}&E_{66}&E_{67}\\E_{71}&E_{75}&E_{76}&E_{77}\\E_{81}&E_{85}&E_{86}&E_{87}\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \tag{3.0.23}$$

By using the equation (3.0.22), we can write:

$$\begin{pmatrix}E_{13}-E_{12}&E_{14}-E_{12}&E_{15}-E_{12}&E_{18}-E_{12}\\E_{23}-E_{22}&E_{24}-E_{22}&E_{25}-E_{22}&E_{28}-E_{22}\\E_{33}-E_{32}&E_{34}-E_{32}&E_{35}-E_{32}&E_{38}-E_{32}\\E_{43}-E_{42}&E_{44}-E_{42}&E_{45}-E_{42}&E_{48}-E_{42}\\E_{53}-E_{52}&E_{54}-E_{52}&E_{55}-E_{52}&E_{58}-E_{52}\\E_{63}-E_{62}&E_{64}-E_{62}&E_{65}-E_{62}&E_{68}-E_{62}\\E_{73}-E_{72}&E_{74}-E_{72}&E_{75}-E_{72}&E_{78}-E_{72}\\E_{83}-E_{82}&E_{84}-E_{82}&E_{85}-E_{82}&E_{88}-E_{82}\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{3.0.24}$$

Obtaining each elements of $\mathbf{E}_1$ by equating both sides in equations (3.0.23) and (3.0.24):

	$$\mathbf{E}_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&1&1&1&1&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad (3.0.25)$$ Here, $\mathbf{E}_1^2 = \mathbf{E}_1$. thus the obtained $\mathbf{E}_1$ is valid projection.
Finding the projection $\mathbf{E}_2$ on the primary component $\mathbf{W}_2$	Finding the projection $\mathbf{E}_2$ that projects $\mathbf{R^8}$ on $\mathbf{W}_2$: We know that projection $\mathbf{E}_2$ will satisfy the following: $$\mathbf{E}_2(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{for } \mathbf{v} \in \mathbf{W}_2 \\ 0 & \text{for } \mathbf{v} \notin \mathbf{W}_2 \end{cases} \quad (3.0.26)$$ Using this and the equations (3.0.14) and (3.0.19), we can write: $$\mathbf{E}_2 \begin{pmatrix} 0&0&0&0\\ -1&-1&-1&-1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ -1&-1&-1&-1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix} \quad (3.0.27)$$ $$\mathbf{E}_2 \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \quad (3.0.28)$$ By using the equation (3.0.27), we can write:

	$$\begin{pmatrix} E_{13}-E_{12} & E_{14}-E_{12} & E_{15}-E_{12} & E_{18}-E_{12} \\ E_{23}-E_{22} & E_{24}-E_{22} & E_{25}-E_{22} & E_{28}-E_{22} \\ E_{33}-E_{32} & E_{34}-E_{32} & E_{35}-E_{32} & E_{38}-E_{32} \\ E_{43}-E_{42} & E_{44}-E_{42} & E_{45}-E_{42} & E_{48}-E_{42} \\ E_{53}-E_{52} & E_{54}-E_{52} & E_{55}-E_{52} & E_{58}-E_{52} \\ E_{63}-E_{62} & E_{64}-E_{62} & E_{65}-E_{62} & E_{68}-E_{62} \\ E_{73}-E_{72} & E_{74}-E_{72} & E_{75}-E_{72} & E_{78}-E_{72} \\ E_{83}-E_{82} & E_{84}-E_{82} & E_{85}-E_{82} & E_{88}-E_{82} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.29)$$ By using the equation (3.0.28), we can write: $$\begin{pmatrix} E_{11} & E_{15} & E_{16} & E_{17} \\ E_{21} & E_{25} & E_{26} & E_{27} \\ E_{31} & E_{35} & E_{36} & E_{37} \\ E_{41} & E_{45} & E_{46} & E_{47} \\ E_{51} & E_{55} & E_{56} & E_{57} \\ E_{61} & E_{65} & E_{66} & E_{67} \\ E_{71} & E_{75} & E_{76} & E_{77} \\ E_{81} & E_{85} & E_{86} & E_{87} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.0.30)$$ Obtaining each elements of $\mathbf{E}_2$ by equating both sides in equations (3.0.29) and (3.0.30):: $$\mathbf{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.31)$$ Here, $\mathbf{E}_2^2 = \mathbf{E}_2$. thus the obtained $\mathbf{E}_2$ is valid projection. Also, from equations (3.0.25) and (3.0.31), It is also verified that $\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I}$
Jordan form	
Finding Jordan form	The characteristic polynomial of the matrix **A** is: $$(\lambda-1)^4\lambda^4 = \lambda^8 - 4\lambda^7 + 6\lambda^6 - 4\lambda^5 + \lambda^4 \quad (3.0.32)$$ Thus, the eigen values are $1, 1, 1, 1, 0, 0, 0, 0$

The eigen space corresponding to the eigenvalue 1 is the null space of $(\mathbf{A}-\mathbf{I})$:

$$(\mathbf{A}-\mathbf{I})\mathbf{v}=0 \tag{3.0.33}$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\mathbf{v}=0 \tag{3.0.34}$$

Row reduced echelon form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{v}=0 \tag{3.0.35}$$

$$\therefore Nul(\mathbf{A}-\mathbf{I})=\begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -v_7 \\ v_7 \\ 0 \end{pmatrix} \tag{3.0.36}$$

$$\therefore Nul(\mathbf{A}-\mathbf{I})= \text{ Span }\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}\right\} \tag{3.0.37}$$

- Since the eigenspace corresponding to eigen value 1 is 2-dimensional, there are 2 Jordan blocks for eigen value 1;
- Also this eigenvalue has algebraic multiplicity 4 (from characteristic polynomial), thus the two blocks have to have sizes adding to 4. Hence, there are two 2×2 blocks.

The eigen space corresponding to the eigenvalue 0 is the null space of $\mathbf{A}$

$$\mathbf{A}\mathbf{v} = 0 \tag{3.0.38}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{v} = 0 \tag{3.0.39}$$

Row reduced echelon form:

$$\therefore \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{v} = 0 \tag{3.0.40}$$

$$\therefore Nul(\mathbf{A} - \mathbf{I}) = \begin{pmatrix} 0 \\ -v_3 \\ v_3 \\ -v_5 \\ v_5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.41}$$

$$\therefore Nul(\mathbf{A} - \mathbf{I}) = \text{Span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \tag{3.0.42}$$

- Here also, the eigenspace corresponding to eigen value 0 is 2-dimensional, thus there are 2 Jordan blocks for eigen value 0;
- Also this eigenvalue has algebraic multiplicity 4 (from characteristic polynomial), thus the two blocks have to have sizes adding to 4. Hence, there are two 2×2 blocks.

Using all the Jordan blocks, The Jordan form of **A** can be written as:

<table>
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$$\text{Jordan}(\mathbf{A}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.43}$$

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<tr><td colspan="2">Direct sum decomposition into T-cyclic subspaces.</td></tr>
<tr><td>Finding direct-sum decomposition of $\mathbf{R^8}$ into T-cyclic subspaces</td><td>

$\mathbf{T}$ (represented by matrix $\mathbf{A}$) is a linear operator on vector space $\mathbf{R^8}$ (represented by $\mathbf{V}$). The T-annihilators $p_1, p_2, \cdots, p_k$, such that p_k divides p_{k-1}, $k = 2, \ldots, r$, are:

$$p_1 = (\lambda - 1)^2\lambda^2 \tag{3.0.44}$$
$$p_2 = (\lambda - 1)^2\lambda \tag{3.0.45}$$
$$p_3 = \lambda \tag{3.0.46}$$

By cyclic decomposition theorem, There will exists non-zero vectors α_1, α_2 and α_3 in $\mathbf{V}$ with respective T-annihilators p_1, p_2 and p_3 such that:

$$\mathbf{V} = \mathbf{W}_0 \oplus \mathbf{Z}(\alpha_1; \mathbf{A}) \oplus \mathbf{Z}(\alpha_2; \mathbf{A}) \oplus \mathbf{Z}(\alpha_3; \mathbf{A}) \tag{3.0.47}$$

Where, the T-cyclic subspace $\mathbf{Z}(\alpha_i; T)$ is defined as :

$$\mathbf{Z}(\alpha_i; T) = \text{Span}\left\{\alpha_i, \mathbf{T}\alpha_i, \ldots, \mathbf{T}^{k-1}\alpha_i\right\} \tag{3.0.48}$$
$$\text{Where } k = \text{Degree of } p_i \tag{3.0.49}$$

Here, $\mathbf{W}_0$ is zero subspace:

$$\mathbf{W}_0 = \mathbf{0} \tag{3.0.50}$$

For defining $\mathbf{Z}(\alpha_1; \mathbf{A})$, we need to find non-zero vector α_1 such that:

$$p_1(\mathbf{A})(\alpha_1) = 0 \tag{3.0.51}$$
$$\text{Here, } p_1 = (\lambda - 1)^2\lambda^2 \tag{3.0.52}$$
$$\text{and } p_1(\mathbf{A}) = (\mathbf{A} - \mathbf{I})^2\mathbf{A}^2 = 0 \tag{3.0.53}$$
$$\implies \text{Any vector } \alpha_1 \in \mathbf{R^8} \text{ will satisfy (3.0.51)}$$

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	$$\text{Let,} \quad \alpha_1 = \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix} \tag{3.0.54}$$ $$\therefore \mathbf{Z}(\alpha_1;\mathbf{A}) = \text{Span}\left\{\alpha_1, \mathbf{A}\alpha_1, \mathbf{A}^2\alpha_1, \mathbf{A}^3\alpha_1\right\} \tag{3.0.55}$$ $$\therefore \mathbf{Z}(\alpha_1;\mathbf{A}) = \text{Span}\left\{\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}8\\1\\-1\\3\\2\\6\\-4\\0\end{pmatrix}, \begin{pmatrix}15\\0\\0\\0\\5\\11\\-9\\0\end{pmatrix}, \begin{pmatrix}22\\0\\0\\0\\5\\16\\-14\\0\end{pmatrix}\right\} \tag{3.0.56}$$ $$\therefore \dim(\mathbf{Z}(\alpha_1;\mathbf{A})) = \text{Degree of } p_1 = 4 \tag{3.0.57}$$
	For defining $\mathbf{Z}(\alpha_2;\mathbf{A})$, we need to find non-zero vector α_2 such that: $$\alpha_2 \notin \mathbf{Z}(\alpha_1;\mathbf{A}), p_2(\mathbf{A})(\alpha_2) = 0 \tag{3.0.58}$$ $$\text{Here, } p_2 = (\lambda-1)^2\lambda \tag{3.0.59}$$ $$\frac{p_1}{p_2} = \lambda \implies p_2 \text{ divides } p_1 \tag{3.0.60}$$ $$p_2(\mathbf{A}) = (\mathbf{A}-\mathbf{I})^2\mathbf{A} = \begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&-1\\0&1&1&0&0&0&0&1\\0&-1&-1&0&0&0&0&-1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix} \tag{3.0.61}$$

One such vector that satisfies (3.0.58) is:

$$\text{Let,} \quad \alpha_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \tag{3.0.62}$$

$$\therefore \mathbf{Z}(\alpha_2; \mathbf{A}) = \text{Span}\left\{\alpha_2, \mathbf{A}\alpha_2, \mathbf{A}^2\alpha_2\right\} \tag{3.0.63}$$

$$\therefore \mathbf{Z}(\alpha_2; \mathbf{A}) = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 2 \\ 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 2 \\ 5 \\ -3 \\ 0 \end{pmatrix}\right\} \tag{3.0.64}$$

$$\therefore \dim(\mathbf{Z}(\alpha_2; \mathbf{A})) = \text{Degree of } p_2 = 3 \tag{3.0.65}$$

For defining $\mathbf{Z}(\alpha_3; \mathbf{A})$, we need to find non-zero vector α_3 such that:

$$\alpha_3 \notin \mathbf{Z}(\alpha_1; \mathbf{A}), \alpha_3 \notin \mathbf{Z}(\alpha_2; \mathbf{A}) \text{ and } p_3(\mathbf{A})(\alpha_3) = 0 \tag{3.0.66}$$

$$\text{Here, } p_3 = \lambda \tag{3.0.67}$$

$$\frac{p_2}{p_3} = (\lambda - 1)^2 \implies p_3 \text{ divides } p_2 \tag{3.0.68}$$

$$p_3(\mathbf{A}) = \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.69}$$

	One such vector that satisfies (3.0.66) is: $$\text{Let,} \quad \alpha_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.70}$$ $$\therefore \mathbf{Z}(\alpha_2; \mathbf{A}) = \text{Span}\{\alpha_2\} \tag{3.0.71}$$ $$\therefore \mathbf{Z}(\alpha_2; \mathbf{A}) = \text{Span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} \tag{3.0.72}$$ $$\therefore \dim(\mathbf{Z}(\alpha_2; \mathbf{A})) = \text{Degree of } p_3 = 1 \tag{3.0.73}$$ $$\dim(\mathbf{Z}(\alpha_1; \mathbf{A})) + \dim(\mathbf{Z}(\alpha_2; \mathbf{A})) + \dim(\mathbf{Z}(\alpha_2; \mathbf{A})) = 8 \tag{3.0.74}$$ Thus, $\mathbf{V} = \mathbf{Z}(\alpha_1; \mathbf{A}) \oplus \mathbf{Z}(\alpha_2; \mathbf{A}) \oplus \mathbf{Z}(\alpha_3; \mathbf{A})$ is the required direct sum decomposition into T-cyclic subspaces.