Assignment 9

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Download all python codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment9/codes
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and latex-tikz codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment9
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1 Problem

(Hoffman/Page27/12)
Prove that the given matrix is invertible and $\mathbf{A}^{-1}$ has integer values.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & . & . & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & . & . & \frac{1}{n+1} \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n+1} & . & . & \frac{1}{2n-1} \end{pmatrix} \tag{1.0.1}$$

2 Solution

Proof that A is Invertible.

The elements of the given matrix is of the form:

$$A_{ij} = \frac{1}{i+j-1} = \int_0^1 t^{i+j-2}dt = \int_0^1 t^{i-1}t^{j-1}dt \tag{2.0.1}$$

We will prove that the matrix $\mathbf{A}$ is positive definite:
$\mathbf{A}$ is Positive definite, if $\mathbf{XAX}^T > 0$

$$\text{Let } \mathbf{X} = (x_i)_{1\le i\le n} \in \mathbb{R}^N \tag{2.0.2}$$

$$\mathbf{XAX}^T = \sum_{1\le i,j\le n} \frac{x_i x_j}{i+j-1} \tag{2.0.3}$$

From (2.0.1),

$$\mathbf{XAX}^T = \sum_{1\le i,j\le n} x_i x_j \int_0^1 t^{i+j-2}dt \tag{2.0.4}$$

$$\mathbf{XAX}^T = \int_0^1 \left(\sum_{i=1}^{n} x_i t^{i-1}\right)\left(\sum_{j=1}^{n} x_j t^{j-1}\right)dt \tag{2.0.5}$$

$$\implies \mathbf{XAX}^T = \int_0^1 \left(\sum_{i=1}^{n} x_i t^{i-1}\right)^2 dt > 0 \tag{2.0.6}$$

Thus, Matrix $\mathbf{A}$ is Positive definite.
Now, let's say λ is an eigen value of $\mathbf{A}$. Then, for the corresponding eigen vector $\mathbf{X} = (x_1, x_2, ..., x_n)$, we can write:

$$\mathbf{XAX}^T = \mathbf{X}\lambda\mathbf{X}^T \quad [\because \mathbf{AX}^T = \lambda\mathbf{X}^T] \tag{2.0.7}$$

$$\implies \mathbf{XAX}^T = \|\mathbf{X}\|^2 \lambda \tag{2.0.8}$$

$$\implies \lambda = \frac{\mathbf{XAX}^T}{\|\mathbf{X}\|^2} > 0 \tag{2.0.9}$$

So, all of the eigenvalues belonging to $\mathbf{A}$ must be positive. The product of the eigenvalues of a matrix equals the determinant.

$$\boxed{\therefore \det(\mathbf{A}) > 0} \tag{2.0.10}$$

Thus, the given matrix $\mathbf{A}$ is non-singular and its inverse exist (Invertible).

Proof that $\mathbf{A}^{-1}$ has integer values.

Let us consider a set of shifted legendre polynomials as follow:

$$P_i(x) = \begin{cases} P_1(x) = P_{11} \\ P_2(x) = P_{21} + P_{22}x \\ P_3(x) = P_{31} + P_{32}x + p_{33}x^2 \\ . \\ . \\ P_n(x) = P_{n1} + P_{n2}x + P_{n3}x^2 + .. + P_{nn}x^{n-1} \end{cases} \tag{2.0.11}$$

Where, the coefficients P_{ij} are given as:

$$P_{ij} = (-1)^{i+j-1}\binom{j-1}{i-1}\binom{i+j-2}{i-1} \tag{2.0.12}$$

The shifted legendre polynomials are analogous to legendre polynomials, but defined on the interval $[0, 1]$ (whereas the interval is $[-1, 1]$ for legendre polynomial).

A set of shifted legendre polynomials obey the following orthogonal relationship:

$$\int_0^1 P_i(x)P_j(x)dx = 0 \text{ for } i \neq j \tag{2.0.13}$$

$$\int_0^1 P_i(x)P_j(x)dx = \frac{1}{2i+1} \text{ for } i = j \tag{2.0.14}$$

Forming a matrix $\mathbf{P}$ whose elements are the coefficients of polynomials in (2.0.11)

$$\mathbf{P} = \begin{pmatrix} P_{11} & 0 & . & . & 0 \\ P_{21} & P_{22} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ P_{n1} & P_{n2} & . & . & P_{nn} \end{pmatrix} \tag{2.0.15}$$

Forming a matrix $\mathbf{PAP}^T$, the elements of the matrix $\mathbf{PAP}^T$ can be written as:

$$\mathbf{PAP}^T_{ij} = \sum_{s=1}^{N}\sum_{r=1}^{N} P_{ir}P_{js}A_{rs} \tag{2.0.16}$$

From (2.0.1) with suitable change in variable notations, we can write:

$$A_{rs} = \int_0^1 x^{r-1}x^{s-1}dx \tag{2.0.17}$$

From (2.0.16) and (2.0.17),

$$\mathbf{PAP}^T_{ij} = \int_0^1 \sum_{s=1}^{N}\sum_{r=1}^{N} P_{ir}P_{js}x^{r-1}x^{s-1}dx \tag{2.0.18}$$

$$\mathbf{PAP}^T_{ij} = \int_0^1 \sum_{s=1}^{N} P_{ir}x^{r-1} \sum_{r=1}^{N} P_{js}x^{s-1}dx \tag{2.0.19}$$

$$\mathbf{PAP}^T_{ij} = \int_0^1 P_i(x)P_j(x)dx \tag{2.0.20}$$

From (2.0.14)

$$\mathbf{PAP}^T_{ij} = \begin{cases} 0 & i \neq j \\ \frac{1}{2i+1} & i = j \end{cases} \tag{2.0.21}$$

Thus, Matrix $\mathbf{PAP}^T$ is diagonal matrix:

$$\mathbf{PAP}^T = \mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & . & . & 0 \\ 0 & \frac{1}{5} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \frac{1}{2n+1} \end{pmatrix} \tag{2.0.22}$$

From (2.0.22), the inverse of matrix $\mathbf{A}$ can be written as:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}(\mathbf{P}^T)^{-1} \tag{2.0.23}$$

$$\implies \mathbf{A}^{-1} = \mathbf{P}^T\mathbf{D}^{-1}\mathbf{P} \tag{2.0.24}$$

From (2.0.12) and (2.0.22), It can be clearly observed that the elements of matrix $\mathbf{P}$, $\mathbf{P}^T$ and $\mathbf{D}^{-1}$ are all integers given as:

$$\mathbf{P}_{ij} = (-1)^{i+j-1}\binom{j-1}{i-1}\binom{i+j-2}{i-1} \tag{2.0.25}$$

$$\mathbf{D}^{-1}_{ij} = \frac{1}{\mathbf{D}_{ij}} = \begin{cases} 0 & i \neq j \\ 2i+1 & i = j \end{cases} \tag{2.0.26}$$

Since, matrix $\mathbf{P}$, $\mathbf{P}^T$ and $\mathbf{D}^{-1}$ are integer matrices, therefore $\mathbf{A}^{-1}$ is also an integer matrix.
Hence proved.

Observations:

1) The given matrix is a $n \times n$ Hilbert matrix. Which is always invertible with its inverse having integer values.
2) The Hilbert matrix is symmetric and positive definite.