

Problem

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1 PROBLEM

For $\lambda_1 = 0$

What conic does the following equation represent.

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 \quad (1.0.1)$$

Find the center and equation referred to centre.

2 SOLUTION

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

where,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.0.3)$$

$$f = 5 \quad (2.0.4)$$

Expanding the determinant of \mathbf{V} we observe,

$$\begin{vmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \quad (2.0.5)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \neq 0 \quad (2.0.6)$$

Hence we conclude that given equation is a parabola. The characteristic equation of \mathbf{V} is given as follows,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.0.7)$$

$$\Rightarrow \lambda^2 - 4\lambda = 0 \quad (2.0.8)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 4 \quad (2.0.9)$$

The eigen vector \mathbf{p} is defined as,

$$(\mathbf{V} - \lambda \mathbf{I})\mathbf{p} = 0 \quad (2.0.10)$$

$$\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \mathbf{p}_1 = 0 \quad (2.0.11)$$

$$R_2 \leftrightarrow \frac{1}{\sqrt{3}}R_1 + R_2$$

$$\begin{pmatrix} 3 & -\sqrt{3} \\ 0 & 0 \end{pmatrix} \mathbf{p}_1 = 0 \quad (2.0.12)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \quad (2.0.13)$$

[Choosing Orthonormal eigen vectors]

For $\lambda_2 = 4$

$$\begin{pmatrix} -1 - \sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix} \mathbf{p}_2 = 0 \quad (2.0.14)$$

$$R_2 \leftrightarrow -\sqrt{3}R_1 + R_2$$

$$\begin{pmatrix} 3 & -\sqrt{3} \\ 0 & 0 \end{pmatrix} \mathbf{p}_2 = 0 \quad (2.0.15)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad (2.0.16)$$

[Choosing Orthonormal eigen vectors]

The matrix \mathbf{p} is:

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \quad (2.0.17)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \quad (2.0.18)$$

$$\eta = 2\mathbf{p}_1^T \mathbf{u} = 2 \begin{pmatrix} 3 & -2 \end{pmatrix} \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = 3 - 2\sqrt{3} \quad (2.0.19)$$

The focal length of the parabola is given by:

$$\left| \frac{\eta}{\lambda_2} \right| = 0.116 \quad (2.0.20)$$

When $\text{Det}(\mathbf{V}) = 0$, (2.0.1) can be written as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (2.0.21)$$

And the vertex \mathbf{c} is given by :

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.22)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad (2.0.23)$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.0.24)$$

$$\mathbf{p}_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \quad (2.0.25)$$

$$f = 5 \quad (2.0.26)$$

Putting these value we get:

$$\begin{pmatrix} (3.75 - 0.5\sqrt{3}) & (-3.5 + 0.75\sqrt{3}) \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ -2.25 - 0.5\sqrt{3} \\ 0.5 + 0.75\sqrt{3} \end{pmatrix} \quad (2.0.27)$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} (3.75 - 0.5\sqrt{3}) & (-3.5 + 0.75\sqrt{3}) & -5 \\ 3 & -\sqrt{3} & -2.25 - 0.5\sqrt{3} \\ -\sqrt{3} & 1 & 0.5 + 0.75\sqrt{3} \end{pmatrix} \quad (2.0.28)$$

$$R_3 \leftrightarrow -\sqrt{3}R_3 - R_2$$

$$\begin{pmatrix} (3.75 - 0.5\sqrt{3}) & (-3.5 + 0.75\sqrt{3}) & -5 \\ 3 & -\sqrt{3} & -2.25 - 0.5\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.29)$$

$$\begin{pmatrix} 2.8840 & -2.2001 & -5 \\ 3 & -1.7320 & -3.1160 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.30)$$

$$\begin{pmatrix} 1 & -0.7632 & -1.7337 \\ 3 & -1.7320 & -3.1160 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.31)$$

$$\begin{pmatrix} 1 & -0.7632 & -1.7337 \\ 0 & 0.5576 & 2.0851 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.32)$$

$$\begin{pmatrix} 1 & -0.7632 & -1.7337 \\ 0 & 1 & 3.7394 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.33)$$

$$\begin{pmatrix} 1 & 0 & 1.1202 \\ 0 & 1 & 3.7394 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.34)$$

Thus the vertex \mathbf{c} is:

$$\mathbf{c} = \begin{pmatrix} 1.1202 \\ 3.7394 \end{pmatrix} \quad (2.0.35)$$

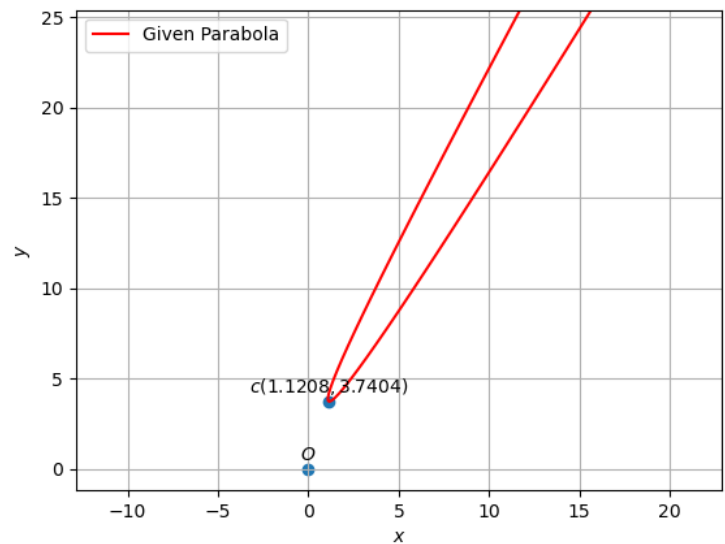


Fig. 0: Plot of the Parabola and its vertex