Assignment 10

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Download the latex-tikz codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment10
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1 Problem

(Hoffman/Page123/8) : If F is a field and h is a polynomial over F of degree ≥ 1, show that the mapping $f \to f(h)$ is a one-one linear transformation of $F[x]$ into $F[x]$. Show that this transformation is an isomorphism of $F[x]$ onto $F[x]$ if and only if deg $h = 1$.

2 Definition and Result used

Linear Transformation	A Transformation $T(x)$ from $V \to W$ is said to be linear if it satisfies the following: $T(ax+y) = aT(x) + T(y)$ where, $x, y \in V$ and $a \in F$.
One to one transformation	A transformation $T(x)$ from $V \to W$ is one to one, if: $T(x_1) = T(x_2) \implies x_1 = x_2$ i.e. The mapping is unique.
Isomorphism	A transformation $T(x)$ from $V \to W$ is isomorphism of V onto W, if there exists an inverse $T'(x)$ of $T(x)$ such that $T'T(x) = TT'(x) = x$
Degree of a polynomial	The degree of a polynomial $p(x)$ is the highest power of x in the polynomial $p(x)$.

3 Solution

Proving that the given mapping is linear.	Here, $F[x]$ is a set of polynomials over field F, written as: $F[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \quad \mid \quad a_i \in F\right\}$ Let the mapping $f \to f(h)$ represented as: $G(f) = f(h)$ Thus, $G(f)$ is clearly a function from $F[x]$ into $F[x]$. Let, $f, g \in F[x]$ and $\alpha \in F$ $G(\alpha f + g) = (\alpha f + g)(h)$ $= (\alpha f)(h) + g(h)$ $= \alpha f(h) + g(h)$ $= \alpha G(f) + G(g)$ Thus, G is a linear transformation
Proving that G is one-one transformation	Now to show that G is one to one, $G(f) = G(g)$ $\therefore f(h) = g(h)$ $\sum_{i=0}^{n} c_i [h(x)]^i = \sum_{j=0}^{m} c_j [h(x)]^j$ Here, $h(x)$ is parameter to both f and g, hence $i = j$ and $c_i = c_j$. Thus, $f = g$ Therefore, G is one-one linear transformation
Proving G is Isomorphism if and only if deg $h = 1$.	Let $\deg(h) = 1$, then h can be written as: $h = a + bx$, Where, $b \neq 0$ Let us define h' such that: $h' = \frac{1}{b}x - \frac{a}{b}$ Let G' be the linear transformation from $F(x)$ to $F(x)$ given by: $G'(f) = f\left(\frac{1}{b}x - \frac{a}{b}\right)$ It can be shown that G' is inverse of G as follow: $G(G'(f)) = G\left(f\left(\frac{1}{b}x - \frac{a}{b}\right)\right)$

$$
\begin{aligned}
&= f\left(a\left(\tfrac{1}{a}x - \tfrac{b}{a}\right) + b\right) \\
&= f(x)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
G'(G(f)) &= G'\left(f\left(ax+b\right)\right) \\
&= f\left(\tfrac{1}{a}\left(ax+b\right) - \tfrac{b}{a}\right) \\
&= f(x)
\end{aligned}
$$

Thus, G' is inverse of G. Therefore, G is isomorphism and we can say:

$$\boxed{\deg(h) = 1 \implies G \text{ is isomorphism.}}$$

Let $\deg(h) > 1$, then

$$
\begin{aligned}
&\deg f(h) = \deg f \cdot \deg h \\
&\implies \deg f(h) \geq 1 \\
&\implies G(f) = f(h) \neq x
\end{aligned}
$$

This means the image of G does not contain polynomials of degree one. Hence G is not onto and therefore G can not be an isomorphism.
Thus we can write:

$$\boxed{\deg(h) > 1 \implies G \text{ is not isomorphism.}}$$

Thus, We can conclude:

$$\boxed{G \text{ is isomorphism.} \iff \deg(h) = 1}$$