Assignment 5

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Download all python codes from

https://github.com/sachinomdubey/Matrix−theory/Assignment5/codes

and latex-tikz codes from

https://github.com/sachinomdubey/Matrix−theory/Assignment5

0.1 Problem

(Geolin 1.9) AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.

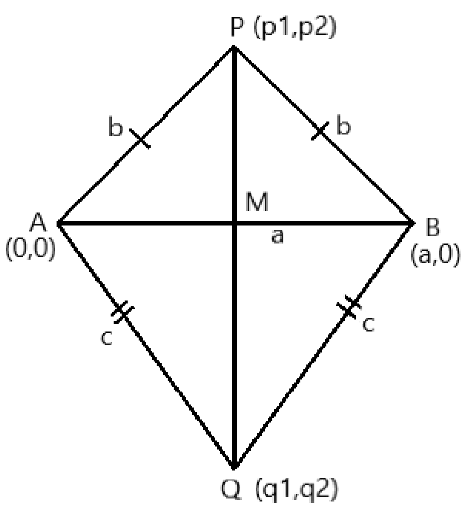

Fig. 0

0.2 Explanation

In order to prove that line PQ is the perpendicular bisector of AB, two conditions need to be met:

1) Angle between line AB and line PQ should be $90°$.
2) AM=BM

Steps to prove the first condition are as follow:

1) Here, the following information is given:

$$\|\mathbf{P}-\mathbf{A}\| = \|\mathbf{P}-\mathbf{B}\| \tag{0.2.1}$$
$$\|\mathbf{Q}-\mathbf{A}\| = \|\mathbf{Q}-\mathbf{B}\| \tag{0.2.2}$$

2) Take the norms and simplify to get $\mathbf{P}$ and $\mathbf{Q}$. Using $\mathbf{P}$ and $\mathbf{Q}$, prove the following condition to show that lines PQ is the perpendicular to line AB:

$$\left(\mathbf{P}-\mathbf{Q}\right)^T\left(\mathbf{A}-\mathbf{B}\right) = 0 \tag{0.2.3}$$

Steps to prove the second condition are as follow:

1) Finding the normal vectors of line PQ. Further using the normal vector to find the equation of line PQ.
2) The solution of equation of line AB and line PQ will give the co-ordinates of point M.
3) Using the distance formula to prove that AM=BM.

0.3 Solution

Let the Points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{Q}$ be:

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}\mathbf{B} = \begin{pmatrix}a\\0\end{pmatrix} \quad \mathbf{P} = \begin{pmatrix}p_1\\p_2\end{pmatrix}\mathbf{Q} = \begin{pmatrix}q_1\\q_2\end{pmatrix} \tag{0.3.1}$$

The points $\mathbf{P}$ and $\mathbf{Q}$ are equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then we can write:

$$\|\mathbf{P}-\mathbf{A}\| = \|\mathbf{P}-\mathbf{B}\| \tag{0.3.2}$$
$$\|\mathbf{Q}-\mathbf{A}\| = \|\mathbf{Q}-\mathbf{B}\| \tag{0.3.3}$$

The equation 0.3.2 can be further written as:

$$\left\|\begin{pmatrix}p_1\\p_2\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}\right\| = \left\|\begin{pmatrix}p_1\\p_2\end{pmatrix}-\begin{pmatrix}a\\0\end{pmatrix}\right\| \tag{0.3.4}$$
$$\sqrt{p_1^2+p_2^2} = \sqrt{(p_1-a)^2+p_2^2} \tag{0.3.5}$$
$$p_1^2+p_2^2 = p_1^2+p_2^2-2ap_1+a^2 \tag{0.3.6}$$
$$\therefore a^2-2ap_1 = 0 \tag{0.3.7}$$
$$\implies p_1 = a/2 \tag{0.3.8}$$
$$\therefore \mathbf{P} = \begin{pmatrix}a/2\\p_2\end{pmatrix} \tag{0.3.9}$$

Similarly, the equation 0.3.3 can be further written as:

$$\left\|\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \begin{pmatrix} a \\ 0 \end{pmatrix}\right\| \tag{0.3.10}$$

$$\sqrt{q_1^2 + q_2^2} = \sqrt{(q_1 - a)^2 + q_2^2} \tag{0.3.11}$$

$$q_1^2 + q_2^2 = q_1^2 + q_2^2 - 2aq_1 + a^2 \tag{0.3.12}$$

$$\therefore a^2 - 2aq_1 = 0 \tag{0.3.13}$$

$$\implies q_1 = a/2 \tag{0.3.14}$$

$$\therefore \mathbf{Q} = \begin{pmatrix} a/2 \\ q_2 \end{pmatrix} \tag{0.3.15}$$

The lines AB and PQ will be perpendicular, when the following condition is met:

$$\left(\mathbf{P} - \mathbf{Q}\right)^T \left(\mathbf{A} - \mathbf{B}\right) = 0 \tag{0.3.16}$$

Using equation 0.3.9 and 0.3.15, The RHS can be written as:

$$RHS = \begin{pmatrix} 0 \\ (p_2 - q_2) \end{pmatrix}^T \begin{pmatrix} -a \\ 0 \end{pmatrix} \tag{0.3.17}$$

$$RHS = \begin{pmatrix} 0 & (p_2 - q_2) \end{pmatrix} \begin{pmatrix} -a \\ 0 \end{pmatrix} \tag{0.3.18}$$

$$RHS = 0 \tag{0.3.19}$$

$$\therefore RHS = LHS \tag{0.3.20}$$

Therefore, The lines AB and PQ are perpendicular to each other as the equation 0.3.16 is satisfied.
The vector equations of lines AB and PQ can be written as:

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -a \\ 0 \end{pmatrix} \tag{0.3.21}$$

$$\mathbf{x} = \begin{pmatrix} a/2 \\ p_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ (p_2 - q_2) \end{pmatrix} \tag{0.3.22}$$

The lines intersect at point M, Equating equations 0.3.21 and 0.3.22 and solving the equations in matrix form:

$$-a\lambda_1 = a/2 \tag{0.3.23}$$

$$0 = p_2 + \lambda_2(p_2 - q_2) \tag{0.3.24}$$

$$\left(\begin{array}{cc|c} -a & 0 & a/2 \\ 0 & (p_2 - q_2) & -p_2 \end{array}\right) \tag{0.3.25}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/-a} \left(\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & (p_2 - q_2) & -p_2 \end{array}\right) \tag{0.3.26}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/(p_2 - q_2)} \left(\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & -p_2/(p_2 - q_2) \end{array}\right) \tag{0.3.27}$$

Therfore, $\lambda_1 = -1/2$ and $\lambda_2 = -p_2/(p_2 - q_2)$
Putting these values in equation 0.3.21, We get point M as:

$$\mathbf{M} = \begin{pmatrix} a/2 \\ 0 \end{pmatrix} \tag{0.3.28}$$

The segments AM and BM are given as:

$$AM = \|\mathbf{A} - \mathbf{M}\| \tag{0.3.29}$$

$$\therefore AM = \left\|\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} a/2 \\ 0 \end{pmatrix}\right\| \tag{0.3.30}$$

$$\therefore AM = \sqrt{a^2/4 + 0} = a/2 \tag{0.3.31}$$

$$Similarly, BM = \|\mathbf{B} - \mathbf{M}\| \tag{0.3.32}$$

$$\therefore BM = \left\|\begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} a/2 \\ 0 \end{pmatrix}\right\| \tag{0.3.33}$$

$$\therefore BM = \sqrt{a^2/4 + 0} = a/2 \tag{0.3.34}$$

From equation 0.3.31 and 0.3.34 :

$$AM = BM \tag{0.3.35}$$

Thus, Both conditions are satisfied, which make line PQ perpendicular bisector of AB. Hence proved.