

Assignment 9

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Download all python codes from

<https://github.com/sachinomdubey/Matrix-theory/Assignment9/codes>

and latex-tikz codes from

<https://github.com/sachinomdubey/Matrix-theory/Assignment9>

1 PROBLEM

(Hoffman/Page27/12)

Prove that the given matrix is invertible and \mathbf{A}^{-1} has integer values.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \cdot & \cdot & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdot & \cdot & \frac{1}{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \cdot & \cdot & \frac{1}{2n-1} \end{pmatrix} \quad (1.0.1)$$

2 SOLUTION 1

Proof that \mathbf{A} is Invertible.

The elements of the given matrix is of the form:

$$A_{ij} = \frac{1}{i+j-1} = \int_0^1 t^{i+j-2} dt = \int_0^1 t^{i-1} t^{j-1} dt \quad (2.0.1)$$

We will prove that the matrix \mathbf{A} is positive definite:
 \mathbf{A} is Positive definite, if $\mathbf{XAX}^T > 0$

$$\text{Let } \mathbf{X} = (x_i)_{1 \leq i \leq n} \in \mathbb{R}^N \quad (2.0.2)$$

$$\mathbf{XAX}^T = \sum_{1 \leq i, j \leq n} \frac{x_i x_j}{i+j-1} \quad (2.0.3)$$

From (2.0.1),

$$\mathbf{XAX}^T = \sum_{1 \leq i, j \leq n} x_i x_j \int_0^1 t^{i+j-2} dt \quad (2.0.4)$$

$$= \int_0^1 \left(\sum_{i=1}^n x_i t^{i-1} \right) \left(\sum_{j=1}^n x_j t^{j-1} \right) dt \quad (2.0.5)$$

$$\Rightarrow \mathbf{XAX}^T = \int_0^1 \left(\sum_{i=1}^n x_i t^{i-1} \right)^2 dt > 0 \quad (2.0.6)$$

Thus, Matrix \mathbf{A} is Positive definite.

Now, let's say λ is an eigen value of \mathbf{A} . Then, for the corresponding eigen vector $\mathbf{X} = (x_1, x_2, \dots, x_n)$, we can write:

$$\mathbf{XAX}^T = \mathbf{X}\lambda\mathbf{X}^T \quad [\because \mathbf{AX}^T = \lambda\mathbf{X}^T] \quad (2.0.7)$$

$$= \|\mathbf{X}\|^2 \lambda \quad (2.0.8)$$

$$\Rightarrow \lambda = \frac{\mathbf{XAX}^T}{\|\mathbf{X}\|^2} > 0 \quad (2.0.9)$$

So, all of the eigenvalues belonging to \mathbf{A} must be positive. The product of the eigenvalues of a matrix equals the determinant.

$$\therefore \det(\mathbf{A}) > 0 \quad (2.0.10)$$

Thus, the given matrix \mathbf{A} is non-singular and its inverse exist (Invertible).

Proof that \mathbf{A}^{-1} has integer values.

Let us consider a set of shifted legendre polynomials as follow:

$$P_i(x) = \begin{cases} P_1(x) = P_{11} \\ P_2(x) = P_{21} + P_{22}x \\ P_3(x) = P_{31} + P_{32}x + P_{33}x^2 \\ \vdots \\ P_n(x) = P_{n1} + P_{n2}x + P_{n3}x^2 + \dots + P_{nn}x^{n-1} \end{cases} \quad (2.0.11)$$

Where, the coefficients P_{ij} are given as:

$$P_{ij} = (-1)^{i+j-1} \binom{j-1}{i-1} \binom{i+j-2}{i-1} \quad (2.0.12)$$

The shifted legendre polynomials are analogous to legendre polynomials, but defined on the interval $[0, 1]$ (whereas the interval is $[-1, 1]$ for legendre polynomial).

A set of shifted legendre polynomials obey the following orthogonal relationship:

$$\int_0^1 P_i(x)P_j(x)dx = 0 \text{ for } i \neq j \quad (2.0.13)$$

$$\int_0^1 P_i(x)P_j(x)dx = \frac{1}{2i+1} \text{ for } i = j \quad (2.0.14)$$

Forming a matrix \mathbf{P} whose elements are the coefficients of polynomials in (2.0.11)

$$\mathbf{P} = \begin{pmatrix} P_{11} & 0 & \cdot & \cdot & 0 \\ P_{21} & P_{22} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{n1} & P_{n2} & \cdot & \cdot & P_{nn} \end{pmatrix} \quad (2.0.15)$$

Forming a matrix \mathbf{PAP}^T , the elements of the matrix \mathbf{PAP}^T can be written as:

$$\mathbf{PAP}^T_{ij} = \sum_{s=1}^N \sum_{r=1}^N P_{ir}P_{js}A_{rs} \quad (2.0.16)$$

From (2.0.1) with suitable change in variable notations, we can write:

$$A_{rs} = \int_0^1 x^{r-1}x^{s-1}dx \quad (2.0.17)$$

From (2.0.16) and (2.0.17),

$$\mathbf{PAP}^T_{ij} = \int_0^1 \sum_{s=1}^N \sum_{r=1}^N P_{ir}P_{js}x^{r-1}x^{s-1}dx \quad (2.0.18)$$

$$= \int_0^1 \sum_{s=1}^N P_{ir}x^{r-1} \sum_{r=1}^N P_{js}x^{s-1}dx \quad (2.0.19)$$

$$= \int_0^1 P_i(x)P_j(x)dx \quad (2.0.20)$$

From (2.0.14)

$$\mathbf{PAP}^T_{ij} = \begin{cases} 0 & i \neq j \\ \frac{1}{2i+1} & i = j \end{cases} \quad (2.0.21)$$

Thus, Matrix \mathbf{PAP}^T is diagonal matrix:

$$\mathbf{PAP}^T = \mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & \cdot & \cdot & 0 \\ 0 & \frac{1}{5} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \frac{1}{2n+1} \end{pmatrix} \quad (2.0.22)$$

From (2.0.22), the inverse of matrix \mathbf{A} can be

written as:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}(\mathbf{P}^T)^{-1} \quad (2.0.23)$$

$$\implies \mathbf{A}^{-1} = \mathbf{P}^T\mathbf{D}^{-1}\mathbf{P} \quad (2.0.24)$$

From (2.0.12) and (2.0.22), It can be clearly observed that the elements of matrix \mathbf{P} , \mathbf{P}^T and \mathbf{D}^{-1} are all integers given as:

$$\mathbf{P}_{ij} = (-1)^{i+j-1} \binom{j-1}{i-1} \binom{i+j-2}{i-1} \quad (2.0.25)$$

$$\mathbf{D}^{-1}_{ij} = \frac{1}{\mathbf{D}_{ij}} = \begin{cases} 0 & i \neq j \\ 2i+1 & i = j \end{cases} \quad (2.0.26)$$

Since, matrix \mathbf{P} , \mathbf{P}^T and \mathbf{D}^{-1} are integer matrices, therefore \mathbf{A}^{-1} is also an integer matrix.

Hence proved.

3 SOLUTION 2

https://github.com/Arko98/EE5609/blob/master/Assignment_14

Let \mathbf{A}_3 be 3×3 matrix i.e

$$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad (3.0.1)$$

Now we find the inverse of the matrix \mathbf{A}_3 as follows,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{pmatrix} \quad (3.0.2)$$

$$\xleftrightarrow[R_3=R_3-\frac{1}{3}R_1]{R_2=R_2-\frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{1}{45} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \quad (3.0.3)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{pmatrix} \quad (3.0.4)$$

$$\xleftrightarrow[R_3=180R_3]{R_2=12R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (3.0.5)$$

$$\xleftrightarrow[R_1=R_1-R_3]{R_2=R_2-R_3} \begin{pmatrix} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (3.0.6)$$

$$\xleftrightarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (3.0.7)$$

Hence we see that \mathbf{A}_3 is invertible and the inverse contains integer entries and \mathbf{A}_3^{-1} is given by,

$$\mathbf{A}_3^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \quad (3.0.8)$$

Let, \mathbf{A}_4 be 4×4 matrix as follows,

$$\mathbf{A}_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \quad (3.0.9)$$

Now, expressing \mathbf{A}_4 using \mathbf{A}_3 we get,

$$\mathbf{A}_4 = \begin{pmatrix} \mathbf{A}_3 & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (3.0.10)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{7} \end{pmatrix} \quad (3.0.11)$$

$$d = \frac{1}{7} \quad (3.0.12)$$

Now assuming \mathbf{A}_4 has an inverse, then from (3.0.10), the inverse of \mathbf{A}_4 can be written using block matrix inversion,

Block matrix inversion

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

If $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ then,

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix} \quad (3.0.13)$$

$$\therefore \mathbf{A}_4^{-1} = \begin{pmatrix} \mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & -\mathbf{A}_3^{-1}\mathbf{u}x_4^{-1} \\ -x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & x_4^{-1} \end{pmatrix} \quad (3.0.14)$$

$$= x_4^{-1} \begin{pmatrix} \mathbf{A}_3^{-1}x_4 + \mathbf{A}_3^{-1}\mathbf{u}\mathbf{u}^T\mathbf{A}_3^{-1} & -\mathbf{A}_3^{-1}\mathbf{u} \\ \mathbf{u}^T\mathbf{A}_3^{-1} & 1 \end{pmatrix} \quad (3.0.15)$$

$$\text{where, } x_4 = d - \mathbf{u}^T\mathbf{A}_3^{-1}\mathbf{u} \quad (3.0.16)$$

Now, the assumption of \mathbf{A}_4 being invertible will hold if and only if \mathbf{A}_3 is invertible, which has been proved in (3.0.8) and x_4 from (3.0.16) is invertible or x_4 is a nonzero scalar. We now prove that x_4 is invertible

as follows,

$$x_4 = \frac{1}{7} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (3.0.17)$$

$$\Rightarrow x_4 = \frac{1}{2800} \quad (3.0.18)$$

Hence, x_4 is a scalar, hence x_4^{-1} exists and is given by,

$$x_4^{-1} = 2800 \quad (3.0.19)$$

Hence, \mathbf{A}_4 is invertible. Now putting the values of \mathbf{A}_3^{-1} , x_4^{-1} and \mathbf{u} we get,

$$\mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \quad (3.0.20)$$

$$-\mathbf{A}_3^{-1}\mathbf{u}x_4^{-1} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \quad (3.0.21)$$

$$x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \quad (3.0.22)$$

$$x_4^{-1} = 2800 \quad (3.0.23)$$

Putting values from (3.0.20), (3.0.21), (3.0.22) and (3.0.23) into (3.0.14) we get,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \quad (3.0.24)$$

Hence, from (3.0.24) we proved that, \mathbf{A}_4 is invertible and has integer entries.

By successively repeating this method, we can prove that \mathbf{A}_5 , \mathbf{A}_6 , \mathbf{A}_7 , and so on, are invertible and have integer values. Thus, we can say, \mathbf{A}_{n-1} will be invertible with integer entries. Then we can represent \mathbf{A}_n as follows,

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (3.0.25)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (3.0.26)$$

$$d = \frac{1}{2n-1} \quad (3.0.27)$$

Now assuming \mathbf{A}_n has an inverse, then from (3.0.32), the inverse of \mathbf{A}_n can be written using block matrix inversion as follows,

$$\mathbf{A}_n^{-1} = \begin{pmatrix} \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1} \mathbf{u} x_n^{-1} \mathbf{u}^T \mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1} \mathbf{u} x_n^{-1} \\ -x_n^{-1} \mathbf{u}^T \mathbf{A}_{n-1}^{-1} & x_n^{-1} \end{pmatrix} \quad (3.0.28)$$

$$= x_n^{-1} \begin{pmatrix} x_n \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1} \mathbf{u}^T \mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1} \mathbf{u} \\ -\mathbf{u}^T \mathbf{A}_{n-1}^{-1} & 1 \end{pmatrix} \quad (3.0.29)$$

where,

$$x_n = d - \mathbf{u}^T \mathbf{A}_{n-1}^{-1} \mathbf{u} \quad (3.0.30)$$

Now, the assumption of \mathbf{A}_n being invertible will hold if and only if \mathbf{A}_{n-1} is invertible, which is intuitively proved and x from (3.0.37) is invertible or x_n is a nonzero scalar. We now prove that x_n is invertible as follows,

$$x_n = \frac{1}{2n-1} - \left(\frac{1}{4} \quad \frac{1}{5} \quad \cdots \quad \frac{1}{2n-2} \right) \mathbf{A}_{n-1}^{-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (3.0.31)$$

In equation (3.0.38) \mathbf{u} contains no negative or zero entries, again \mathbf{A}_{n-1}^{-1} has non zero integer entries, hence $\mathbf{u}^T \mathbf{A}_{n-1}^{-1} \mathbf{u}$ is a non zero scalar. Moreover d is not equal to $\mathbf{u}^T \mathbf{A}_{n-1}^{-1} \mathbf{u}$ hence in (3.0.38) x is non-zero scalar and invertible and hence it has an inverse. Hence \mathbf{A}_n is invertible, proved.

Proof for \mathbf{A}_{n+1} :

Expressing \mathbf{A}_{n+1} using \mathbf{A}_n we get:

$$\mathbf{A}_{n+1} = \begin{pmatrix} \mathbf{A}_n & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (3.0.32)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-1} \end{pmatrix} \quad (3.0.33)$$

$$d = \frac{1}{2n} \quad (3.0.34)$$

Now assuming \mathbf{A}_{n+1} has an inverse, then from (3.0.32), the inverse of \mathbf{A}_{n+1} can be written using block matrix inversion as follows,

$$\mathbf{A}_{n+1}^{-1} = \begin{pmatrix} \mathbf{A}_n^{-1} + \mathbf{A}_n^{-1} \mathbf{u} x_{n+1}^{-1} \mathbf{u}^T \mathbf{A}_n^{-1} & -\mathbf{A}_n^{-1} \mathbf{u} x_{n+1}^{-1} \\ -x_{n+1}^{-1} \mathbf{u}^T \mathbf{A}_n^{-1} & x_{n+1}^{-1} \end{pmatrix} \quad (3.0.35)$$

$$= x_{n+1}^{-1} \begin{pmatrix} x_{n+1} \mathbf{A}_n^{-1} + \mathbf{A}_n^{-1} \mathbf{u}^T \mathbf{A}_n^{-1} & -\mathbf{A}_n^{-1} \mathbf{u} \\ -\mathbf{u}^T \mathbf{A}_n^{-1} & 1 \end{pmatrix} \quad (3.0.36)$$

where,

$$x_{n+1} = d - \mathbf{u}^T \mathbf{A}_n^{-1} \mathbf{u} \quad (3.0.37)$$

Now, the assumption of \mathbf{A}_{n+1} being invertible will hold if and only if \mathbf{A}_n is invertible, which is proved and x from (3.0.37) is invertible or x_{n+1} is a nonzero scalar. We now prove that x_{n+1} is invertible as follows,

$$x_{n+1} = \frac{1}{2n} - \left(\frac{1}{4} \quad \frac{1}{5} \quad \cdots \quad \frac{1}{2n-1} \right) \mathbf{A}_n^{-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-1} \end{pmatrix} \quad (3.0.38)$$

In equation (3.0.38) \mathbf{u} contains no negative or zero entries, again \mathbf{A}_n^{-1} has non zero integer entries, hence $\mathbf{u}^T \mathbf{A}_n^{-1} \mathbf{u}$ is a non zero scalar. Moreover d is not equal to $\mathbf{u}^T \mathbf{A}_n^{-1} \mathbf{u}$ hence in (3.0.38) x is non-zero scalar and invertible and hence it has an inverse. Hence \mathbf{A}_{n+1} is also invertible.

4 OBSERVATIONS:

- 1) The given matrix is a $n \times n$ Hilbert matrix. Which is always invertible with its inverse having integer values.
- 2) The Hilbert matrix is symmetric and positive definite.