Problem Linman 3.6.3

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1 Problem

What conic does the following equation represent.

$$9y^2 - 24xy + 16x^2 - 18x - 101y + 19 = 0 \tag{1.0.1}$$

Find the center and equation refered to centre.

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

where,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \tag{2.0.3}$$

$$f = 19 \tag{2.0.4}$$

1) Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 9 & -12 \\ -12 & 16 \end{vmatrix} = 0 \tag{2.0.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 9 & -12 & -9 \\ -12 & 16 & -\frac{101}{2} \\ -9 & -\frac{101}{2} & 19 \end{vmatrix} \tag{2.0.6}$$

$$\neq 0 \tag{2.0.7}$$

Hence from (2.0.5) and (2.0.7) we conclude that given equation is an parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 9 & 12 \\ 12 & \lambda - 16 \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.0.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.9). The roots of (2.0.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 25 \tag{2.0.10}$$

2) For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.11}$$

Row reducing $\mathbf{V}$ yields

$$\implies \begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + 4R_1]{R_1 = -\frac{R_1}{3}} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

$$\implies \mathbf{p}_1 = \frac{1}{5}\begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{2.0.13}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{2.0.14}$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.0.16}$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{75}{25} = 3 \tag{2.0.17}$$

and its equation is

$$\mathbf{y^T D y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.18}$$

where

$$\eta = 2\mathbf{u}^T\mathbf{p_1} = 75 \tag{2.0.19}$$

and the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.20}$$

using equations (2.0.3),(2.0.4) and (2.0.13)

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \tag{2.0.21}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} -39 & -73 & -19 \\ 9 & -12 & -21 \\ -12 & 16 & 28 \end{pmatrix} \tag{2.0.22}$$

$$R_3 \leftrightarrow R_3 + (4/3)R_2$$

$$\begin{pmatrix} -39 & -73 & -19 \\ 9 & -12 & -21 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.23}$$

$$R_1 \leftrightarrow R_1/(-39)$$

$$\begin{pmatrix} 1 & 73/39 & 19/39 \\ 9 & -12 & -21 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.24}$$

$$R_2 \leftrightarrow R_2 - 9R_1$$

$$\begin{pmatrix} 1 & 73/39 & 19/39 \\ 0 & -1125/39 & -990/39 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.25}$$

$$R_2 \leftrightarrow R_2 \times (-39/1125)$$

$$\begin{pmatrix} 1 & 73/39 & 19/39 \\ 0 & 1 & 22/25 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.26}$$

$$R_1 \leftrightarrow R_1 - (73/39)R_2$$

$$\begin{pmatrix} 1 & 0 & -29/25 \\ 0 & 1 & 22/25 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.27}$$

Thus the vertex $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix} -29/25 \\ 22/25 \end{pmatrix} = \begin{pmatrix} -1.16 \\ 0.88 \end{pmatrix} \tag{2.0.28}$$

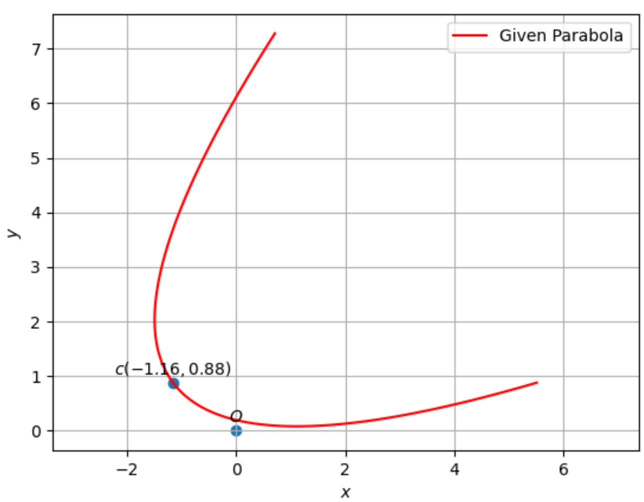

Fig. 0: Plot of the Parabola and its vertex