Problem

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1 Problem

What conic does the following equation represent.

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 \tag{1.0.1}$$

Find the center and equation refered to centre.

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

where,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$f = 5 \tag{2.0.4}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \tag{2.0.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \neq 0 \tag{2.0.6}$$

Hence we conclude that given equation is a parabola. The characteristic equation of V is given as follows,

$$|\mathbf{V} - \lambda\mathbf{I}| = 0 \tag{2.0.7}$$

$$\implies \lambda^2 - 4\lambda = 0 \tag{2.0.8}$$

$$\implies \lambda_1 = 0, \lambda_2 = 4 \tag{2.0.9}$$

The eigen vector $\mathbf{p}$ is defined as,

$$(\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.10}$$

For $\lambda_1 = 0$

$$\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\mathbf{p}_1 = 0 \tag{2.0.11}$$

$$R_2 \leftrightarrow \frac{1}{\sqrt{3}}R_1 + R_2$$

$$\begin{pmatrix} 3 & -\sqrt{3} \\ 0 & 0 \end{pmatrix}\mathbf{p}_1 = 0 \tag{2.0.12}$$

$$\implies \mathbf{p}_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \tag{2.0.13}$$

[Choosing Orthonormal eigen vectors]

For $\lambda_2 = 4$

$$\begin{pmatrix} -1 - \sqrt{3} & \\ -\sqrt{3} & -3 \end{pmatrix}\mathbf{p}_2 = 0 \tag{2.0.14}$$

$$R_2 \leftrightarrow -\sqrt{3}R_1 + R_2$$

$$\begin{pmatrix} 3 & -\sqrt{3} \\ 0 & 0 \end{pmatrix}\mathbf{p}_2 = 0 \tag{2.0.15}$$

$$\implies \mathbf{p}_2 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \tag{2.0.16}$$

[Choosing Orthonormal eigen vectors]

The matrix $\mathbf{p}$ is:

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{2.0.18}$$

$$\eta = 2\mathbf{P}_1^T\mathbf{u} = 2\begin{pmatrix} 3 & -2 \end{pmatrix}\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = 3 - 2\sqrt{3} \tag{2.0.19}$$

The focal length of the parabola is given by:

$$\left|\frac{\eta}{\lambda_2}\right| == 0.116 \tag{2.0.20}$$

When $Det(\mathbf{V}) = 0$, (2.0.1) can be written as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.21}$$

And the vertex c is given by :

$$\begin{pmatrix}\mathbf{u}^T + \frac{\eta}{2}\mathbf{p}_1^T \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u}\end{pmatrix} \tag{2.0.22}$$

Here,

$$\mathbf{V} = \begin{pmatrix}3 & -\sqrt{3} \\ -\sqrt{3} & 1\end{pmatrix} \tag{2.0.23}$$

$$\mathbf{u} = \begin{pmatrix}3 \\ -2\end{pmatrix} \tag{2.0.24}$$

$$\mathbf{p}_1 = \begin{pmatrix}1/2 \\ \sqrt{3}/2\end{pmatrix} \tag{2.0.25}$$

$$f = 5 \tag{2.0.26}$$

Putting these value we get:

$$\begin{pmatrix}(3.75 - 0.5\sqrt{3}) & (-3.5 + 0.75\sqrt{3}) \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1\end{pmatrix}\mathbf{c} = \begin{pmatrix}-5 \\ -2.25 - 0.5\sqrt{3} \\ 0.5 + 0.75\sqrt{3}\end{pmatrix} \tag{2.0.27}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix}(3.75 - 0.5\sqrt{3}) & (-3.5 + 0.75\sqrt{3}) & -5 \\ 3 & -\sqrt{3} & -2.25 - 0.5\sqrt{3} \\ -\sqrt{3} & 1 & 0.5 + 0.75\sqrt{3}\end{pmatrix} \tag{2.0.28}$$

$$R_3 \leftrightarrow -\sqrt{3}R_3 - R_2$$

$$\begin{pmatrix}(3.75 - 0.5\sqrt{3}) & (-3.5 + 0.75\sqrt{3}) & -5 \\ 3 & -\sqrt{3} & -2.25 - 0.5\sqrt{3} \\ 0 & 0 & 0\end{pmatrix} \tag{2.0.29}$$

$$\begin{pmatrix}2.8840 & -2.2001 & -5 \\ 3 & -1.7320 & -3.1160 \\ 0 & 0 & 0\end{pmatrix} \tag{2.0.30}$$

$$\begin{pmatrix}1 & -0.7632 & -1.7337 \\ 3 & -1.7320 & -3.1160 \\ 0 & 0 & 0\end{pmatrix} \tag{2.0.31}$$

$$\begin{pmatrix}1 & -0.7632 & -1.7337 \\ 0 & 0.5576 & 2.0851 \\ 0 & 0 & 0\end{pmatrix} \tag{2.0.32}$$

$$\begin{pmatrix}1 & -0.7632 & -1.7337 \\ 0 & 1 & 3.7394 \\ 0 & 0 & 0\end{pmatrix} \tag{2.0.33}$$

$$\begin{pmatrix}1 & 0 & 1.1202 \\ 0 & 1 & 3.7394 \\ 0 & 0 & 0\end{pmatrix} \tag{2.0.34}$$

Thus the vertex $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix}1.1202 \\ 3.7394\end{pmatrix} \tag{2.0.35}$$