Assignment 9

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Download all python codes from

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https://github.com/sachinomdubey/Matrix-theory/
    Assignment9/codes
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and latex-tikz codes from

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https://github.com/sachinomdubey/Matrix-theory/
    Assignment9
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1 Problem

(Hoffman/Page27/12)
Prove that the given matrix is invertible and $\mathbf{A}^{-1}$ has integer values.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & . & . & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & . & . & \frac{1}{n+1} \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n+1} & . & . & \frac{1}{2n-1} \end{pmatrix} \tag{1.0.1}$$

2 Solution 1

Proof that A is Invertible.
The elements of the given matrix is of the form:

$$A_{ij} = \frac{1}{i+j-1} = \int_0^1 t^{i+j-2} dt = \int_0^1 t^{i-1} t^{j-1} dt \tag{2.0.1}$$

We will prove that the matrix $\mathbf{A}$ is positive definite:
$\mathbf{A}$ is Positive definite, if $\mathbf{XAX}^T > 0$

$$\text{Let } \mathbf{X} = (x_i)_{1\le i\le n} \in \mathbb{R}^N \tag{2.0.2}$$

$$\mathbf{XAX}^T = \sum_{1\le i,j\le n} \frac{x_i x_j}{i+j-1} \tag{2.0.3}$$

From (2.0.1),

$$\mathbf{XAX}^T = \sum_{1\le i,j\le n} x_i x_j \int_0^1 t^{i+j-2} dt \tag{2.0.4}$$

$$= \int_0^1 \left(\sum_{i=1}^n x_i t^{i-1}\right)\left(\sum_{j=1}^n x_j t^{j-1}\right) dt \tag{2.0.5}$$

$$\implies \mathbf{XAX}^T = \int_0^1 \left(\sum_{i=1}^n x_i t^{i-1}\right)^2 dt > 0 \tag{2.0.6}$$

Thus, Matrix $\mathbf{A}$ is Positive definite.
Now, let's say λ is an eigen value of $\mathbf{A}$. Then, for the corresponding eigen vector $\mathbf{X} = (x_1, x_2, ..., x_n)$, we can write:

$$\mathbf{XAX}^T = \mathbf{X}\lambda\mathbf{X}^T \quad [\because \mathbf{AX}^T = \lambda\mathbf{X}^T] \tag{2.0.7}$$

$$= \|\mathbf{X}\|^2 \lambda \tag{2.0.8}$$

$$\implies \lambda = \frac{\mathbf{XAX}^T}{\|\mathbf{X}\|^2} > 0 \tag{2.0.9}$$

So, all of the eigenvalues belonging to $\mathbf{A}$ must be positive. The product of the eigenvalues of a matrix equals the determinant.

$$\boxed{\therefore \det(\mathbf{A}) > 0} \tag{2.0.10}$$

Thus, the given matrix $\mathbf{A}$ is non-singular and its inverse exist (Invertible).

Proof that $\mathbf{A}^{-1}$ has integer values.
Let us consider a set of shifted legendre polynomials as follow:

$$P_i(x) = \begin{cases} P_1(x) = P_{11} \\ P_2(x) = P_{21} + P_{22}x \\ P_3(x) = P_{31} + P_{32}x + p_{33}x^2 \\ . \\ . \\ P_n(x) = P_{n1} + P_{n2}x + P_{n3}x^2 + .. + P_{nn}x^{n-1} \end{cases} \tag{2.0.11}$$

Where, the coefficients P_{ij} are given as:

$$P_{ij} = (-1)^{i+j-1}\binom{j-1}{i-1}\binom{i+j-2}{i-1} \tag{2.0.12}$$

The shifted legendre polynomials are analogous to legendre polynomials, but defined on the interval $[0, 1]$ (whereas the interval is $[-1, 1]$ for legendre polynomial).

A set of shifted legendre polynomials obey the following orthogonal relationship:

$$\int_0^1 P_i(x)P_j(x)dx = 0 \text{ for } i \neq j \tag{2.0.13}$$

$$\int_0^1 P_i(x)P_j(x)dx = \frac{1}{2i+1} \text{ for } i = j \tag{2.0.14}$$

Forming a matrix $\mathbf{P}$ whose elements are the coefficients of polynomials in (2.0.11)

$$\mathbf{P} = \begin{pmatrix} P_{11} & 0 & . & . & 0 \\ P_{21} & P_{22} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ P_{n1} & P_{n2} & . & . & P_{nn} \end{pmatrix} \tag{2.0.15}$$

Forming a matrix $\mathbf{PAP}^T$, the elements of the matrix $\mathbf{PAP}^T$ can be written as:

$$\mathbf{PAP}^T_{ij} = \sum_{s=1}^{N}\sum_{r=1}^{N} P_{ir}P_{js}A_{rs} \tag{2.0.16}$$

From (2.0.1) with suitable change in variable notations, we can write:

$$A_{rs} = \int_0^1 x^{r-1}x^{s-1}dx \tag{2.0.17}$$

From (2.0.16) and (2.0.17),

$$\mathbf{PAP}^T_{ij} = \int_0^1 \sum_{s=1}^{N}\sum_{r=1}^{N} P_{ir}P_{js}x^{r-1}x^{s-1}dx \tag{2.0.18}$$

$$= \int_0^1 \sum_{s=1}^{N} P_{ir}x^{r-1}\sum_{r=1}^{N} P_{js}x^{s-1}dx \tag{2.0.19}$$

$$= \int_0^1 P_i(x)P_j(x)dx \tag{2.0.20}$$

From (2.0.14)

$$\mathbf{PAP}^T_{ij} = \begin{cases} 0 & i \neq j \\ \frac{1}{2i+1} & i = j \end{cases} \tag{2.0.21}$$

Thus, Matrix $\mathbf{PAP}^T$ is diagonal matrix:

$$\mathbf{PAP}^T = \mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & . & . & 0 \\ 0 & \frac{1}{5} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \frac{1}{2n+1} \end{pmatrix} \tag{2.0.22}$$

From (2.0.22), the inverse of matrix $\mathbf{A}$ can be written as:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}(\mathbf{P}^T)^{-1} \tag{2.0.23}$$

$$\implies \mathbf{A}^{-1} = \mathbf{P}^T\mathbf{D}^{-1}\mathbf{P} \tag{2.0.24}$$

From (2.0.12) and (2.0.22), It can be clearly observed that the elements of matrix $\mathbf{P}$, $\mathbf{P}^T$ and $\mathbf{D}^{-1}$ are all integers given as:

$$\mathbf{P}_{ij} = (-1)^{i+j-1}\binom{j-1}{i-1}\binom{i+j-2}{i-1} \tag{2.0.25}$$

$$\mathbf{D}^{-1}_{ij} = \frac{1}{\mathbf{D}_{ij}} = \begin{cases} 0 & i \neq j \\ 2i+1 & i = j \end{cases} \tag{2.0.26}$$

Since, matrix $\mathbf{P}$, $\mathbf{P}^T$ and $\mathbf{D}^{-1}$ are integer matrices, therefore $\mathbf{A}^{-1}$ is also an integer matrix.
Hence proved.

3 Solution 2

https://github.com/Arko98/EE5609/blob/master/Assignment_14

Let $\mathbf{A_3}$ be 3×3 matrix i.e

$$\mathbf{A_3} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \tag{3.0.1}$$

Now we find the inverse of the matrix $\mathbf{A_3}$ as follows,

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right) \tag{3.0.2}$$

$$\xleftrightarrow[R_3=R_3-\frac{1}{3}R_1]{R_2=R_2-\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right) \tag{3.0.3}$$

$$\xleftrightarrow{R_3=R_3-R_2} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right) \tag{3.0.4}$$

$$\xleftrightarrow[R_3=180R_3]{R_2=12R_2} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right) \tag{3.0.5}$$

$$\xleftrightarrow[R_1=R_1-R_3]{R_2=R_2-R3} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right) \tag{3.0.6}$$

$$\xleftrightarrow{R_1=R_1-\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right) \tag{3.0.7}$$

Hence we see that $\mathbf{A_3}$ is invertible and the inverse contains integer entries and $\mathbf{A_3^{-1}}$ is given by,

$$\mathbf{A_3^{-1}} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \tag{3.0.8}$$

Let, $\mathbf{A_4}$ be 4×4 matrix as follows,

$$\mathbf{A_4} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \tag{3.0.9}$$

Now, expressing $\mathbf{A_4}$ using $\mathbf{A_3}$ we get,

$$\mathbf{A_4} = \begin{pmatrix} \mathbf{A_3} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{3.0.10}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \tag{3.0.11}$$

$$d = \frac{1}{7} \tag{3.0.12}$$

Now assuming $\mathbf{A_4}$ has an inverse, then from (3.0.10), the inverse of $\mathbf{A_4}$ can be written using block matrix inversion,

Block matrix inversion

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$\text{If } \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \text{ then,}$$

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & \left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{pmatrix} \tag{3.0.13}$$

$$\therefore \mathbf{A_4^{-1}} = \begin{pmatrix} \mathbf{A_3^{-1}} + \mathbf{A_3^{-1}}\mathbf{u}x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} & -\mathbf{A_3^{-1}}\mathbf{u}x_4^{-1} \\ -x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} & x_4^{-1} \end{pmatrix} \tag{3.0.14}$$

$$= x_4^{-1}\begin{pmatrix} \mathbf{A_3^{-1}}x_4 + \mathbf{A_3^{-1}}\mathbf{u}\mathbf{u^T}\mathbf{A_3^{-1}} & -\mathbf{A_3^{-1}}\mathbf{u} \\ \mathbf{u^T}\mathbf{A_3^{-1}} & 1 \end{pmatrix} \tag{3.0.15}$$

$$\text{where, } x_4 = d - \mathbf{u^T}\mathbf{A_3^{-1}}\mathbf{u} \tag{3.0.16}$$

Now, the assumption of $\mathbf{A_4}$ being invertible will hold if and only if $\mathbf{A_3}$ is invertible, which has been proved in (3.0.8) and x_4 from (3.0.16) is invertible or x_4 is a nonzero scalar. We now prove that x_4 is invertible as follows,

$$x_4 = \frac{1}{7} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \tag{3.0.17}$$

$$\implies x_4 = \frac{1}{2800} \tag{3.0.18}$$

Hence, x_4 is a scalar, hence x_4^{-1} exists and is given by,

$$x_4^{-1} = 2800 \tag{3.0.19}$$

Hence, $\mathbf{A_4}$ is invertible. Now putting the values of $\mathbf{A_3^{-1}}$, x_4^{-1} and $\mathbf{u}$ we get,

$$\mathbf{A_3^{-1}} + \mathbf{A_3^{-1}}\mathbf{u}x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \tag{3.0.20}$$

$$-\mathbf{A_3^{-1}}\mathbf{u}x_4^{-1} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \tag{3.0.21}$$

$$x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \tag{3.0.22}$$

$$x_4^{-1} = 2800 \tag{3.0.23}$$

Putting values from (3.0.20), (3.0.21), (3.0.22) and (3.0.23) into (3.0.14) we get,

$$\mathbf{A_4^{-1}} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \tag{3.0.24}$$

Hence, from (3.0.24) we proved that, $\mathbf{A_4}$ is invertible and has integer entries.

By successively repeating this method, we can prove that $\mathbf{A_5}$, $\mathbf{A_6}$,$\mathbf{A_7}$,.... and so on, are invertible and have integer values. Thus, we can say, $\mathbf{A_{n-1}}$ will be invertible with integer entries. Then we can represent $\mathbf{A_n}$ as follows,

$$\mathbf{A_n} = \begin{pmatrix} \mathbf{A_{n-1}} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{3.0.25}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \tag{3.0.26}$$

$$d = \frac{1}{2n-1} \tag{3.0.27}$$

Now assuming $\mathbf{A_n}$ has an inverse, then from (3.0.25), the inverse of $\mathbf{A_n}$ can be written using block matrix inversion as follows,

$$\mathbf{A_n}^{-1} = \begin{pmatrix} \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}} \mathbf{u} x_n^{-1} \mathbf{u^T} \mathbf{A_{n-1}^{-1}} & -\mathbf{A_{n-1}^{-1}} \mathbf{u} x_n^{-1} \\ -x_n^{-1} \mathbf{u^T} \mathbf{A_{n-1}^{-1}} & x_n^{-1} \end{pmatrix} \tag{3.0.28}$$

$$= x_n^{-1} \begin{pmatrix} x_n \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}} \mathbf{u^T} \mathbf{A_{n-1}^{-1}} & -\mathbf{A_{n-1}^{-1}} \mathbf{u} \\ -\mathbf{u^T} \mathbf{A_{n-1}^{-1}} & 1 \end{pmatrix} \tag{3.0.29}$$

where,

$$x_n = d - \mathbf{u^T} \mathbf{A_{n-1}^{-1}} \mathbf{u} \tag{3.0.30}$$

Now, the assumption of $\mathbf{A_n}$ being invertible will hold if and only if $\mathbf{A_{n-1}}$ is invertible, which is intuitively proved and x from (3.0.30) is invertible or x_n is a nonzero scalar. We now prove that x_n is invertible as follows,

$$x_n = \frac{1}{2n-1} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{2n-2} \end{pmatrix} \mathbf{A_{n-1}^{-1}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \dots \\ \frac{1}{2n-2} \end{pmatrix} \tag{3.0.31}$$

In equation (3.0.31) $\mathbf{u}$ contains no negative or zero entries, again $\mathbf{A_{n-1}^{-1}}$ has non zero integer entries, hence $\mathbf{u^T} \mathbf{A_{n-1}^{-1}} \mathbf{u}$ is a non zero scalar. Moreover d is not equal to $\mathbf{u^T} \mathbf{A_{n-1}^{-1}} \mathbf{u}$ hence in (3.0.31) x is non-zero scalar and invertible and hence it has an inverse. Hence $\mathbf{A_n}$ is invertible, proved.

Proof for $\mathbf{A}_{n+1}$:
Expressing $\mathbf{A}_{n+1}$ using $\mathbf{A}_n$ we get:

$$\mathbf{A_{n+1}} = \begin{pmatrix} \mathbf{A_n} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{3.0.32}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-1} \end{pmatrix} \tag{3.0.33}$$

$$d = \frac{1}{2n} \tag{3.0.34}$$

Now assuming $\mathbf{A_{n+1}}$ has an inverse, then from (3.0.32), the inverse of $\mathbf{A_{n+1}}$ can be written using block matrix inversion as follows,

$$\mathbf{A_{n+1}}^{-1} = \begin{pmatrix} \mathbf{A_n^{-1}} + \mathbf{A_n^{-1}} \mathbf{u} x_{n+1}^{-1} \mathbf{u^T} \mathbf{A_n^{-1}} & -\mathbf{A_n^{-1}} \mathbf{u} x_{n+1}^{-1} \\ -x_{n+1}^{-1} \mathbf{u^T} \mathbf{A_n^{-1}} & x_{n+1}^{-1} \end{pmatrix} \tag{3.0.35}$$

$$= x_{n+1}^{-1} \begin{pmatrix} x_{n+1} \mathbf{A_n^{-1}} + \mathbf{A_n^{-1}} \mathbf{u^T} \mathbf{A_n^{-1}} & -\mathbf{A_n^{-1}} \mathbf{u} \\ -\mathbf{u^T} \mathbf{A_n^{-1}} & 1 \end{pmatrix} \tag{3.0.36}$$

where,

$$x_{n+1} = d - \mathbf{u^T} \mathbf{A_n^{-1}} \mathbf{u} \tag{3.0.37}$$

Now, the assumption of $\mathbf{A_{n+1}}$ being invertible will hold if and only if $\mathbf{A_n}$ is invertible, which is proved and x from (3.0.37) is invertible or x_{n+1} is a nonzero scalar. We now prove that x_{n+1} is invertible as follows,

$$x_{n+1} = \frac{1}{2n} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{2n-1} \end{pmatrix} \mathbf{A_n^{-1}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \dots \\ \frac{1}{2n-1} \end{pmatrix} \tag{3.0.38}$$

In equation (3.0.38) $\mathbf{u}$ contains no negative or zero entries, again $\mathbf{A_n^{-1}}$ has non zero integer entries, hence $\mathbf{u^T} \mathbf{A_n^{-1}} \mathbf{u}$ is a non zero scalar. Moreover d is not equal to $\mathbf{u^T} \mathbf{A_n^{-1}} \mathbf{u}$ hence in (3.0.38) x is non-zero scalar and invertible and hence it has an inverse. Hence $\mathbf{A_{n+1}}$ is also invertible.

Problem statement: If $\mathbf{A_{n-1}^{-1}}$ is invertible and has integer values, Then $\mathbf{A_n^{-1}}$ also has integer values.

Proof:
The matrix $\mathbf{A_n^{-1}}$ can be expressed as:

$$\mathbf{A_n} = \begin{pmatrix} \mathbf{A_{n-1}} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{3.0.39}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n+1} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \tag{3.0.40}$$

$$d = \frac{1}{2n-1} \tag{3.0.41}$$

The inverse of $\mathbf{A_n}$ can be written using block matrix inversion,

Block matrix inversion

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$\text{If } \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \text{ then,}$$

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}\right)^{-1}\mathbf{CA}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}\right)^{-1}\mathbf{CA}^{-1} & \left(\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}\right)^{-1} \end{pmatrix} \tag{3.0.42}$$

$$\mathbf{A_n}^{-1} = \begin{pmatrix} \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}u}x_n^{-1}\mathbf{u^T A_{n-1}^{-1}} & -\mathbf{A_{n-1}^{-1}u}x_n^{-1} \\ -x_n^{-1}\mathbf{u^T A_{n-1}^{-1}} & x_n^{-1} \end{pmatrix} \tag{3.0.43}$$

where,

$$x_n = d - \mathbf{u^T A_{n-1}^{-1} u} \tag{3.0.44}$$

For $\mathbf{A_n^{-1}}$ to have integer values, each of the four blocks in (3.0.43) should have integer values.
Let us first prove that x_n^{-1} have integer values as follow:

$$x_n = d - \mathbf{u^T A_{n-1}^{-1} u} \tag{3.0.45}$$

$$= \frac{1}{2n-1} - \sum_i^{n-1}\sum_j^{n-1} u_i u_j (A_{n-1}^{-1})_{ij} \tag{3.0.46}$$

$$= \frac{1}{2n-1} - \sum_i^{n-1}\sum_j^{n-1} \frac{1}{n+i-1}\frac{1}{n+j-1}(A_{n-1}^{-1})_{ij} \tag{3.0.47}$$

Thus, x_n is a scalar of the form $\frac{p}{q}$. Also, by calculation, $x_2 = \frac{1}{12}$, $x_3 = \frac{1}{180}$ and $x_3 = \frac{1}{2800}$. Thus, by induction, we can say $p = 1$ for any value of n. Thus, x_n^{-1} can be written as:

$$x_n^{-1} = \frac{1}{x_n} = \frac{q}{p} \tag{3.0.48}$$

Since, $p = 1$ for any value of n, Hence $x_n^{-1} = q$ is always integer.

Now, let us consider the block $-\mathbf{A_{n-1}^{-1}u}x_n^{-1}$, we can write:

$$(-\mathbf{A_{n-1}^{-1}u}x_n^{-1})_{i,1} = -q\sum_j^{n-1}\frac{1}{n+j-1}(A_{n-1}^{-1})_{i,j} \tag{3.0.49}$$

Here, the considered block is a $(n-1)\times 1$ matrix, with each element of rational form $\frac{p_1}{q_1}$. For $n = 2, 3, 4$ the value of q_1 comes as 1. Thus by induction, the value of q_1 is always 1 for any value of n. Hence, this matrix always has integer values.

Now, let us consider the block $-x_n^{-1}\mathbf{u}^T\mathbf{A_{n-1}^{-1}}$, we can write:

$$(-x_n^{-1}\mathbf{u}^T\mathbf{A_{n-1}^{-1}})_{1,j} = -q\sum_i^{n-1}\frac{1}{n+i-1}(A_{n-1}^{-1})_{i,j} \tag{3.0.50}$$

Here, the considered block is a $1\times(n-1)$ matrix, with each element of rational form $\frac{p_2}{q_2}$. For $n = 2, 3, 4$ the value of q_2 comes as 1. Thus by induction, the value of q_2 is always 1 for any value of n. Hence, this matrix always has integer values.

Considering the block $\mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}u}x_n^{-1}\mathbf{u^T A_{n-1}^{-1}}$, let denote it as V_{n-1}^{-1}:

$$V_{n-1}^{-1} = \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}u}x_n^{-1}\mathbf{u^T A_{n-1}^{-1}} \tag{3.0.51}$$

$$= \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}u}\left(d - \mathbf{u^T A_{n-1}^{-1} u}\right)^{-1}\mathbf{u^T A_{n-1}^{-1}} \tag{3.0.52}$$

The Woodbury matrix identity is given by:

$$\boxed{(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}} \tag{3.0.53}$$

Comparing (3.0.52) and (3.0.53), we can write:

$$V_{n-1}^{-1} = \left(A_{n-1} - (2n-1)\mathbf{uu}^T\right)^{-1} \tag{3.0.54}$$

$$\therefore V_{n-1} = A_{n-1} - (2n-1)\mathbf{uu}^T \tag{3.0.55}$$

Here, V_{n-1} can be expanded as:

$$V_{n-1} = A_{n-1} - (2n-1)\mathbf{u}\mathbf{u}^T \tag{3.0.56}$$

$$= \frac{1}{i+j-1} - \frac{(2n-1)}{(n+i-1)(n+j-1)} \tag{3.0.57}$$

$$= \frac{n^2 - ni - nj - ij}{(i+j-1)(n+i-1)(n+j-1)} \tag{3.0.58}$$

$$= \frac{(n-i)(n-j)}{(i+j-1)(n+i-1)(n+j-1)} \tag{3.0.59}$$

$$\therefore (V_{n-1})_{ij} = \left((A_{n-1})_{ij} \frac{(n-i)(n-j)}{(n+i-1)(n+j-1)}\right) \tag{3.0.60}$$

Using (3.0.60), The inverse V_{n-1}^{-1} can be written as:

$$(V_{n-1}^{-1})_{ij} = \left((A_{n-1}^{-1})_{ij} \frac{(n+i-1)(n+j-1)}{(n-i)(n-j)}\right) \tag{3.0.61}$$

Here, the considered block is a $(n-1) \times (n-1)$ matrix, with each element of rational form $\frac{p_3}{q_3}$. For $n = 2, 3, 4$ the value of q_3 comes as 1. Thus by induction, the value of q_3 is always 1 for any value of n. Hence, this matrix always has integer values. Since, all the four blocks has integer values, the inverse A_n^{-1} has integer values.

4 Observations:

1) The given matrix is a $n \times n$ Hilbert matrix. Which is always invertible with its inverse having integer values.
2) The Hilbert matrix is symmetric and positive definite.