Assignment 8

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Download all python codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment8/codes
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and latex-tikz codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment8
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1 Problem

(Dresden/Page80/Example1/D)
Determine the distance of the point $D(-1, 2, -4)$ from the plane given below. Also find the foot of perpendicular drawn from the point D to the given plane using SVD.

$$3x + 2y - 6z - 2 = 0 \tag{1.0.1}$$

2 Solution

Equation of plane can be written in the form:

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.1}$$

Writing the given plane equation (1.0.1) in the form (2.0.1):

$$\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \tag{2.0.2}$$

Where,

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad c = 2 \tag{2.0.4}$$

A vector from the plane to the point $D(-1, 2, -4)$ is given by:

$$\mathbf{w} = \mathbf{d} - \mathbf{x} \tag{2.0.5}$$

Where,

$$\mathbf{d} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.0.7}$$

The projection of $\mathbf{w}$ onto the normal vector $\mathbf{n}$ can be written as:

$$\text{proj}_{\mathbf{n}}\mathbf{w} = \frac{\mathbf{n}^T\mathbf{w}}{\mathbf{n}^T\mathbf{n}} \cdot \mathbf{n} \tag{2.0.8}$$

$$\text{proj}_{\mathbf{n}}\mathbf{w} = \frac{\mathbf{n}^T(\mathbf{d} - \mathbf{x})}{\mathbf{n}^T\mathbf{n}} \cdot \mathbf{n} \tag{2.0.9}$$

$$\text{proj}_{\mathbf{n}}\mathbf{w} = \frac{\mathbf{n}^T\mathbf{d} - \mathbf{n}^T\mathbf{x}}{\mathbf{n}^T\mathbf{n}} \cdot \mathbf{n} \tag{2.0.10}$$

$$\tag{2.0.11}$$

From equation (2.0.1),

$$\text{proj}_{\mathbf{n}}\mathbf{w} = \frac{\mathbf{n}^T\mathbf{d} - c}{\mathbf{n}^T\mathbf{n}} \cdot \mathbf{n} \tag{2.0.12}$$

$$\tag{2.0.13}$$

Putting the values of $\mathbf{n}$, $\mathbf{d}$ and c, we get:

$$\text{proj}_{\mathbf{n}}\mathbf{w} = \frac{\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} - 2}{\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \tag{2.0.14}$$

$$\text{proj}_{\mathbf{n}}\mathbf{w} = \frac{23}{49} \cdot \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \tag{2.0.15}$$

The distance d_{min} between point $D(-1, 2, -4)$ and the given plane is obtained as:

$$d_{min} = \left\|\text{proj}_{\mathbf{n}}\mathbf{w}\right\| \tag{2.0.16}$$

$$d_{min} = \frac{23}{49} \cdot \left\|\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}\right\| \tag{2.0.17}$$

$$\therefore d_{min} = \frac{23}{49} \times \sqrt{(3)^2 + (2)^2 + (-6)^2} \tag{2.0.18}$$

$$\therefore d_{min} = \frac{23}{49} \times 7 \tag{2.0.19}$$

$$\implies \boxed{d_{min} = \frac{23}{7} = 3.2857} \tag{2.0.20}$$

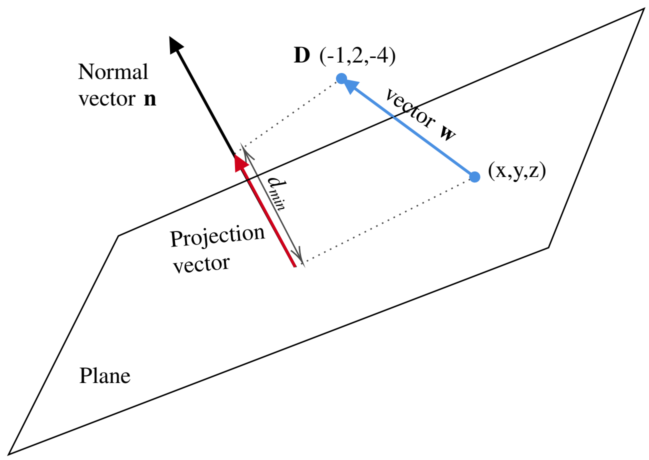

Fig. 0: Distance between a point and a plane

Finding the foot of perpendicular from point D to the given plane using SVD:

We need to represent the equation of plane in parametric form,

$$\mathbf{Q} = \mathbf{p} + \alpha_1\mathbf{m_1} + \alpha_2\mathbf{m_2} \tag{2.0.21}$$

Here $\mathbf{p}$ is any point on plane and $\mathbf{m_1}, \mathbf{m_2}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the vector $\mathbf{n}$.

Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0$$

$$\implies 3a + 2b - 6c = 0 \tag{2.0.22}$$

By substituting $a = 1; b = 0$ in (2.0.22),

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix} \tag{2.0.23}$$

By substituting $a = 0; b = 1$ in (2.0.22),

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix} \tag{2.0.24}$$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, z = 0$ in (2.0.2), we get $\mathbf{p} = \begin{pmatrix} 1 \\ \frac{-1}{2} \\ 0 \end{pmatrix}$

Let $\mathbf{Q}$ be the point on plane with shortest distance to point $\mathbf{d}$. $\mathbf{Q}$ can be expressed in (2.0.21) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ \frac{-1}{2} \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \tag{2.0.25}$$

Computation of Pseudo Inverse using SVD in order to determine the value of α_1 and α_2 :

$$\begin{pmatrix} 1 \\ \frac{-1}{2} \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} \tag{2.0.26}$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{5}{2} \\ -4 \end{pmatrix} \tag{2.0.27}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{5}{2} \\ -4 \end{pmatrix} \tag{2.0.28}$$

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.29}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \tag{2.0.30}$$

$$\mathbf{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{b} = \begin{pmatrix} -2 \\ \frac{5}{2} \\ -4 \end{pmatrix} \tag{2.0.32}$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.33}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \tag{2.0.34}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \tag{2.0.35}$$

From (2.0.29) and (2.0.33),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$
$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{2.0.36}$$

Where $\mathbf{S}_{+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36}; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{2.0.37}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} 18 \\ 12 \\ 13 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix} \tag{2.0.38}$$

Normalizing the above eigenvectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{13}{7\sqrt{13}} \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} \frac{-3}{7} \\ \frac{-2}{7} \\ \frac{6}{7} \end{pmatrix} \tag{2.0.39}$$

From (2.0.39) we obtain $\mathbf{U}$ as,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{13}{7\sqrt{13}} & 0 & \frac{6}{7} \end{pmatrix} \tag{2.0.40}$$

Using values from (2.0.37),

$$\mathbf{S} = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.41}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9}-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{49}{36}; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v_1} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}; \quad \mathbf{v_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix};$$

Normalizing the above eigenvectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{2.0.42}$$

From(2.0.42) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{2.0.43}$$

From (2.0.33) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{13}{7\sqrt{13}} & 0 & \frac{6}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{2.0.44}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_{+} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.45}$$

From (2.0.36),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-58}{7\sqrt{13}} \\ \frac{23}{2\sqrt{13}} \\ \frac{-23}{7} \end{pmatrix}$$

$$\mathbf{S}_{+}\mathbf{U}^T\mathbf{d} = \begin{pmatrix} \frac{-348}{49\sqrt{13}} \\ \frac{23}{2\sqrt{13}} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^T\mathbf{d} = \boxed{\begin{pmatrix} \frac{-167}{49} \\ \frac{153}{98} \end{pmatrix}} \tag{2.0.46}$$

To verify the value of $\mathbf{x}$ obtained from (2.0.46),

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.47}$$

Substituting the values from (2.0.34) in (2.0.47),

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -4 \\ \frac{7}{6} \end{pmatrix}$$

Converting the above equation into augmented form and solving for $\mathbf{x}$,

$$\begin{aligned} &\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & -4 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{6} \end{pmatrix} \\ \xleftrightarrow{R_1 \leftarrow \frac{4}{5} R_1} &\begin{pmatrix} 1 & \frac{2}{15} & \frac{-16}{5} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{6} \end{pmatrix} \\ \xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{6} R_1} &\begin{pmatrix} 1 & \frac{2}{15} & \frac{-12}{5} \\ 0 & \frac{49}{45} & \frac{17}{10} \end{pmatrix} \\ \xleftrightarrow{R_2 \leftarrow \frac{45}{49} R_2} &\begin{pmatrix} 1 & \frac{2}{15} & \frac{-12}{5} \\ 0 & 1 & \frac{153}{98} \end{pmatrix} \\ \xleftrightarrow{R_1 \leftarrow R_1 - \frac{2}{15} R_2} &\begin{pmatrix} 1 & 0 & \frac{-167}{49} \\ 0 & 1 & \frac{153}{98} \end{pmatrix} \end{aligned} \tag{2.0.48}$$

From (2.0.48) it can be observed that,

$$\mathbf{x} = \begin{pmatrix} \frac{-167}{49} \\ \frac{153}{98} \end{pmatrix} \tag{2.0.49}$$

Hence verified.
Thus, the point $\mathbf{Q}$ (foot of the perpendicular) can be obtained by putting values of α_1 and α_2 in (2.0.25):

$$\mathbf{Q} = \begin{pmatrix} 1 \\ \frac{-1}{2} \\ 0 \end{pmatrix} + \frac{-167}{49} \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \frac{153}{98} \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \tag{2.0.50}$$

$$\boxed{\mathbf{Q} = \begin{pmatrix} \frac{-118}{49} \\ \frac{52}{49} \\ \frac{-58}{49} \end{pmatrix}} \tag{2.0.51}$$

The distance between the point D and the plane can be obtained as:

$$\|\mathbf{Q} - \mathbf{d}\| = \sqrt{\left(\frac{-118}{49} + 1\right)^2 + \left(\frac{52}{49} - 2\right)^2 + \left(\frac{-58}{49} + 4\right)^2} \tag{2.0.52}$$

$$\boxed{\|\mathbf{Q} - \mathbf{d}\| = \frac{23}{7} = 3.2857} \tag{2.0.53}$$

Thus the distance obtained in equation (2.0.20) by projection method is matches with the distance obtained in equation (2.0.53). Hence verified.