Assignment 6

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Download all python codes from

```
https://github.com/sachinomdubey/Matrix−theory/
    Assignment6/codes
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and latex-tikz codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment6
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1 Problem

(Rams 3.4.1) find the Asymptotes of the following:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{1.0.1}$$

2 Solution

Comparing the given equation with the form:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.0.1}$$

We get:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{2.0.2}$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{2.0.4}$$

$$f = -6 \tag{2.0.5}$$

Here, $\left|\mathbf{V}\right| = -1$. Since $\left|\mathbf{V}\right| < 0$ the given equation represents a hyperbola with center:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{2.0.6}$$

The characteristic equation of $\mathbf{V}$ is:

$$\left|V - \lambda\mathbf{I}\right| = 0 \tag{2.0.7}$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - 1 = 0 \tag{2.0.9}$$

$$\lambda_1 = 1, \lambda_2 = -1 \tag{2.0.10}$$

Finding the eigen vector matrix $\mathbf{P}$ such that $\mathbf{P}^T = \mathbf{P}^{-1}$:

For $\lambda_1 = 1$

$$(\mathbf{V} - \lambda_1\mathbf{I})\mathbf{p}_1 = 0 \tag{2.0.11}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{p}_1 = 0 \tag{2.0.12}$$

$$\implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.13}$$

[Choosing Orthonormal eigen vectors]

For $\lambda_2 = -1$

$$(\mathbf{V} - \lambda_2\mathbf{I})\mathbf{p}_2 = 0 \tag{2.0.14}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{p}_2 = 0 \tag{2.0.15}$$

$$\implies \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.16}$$

[Choosing Orthonormal eigen vectors]

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.17}$$

By affine transformation $\mathbf{x} = \mathbf{P}\mathbf{y}+\mathbf{c}$, Equation (2.0.1) can be written in the form:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{2.0.18}$$

where,

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{2.0.19}$$

Thus, we can write:

$$\lambda_1 y_1^2 - (-\lambda_2)\, y_1^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{2.0.20}$$

The equation (2.0.20) represents a modified hyperbola, The equation of the asymptotes for (2.0.20) is:

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{y} = 0 \tag{2.0.21}$$

Putting the values of λ_1 and λ_2 in equation (2.0.21), we get the two asymptotes for (2.0.20):

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{y} = 0 \tag{2.0.22}$$

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{y} = 0 \tag{2.0.23}$$

These are the asymptotes of our modified hyperbola. The asymptotes of our original hyberbola in equation (2.0.2) can be obtained using:

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^T (\mathbf{x} - \mathbf{c}) = 0 \tag{2.0.24}$$

Putting the values of λ_1, λ_2 and $\mathbf{P}$ in equation (2.0.24), we get the equations of the asymptotes of the original hyperbola with center at $\mathbf{c}$:

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\mathbf{x} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right) = 0 \tag{2.0.25}$$

$$\implies \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 5} \tag{2.0.26}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\mathbf{x} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right) = 0 \tag{2.0.27}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -1} \tag{2.0.28}$$

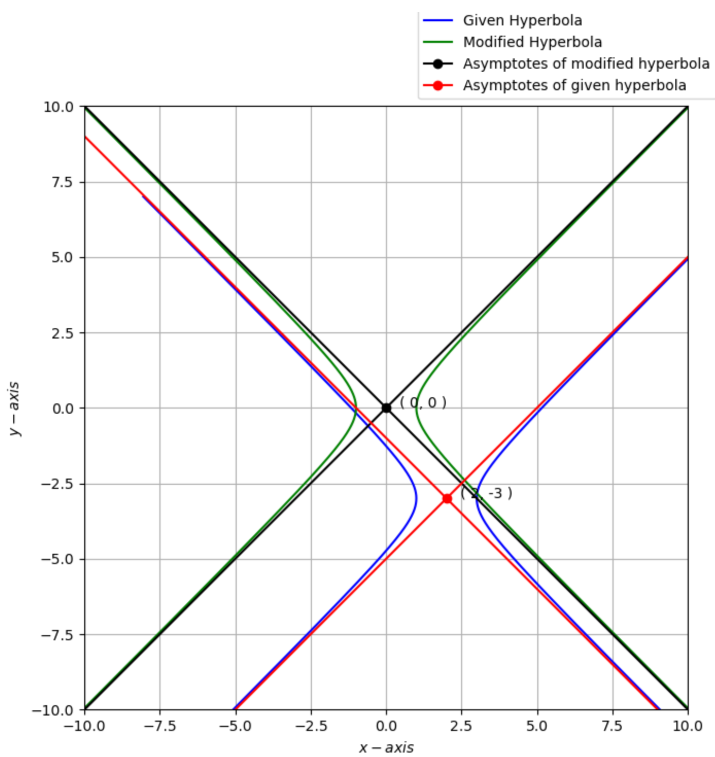

Fig. 0: Plot of the Asymtotes.