Assignment 5

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Download all python codes from

```
https://github.com/sachinomdubey/Matrix−theory/
    Assignment5/codes
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and latex-tikz codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment5
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0.1 Problem

(Geolin 1.9) AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.

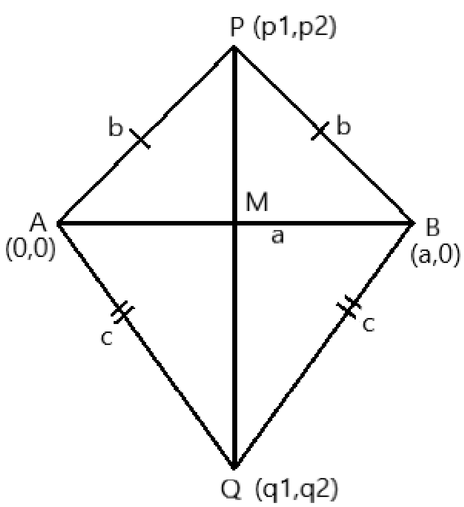

Fig. 0

0.2 Explanation

In order to prove that line PQ is the perpendicular bisector of AB, two conditions need to be met:

1) AM=BM
2) Angle between line AB and line PQ should be 90°.

Steps to prove the first condition are as follow:

1) Finding the normal vectors of line PQ. Further using the normal vector to find the equation of line PQ.
2) The solution of equation of line AB and line PQ will give the co-ordinates of point M.
3) Using the distance formula to prove that AM=BM.

Steps to prove the second condition are as follow:

1) Finding the normal vectors of line AB and PQ.
2) Taking their dot product and checking whether it is zero. If the dot product of the normal vectors is zero, the lines will be perpendicular to each other.

0.3 Solution

Let the Points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{Q}$ be:

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{0.3.1}$$

The point $\mathbf{P}$ is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Let the distance be b. Then we can write:

$$\|\mathbf{A} - \mathbf{P}\| = b \tag{0.3.2}$$

$$\|\mathbf{B} - \mathbf{P}\| = b \tag{0.3.3}$$

The equation 0.3.2 can be further written as:

$$\left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\| = b \tag{0.3.4}$$

$$\sqrt{p_1^2 + p_2^2} = b \tag{0.3.5}$$

$$p_1^2 + p_2^2 = b^2 \tag{0.3.6}$$

Similarly, the equation 0.3.3 can be further written as:

$$\left\| \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\| = b \tag{0.3.7}$$

$$\sqrt{(a - p_1)^2 + p_2^2} = b \tag{0.3.8}$$

$$p_1^2 + p_2^2 - 2ap_1 + a^2 = b^2 \tag{0.3.9}$$

From equations 0.3.6 and 0.3.9, we get:

$$p_1^2 + p_2^2 - 2ap_1 + a^2 = p_1^2 + p_2^2 \tag{0.3.10}$$
$$\therefore a^2 - 2ap_1 = 0 \tag{0.3.11}$$
$$\implies p_1 = a/2 \tag{0.3.12}$$

Putting p_1 in equation 0.3.6:

$$(a/2)^2 + p_2^2 = b^2 \tag{0.3.13}$$
$$p_2 = \pm\frac{\sqrt{4b^2 - a^2}}{2} \tag{0.3.14}$$

Taking positive value of p_2 as the point $\mathbf{P}$ lies in the first quadrant.

$$\therefore p_2 = \frac{\sqrt{4b^2 - a^2}}{2} \tag{0.3.15}$$

Thus, point $\mathbf{P}$ is:

$$\mathbf{P} = \begin{pmatrix} a/2 \\ \sqrt{4b^2 - a^2}/2 \end{pmatrix} \tag{0.3.16}$$
$$\tag{0.3.17}$$

Similarly, The point $\mathbf{Q}$ equidistant from the points $\mathbf{A}$ and $\mathbf{B}$ with distance c can be calculated as:

$$\mathbf{Q} = \begin{pmatrix} a/2 \\ -\sqrt{4c^2 - a^2}/2 \end{pmatrix} \tag{0.3.18}$$

The directional vector of the line PQ is given by:

$$\mathbf{m}_{PQ} = (\mathbf{P} - \mathbf{Q}) \tag{0.3.19}$$
$$\mathbf{m}_{PQ} = \begin{pmatrix} a/2 \\ \sqrt{4b^2 - a^2}/2 \end{pmatrix} - \begin{pmatrix} a/2 \\ -\sqrt{4c^2 - a^2}/2 \end{pmatrix} \tag{0.3.20}$$
$$\mathbf{m}_{PQ} = \begin{pmatrix} 0 \\ \dfrac{\sqrt{4b^2 - a^2} + \sqrt{4c^2 - a^2}}{2} \end{pmatrix} \tag{0.3.21}$$

The normal vector of line PQ is:

$$\mathbf{n}_{PQ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m}_{PQ} \tag{0.3.22}$$
$$\mathbf{n}_{PQ} = \begin{pmatrix} \dfrac{-\sqrt{4b^2 - a^2} - \sqrt{4c^2 - a^2}}{2} \\ 0 \end{pmatrix} \tag{0.3.23}$$

Equation of a line PQ in terms of normal vector is by:

$$\mathbf{n}_{PQ}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{0.3.24}$$
$$\therefore \begin{pmatrix} a/2 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{0.3.25}$$

Since the line AB is the X-axis, its equation and normal vector can be written as:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{0.3.26}$$
$$\mathbf{n}_{AB} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.3.27}$$

Solving equation 0.3.25 and 0.3.26, we get the cordinates of point $\mathbf{M}$:

$$\mathbf{M} = \begin{pmatrix} a/2 \\ 0 \end{pmatrix} \tag{0.3.28}$$

The segments AM and BM are given as:

$$AM = \|\mathbf{A} - \mathbf{M}\| \tag{0.3.29}$$
$$\therefore AM = \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} a/2 \\ 0 \end{pmatrix} \right\| \tag{0.3.30}$$
$$\therefore AM = \sqrt{a^2/4 + 0} = a/2 \tag{0.3.31}$$
$$Similarly, BM = \|\mathbf{B} - \mathbf{M}\| \tag{0.3.32}$$
$$\therefore BM = \left\| \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} a/2 \\ 0 \end{pmatrix} \right\| \tag{0.3.33}$$
$$\therefore BM = \sqrt{a^2/4 + 0} = a/2 \tag{0.3.34}$$

From equation 0.3.31 and 0.3.34 :

$$AM = BM \tag{0.3.35}$$

Using equation 0.3.23 and 0.3.27, dot product of normal vectors of line AB and PQ is:

$$\mathbf{n}_{AB}^T \mathbf{n}_{PQ} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{-\sqrt{4b^2 - a^2} - \sqrt{4c^2 - a^2}}{2} \\ 0 \end{pmatrix} \tag{0.3.36}$$
$$\therefore \mathbf{n}_{AB}^T \mathbf{n}_{PQ} = 0 \tag{0.3.37}$$

From equation 0.3.35 and 0.3.37, PQ is perpendicular bisector of AB. Hence proved.