Assignment 2

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Download all python codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment2/codes
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and latex-tikz codes from

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https://github.com/sachinomdubey/Matrix−theory/
    Assignment2
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Q no. 73. Find the angle between the following pair of lines: Also find the closest points and minimum distance between them.

1)

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{0.0.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{0.0.2}$$

2)

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \tag{0.0.3}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \tag{0.0.4}$$

Solution:

1) The direction vectors of the lines are $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.
   Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{0.0.5}$$

$$= \frac{19}{3 \times 7} \tag{0.0.6}$$

$$\implies \theta = 25.21^\circ \tag{0.0.7}$$

2) The direction vectors of the lines are $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}$.
   Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{0.0.8}$$

$$= \frac{16}{\sqrt{6} \times \sqrt{50}} \tag{0.0.9}$$

$$\implies \theta = 22.52^\circ \tag{0.0.10}$$

Note : In both problems, the respective pair of lines do not intersect each other (called skew lines), The obtained angle is the angle between the direction vectors of the lines. The proof that the pair of lines do not intersect in Problem 1 is as follows:

Problem 1 : Equating the x, y and z components of both lines and forming equation in the augmented matrix form. The Matrix is row reduced as follows:

$$\begin{pmatrix} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{pmatrix} \tag{0.0.11}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix} 3 & -1/3 & 5/3 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{pmatrix} \tag{0.0.12}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 3 & -1/3 & 5/3 \\ 0 & -4/3 & -13/3 \\ 6 & -2 & -1 \end{pmatrix} \tag{0.0.13}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 6R_1} \begin{pmatrix} 3 & -1/3 & 5/3 \\ 0 & -4/3 & -13/3 \\ 0 & 0 & -11 \end{pmatrix} \tag{0.0.14}$$

Here, Rank(A) $\neq$ Rank(A|B). Hence, these three equations are inconsistent, which proves that the two lines do not intersect in the 3D plane. (Where A is the coefficient matrix and A|B is the augmented matrix.)

Finding the closest points on the lines and minimum distance (Problem 1) :
The given equations are in the form:

$$\mathbf{x} = p_1 + \lambda_1 d_1 \tag{0.0.15}$$
$$\mathbf{x} = p_2 + \lambda_2 d_2 \tag{0.0.16}$$

Where, p_1 and p_2 are points on line1 and line2 respectively. Also, d_1 and d_2 are the direction vectors of respective lines. The cross product of d_1 and d_2 is perpendicular to both lines.

$$\mathbf{n} = d_1 \times d_2 \tag{0.0.17}$$
$$n = \begin{vmatrix} i & j & k \\ 3 & 2 & 6 \\ 1 & 2 & 2 \end{vmatrix} \implies n = \begin{pmatrix} -8 \\ 0 \\ 4 \end{pmatrix} \tag{0.0.18}$$

When the line2 is translated along the vector n, it forms a plane. This plane intersects line1 at a single point c_1 which is nearest to the line2. Point c_1 is given by:

$$c_1 = p_1 + \frac{(p_2 - p_1) \cdot n_2}{d_1 \cdot n_2} d_1 \tag{0.0.19}$$
$$where, n_2 = d_2 \times n \tag{0.0.20}$$
$$\therefore n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ -8 & 0 & 4 \end{vmatrix} \implies n_2 = \begin{pmatrix} 8 \\ -20 \\ 16 \end{pmatrix} \tag{0.0.21}$$
$$\therefore c_1 = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -20 \\ 16 \end{pmatrix}}{\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -20 \\ 16 \end{pmatrix}} \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{0.0.22}$$
$$\implies c_1 = \begin{pmatrix} 3.65 \\ -3.9 \\ 4.3 \end{pmatrix} \tag{0.0.23}$$

Similarly, the point on Line2 nearest to Line 1 is c_2 given by:

$$c_2 = p_2 + \frac{(p_1 - p_2) \cdot n_1}{d_2 \cdot n_1} d_2 \tag{0.0.24}$$
$$where, n_1 = d_1 \times n \tag{0.0.25}$$
$$\therefore n_1 = \begin{vmatrix} i & j & k \\ 3 & 2 & 6 \\ -8 & 0 & 4 \end{vmatrix} \implies n_1 = \begin{pmatrix} 8 \\ -60 \\ 16 \end{pmatrix} \tag{0.0.26}$$
$$\therefore c_2 = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -60 \\ 16 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -60 \\ 16 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{0.0.27}$$
$$\implies c_2 = \begin{pmatrix} 8.05 \\ -3.9 \\ 2.1 \end{pmatrix} \tag{0.0.28}$$

The minimum distance can be found using points c_1 and c_2 as **4.92** units.

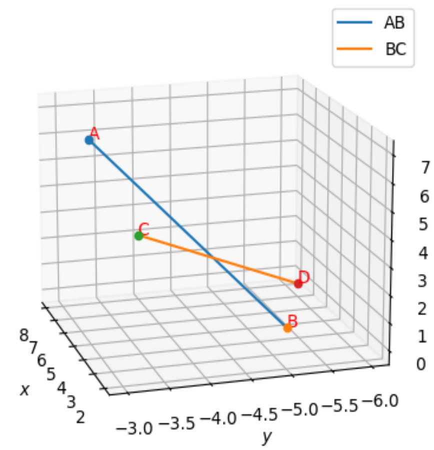

Fig. 2: Problem 1 : Lines crossing each other, but not intersecting

The proof that the pair of lines do not intersect in Problem 2 is as follows:

Problem 2 : Equating the x, y and z components of both lines and forming equation in the augmented matrix forms. The Matrix is row reduced as follows:

$$\begin{pmatrix} 1 & -3 & -1 \\ -1 & 5 & -2 \\ -2 & 4 & -54 \end{pmatrix} \tag{0.0.29}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -3 & -1 \\ 0 & 2 & -3 \\ -2 & 4 & -54 \end{pmatrix} \tag{0.0.30}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 2R_1} \begin{pmatrix} 1 & -3 & -1 \\ 0 & 2 & -3 \\ 0 & -2 & -56 \end{pmatrix} \tag{0.0.31}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & -3/2 \\ 0 & -2 & -56 \end{pmatrix} \tag{0.0.32}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & -3/2 \\ 0 & 0 & -59 \end{pmatrix} \tag{0.0.33}$$

Here, Rank(A) $\neq$ Rank(A|B).
Hence, the equations are inconsistent, which proves that the two lines do not intersect in the 3D plane. (Where A is the coefficient matrix and A|B is the augmented matrix.)

Finding the closest points on the lines and minimum distance (Problem 2) :

The given equations are in the form:

$$\mathbf{x} = p_1 + \lambda_1 d_1 \tag{0.0.34}$$

$$\mathbf{x} = p_2 + \lambda_2 d_2 \tag{0.0.35}$$

Where, p_1 and p_2 are points on line1 and line2 respectively. Also, d_1 and d_2 are the direction vectors of respective lines. The cross product of d_1 and d_2 is perpendicular to both lines.

$$\mathbf{n} = d_1 \times d_2 \tag{0.0.36}$$

$$n = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ 3 & -5 & -4 \end{vmatrix} \implies n = \begin{pmatrix} -6 \\ -2 \\ -2 \end{pmatrix} \tag{0.0.37}$$

When the line2 is translated along the vector n, it forms a plane. This plane intersects line1 at a single point c_1 which is nearest to the line2. Point c_1 is given by:

$$c_1 = p_1 + \frac{(p_2 - p_1) \cdot n_2}{d_1 \cdot n_2} d_1 \tag{0.0.38}$$

$$where, n_2 = d_2 \times n \tag{0.0.39}$$

$$\therefore n_2 = \begin{vmatrix} i & j & k \\ 3 & -5 & -4 \\ -6 & -2 & -2 \end{vmatrix} \implies n_2 = \begin{pmatrix} 2 \\ 30 \\ -36 \end{pmatrix} \tag{0.0.40}$$

$$\therefore c_1 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + \frac{\begin{pmatrix} -1 \\ -2 \\ -54 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 30 \\ -36 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 30 \\ -36 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \tag{0.0.41}$$

$$\implies c_1 = \begin{pmatrix} 45.77 \\ -41.77 \\ -87.54 \end{pmatrix} \tag{0.0.42}$$

Similarly, the point on Line2 nearest to Line 1 is c_2 given by:

$$c_2 = p_2 + \frac{(p_1 - p_2) \cdot n_1}{d_2 \cdot n_1} d_2 \tag{0.0.43}$$

$$where, n_1 = d_1 \times n \tag{0.0.44}$$

$$\therefore n_1 = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ -6 & -2 & -2 \end{vmatrix} \implies n_1 = \begin{pmatrix} -2 \\ 14 \\ -8 \end{pmatrix} \tag{0.0.45}$$

$$\therefore c_2 = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \frac{\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 14 \\ -8 \end{pmatrix}}{\begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 14 \\ -8 \end{pmatrix}} \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \tag{0.0.46}$$

$$\implies c_2 = \begin{pmatrix} 29.69 \\ -47.15 \\ -92.92 \end{pmatrix} \tag{0.0.47}$$

The minimum distance can be found using points c_1 and c_2 as **17.79** units.

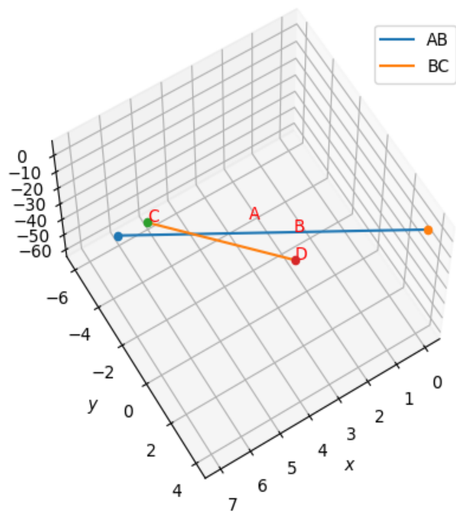

Fig. 2: Problem 2 : Lines crossing each other, but not intersecting