23IICS030210

Assignment - 2

1) If $u = \log(x^3+y^3+z^3-3xyz)$ prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Ans: Given $u = \log(x^3+y^3+z^3-3xyz)$

Since $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)$

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \text{ --- (P)}$$

Let's take $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$ --- (1)

$$\frac{\partial u}{\partial y} = \frac{1}{(x^3+y^3+z^3-3xyz}(3y^2-3xz) \text{ --- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3yx) \text{ --- (3)}$$

Adding (1), (2). and (3)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$[\because a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)]$

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$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z} \quad \text{(II)}$$

Sub in (I)

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

then

$$\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) = 3\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)$$

$$= 3\left(\frac{-1}{(x+y+z)^2}(1)\right)$$

$$\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) = 3\left(\frac{-1}{(x+y+z)^2}(1)\right)$$

$$\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right) = 3\left(\frac{-1}{(x+y+z)^2}(1)\right) = \frac{-3}{(x+y+z)^2}$$

adding above 3 equations:

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$\therefore$ LHS = RHS

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2) If $U = \frac{x^2y^2}{x+y}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \pm 3u$

Ans: Given $U = \frac{x^2y^2}{x+y}$

$$U = \frac{(kx)^2 \times (ky)^2}{(kx)+(ky)}$$

$$U = \frac{k^2x^2 \cdot k^2y^2}{(kx)+(ky)}$$

$$U = \frac{k^4(x^2y^2)}{k(x+y)}$$

$$U = k^3\left[\frac{x^2y^2}{x+y}\right]$$

$\therefore$ U is a homogenous function of degree '3' = n

$\therefore$ By Euler's then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot u$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3 \cdot u}$$

3) If $U = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Ans: Given $U = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$

multiplying 'tan u' on both sides

$$\Rightarrow \tan u = \left[\frac{x^3+y^3}{x+y}\right]$$

Put $\tan u = z$

$$z = \left[\frac{x^3+y^3}{x+y}\right] \quad \text{—— (I)}$$

then By Euler's theorem then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n \cdot z \quad \text{—— (II)}$$

Since z is a homogenous function of degree

'2' = n

Sub in (II)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

replace 'z' by $\tan u$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2 \cdot \tan u$$

$$\Rightarrow x \sec^2 u \frac{\partial u}{\partial x} + y \cdot \sec^2 u \frac{\partial u}{\partial y} = 2 \cdot \tan u$$

$$\Rightarrow \sec^2 u \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2 \cdot \tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 \cdot \frac{\sin u}{\cos u} \cdot \frac{1}{\sec^2 u}$$

$[\because \cos\theta \times \sec\theta = 1]$

$[\because \frac{1}{\sec^2\theta} = \cos^2\theta]$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cancel{\cos u}} \cos^{\cancel{2}} u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u \cdot \cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

$[\because 2\sin\theta\cos\theta = \sin 2\theta]$

4) If $u = f(y-z, z-x, x-y)$ show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Ans:- Given $u = f(y-z, z-x, x-y)$

put $y-z = r$

$z-x = s$

$x-y = t$

then $u = f(r, s, t)$

then $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

then

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-1) + \frac{\partial f}{\partial t}(1)$$

$$= -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-1)$$

$$= \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \quad \text{--- } (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}(-1) + \frac{\partial f}{\partial s}(1) + \frac{\partial f}{\partial t}(0)$$

$$= -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \quad \text{--- } (3)$$

By adding (1), (2) and (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\cancel{\frac{\partial f}{\partial s}} + \cancel{\frac{\partial f}{\partial t}} + \cancel{\frac{\partial f}{\partial r}} - \cancel{\frac{\partial f}{\partial t}} - \cancel{\frac{\partial f}{\partial r}} + \cancel{\frac{\partial f}{\partial s}}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

LHS = RHS.

5) If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$

find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Ans:- Since by Jacobian Transformation

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$= 2x\,[3+2] + 2\,[3-1]$

$= 2x\,[5] + 2\,[2]$

$= 10x + 4$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 10x + 4$$

6) If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ then find $\frac{\partial(u,v)}{\partial(x,y)}$. Hence prove that u and v are functionally dependent. Find the functional relational between them.

Ans: Given:

$U = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

we know that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} \quad \text{—— (I)}$$

$$\left[\because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$$

Thus $U_x = \frac{\partial u}{\partial x}$

$$\frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2}$$

$$= \frac{(1-xy)+y(x+y)}{(1-xy)^2}$$

$$U_x = \frac{1-xy+yx+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$U_y = \frac{\partial v}{\partial y}$$

$$= \frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2}$$

$$= \frac{(1-xy)+x(x+y)}{(1-xy)^2}$$

$$= \frac{1-xy+xy+x^2}{(1-xy)} = \frac{1+x^2}{(1-xy)^2}$$

$$V_x = \frac{\partial}{\partial x}(\tan^{-1}x+\tan^{-1}y)$$

$$V_x = \frac{1}{1+x^2}$$

$$V_y = \frac{1}{1+y^2}$$

Substitute in (I)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\[2ex] \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{\cancel{1+y^2}}{(1-xy)^2} \times \frac{1}{\cancel{1+y^2}} = \frac{\cancel{1+x^2}}{(1-xy)^2} \times \frac{1}{\cancel{1+x^2}}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

$\therefore$ u and v are (fundamen)× functionally dependent

$u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

Put $x = \tan A$, $A = \tan^{-1}x$

$y = \tan B$, $B = \tan^{-1}y$

then, $u = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$u = \tan(A+B)$

$\tan^{-1}u = A + B$

$\tan^{-1}u = \tan^{-1}x + \tan^{-1}y$

$$\boxed{\tan^{-1}u = v}$$

7) Show that the functions $u = xy+yz+zx$, $v = x^2+y^2+z^2$ and $w = x+y+z$ are functionally ~~relatio~~ related. find the relation between them.

Ans: Given $u = xy+yz+zx$, $v = x^2+y^2+z^2$ and $w = x+y+z$

To statement that u, v, w are functionally related.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$= 2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad R_1 \to R_1 - R_3$$

$$2(x+y+z)\begin{vmatrix} 0 & 0 & 0 \\ x & y & z \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\therefore$ u, v, w are functionally related

$\therefore (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$

$$\boxed{w^2 = v + 2u}$$

8) Find the maximum and minimum values of $x^3+3xy^2-15x^2-15y^2+72x$.

Ans. given $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$

The necessary conditions:-

$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2+3y^2-30x+72 = 0$ —— (1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 30y = 0$

~~6ty~~ $6y(x-5) = 0$

$y = 0$ & $x = 5$

for $y = 0$, Substitute in (1)

$3x^2 - 30x + 72 = 0$

$3(x^2 - 10x + 24) = 0$

$x^2 - 10x + 24 = 0$

$x^2 - 6x - 4x + 24 = 0$

$x(x-6) - 4(x-6) = 0$

$(x-6)\ (x-4) = 0$

$x = 6, 4$

$(6,0)\ (4,0)$

for $x=5$ Substitute in (1)

$3(5)^2 + 3y^2 - 30(5) + 72 = 0$

$75 + 3y^2 - 150 + 72 = 0$

$3y^2 - 3 = 0$

$3(y^2 - 1) = 0$

$y^2 - 1 = 0$

$y = 1, -1$

$(5,1)\ (5,-1)$

$\therefore$ Stationary points $(6,0)\ (4,0)\ (5,1)$ and $(5,-1)$

Sufficient Conditions:-

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + 3y^2 - 30x + 72)$$

$$\boxed{l = 6x - 30}$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(6xy - 30y)$$

$$\boxed{m = 6y}$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(6xy - 30y)$$

(1) $ln - m^2 = (6x - 30)^2 - (6y)^2$

$= 6^2\left[(x-5)^2 - (y)^2\right]$

$l = 6x - 30$

at $(6,0)$

$ln - m^2 = 6^2\left[(6-5)^2 - (0)^2\right]$

$= 36\,[1 - 0] = 36 > 0$

$l = 6(6) - 30$

$= 36 - 30 = 6 > 0$

$\therefore ln - m^2 > 0$, $l > 0$ at $(6,0)$ then 'f' is min at $(6,0)$ and the min value is $f(6,0)$

$f(x,y) = x^3 + 3xy - 15x^2 - 15y^2 + 72x$

$f(6,0) = (6)^3 + 3(6)(0)^2 - 15(6)^2 - 15(0)^2 + 72(6)$

$= (6)^3 - 15(6)^2 + 72(6)$

$= 108$

at $(4,0)$

$ln - m^2 = 6^2\left[(4-5)^2 - 0^2\right]$

$= 36\,[1 - 0] = 36 > 0$

$l = 6(4) - 30$

$= 24 - 30 = 6 < 0$

$\therefore ln - m^2 > 0$, $l < 0$

then 'f' is max at $(4,0)$

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f(4,0) = (4)^3 + 3(4)(0) - 15(4)^2 - 15(0) + 72(4)$

$= (4)^3 - 15(4)^2 + 72(4)$

$= 112$

at $(5,1)$

$ln - m^2 < 0$

at $(+5,-1)$

$ln - m^2 < 0$

$\therefore$ $(5,1)$ & $(5,-1)$ are saddle points.

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9) Divide 24 into three parts such that the continued product of the first square of the second and cube of the third is maximum.

A) Let 24 be divided into three parts x, y, z then $x + y + z = 24$ — ①

take $f = x^3 y^2 z$ — ②

from ① $z = 24 - (x+y)$

$\therefore f(x,y) = x^3 y^2 (24 - (x+y))$

$\Rightarrow f(x,y) = 24x^3y^2 - x^4y^2 - x^3y^3$

then $\frac{\partial f}{\partial x} = 72x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{\partial f}{\partial y} = 48x^3y - 2x^4y - 3x^3y^2$

$l = \frac{\partial^2 f}{\partial x} = 144xy^2 - 12x^2y^2 - 6xy^3 = l$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(48x^3y - 2x^4y - 3x^3y^2)$

$m = 144x^2y - 8x^3y - 9x^2y^2$

$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y^2}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}(48x^3y - 2x^4y - 3x^3y^2)$

$n = 48x^3 - 2x^4 - 6x^3y$

To find the maxima (or) minima

$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 72x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$x^2y^2[72 - 4x - 3y] = 0$

$\Rightarrow 72 - 4x - 3y = 0, \quad x = 0, y = 0$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 48x^3y - 2x^4y - 3x^3y^2$

$\Rightarrow x^3y[48 - 2x - 3y] = 0$

$\Rightarrow 48 - 2x - 3y = 0, \quad x = 0, \quad y = 0$

Solving $72 - 4x - 3y = 0$ and $48 - 2x - 3y = 0$

$x = 12,\ y = 8$

At $(12, 8)$

$l = 144xy^2 - 12x^2y^2 - 6xy^3$

$= 144(12)(8)^2 - 12(12)^2(8)^2 - 6(12)(8)^3$

$= 12(8)^2\ (-48) < 0$

$n = 48(12)^3 - 2(12)^4 - 6(12)^3 \cdot 8$

$= (12)^3\ [48 - 2(12) - 48]$

$= (12)^3\ [-24] = -\text{ve value}.$

$m = 144(12)^2 \cdot 8 - 8(12)^3 \cdot 8 - 19(12)^2(8)^2$

$= (12)^2 \cdot 8\ [144 - 96 - 72]$

$= (12)^2 \cdot 8\ [-24] = -\text{ve value}$

$ln - m^2 = [12(8)^2(-48)]\,[(12)^3(-24)] - [12^2 \cdot 8(-24)]^2$

$\Rightarrow (12)^4(8)^2(24)(48) - (12)^4(8)^2(24)^2$

$\Rightarrow (12)^4(8)^2(24)\ [2-1]$

$\Rightarrow (12)^4(8)^2(24)^2 > 0$

at point $A(12, 8)$ $l < 0,\ ln - m^2 > 0$

$\therefore f$ is maximum at $(12, 8)$

Since we have $x + y + z = 24$

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$12 + 8 + z = 24$

$z = 24 - 20$

$\boxed{z = 4}$

$\therefore$ The values of $(x, y, z) = (12, 8, 4)$.

10) Find the minimum value of $x^2 + y^2 + z^2$ given $x + y + z = 3a$.

Ans step - 1 : from lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$$

$$= (x^2 + y^2 + z^2) + \lambda (x + y + z - 3a) \quad \text{— (1)}$$

Step 2 :- obtain the equations

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$= 2x + \lambda = 0$$

$$x = -\lambda/2 \quad \text{— (2)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}$$

$$\Rightarrow 2y + \lambda = 0 \Rightarrow y = -\lambda/2 \quad \text{— (3)}$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}$$

$$\Rightarrow 2z + \lambda = 0 \Rightarrow z = -\lambda/2 \quad \text{— (4)}$$

Step - III :

Substitute the values in $x+y+z = 3a$

$$-\lambda/2 - \lambda/2 - \lambda/2 = 3a$$

$$-\frac{\cancel{3}\lambda}{2} = \cancel{3}a$$

$$\boxed{\lambda = -2a}$$

Substitute in ②, ③ & ④

$$x = y = z = -\frac{(-2a)}{2} = a$$

$\Rightarrow x = y = z = a$

∴ The possible extreme point is (a, a, a)

Thus, the minimum point is $f = x^2 + y^2 + z^2$

is $f = a^2 + a^2 + a^2 = 3a^2$

$$\boxed{f = 3a^2}$$

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Assignment-1

1) Verify Rolle's theorem for the function $f(x) = (x-a)^m (x-b)^n$ where m,n are positive integers in $[a,b]$

A) Given $f(x) = (x-a)^m (x-b)^n$ in $[a,b]$ —— (I)

(i) $f(x)$ is continous in $[a,b]$

(ii) $f(x)$ is derivable in $[a,b]$

(iii) $f(a) = (a-a)^m (a-b)^n$

$= 0 \cdot (a-b)^n = 0$

$f(a) = 0$

$f(b) = (b-a)^m (b-b)^n$

$= (b-a)^m \cdot 0 = 0$

$f(b) = 0 = f(a)$

To check for $c, f'(c) = 0$

differentiate (I) w.r.t x

$f'(x) = m(x-a)^{m-1} \cdot (x-b)^n + (x-a)^m \cdot n(x-b)^{n-1}$

$f'(x) = (x-a)^{m-1} (x-b)^{n-1} [m(x-b) + n(x-a)]$

then $f'(c) = 0$

$(c-a)^{m-1} (c-b)^{n-1} [m(c-b) + n(c-a)] = 0$

$\Rightarrow [m(c-b) + n(c-a)] = 0$

$\Rightarrow c(m+n) = mb + na$

$\Rightarrow c = \frac{mb+na}{m+n} \in (a,b)$

Here Rolle's theorem is verified.

a) State the Rolle's theorem and verify Rolle's theorem for the function.

$f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $a>0, b>0$

Ans: Given $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$

$\left[\because \log\left[\frac{a}{b}\right] = \log a - \log b\right]$

$= \log[x^2+ab] - \log[x(a+b)]$

$= \log[x^2+ab] - [\log x + \log(a+b)]$

$f(x) = \log(x^2+ab) - \log x - \log(a+b)$ — (I)

i) $f(x)$ is continous in $[a,b]$

ii) $f(x)$ is derivable in $[a,b]$

iii) $f(a) = \log(a^2+ab) - \log a - \log(a+b)$

$= \log(a(a+b)) - \log a - \log(a+b)$

$= \cancel{\log a} + \cancel{\log(a+b)} - \cancel{\log a} - \cancel{\log(a+b)}$

$f(a) = 0$

$f(b) = \log(b^2+ab) - \log b - \log(a+b)$ — (II)

$= \log(b(b+a)) - \log b - \log(a+b)$

$= \cancel{\log b} + \cancel{\log(b+a)} - \cancel{\log b} - \cancel{\log(a+b)}$

$f(b) = 0$

$f(a) = 0 = f(b)$

To check $c \in (a,b)$ we have $f'(c) = 0$

Differentiate (1) w.r.t

$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$

$$= \frac{2x^2 - (x^2+ab)}{x(x^2+ab)}$$

$$= \frac{x^2 - ab}{x(x^2+ab)}$$

Put x by c

$$f'(c) = \frac{c^2 - ab}{c(c+ab)} = 0$$

$\Rightarrow c^2 = ab$

$\Rightarrow c = \sqrt{ab} \in (a,b)$

$\because$ c is Geomentric mean of (a,b)

$\therefore c \in (a,b)$

Hence Rolle's theorem is verified.

3) Verify Lagrange's mean value theorem for the function $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

A) Given $f(x) = x^3 - x^2 - 5x + 3$ —— (I)

i) $f(x)$ is continous in $[0,4)$ Since $f(x)$ is polynomial of x

ii) $f(x)$ is derivable in $[0,4]$

for checking $c \in (a,b)$:-

we have $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{f(4)-f(0)}{4-0}$ —— (II)

Differentiate (I) w.r.t x

$f'(x) = 3x^2 - 2x - 5$

then $f'(c) = 3c^2 - 2c - 5$

Substitute in (ii)

$$3c^2-2c-5=\frac{[(4)^3-(4)^2-5(4)+3]-[0^3-0^2-5(0)+3]}{4}$$

$$3c^2-2c-5=\frac{[64-16-20+3]-3}{4}$$

$$3c^2-2c-5=\frac{\cancel{28}\,7}{\cancel{4}}$$

$$3c^2-2c-5=7$$

$$3c^2-2c-5-7=0$$

$$3c^2-2c-12=0 \quad [ax^2+bx+c=0]$$

$$x=-\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{2\pm\sqrt{4\pm4(3)(-12)}}{2(3)}$$

$$x=\frac{2\pm\sqrt{148}}{6}\Rightarrow\frac{2\pm\sqrt{4\times37}}{2\cdot3}\Rightarrow\frac{\cancel{2}\pm\cancel{2}\sqrt{37}}{3\times\cancel{2}}$$

$$=\frac{1\pm\sqrt{37}}{3}=\frac{7.08}{3}$$

$$=2.36\in[0,4]$$

4) If $a<b$, prove that $\frac{b-a}{1+b^2}<\tan^{-1}b-\tan^{-1}a<\frac{b-a}{1+a^2}$ using Lagrange's mean value theorem. Deduce the following.

$$\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$$

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1) Let take $f(x) = \tan^{-1}x$ in $[a,b]$

(i) $f(x)$ is continous in $[a,b]$

ii) $f(x)$ is derivable in (a,b)

then $\exists\, c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Since $c \in (a,b)$

$\Rightarrow a < c < b$

Taking Squares

$\Rightarrow a^2 < c^2 < b^2$

$\Rightarrow \frac{1}{1+a^2} \cancel{<} \frac{1}{1+c^2} \cancel{<} \frac{1}{1+b^2}$

$\Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$ — (ii)

Since $f(x) = \tan^{-1}(x)$

$f'(x) = \frac{1}{1+x^2}$

then $f'(c) = \frac{1}{1+c^2}$ — (iii)

Substitute (iii) in (ii)

$\frac{1}{1+b^2} < f'(c) < \frac{1}{1+a^2}$

from (i)

$\frac{1}{1+b^2} < \frac{f(b)-f(a)}{b-a} < \frac{1}{1+a^2}$

multiply by $b-a$

$\Rightarrow \frac{b-a}{1+b^2} < f(b) - f(a) < \frac{b-a}{1+a^2}$

$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

$\left[\because f(x) = \tan^{-1}(x)\right]$

(i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Sub $b = \frac{4}{3}$ and $a = 1$ in (II)

$\Rightarrow \frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^3} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+(1)^2}$

$\Rightarrow \frac{4-3}{3\left[9+\frac{16}{9}\right]} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{4-3}{3(1+1)}$

$\Rightarrow \frac{1\cdot 3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$

$\Rightarrow \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$

add $\frac{\pi}{4}$

$\Rightarrow \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} + \frac{\pi}{4} < \frac{1}{6} + \frac{\pi}{4}$

$\Rightarrow \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$

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5) Discuss the applicability of cauchy's mean value theorem for the function $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in (a,b)

Ans:- (i) $f(x), g(x)$ are continous in $[a,b]$

(ii) $f(x), g(x)$ are derivable in (a,b)

Since $f'(x) = \frac{-2}{x^3}$; $g'(x) = \frac{-1}{x^2}$ are differentiable $\forall\, x \in (a,b)$ where $0 < a < b$

iii) $g'(x) \neq 0 \;\forall\, x \in (a,b)$

$\because 0 < a < b$, $g'(x) = \frac{-1}{x^2} \neq 0$

Then checking for $c \in (a,b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \frac{-2c^{x}}{c^3(-1)} = \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}}$$

$$\Rightarrow 2 = \frac{(a^2-b^2)}{(a^2-b^2)\frac{[a-b]}{ab}} \Rightarrow \frac{2}{c} = \frac{a^2-b^2}{(a-b)ab}$$

$$\Rightarrow \frac{(a+b)(a-b)}{(a-b)ab}$$

$$\Rightarrow \cancel{\frac{2}{b}}\ \frac{2}{c} = \frac{a+b}{ab}$$

$$\Rightarrow \frac{c}{2} = \frac{ab}{a+b}$$

$$c = \frac{2ab}{a+b} \in (a,b)$$

$\therefore$ c is Harmonic mean of (a,b).

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6) Verify cauchy's mean value theorem for the function $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in (a,b)

Ans: (i) $f(x)$, $g(x)$ is continous in $[a,b]$

(ii) $f(x)$, $g(x)$ is derivable in (a,b)

Since $f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{1}{2x\sqrt{x}}$

$$g'(x) = x^{-1/2}$$

$$\Rightarrow g'(x) = -\frac{1}{2}x^{-1/2-1}$$

$\Rightarrow g'(x) = -\frac{1}{2} x^{-3/2}$

$\Rightarrow g'(x) = \frac{-1}{2x^{3/2}}$

$\Rightarrow g'(x) = \frac{-1}{2x\sqrt{x}}$

are differentiable $\forall x \in (a,b)$ where $0<a<b$

(iii) $g'(x) \neq 0 \ \forall x \in (a,b)$

$\because 0<a<b$, $g'(x) = \frac{-1}{2x\sqrt{x}} \neq 0$

Then checking for $c \in (a,b)$:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{2c\sqrt{c}}{(2\sqrt{c})(-1)} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\Rightarrow -c = \frac{\sqrt{b}-\sqrt{a}}{(\sqrt{a}-\sqrt{b})/\sqrt{ab}}$$

$$c = \frac{\cancel{(\sqrt{a}-\sqrt{b})}\sqrt{ab}}{\cancel{\sqrt{a}-\sqrt{b}}}$$

$$c = \sqrt{ab} \in (a,b)$$

$c = \sqrt{ab}$ is Geometric mean of (a,b)

$\because c \in (a,b)$

Hence Cauchy mean value theorem is verified.

1) Obtain the Taylor's Series expansion of $\sin x$ in powers of $x - \frac{\pi}{4}$

Ans: $f(x) = \sin x$

in powers of $x - \frac{\pi}{4} = b - a \Rightarrow b - x,\ a = \frac{\pi}{4}$

By Taylor's Series:-

$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{b-a}{2!} f''(a) + \frac{b-a}{3!} f'''(a) + \dots$

Since $f(x) = \sin x$

So, $f'(x), f''(x) \dots \cancel{f} f^{(n-1)}(x)$ are continous in $[a, b]$

$\Rightarrow \left[\frac{\pi}{4}, x\right]$ & $f'(x), f''(x) \dots f^{n-1}(x)$ are derivable in $(a, b) \Rightarrow \left(\frac{\pi}{4}, x\right)$

$\Rightarrow$ They by Taylor's Series then for then $\left(\frac{\pi}{4}, x\right)$ is (from (I))

$\Rightarrow f(x) = f\left(\frac{\pi}{4}\right) + \frac{x - \frac{\pi}{4}}{1!} f'\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^4}{4!} f^{iv}\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^5}{5!} f^{v}\left(\frac{\pi}{4}\right) + \dots \rightarrow$ (II)

$\because f(x) = \sin x \rightarrow f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x \rightarrow f\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f^{iv}(x) = -(-\sin x) = f^{iv}(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f^{v}(x) = \cos x = f^{v}(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Sub in (II)

$$\sin x = \frac{1}{\sqrt{2}} + \frac{(x-\frac{\pi}{4})}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^3}{3!}\left(\frac{-1}{\sqrt{2}}\right)$$
$$+ \frac{(x-\frac{\pi}{4})^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^5}{5!}\left(\frac{1}{\sqrt{2}}\right) + \dots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{x-\frac{\pi}{4}}{1!} - \frac{(x-\frac{\pi}{4})^2}{2!} - \frac{(x-\frac{\pi}{4})^3}{3!} + \frac{(x-\frac{\pi}{4})^4}{4!} + \frac{(x-\frac{\pi}{4})^5}{5!}\right]$$

3) Obtain the Taylor's series expansion of $\cos x$ in powers of $x-\frac{\pi}{4}$

Ans) $f(x) = f(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})}{1!} f'(x) + \frac{(x-\frac{\pi}{4})}{2!} f''(x) + \frac{x-\frac{\pi}{4}}{3!} f'''(x) + \dots$ — (1)

$\therefore f(x) = \cos x = f(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f'(x) = -\sin x = f(\frac{\pi}{4}) = -\sin(\frac{\pi}{4}) = \frac{-1}{\sqrt{2}}$

$f''(x) = -\cos x = f(\frac{\pi}{4}) = -\cos(\frac{\pi}{4}) = \frac{-1}{\sqrt{2}}$

$f'''(x) = -(-\sin x) = f(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f^{iv}(x) = \cos x = f(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f^{v}(x) = -\sin x = f(\frac{\pi}{4}) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

Sub in (1)

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$$\cos x = \frac{1}{\sqrt{2}} + \frac{x-\frac{\pi}{4}}{1!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^3}{3!}\left(\frac{1}{\sqrt{2}}\right)$$

$$+ \frac{(x-\frac{\pi}{4})^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^5}{5!}\left(\frac{-1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} - \frac{x-\frac{\pi}{4}}{1!}\left(\frac{1}{\sqrt{2}}\right) - \frac{(x-\frac{\pi}{4})^2}{2!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^3}{3!}\left(\frac{1}{\sqrt{2}}\right)$$

$$+ \frac{(x-\frac{\pi}{4})^4}{4!}\left(\frac{1}{\sqrt{2}}\right) - \frac{(x-\frac{\pi}{4})^5}{5!}\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

$$\cos x = \frac{1}{\sqrt{2}}\left[\frac{x-\frac{\pi}{4}}{1!} - \frac{(x-\frac{\pi}{4})^2}{2!} + \frac{(x-\frac{\pi}{4})^3}{3!} + \frac{(x-\frac{\pi}{4})^4}{4!} - \frac{(x-\frac{\pi}{4})^5}{5!} + \cdots\right]$$

9) Obtain the Maclaurin's series expansion of a) $\sin x$ and b) e^x

Ans: a) $f(x) = f(x) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{iv}(x) = -(-\sin x) \Rightarrow f^{iv}(0) = \sin 0 = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = \cos 0 = 1$

$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) +$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

b) e^x

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x}{2!} f''(0) + \frac{x}{3!} f'''(0) + \dots$

$\Rightarrow f(x) = e^x, \ f'(x) = e^x, \ f''(x) = e^x$

$f'''(x) = e^x; \ f^{iv}(x) = e^x, \ f^{v}(x) = e^x$

$f(0) = e^0 = 1$

$f'(0) = e^0 = 1$

$f''(0) = e^0 = 1$

$f'''(0) = e^0 = 1$

$f^{iv}(0) = e^0 = 1$

$f^{v}(0) = e^0 = 1$

$\Rightarrow e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(1) + \dots$

$\Rightarrow e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

10) Obtain the Meclaurin's series expansion of $\log(1+x)$

Ans: $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x}{2!} f''(0) + \frac{x}{3!} f'''(0) + \frac{x}{4!} f^{iv}(0) + \frac{x}{5!} f^{v}(0) + \dots$

Given $f(x) = \log(1+x)$ is continous in $(0, x)$

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$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f'''(x) = \frac{(-1)(-2)}{(1+x)^3}$$

$$f^{iv}(x) = \frac{(-1)(-2)(-3)}{(1+x)^4}$$

$$f^{v}(x) = \frac{(-1)(-2)(-3)(-4)}{(1+x)^5}$$

Then $f(0) = \log(1+0) = \log 1 = 0$

$$f'(0) = \frac{1}{1+x} \Rightarrow f'(0) = \frac{1}{1+0} = 1$$

$$f''(0) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = \frac{-1}{(1+0)^2} = -1$$

$$f'''(0) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = \frac{2}{(1+0)^3} = 2$$

$$f^{iv}(0) = \frac{-6}{(1+x)^4} \Rightarrow f^{iv} = \frac{-6}{(1+0)^4} = -6$$

$$f^{v}(0) = \frac{24}{(1+x)^5} \Rightarrow f^{v} = \frac{24}{(1+0)^5} = 24$$

Sub in I

$$\log(1+x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) - \frac{x^5}{5!}(24)$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$$

$$+ \frac{x^5}{5!} f^{v}(0) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{\cancel{6}_3} \times \cancel{2} - \frac{x^4}{\cancel{24}_4} \times \cancel{6} + \frac{x^5}{\cancel{4!} \times 5} \cancel{24} + \dots$$

$$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$