CHAPTER-XIII

PROBABILITY

2 MARK QUESTIONS

1. Given that E and F are events, such that P (E) = 0.6, P (F) = 0.3 and P (E ∩ F) = 0.2, find P (E|F) and P (F|E).

Solution:

Given P (E) = 0.6, P (F) = 0.3 and P (E ∩ F) = 0.2

We know that by the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

By substituting the values we get

$$\Rightarrow P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.2}{0.3} = \frac{2}{3}$$

And $\Rightarrow P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.6} = \frac{2}{6} = \frac{1}{3}$

2. Compute P (A|B), if P (B) = 0.5 and P (A ∩ B) = 0.32

Solution:

Given: P (B) = 0.5 and P (A ∩ B) = 0.32

We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now by substituting the values we get

$$\Rightarrow P(A|B) = \frac{0.32}{0.5} = \frac{32}{50} = \frac{16}{25}$$

3. If P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4, find
(i) P (A ∩ B)

(ii) P (A|B)

(iii) P (A ∪ B)

Solution:

Given P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4

(i) We know that by definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$\Rightarrow$ P (A ∩ B) = P (B|A) P (A)

$\Rightarrow$ P (A ∩ B) = 0.4 × 0.8

$\Rightarrow$ P (A ∩ B) = 0.32

(ii) We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now substituting the values we get

$$\Rightarrow P(A|B) = \frac{0.32}{0.5} = 0.64$$

$\Rightarrow$ P (A|B) = 0.64

(iii) Now, $\because$ P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

Substituting the values we get

$\Rightarrow$ P (A ∪ B) = 0.8 + 0.5 – 0.32 = 1.3 – 0.32

$\Rightarrow$ P (A ∪ B) = 0.98

4. Evaluate P (A ∪ B), if 2P (A) = P (B) = 5/13 and P (A|B) = 2/5.

Solution:

Given $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$

$\Rightarrow P(B) = \frac{5}{13}, P(A) = \frac{5}{13\times 2} = \frac{5}{26}, P(A|B) = \frac{2}{5}$ .... (i)

We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$\Rightarrow$ P (A ∩ B) = P (A|B) P (B)

$\Rightarrow$ $P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$ .... (ii)

Now, ∵ P (A * B) = P (A) + P (B) – P (A ∩ B)

$$\Rightarrow P(A \cup B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{5 + 10 - 4}{26} = \frac{15 - 4}{26}$$

$$\Rightarrow P(A \cup B) = \frac{11}{26}$$

4 MARK QUESTIONS

1: A die is thrown twice and the sum of the numbers rising is noted to be 6. Calculate the is the conditional probability that the number 4 has arrived at least once?

Solution:

If a dice is thrown twice, then the sample space obtained is:

S = {(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)

(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)

(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)

(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)

(5,1)(5,2)(5,3)(5,4)(5,5)(5,6)

(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)}

From the given data, it is needed to find the Probability that 4 has appeared at least once, given the sum of nos. is observed to be 6

Assume that, F: Addition of numbers is 6

and take E: 4 has appeared at least once

So, that, we need to find P(E|F)

Finding P (E):

The probability of getting 4 atleast once is:

E = {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)}

Thus , P(E) = 11/ 36

Finding P (F):

The probability to get the addition of numbers is 6 is:

F = {(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)}

Thus, P(F) = 5/ 36

Also, E ∩ F = {(2,4), (4,2)}

P(E ∩ F) = 2/36

Thus, P(E|F) = (P(E ∩ F)) / (P (F))

Now, subsbtitute the probability values obtained= (2/36)/ (5/36)

Hence, Required probability is 2/5.

2: The probability of solving the specific problem independently by the persons' A and B are 1/2 and 1/3 respectively. In case, if both the persons try to solve the problem independently, then calculate the probability that the problem is solved.

Solution:

Given that, the two events say A and B are independent if P(A ∩ B) = P(A). P(B)

From the given data, we can observe that P(A) = 1/2 & P(B) = 1/3

The probability that the problem is solved = Probability that person A solves the problem or the person B solves the Problem

This can be written as:

= P(A ∪ B) = P(A) + P(B) – P(A ∩ B)

If A and B are independent, then P(A ∩ B) = P(A). P(B)

Now, substitute the values,

= (1/2) × (1/3)

P(A ∩ B) = 1/6

Now, the probability of problem solved is written as

P(Problem is solved) = $P(A) + P(B) - P(A \cap B)$

$= (1/2) + (1/3) - (1/6)$

$= (3/6) + (2/6) - (1/6)$

$= 4/6$

$= 2/3$

Hence, the probability of the problem solved is 2/3.

3: 5 cards are drawn successively from a well-shuffled pack of 52 cards with replacement. Determine the probability that (i) all the five cards should be spades? (ii) only 3 cards should be spades? (iii) none of the cards is a spade?

Solution:

Let us assume that X be the number of spade cards

Using the Bernoulli trial, X has a binomial distribution

$P(X = x) = {}^nC_x\, q^{n-x}\, p^x$

Thus, the number of cards drawn, $n = 5$

Probability of getting spade card, $p = 13/52 = 1/4$

Thus the value of the q can be found using

$q = 1 - p = 1 - (1/4) = 3/4$

Now substitute the p and q values in the formula,

Hence, $P(X = x) = {}^5C_x\, (3/4)^{5-x}(1/4)^x$

(1) Probability of Getting all the spade cards:

P(all the five cards should be spade) = ${}^{5}C_{5}\,(1/4)^5(3/4)^0$

$= (1/4)^5$

$= 1/1024$

(2) Probability of Getting only three spade cards:

P(only three cards should be spade) = ${}^{5}C_{3}\,(1/4)^3(3/4)^2$

$= (5!/3!\,2!) \times (9/1024)$

$= 45/\,512$

(3) Probability of Getting no spades:

P(none of the cards is a spade) = ${}^{5}C_{0}(1/4)^0(3/4)^5$

$= (3/4)^5$

$= 243/\,1024$

4: An fair die is thrown double times. Assume that the event A is "odd number on the first throw" and B the event "odd number on the second throw". Compare the independence of the events A and B.

Solution:

Let us consider two independent events A and B, then $P(A \cap B) = P(A).\ P(B)$

when an unbiased die is thrown twice

S = {(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)

(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)

(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)

(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)

(5,1)(5,2)(5,3)(5,4)(5,5)(5,6)

(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)}

Let us describe two events as

A: odd number on the first throw

B: odd number on the second throw

To find P(A)

A = {(1, 1), (1, 2), (1, 3), ..., (1, 6)

(3, 1), (3, 2), (3, 3), ..., (3, 6)

(5, 1), (5, 2), (5, 3), ..., (5, 6)}

Thus, P (A) = 18/36 = 1/2

To find P(B)

B = {(1, 1), (2, 1), (3, 1), ..., (6, 1)

(1, 3), (2, 3), (3, 3), ..., (6, 3)

(1, 5), (2, 5), (3, 5), ..., (6, 5)}

Thus, P (B) = 18/36 = 1/2

$A \cap B$ = odd number on the first & second throw = { (1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)}

So, $P(A \cap B) = 9/36 = 1/4$

Now, P(A). P(B) = (1/2) × (1/2) = 1/4

As $P(A \cap B)$ = P(A). P(B),

Hence, the two events A and B are independent events.

5. If P (A) = 6/11, P (B) = 5/11 and P (A ∪ B) = 7/11, find

(i) P (A∩B)

(ii) P (A|B)

(iii) P (B|A)

Solution:

Given: $P(A) = \frac{6}{11}, P(B) = \frac{5}{11}, P(A \cup B) = \frac{7}{11}$

(i) We know that P (A * B) = P (A) + P (B) – P (A ∩ B)

⇒ P (A ∩ B) = P (A) + P (B) – P (A ∪ B)

$$\Rightarrow P(A \cap B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{11 - 7}{11}$$

$$\Rightarrow P(A \cap B) = \frac{4}{11}$$

(ii) Now, by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A|B) = \frac{4/11}{5/11}$$

$$\Rightarrow P(A|B) = \frac{4}{5}$$

(iii) Again, by definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(B|A) = \frac{4/11}{6/11} = \frac{4}{6} = \frac{2}{3}$$

$$\Rightarrow P(B|A) = \frac{2}{3}$$

Determine P (E|F) in Exercises 6 to 9.

6. A coin is tossed three times, where
(i) E : head on the third toss, F : heads on first two tosses
(ii) E : at least two heads, F : at most two heads
(iii) E : at most two tails, F : at least one tail

Solution:

The sample space of the given experiment will be:

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(i) Here, E: head on third toss

And F: heads on first two tosses

⇒ E = {HHH, HTH, THH, TTH} and F = {HHH, HHT}

⇒ E ∩ F = {HHH}

So, $P(E) = \frac{4}{8} = \frac{1}{2}, P(F) = \frac{2}{8} = \frac{1}{4}, P(E \cap F) = \frac{1}{8}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

(ii) Here, E: at least two heads

And F: at most two heads

⇒ E = {HHH, HHT, HTH, THH} and F = {HHT, HTH, THH, HTT, THT, TTH, TTT}

⇒ E ∩ F = {HHT, HTH, THH}

So, $P(E) = \frac{4}{8} = \frac{1}{2}, P(F) = \frac{7}{8}, P(E \cap F) = \frac{3}{8}$

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/8}{7/8} = \frac{3}{7}$$

(iii) Here, E: at most two tails

And F: at least one tail

⇒ E = {HHH, HHT, HTH, THH, HTT, THT, TTH}

And F = {HHT, HTH, THH, HTT, THT, TTH, TTT}

So, $P(E) = \frac{7}{8}, P(F) = \frac{7}{8}, P(E \cap F) = \frac{6}{8} = \frac{3}{4}$

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/4}{7/8} = \frac{6}{7}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/4}{7/8} = \frac{6}{7}$$

7 MARK QUESTIONS

1: Given that the events A and B are such that $P(A) = 1/2$, $P(A \cup B) = 3/5$, and $P(B) = p$. Find p if they are

(i) mutually exclusive

(ii) independent

Solution:

Given, $P(A) = 1/2$,

$P(A \cup B) = 3/5$

and $P(B) = p$.

(1) For Mutually Exclusive

Given that, the sets A and B are mutually exclusive.

Thus, they do not have any common elements

Therefore, $P(A \cap B) = 0$

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Substitute the formulas in the above-given formula, we get

$3/5 = (1/2) + p - 0$

Simplify the expression, we get

$(3/5) - (1/2) = p$

$(6 - 5)/10 = p$

$1/10 = p$

Therefore, $p = 1/10$

Hence, the value of p is 1/10, if they are mutually exclusive.

(ii) For Independent events:

If the two events A & B are independent,

we can write it as $P(A \cap B) = P(A)\, P(B)$

Substitute the values,

$= (1/2) \times p$

$= p/2$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Now, substitute the values in the formula,

$(3/5) = (1/2) + p - (p/2)$

$(3/2) - (1/2) = p - (p/2)$

$(6 - 5)/10 = p/2$

$1/10 = p/2$

$p = 2/10$

$P = 1/5$

Thus, the value of p is 1/5, if they are independent.

2. Two coins are tossed once, where
(i) E: tail appears on one coin, F: one coin shows the head
(ii) E: no tail appears, F: no head appears

Solution:

The sample space of the given experiment is S = {HH, HT, TH, TT}

(i) Here, E: tail appears on one coin

And F: one coin shows head

⇒ E = {HT, TH} and F = {HT, TH}

⇒ $E \cap F$ = {HT, TH}

So, $P(E) = \frac{2}{4} = \frac{1}{2}, P(F) = \frac{2}{4} = \frac{1}{2}, P(E \cap F) = \frac{2}{4} = \frac{1}{2}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/2}{1/2}$$

⇒ P (E|F) = 1

(ii) Here, E: no tail appears

And F: no head appears

⇒ E = {HH} and F = {TT}

⇒ $E \cap F = \varphi$

So, $P(E) = \frac{1}{4}, P(F) = \frac{1}{4}, P(E \cap F) = \frac{0}{4} = 0$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Substituting the values we get

$$\Rightarrow P(E|F) = \frac{0}{1/4}$$

$\Rightarrow P(E|F) = 0$

3. A die is thrown three times, E: 4 appears on the third toss, F: 6 and 5 appear, respectively, on the first two tosses.

Solution:

The sample space has 216 outcomes, where each element of the sample space has 3 entries and is of the form (x, y, z) where $1 \le x, y, z \le 6$.

Here, E: 4 appears on the third toss

$$\Rightarrow E = \begin{Bmatrix} (1,1,4),(1,2,4),(1,3,4),(1,4,4),(1,5,4),(1,6,4), \\ (2,1,4),(2,2,4),(2,3,4),(2,4,4),(2,5,4),(2,6,4), \\ (3,1,4),(3,2,4),(3,3,4),(3,4,4),(3,5,4),(3,6,4), \\ (4,1,4),(4,2,4),(4,3,4),(4,4,4),(4,5,4),(4,6,4), \\ (5,1,4),(5,2,4),(5,3,4),(5,4,4),(5,5,4),(5,6,4), \\ (6,1,4),(6,2,4),(6,3,4),(6,4,4),(6,5,4),(6,6,4) \end{Bmatrix}$$

Now, F: 6 and 5 appears respectively on first two tosses

$\Rightarrow F = \{(6, 5, 1), (6, 5, 2), (6, 5, 3), (6, 5, 4), (6, 5, 5), (6, 5, 6)\}$

$\Rightarrow E \cap F = \{(6, 5, 4)\}$

So, $P(E) = \frac{36}{216}, P(F) = \frac{6}{216}, P(E \cap F) = \frac{1}{216}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/216}{6/216} = \frac{1}{6}$$

$$\Rightarrow P(E|F) = \frac{1}{6}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Now by substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/216}{6/216} = \frac{1}{6}$$

$$\Rightarrow P(E|F) = \frac{1}{6}$$

4. Mother, father and son line up at random for a family picture
E: son on one end, F: father in the middle.

Solution:

Let M denotes mother, F denotes father, and S denotes son.

Then, the sample space for the given experiment will be

S = {MFS, SFM, FSM, MSF, SMF, FMS}

Here, E: Son on one end

And F: Father in the middle

⇒ E = {MFS, SFM, SMF, FMS} and F = {MFS, SFM}

⇒ E ∩ F = {MFS, SFM}

So, $P(E) = \frac{4}{6} = \frac{2}{3}, P(F) = \frac{2}{6} = \frac{1}{3}, P(E \cap F) = \frac{2}{6} = \frac{1}{3}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Now by substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/3}{1/3} = 1$$

⇒ P (E|F) = 1

5. A fair die is rolled. Consider events E = {1,3,5}, F = {2,3} and G = {2,3,4,5}
Find
(i) P (E|F) and P (F|E)
(ii) P (E|G) and P (G|E)
(iii) P ((E ∪ F)|G) and P ((E ∩ F)|G)

Solution:

The sample space for the given experiment is S = {1, 2, 3, 4, 5, 6}

Here, E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5} .... (i)

$\Rightarrow P(E) = \frac{3}{6} = \frac{1}{2}, P(F) = \frac{2}{6} = \frac{1}{3}, P(G) = \frac{4}{6} = \frac{2}{3}$ .... (ii)

Now, $E \cap F = \{3\}$, $F \cap G = \{2, 3\}$, $E \cap G = \{3, 5\}$ .... (iii)

$\Rightarrow P(E \cap F) = \frac{1}{6}, P(F \cap G) = \frac{2}{6} = \frac{1}{3}, P(E \cap G) = \frac{2}{6} = \frac{1}{3}$ .... (iv)

(i) We know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/6}{1/3} = \frac{3}{6} = \frac{1}{2} \text{ [Using (II) and (IV)]}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

Similarly, we have

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/6}{1/2} = \frac{2}{6} = \frac{1}{3} \text{ [Using (ii) and (iv)]}$$

$$\Rightarrow P(F|E) = \frac{1}{3}$$

(ii) We know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|G) = \frac{P(E \cap G)}{P(G)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$\Rightarrow P(E|G) = \frac{1}{2}$$

Similarly, we have

$$P(G|E) = \frac{P(G \cap E)}{P(E)} = \frac{1/3}{1/2} = \frac{2}{3}$$

$$\Rightarrow P(G|E) = \frac{2}{3}$$

(iii) Clearly, from (i), we have

$E = \{1, 3, 5\}$, $F = \{2, 3\}$ and $G = \{2, 3, 4, 5\}$

$\Rightarrow E \cup F = \{1, 2, 3, 5\}$

$\Rightarrow (E \cup F) \cap G = \{2, 3, 5\}$

(iii) Clearly, from (i), we have

$E = \{1, 3, 5\}$, $F = \{2, 3\}$ and $G = \{2, 3, 4, 5\}$

$\Rightarrow E \cup F = \{1, 2, 3, 5\}$

$\Rightarrow (E \cup F) \cap G = \{2, 3, 5\}$

$$\Rightarrow P((E \cup F) \cap G) = \frac{3}{6} = \frac{1}{2}$$

$\Rightarrow P((E \cup F) \cap G) = \frac{1}{2}$ .... (v)

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$\Rightarrow P((E \cup F)|G) = \frac{P((E \cup F) \cap G)}{P(G)} = \frac{1/2}{2/3} = \frac{3}{4}$ [Using (ii) and (v)]

$$\Rightarrow P((E \cup F)|G) = \frac{3}{4}$$

Similarly, we have $E \cap F = \{3\}$ [Using (iii)]

And $G = \{2, 3, 4, 5\}$ [Using (i)]

$\Rightarrow (E \cap F) \cap G = \{3\}$

$\Rightarrow P((E \cap F) \cap G) = \frac{1}{6}$ .... (vi)

So,

$P((E \cap F)|G) = \frac{P((E \cap F) \cap G)}{P(G)} = \frac{1/6}{2/3} = \frac{1}{4}$ [Using (ii) and (vi)]

$$\Rightarrow P((E \cap F)|G) = \frac{1}{4}$$