photon and matter waves
chapter-9

Date . .

The light exhibits the phenomenon of interference, diffraction polarization, photoelectric effect, compton effect and emission and absorption. The phenomenon like interference, diffraction and polarization can only be explained on the basis of wave nature of light. This phenomenon shows that light passes wave nature.

On the other hand the phenomenon like photoelectric effect, compton effect, discrete emission and absorption can only be explained on the basis of the quantum theory of light, according to which light is propagated in a small packets or bundle of eneroy called photon or quanta. The energy of each quanta is given by

$$E = h\nu \quad --(1)$$

where h is a plank's constant, $h = 6.6 \times 10^{-34}$ J sec

and ν = frequency of photon.

The photon acts as particles. This phenomena indicates that light possess particle nature. From above discussion we say that light shows dual nature.

* De-Broglie wave-

As the electromagnetic wave possess dual nature i.e wave nature and particle nature, matters (electron, proton) also should possess dual nature. According to De-Broglie, a moving particle whatever its nature has wave properties associated with it. The wave associated with the particle is called De-Broglie wave or matter wave, and the wavelength associated with matter is called De-Broglie wavelength.

we know, the energy of photon

$$E = h\nu \quad ---(1)$$

If photon is considered as a particle of mass 'm' moving with velocity 'c' then according to mass energy relation.

$$E = mc^2 \quad ---(2)$$

From (1) and (2)

$$h\nu = mc^2$$

$$\Rightarrow \frac{hc}{\lambda} = mc^2$$

$$\Rightarrow \lambda = \frac{h}{mc} \quad ---(3)$$

where mc = momentum of photon.

eq^n (3) is the expression of De-Broglie wavelength.

For other particles eq^n (3) becomes:

$$\lambda = \frac{h}{mv} \quad ---(4)$$

* phase velocity (wave velocity):

A particle of mass 'm' having velocity 'v' has a wave associated with it whose wavelength is given by

$$\lambda = \frac{h}{mv} \quad ---(1)$$

let E be the total energy of the particle and ν be the frequency of the associated wave then.

$$E = h\nu \quad ---(2)$$

Also, energy associated with relativistic formula is

$$E = mc^2 \quad ---(3)$$

From (2) and (3)

$$h\nu = mc^2$$

$$\Rightarrow \nu = \frac{mc^2}{h} \quad ----(4)$$

Let V_p be the De-Broglie wave velocity (phase velocity) then,

v_p = frequency x wavelength

$$V_p = \nu\lambda \quad ---(5)$$

substituting ν and λ from (1) and (4) in (5) we get,

$$V_p = \frac{mc^2}{h} \frac{h}{mv}$$

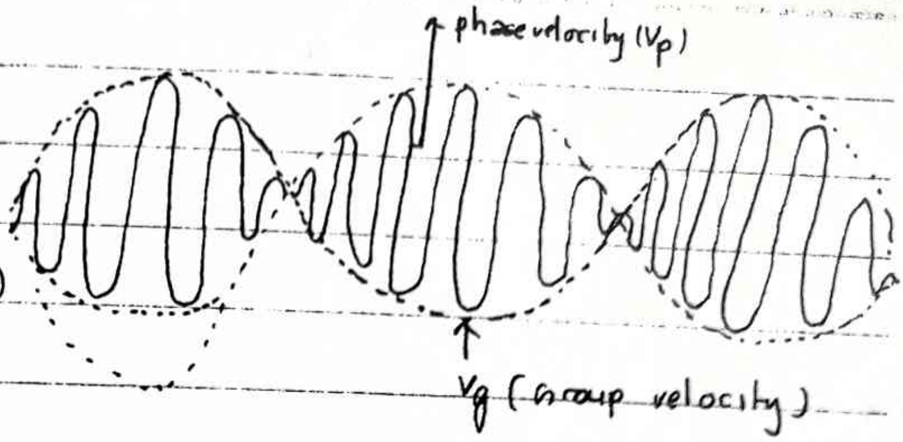
$$V_p = \frac{c^2}{v} \quad --(6)$$

As particle velocity is always less than 'C' but eq^n (6) shows that $V_p > C$, which is not possible. However this problem is solved by De-Broglie by showing that the wave always travel in the form of group with velocity called group velocity.

* Group velocity -

In general real waves are of complex form. In practice waves are far from monochromatic and can be regarded as the result of superposition of waves of number of frequencies. The propagating velocity of a wave varies with frequency. The superposition of a very large number of harmonic waves differing small in frequency will produced a single wave packet. The wave packet amplitude varies with position and time. such variation in amplitude is called modulation of the wave. The velocity of propagation of the modulation is known as group velocity. It is denoted by V_g and given as

$$V_g = \frac{d\omega}{dk} \quad \text{--- (1)}$$



where $W = 2\pi\nu$ ...(2)

and $k = \frac{2\pi}{\lambda}$ ---(3)

Also, $\nu = \frac{mc^2}{h}$ --- (4)

The rest mass ~~energy~~ interms of moving mass can be written as (using relativistic law)

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad \text{---(5)}$$

where m = moving mass

m_0 = rest mass

∴ From eq^n (2) and (4)

$$\omega = 2\pi \frac{mc^2}{h} \quad \text{...(6)}$$

From (5), substitue m in eq^n (6)

$$\omega = \frac{2\pi c^2 m_0}{h\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{--(7)}$$

Now differentiating eq^n (7) w.r.to v

$$\frac{d\omega}{dv} = \frac{2\pi c^2 m_0}{h} \frac{d}{dv}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$= \frac{2\pi c^2 m_0}{h}\left(\frac{-2v}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{1}{2}\right)$$

$$\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h\left(1 - v^2/c^2\right)^{3/2}} \quad \text{---- (8)}$$

we have, wavelength of De-Broglie wave

$$\lambda = \frac{h}{mv} \quad ---(9)$$

From (3) and (9)

$$k = \frac{2\pi mv}{h}$$

In terms of rest mass,

$$k = \frac{2\pi v}{h} \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad ---(10)$$

Now Differentiating with respect to v, we get

$$\frac{dk}{dv} = \frac{2\pi m_0}{h} \frac{d}{dv}\left(\frac{v}{[1-v^2/c^2]^{1/2}}\right)$$

$$= \frac{2\pi m_0}{h}\left[\frac{d}{dv}\left\{v\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right\}\right]$$

$$= \frac{2\pi m_0}{h}\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} + v\left(\frac{-1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(\frac{-2v}{c^2}\right)\right]$$

$$= \frac{2\pi m_0}{h}\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} + \frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\right]$$

$$= \frac{2\pi m_0}{h}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}\right]$$

$$\frac{dk}{dv} = \frac{2\pi m_0}{h(1-v^2/c^2)^{3/2}} \quad ---(11)$$

From (8) and (11)

$$\frac{d\omega}{dk} = v$$

$$\Rightarrow \quad v_g = v \quad ----(12)$$

* Relation between phase velocity and group velocity:

we have phase velocity,

$$v_p = \nu \lambda \quad \text{---(1)}$$

Group velocity,

$$V_g = \frac{d\omega}{dk} \quad \text{---(2)}$$

but, $\omega = 2\pi\nu$

$$\Rightarrow \omega = 2\pi \frac{v_p}{\lambda} \quad \text{---(3)} \quad [\text{From (1)}]$$

Now, differentiating eqn (3) w.r.to λ

$$\frac{d\omega}{d\lambda} = 2\pi \left[-\frac{v_p}{\lambda^2} + \frac{1}{\lambda}\frac{dv_p}{d\lambda} \right]$$

$$\Rightarrow \frac{d\omega}{d\lambda} = -\frac{2\pi}{\lambda^2}\left[v_p - \lambda \frac{dv_p}{d\lambda} \right] \quad \text{----(4)}$$

Also,

$$k = \frac{2\pi}{\lambda}$$

$\Rightarrow$ Differentiating w.r.to λ,

$$\frac{dk}{d\lambda} = -\frac{2\pi}{\lambda^2} \quad \text{---(5)}$$

From (4) and (5)

$$\frac{d\omega/d\lambda}{dk/d\lambda} = \frac{-2\pi/\lambda^2 \left[v_p - \lambda \frac{dv_p}{d\lambda} \right]}{-2\pi/\lambda^2}$$

$$\Rightarrow \frac{d\omega}{dk} = v_p - \lambda \frac{dv_p}{d\lambda}$$

$$\Rightarrow v_g = v_p - \lambda \frac{dv_p}{d\lambda} \quad ---(6)$$

Group velocity will be the same as the phase velocity if the entire constituent waves travel with same velocity. It means in a non-dispersive medium, $v_g = v_p$. However the waves of different wavelengths travel in a medium with different velocity. Therefore the group velocity is in general less than phase velocity.

* schrodinger wave equation - (Time dependent)

The quantity that characterises the De-Broglie wave is called wave function. It is denoted by ψ. It may be the complex function. Let us assume that ψ is specified in the x-direction by

$$\psi = Ae^{-i(\omega t - kx)} \quad ----(1)$$

If ν be the frequency and λ be the wavelength then

$$\omega = 2\pi\nu \quad \text{and} \quad k = 2\pi/\lambda$$

then eq^n (1) becomes

$$\psi = Ae^{-i[2\pi\nu t - 2\pi/_\lambda x]}$$

$$\psi = Ae^{-2\pi i(\nu t - x/\lambda)} \quad --(2)$$

Let E be the total energy and p be the momentum of the particles then,

$$E = h\nu$$

$$\Rightarrow \nu = E/h \quad ---(3)$$

and

$$\lambda = h/mv$$

$$\lambda = \frac{h}{p} \quad ---(4)$$

substituting (3) and (4) in (2)

Date

$$\psi = Ae^{-2\pi i\left[\frac{E}{h}t - \frac{x}{h}p\right]}$$

$$\Rightarrow \psi = Ae^{-2\pi i/h\,[Et - px]} \quad ----(5)$$

since, the total energy E, of the particle is the sum of kinetic energy and potential energy.

$$E = \frac{1}{2}mv^2 + V$$

$$\Rightarrow E = \frac{1}{2m}(mv)^2 + V$$

$$\Rightarrow E = \frac{1}{2m}p^2 + V$$

multiplying both sides by ψ

$$\Rightarrow E\psi = \frac{1}{2m}p^2\psi + V\psi \quad ---(6)$$

to find $E\psi$ and $p^2\psi$ we use eqn (5)

Now, differentiating eqn (5) w.r.to x

$$\frac{\partial\psi}{\partial x} = Ae^{-2\pi i/h\,[Et - px]}\left(-\frac{2\pi i}{h}\right)(-p)$$

Again, differentiating w.r.to x

$$\frac{\partial^2\psi}{\partial x^2} = Ae^{-2\pi i/h\,[Et - px]}\left(\frac{-2\pi i}{h}\right)^2(-p)^2$$

$$= \frac{i^2 4\pi^2}{h^2}p^2 A e^{-2\pi i/h\,(Et - px)}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{i^2 4\pi^2}{h^2}p^2\psi$$

$$\Rightarrow p^2\psi = \frac{h^2}{i^2 4\pi^2}\frac{\partial^2\psi}{\partial x^2} \quad ----(7)$$

Now, differentiating (5) w.r.t 't'

$$\Rightarrow \frac{\partial \psi}{\partial t} = A e^{-2\pi i/h [Et - px]} \left(-\frac{2\pi i}{h}\right) E$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = -\frac{2\pi i}{h} E \psi$$

$$\Rightarrow E\psi = \frac{-h}{2\pi i} \frac{\partial \psi}{\partial t} \quad \text{---} (8)$$

Now substituting (7) and (8) in (6)

$$\frac{-h}{2\pi i} \frac{\partial \psi}{\partial t} = \frac{1}{2m} \frac{h^2}{i^2 4\pi^2} \frac{\partial^2 \psi}{\partial x^2} + V\psi \qquad [\because -1 = i^2]$$

$$\Rightarrow \frac{ih}{2\pi} \frac{\partial \psi}{\partial t} = \frac{-h^2}{8\pi^2 m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \quad \text{---} (9) \qquad \left[\because \hbar = \frac{h}{2\pi}\right]$$

This is the time dependent form of schrodinger wave equation.

* schrodinger wave equation (Time independent)

We have, the time dependent form of schrodinger wave eqn.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \quad \text{---} (1)$$

where, $\psi = A e^{-2\pi i/h [Et - px]}$ --- (2)

since the potential is the function of space ie very with position only, the Schrodinger wave equation may be simplified by removing ~~all~~ the reference to 't'.

From eqn (2)

$$\psi = A e^{-2\pi i/h \, Et} \cdot e^{2\pi i/h \, px}$$

$$\Rightarrow \psi = \psi_0 e^{-2\pi i E t/h} \quad --- (3)$$

where, $\psi_0 = A e^{2\pi i p x/h} \quad --- (4)$

This means eq^n (3) is the product of a position dependent and time dependent function.

Now, differentiating eq^n (3) w.r.to x

$$\Rightarrow \frac{\partial \psi}{\partial x} = e^{-2\pi i E t/h} \frac{\partial \psi_0}{\partial x}$$

Again, differentiating w.r.to x,

$$\frac{\partial^2 \psi}{\partial x^2} = e^{-2\pi i E t/h} \frac{\partial^2 \psi_0}{\partial x} \quad ---(5)$$

Now, differentiating eq^n (3) w.r.to time 't'

$$\frac{\partial \psi}{\partial t} = \psi_0 e^{-2\pi i E t/h} \left(-\frac{2\pi i E}{h}\right) \quad --(6)$$

Now, substituting eq^n (3), (5) and (6) in eq^n (1) we get

$$i\hbar \psi_0 e^{-2\pi i E t/h} \left(-\frac{2\pi i E}{h}\right) = -\frac{\hbar^2}{2m} e^{-2\pi i E t/h} \frac{\partial^2 \psi_0}{\partial x^2} + V \psi_0 e^{-2\pi i E t/h}$$

$$\Rightarrow i\hbar \psi_0 e^{-2\pi i E t/h} \left(-\frac{iE}{\hbar}\right) = -\frac{\hbar^2}{2m} e^{-2\pi i E t/h} \frac{\partial^2 \psi_0}{\partial x^2} + V \psi_0 e^{-2\pi i E t/h}$$

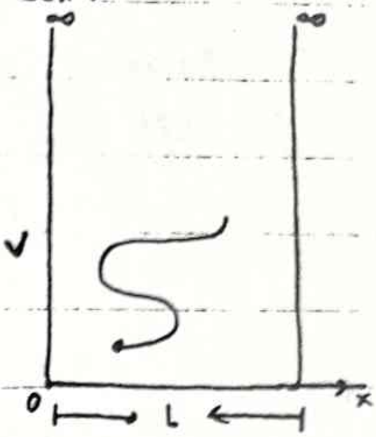
$$\Rightarrow E\psi_0 = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_0}{\partial x^2} + V\psi_0$$

$$\Rightarrow \frac{\hbar^2}{2m} \frac{\partial^2 \psi_0}{\partial x^2} + E\psi_0 - V\psi_0 = 0$$

$$\Rightarrow \frac{\partial^2 \psi_0}{\partial x^2} + \frac{2m}{\hbar^2} [E - V] \psi_0 = 0 \quad ---(7)$$

This is the time independent form of schrodinger wave equation.

* one dimensional potential well : (particle in a box)



consider a particle of mass m moving inside a box along x-axis (direction) and it is confined to move freely in the region $0 < x < L$

The potential energy 'V' of the particle is infinite on both sides of a box and zero can be assumed between $x=0$ and $x=L$.

$$\text{i.e}\quad \left.\begin{array}{ll} V = 0 & 0 < x < L \\ V = \infty & 0 \geq x \geq L \end{array}\right\} \quad ---(1)$$

since there is infinite potential at the boundary and particle can not exist outside the box so wave function ψ is zero for $0 \geq x \geq L$

$\therefore$ schrodinger wave equation within the region $0<x<L$ is

$$\frac{\partial^2 \psi_0}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi_0 = 0 \quad ---(2)$$

but in region $0<x<L$, $V=0$

$\therefore$ eq^n (2) becomes

$$\frac{\partial^2 \psi_0}{\partial x^2} + \frac{2mE}{\hbar^2}\psi_0 = 0$$

$$\Rightarrow \frac{\partial^2 \psi_0}{\partial x^2} + k^2 \psi_0 = 0 \quad ---(3)$$

where $k^2 = \frac{2mE}{\hbar^2}$ ---(4)

The solution of equation (3) can be writen as

$$\psi_0 = A\sin kx + B\cos kx \quad ---(5)$$

To find the values of A and B we can use boundary condition.

At $x=0$, $\psi_0 = 0$

$\therefore$ From (5)

$$0 = A\sin 0 + B\cos 0$$

$$\Rightarrow B = 0$$

Then eq^n (5) becomes

$$\psi_0 = A\sin kx \text{ ---(6)}$$

Again,

at $x = L$, $\psi_0 = 0$

$\therefore$ From (6) $A\sin kL = 0$

$$A \neq 0$$

$$\therefore \sin kL = 0$$

$$\Rightarrow \sin kL = \sin n\pi$$

$$\Rightarrow kL = n\pi$$

$$\Rightarrow k = \frac{n\pi}{L} \text{ ----(7)}$$

From eq^n (6) and (7)

$$\psi_0 = A\sin\frac{n\pi}{L}x \text{ ---(8)} \qquad \left[\text{or } \psi_n(x) = A\sin\frac{n\pi x}{L}\right]$$

From (4) and (7) $\frac{n^2\pi^2}{L^2} = \frac{2mE_n}{\hbar^2}$

$$\Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \text{ ---(9)}$$

This gives the total energy of particles

$\therefore$ we can say that for each value of n there is an energy level and corresponding wave function is given by $\psi_n(x)$

The values of E_n is called eigenvalue and Corresponding wave function ψ_n is called eigen function. Thus inside the box, the particle can only have descrete energy values specified by E_n.

Note that particle can not come out of the box and also the particle can not have zero energy. It is certain that the particle is somewhere inside the box. Hence there is a probability of finding the particle inside the box. It is given as

$$\int_0^L \psi^* \psi \, dx = 1 \quad \text{-----}(10)$$

It is called normalized wave function

$\therefore$ From (8) and (10)

$$\int_0^L A \sin \frac{n\pi x}{L} \cdot A \sin \frac{n\pi x}{L} \, dx = 1$$

$$\Rightarrow A^2 \int_0^L \sin^2 \frac{n\pi x}{L} \, dx = 1$$

$$\Rightarrow A^2 \int_0^L \left[\frac{1 - \cos 2n\pi x/L}{2} \right] dx = 1$$

$$\Rightarrow A^2 \left[\frac{1}{2} \int_0^L dx - \frac{1}{2} \int_0^L \cos \frac{2n\pi x}{L} \, dx \right] = 1$$

$$\Rightarrow A^2 \left[\frac{L}{2} - \frac{1}{2} \left\{ \sin \frac{2n\pi x}{L} \cdot \frac{L}{2n\pi} \right\}_0^L \right] = 1$$

$$\Rightarrow A^2 \left[\frac{L}{2} - 0 \right] = 1$$

$$\Rightarrow \frac{A^2 L}{2} = 1$$

$$\Rightarrow A^2 = \frac{2}{L}$$

$$\Rightarrow A = \sqrt{2/L} \quad \text{---} \quad (11)$$

$\therefore$ eqⁿ (8) becomes.

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \quad \text{-----} (12)$$

* physical significance of ψ -

The probability of that a particle will be found at a given place in space at given instant of time is characterised by the function ψ. It is called wave function. The function can be either real or complex. The only the quantity having a physical meaning is that the square of its magnitude

$$P = |\psi|^2$$

$$\Rightarrow P = \psi^* \psi$$

The quantity P is the probability density. The probability of finding the particle in a volume $dx\,dy\,dz$ is

$$|\psi|^2 \, dx \, dy \, dz.$$

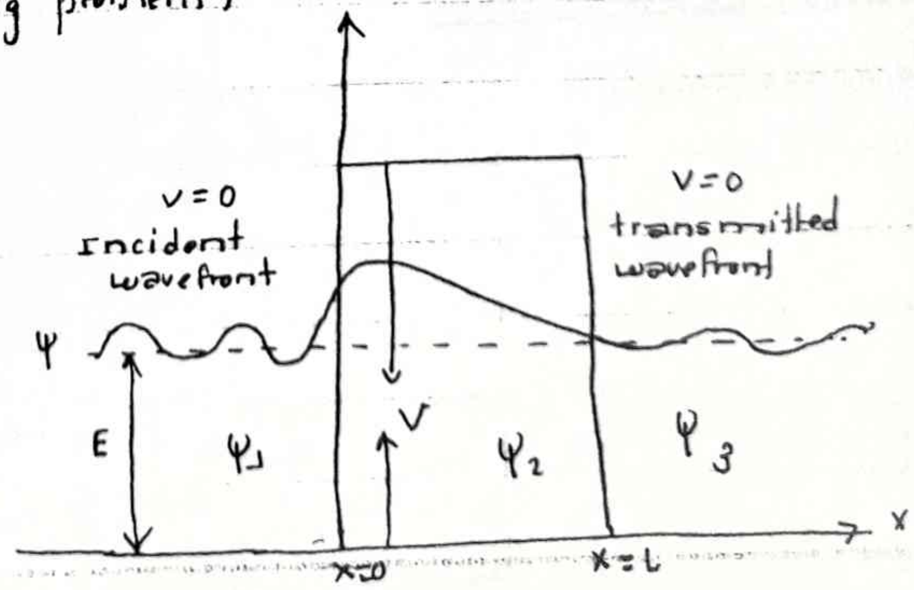
Further the particle is certainly to be found somewhere in space.

* potential barrier

(The barrier penetrating problem)

(Tunneling effect)

consider a beam of particles of kinetic energy E incident from the left on a potential barrier of height V and



width 'L', with V > E and on the sides of the barrier V = 0 which means that no forces acts upon the particles, there.

The potential is described as

$V = 0, \quad x < 0$, region I

$V = V, \quad 0 < x < L$ region II

$V = 0, \quad x > L$ region III

Let ψ_1, ψ_2 and ψ_3 be the respective wave function in region I, II and III as indicating in the figure.

The corresponding schrodinger wave equations are

In region I,

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\psi_1 = 0$$

but $V = 0$

$$\therefore \quad \frac{\partial^2 \psi_1}{\partial x^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\Rightarrow \quad \frac{\partial^2 \psi_1}{\partial x^2} + \alpha^2 \psi = 0 \quad \text{---(1)}$$

where $\alpha = \frac{2mE}{\hbar^2}$ ---(2)

similarly, in region II,

$$\frac{\partial^2 \psi_2}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\psi_2 = 0$$

$$\Rightarrow \quad \frac{\partial^2 \psi_2}{\partial x^2} - \beta^2 \psi_2 = 0 \quad \text{---(3)}$$

where, $\beta^2 = \frac{2m}{\hbar^2}(V - E)$ ---(4)

In region III

$$\frac{\partial^2 \psi_3}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi_3 = 0$$

but $V=0$,

$$\therefore \frac{\partial^2 \psi_3}{\partial x^2} + \frac{2mE}{\hbar^2}\psi_3 = 0$$

$$\Rightarrow \frac{\partial^2 \psi_3}{\partial x^2} + \alpha^2 \psi_3 = 0 \quad ----(5)$$

The solution of eq^n (1), (3) and (5) can be written as

$$\psi_1 = Ae^{i\alpha x} + Be^{-i\alpha x}$$

$$\psi_2 = Fe^{-\beta x} + Ge^{\beta x}$$

$$\psi_3 = Ce^{i\alpha x} + De^{-i\alpha x}$$

where A is the amplitude of incident wave on the barrier from left. B is the amplitude of reflected wave in region I. F is the amplitude of the wave penetrating the barrier in region II, G is the amplitude of reflected wave in region II. C is the amplitude of transmitted wave and D is the amplitude of non-existing reflected wave in the region III.

Since the probability density associated with the wave function is proportional to the square of the amplitude of that function, we define the barrier transmission coefficient as

$$T = \frac{|C|^2}{|A|^2}$$

And the reflected coefficient for the barrier surface at $x=0$ is

$$R = \frac{|B|^2}{|A|^2}$$

Date: . .

The ratio $\frac{|C|^2}{|A|^2}$ is also called penetrability of the barrier. It represents the probability that a particle incidence on the barrier from one side will appear on the other side. such probability is zero classically. But it is finite quantity in quantum mechanics. we thus conclude that if the particle with energy E incident on the thin barrier of hight greater than E, there is a finite probability of the particle penetrating the barrier.

This phenomena is called Tunneling effect.