(prepared by CPS)

Mechanical Oscillation -

* Harmonic motion -

The motion which repeats after a regular interval of time is called periodic motion. In such a motion there is always an equilibrium position or mean position at which body will come to at rest. If the body is displaced from equilibrium position there exists a certain kind of force which tries to keep the body back to its mean position. such a force is called restoring force. The restoring force is the function of displacement. The displacement of the body can also be expressed interms of trigonometric function. Therefore the periodic motion is also called harmonic motion.

* Free oscillation -

Free oscillations are the oscillations that appear in a system as a result of single initial deviation of the system from its state of mean or equilibrium position. If there is no resistance in its motion then object oscillates freely at its own is called free oscillation and the frequency of oscillation is called natural frequency.

* Damped oscillation -

If an object is set into oscillation and observed for a certain time the amplitude of oscillation goes on decreasing and finally dies off. such oscillation is called damped oscillation. This is due to the frictional force i,e resistance which opposes the motion of the body. Therefore, energy given to the body is converted slowly and slowly into heat for doing work against friction. This is called dissipation of energy. Therefore, resultant force is the restoring force plus frictional force. The frictional force is the function of velocity.

* Forced oscillation -

It is known that energy of the damped oscillation decrease with time, but it is possible to compensate for the energy if mechanical force is applied to the system. The oscillation produced when an external oscillating force is applied to a body subject to an elastic force is called forced oscillation. In this case the body oscillates with frequency other than natural frequency. The force applied externally is of periodic type Therefore, forced oscillation is sum of restoring force, frictional force and periodic external force.

* Simple harmonic motion -

simple harmonic motion is an especial type of periodic motion in which body oscillates in a straight line in such a way that restoring force is directly proportional to the displacement from mean position and alway acting towards the mean position.

If F be the restoring force,
x be the displacement then

$$F \propto x \quad \text{----(1)}$$

$\Rightarrow F = -kx \;...(2)$

Also, From Newton's law of motion.

$$F = \text{mass} \times \text{accl}^n \;..(3) \qquad [F = ma]$$

From (2) and (3)

$$ma = -kx$$

$$\Rightarrow a = -\frac{k}{m}x$$

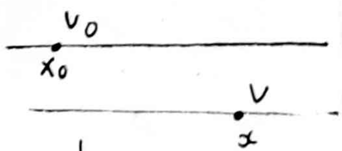
$$\Rightarrow a + \frac{k}{m}x$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \;...(4) \qquad \left[\because \frac{k}{m} = \omega^2 \text{ is constant}; \frac{d^2x}{dt^2} = \text{accl}\right]$$

we can also observed that the accl^n is directly proportional to displacement and directed towards mean position.

Eq^n (4) is the standard differential form of S. H. M.

* Equation of simple harmonic motion -



consider a particle of mass 'm' moving towards positive 'x' direction. suppose at any instant of time ($t=0$) the position of the particle is x_0 and velocity 'v_0'. Let 'v' and 'x' be the velocity and position of particle at time after time 't'.

If F be the restoring force, then

$$F = -kx \;...(1)$$

If 'a' be the accl^n at time 't' then,

$$a = \frac{F}{m} \;...(2)$$

From (1) and (2) $\quad a = -\frac{k}{m}x$

$$\Rightarrow a = -\omega^2 x$$

$$\Rightarrow \frac{dv}{dt} = -\omega^2 x$$

$$\Rightarrow \frac{dv}{dx}\frac{dx}{dt} = -\omega^2 x \qquad \left[\because \frac{dx}{dt} = v\right]$$

$$\Rightarrow v\,dv = -\omega^2 x\,dx \;...(3)$$

As velocity of particle is v_0 at x_0 and becomes v at x, then,

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} -\omega^2 x\,dx$$

$$\Rightarrow \left(\frac{v^2}{2}\right)_{v_0}^{v} = -\omega^2\left(\frac{x^2}{2}\right)_{x_0}^{x}$$

$$\Rightarrow v^2 - v_0^2 = -\omega^2 x^2 + \omega^2 x_0^2$$

$$\Rightarrow \quad v^2 = v_0^2 + \omega^2 x_0^2 - \omega^2 x^2$$

$$\Rightarrow \quad v^2 = \omega^2 \left(\frac{v_0^2}{\omega^2} + x_0^2 - x^2 \right)$$

$$\Rightarrow \quad v = \omega \sqrt{\left(\frac{v_0^2}{\omega^2} + x_0^2 - x^2 \right)}$$

$$\Rightarrow \quad v = \omega \sqrt{A^2 - x^2} \quad \text{....(4)} \qquad \left[\because \frac{v_0^2}{\omega^2} + x_0^2 \text{ is constant replaced by } A^2 \right]$$

Again, from (4)

As $dx/dt = v$ ... (5)

$\therefore$ From (4) and (5)

$$\frac{dx}{dt} = \omega \sqrt{A^2 - x^2}$$

$$\Rightarrow \frac{dx}{\sqrt{A^2 - x^2}} = \omega\, dt$$

Integrating we get,

$$\int_{x_0}^{x} \frac{dx}{\sqrt{A^2 - x^2}} = \int_0^t \omega\, dt$$

$$\Rightarrow \left(\sin^{-1} \frac{x}{A} \right)_{x_0}^{x} = \omega t$$

$$\Rightarrow \sin^{-1} \frac{x}{A} - \sin^{-1} \frac{x_0}{A} = \omega t$$

$$\Rightarrow \sin^{-1} \frac{x}{A} = \sin^{-1} \frac{x_0}{A} + \omega t$$

$$\Rightarrow \sin^{-1} \frac{x}{A} = \omega t + \delta \qquad \left[\sin^{-1} \frac{x_0}{A} = \delta, \text{ phase constant} \right]$$

$$\Rightarrow \frac{x}{A} = \sin(\omega t + \delta)$$

$$\Rightarrow x = A \sin(\omega t + \delta) \quad \text{....(6)}$$

Also, $\frac{dx}{dt} = v$, $\Rightarrow$ $v = A\omega \cos(\omega t + \delta)$ ....(7)

From (6) and (7) we conclude that displacement and velocity of particle which executes periodic motion can be expressed interms of harmonic or trigonometric function. Therefore, periodic motion is also called harmonic motion.

x Some terms associated with simple harmonic motion -

(1) Amplitude:

The maximum displacement of the particle in simple harmonic motion, from mean position is called amplitude of simple harmonic motion.

As we have,

displacement, $x = A \sin(\omega t + \delta)$

and $\sin(\omega t + \delta)$ can take values betn -1 to +1 then maximum displacement of particle will be $x = \pm A$

Therefore, A is amplitude of oscillation.

(2) Time period -

The time taken to complete one oscillation is called time period and it is denoted by T.

we have displacement of particle executing simple harmonic motion.

$x = A\sin(\omega t + \delta)$

If T be the time period, the displacements have same value at 't' and 't+T'

$\therefore \quad A\sin(\omega t+\delta) = A\sin[\omega(t+T)+\delta]$

since $(\omega t+\delta)$ repeats its value if the angle $(\omega t+\delta)$ is increased by 2π or its multiple

$$\therefore \omega(t+T) + \delta = (\omega t+\delta) + 2\pi$$

$$\Rightarrow \omega t + \omega T + \delta = \omega t + \delta + 2\pi$$

$$\Rightarrow \omega T = 2\pi$$

$$\Rightarrow T = \frac{2\pi}{\omega} \quad \cdots (*)$$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}} \quad \cdots (*)(*) \quad [\because \omega^2 = k/m]$$

(3) Frequency -

Frequency is the number of oscillations per sec. It is the reciprocal of time period and ~~depa~~ denoted by f and given as

$$f = \frac{1}{T}$$

$$\Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

(4) Angular frequency (Angular velocity)

Angular velocity is and angular frequency. It is denoted by ω and given by $\omega = \frac{2\pi}{T}$ [From (*)]

$$\Rightarrow \omega = 2\pi f$$

(5) phase and phase constant -

phase is the status of the particle in simple harmonic motion. It is denoted by ϕ.

As the displacement and velocity of particle in S.H.M. is given as

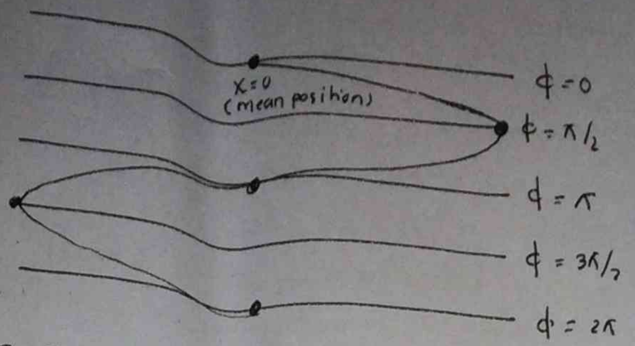
$x = A\sin(\omega t+\delta)$ and $v = A\omega\cos(\omega t+\delta)$

then $\phi = \omega t + \delta$

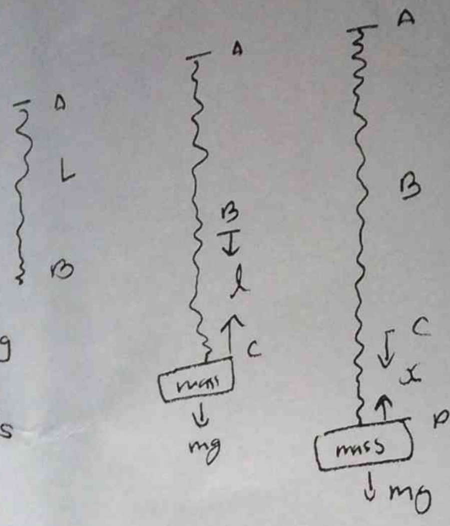
As the time increases, phase increases and δ is called phase constant. The phase constant depends upon the choice of instant time, $t=0$. If we choose the instant time at mean (original) position, then

$\phi = \omega t+\delta$ is zero that means,

$$\delta = 0$$



(• position of particle at different phase)



* spring mass system-

consider a spring of length AB = l, of negligible weight suspended from rigid support. Now, Let a mass 'm' is attached to the free end and then the spring elongates. Let l be the elongation produced on the spring because of mass 'm'.

According to Hook's law, extension produced is directly proportional to the force applied. If F_1 be the force applied for extension then,

$$F_1 = -Cl \quad \text{------} (1)$$

where c is force constant.

Now, pull the mass through a distance 'x' then the mass start to oscillate i,e periodic motion.

If F_2 be the force on the spring then

$$F_2 = -C(l+x) \quad \text{---} (2)$$

The resultant force on spring due to which mass sets into oscil is $F = F_2 - F_1$

$$\Rightarrow F = -C(l+x) + Cl$$

$$\Rightarrow F = -Cx \quad \text{---} (3)$$

Also, According to Newton's Law,

$$F = \text{mass} \times \text{accl}^n$$

$$F = m\frac{d^2x}{dt^2} \quad \text{---} (4)$$

From (3) and (4) $m\dfrac{d^2x}{dt^2} = -Cx$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{c}{m}x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{---} (5)$$

∴ This is analogous to simple harmonic motion.

I,e the spring mass system executes simple harmonic motion.

∴ The time period of spring mass system is

$$T = 2\pi/\omega$$

$\Rightarrow T = 2\pi\sqrt{m/c}$

or, frequency, $f = \frac{1}{2\pi}\sqrt{\frac{c}{m}}$

* Angular harmonic motion.

The motion in which the angular accl^n is directly proportional to the angular displacement is called angular harmonic motion. If θ be the angular displacement and α be the angular acceleration then,

$\alpha \propto \theta$

$\Rightarrow \alpha = -\omega^2 \theta$

$\Rightarrow \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$

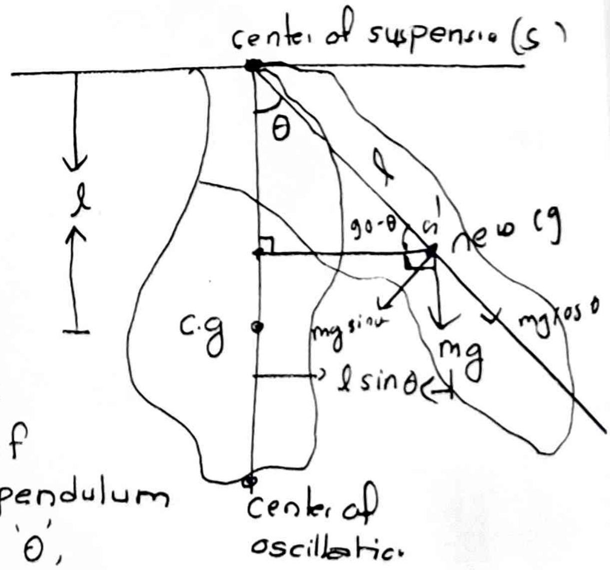
This is the equation of angular harmonic motion.

* Limitations of simple pendulum.

1. The heavy point mass bob is impossible.
2. The weightless, inextensible string is impossible.
3. The string has finite mass and hence it has finite moment of inertia but this inertia is not considered for time period of pendulum.
4. In simple pendulum, the center of oscillation and center of gravity lies at same point but in actual the center of oscillation always lies beyond the center of gravity.

Due to these limitations compound pendulum is prefered than simple pendulum.

* Compound pendulum
(Physical pendulum)



compound pendulum is the rigid body of any shaped capable of oscillating in a horizontal axis in vertical plane not passing through the center of gravity.

Let mg be the weight acting downward through the center of gravity. The distance of center of gravity from axis of rotation (center of suspension) is l. The pendulum is displaced through certain angle 'θ', then the pendulum starts oscillating. If G' be the position of new center of gravity then center of gravity G' and center of suspension 's' constitute a couple i.e weight mg acting vertically downward at G' and its reaction at 's' constitute torque which tend the pendulum back into its original position.

$\therefore$ Torque = force × perpendicular destance from axis of rotation to a'

$\Rightarrow \tau = mg \times l\sin\theta$

$\Rightarrow \tau = mgl\theta$ ---(1) [For small displacement, $\sin\theta \approx \theta$]

This torque provides the restoring force i,e

restoring torque, $\tau = -mgl\theta$ --(2)

If I be the moment of inertia of given axis and α be the angular accln then,

$$\tau = I\alpha \quad \text{---(3)}$$

From (2) and (3)

$$I\alpha = -mgl\theta$$

$$\Rightarrow \alpha = -\frac{mgl}{I}\theta \quad \text{---(4)}$$

i,e angular accln is directly proportional to angular displacemen and acting towards mean position. This means pendulum execute angular harmonic motion.

$\therefore$ comparing eqn (4) to $\alpha = -\omega^2\theta$ we get,

$$\Rightarrow \omega^2 = \frac{mgl}{I}$$

$$\Rightarrow \omega = \sqrt{\frac{mgl}{I}}$$

$$\Rightarrow 2\pi f = \sqrt{\frac{mgl}{I}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{I}{mgl}} \quad \text{---(5)}$$

If I_0 be the moment of inertia about an axis passing through center of gravity, then,
moment of inertia interms of radius of gyration k is

$$I_0 = mk^2 \quad \text{---(6)}$$

but using parallel axis theorem,
moment of inertia about an axis passing through center of suspension will be

$$I = I_0 + ml^2$$

$$\Rightarrow I = mk^2 + ml^2 \quad \text{---(7)}$$

From (5) and (7)

$$T = 2\pi\sqrt{\frac{(mk^2 + ml^2)}{mgl}}$$

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

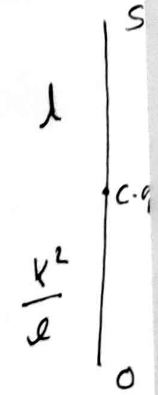
$$\Rightarrow T = 2\pi\sqrt{\frac{k^2/l + l}{g}} \quad \text{---(8)}$$

This is the time period of compound pendulum i,e same as that of simple pendulum of length $L = \frac{k^2}{l} + l$. This is called length of equivalent simple pendulum.

since k^2 is always positive, the length l is always greater the length of compound pendulum l. i,e the center of oscillation is always lies beyond the center of gravity.

* Interchangeability of center of suspension and oscillation -

we have, time period of compound pendulum,



$$T = 2\pi\sqrt{\frac{k^2/l + l}{g}} \quad \ldots (1)$$

Let $\frac{k^2}{l} = l' \ldots (2)$

$\Rightarrow \; k^2 = l l' \ldots (3)$

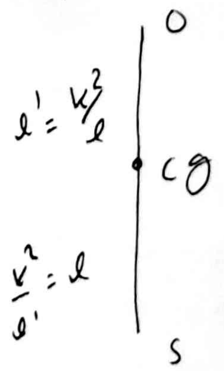
From (1) and (2)

$$T = 2\pi\sqrt{\frac{l' + l}{g}} \quad \ldots (2)$$

Now, the center of oscillation is made center of suspension by inverting the pendulum,

Then, time period will be,

$$T' = 2\pi\sqrt{\frac{\frac{k^2}{l'} + l'}{g}} \quad \ldots (3)$$



From (1) and (3)

$$T' = 2\pi\sqrt{\frac{l + l'}{g}} \quad \ldots (4)$$

From (2) and (4)

$$T = T'$$

i,e center of suspension and oscillation can be interchanged.

* Maximum and minimum time period:

we have,

$$T = 2\pi\sqrt{\frac{\frac{k^2}{l} + l}{g}} \quad \ldots (1)$$

when $l = 0$, $T = \infty$ (maximum value)

$\therefore \; T_{max} = \infty$

Now, squaring eqn (1)

$$T^2 = \frac{4\pi^2}{g}\left[\frac{k^2}{l} + l\right]$$

Now, differentiating w.r. to l we get

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left[-\frac{k^2}{l^2} + 1\right]$$

For minima $\frac{dT}{dl} = 0$

$$\therefore \quad 0 = \frac{4\pi^2}{g}\left[-\frac{k^2}{l^2} + 1\right]$$

$$\Rightarrow -\frac{k^2}{l^2} + 1 = 0$$

$$\Rightarrow \frac{k^2}{l^2} = 1$$

$$\Rightarrow k^2 = l^2$$

$$\Rightarrow k = l \quad \text{....}(*)$$

$\therefore$ time period is minimum when $k = l$.

$\therefore$ minimum. time period,

$$T_{min} = 2\pi\sqrt{\left(\frac{l^2}{l} + l\right)/g}$$

$$\Rightarrow T_{min} = 2\pi\sqrt{\frac{2l}{g}} \quad \text{---}(**)$$

* Determination of value of 'g':-

we have, time period of compound pendulum,

$$T = 2\pi\sqrt{\frac{\frac{k^2}{l} + l}{g}}$$

squaring both sides we get

$$T^2 = \frac{4\pi^2}{g}\left[\frac{k^2}{l} + l\right]$$

$$\Rightarrow T^2 = \frac{4\pi^2 k^2}{gl} + \frac{4\pi^2 l}{g}$$

multiplying by l on both sides

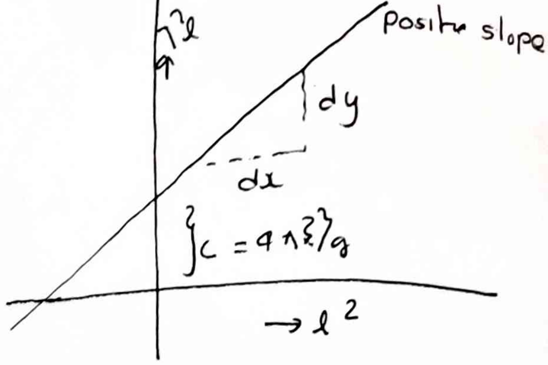
$$T^2 l = \frac{4\pi^2 l^2}{g} + \frac{4\pi^2 k^2}{gl}$$

comparing this eqn with $y = mx + c$, we get

$y = T^2 l$, $m = \frac{4\pi^2}{g}$, $x = l^2$ and $c = \frac{4\pi^2 k^2}{g}$

$\therefore$ slope of straight line, $m = \frac{4\pi^2}{g}$ ....(*)

Also, slope $= dy/dx$ ..(**)



From (*) and (**)

$$\frac{4\pi^2}{g} = \frac{dy}{dx}$$

$$\Rightarrow g = \frac{4\pi^2}{(dy/dx)}$$

This gives the accln due to gravity 'g'

* Energy conservation in simple harmonic motion - simple harmonic motion is defined by the ~~rotation~~, equation,

$$F = -kx \quad ---(1)$$

The work done by force during the displacement from x to $x+dx$ is

$$dw = F\,dx \quad ...(2)$$

From (1) and (2)

$$dw = -kx\,dx$$

The workdone to displace the particle from $x=0$ to x is

$$\int_0^w dw = \int_0^x -kx\,dx$$

$$\Rightarrow w = -\frac{kx^2}{2}$$

This much amount of work is stored in the form of potential energy

$$\therefore P.E. = \frac{1}{2}kx^2$$

$$P.E. = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t+\delta) \qquad [\because k = m\omega^2,\ x = A\sin(\omega t+\delta)]$$

Also, kinetic energy, $K.E. = \frac{1}{2}mv^2$

$$K.E. = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t+\delta)$$

$\therefore$ total energy, $E = K.E + P.E.$

$$= \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t+\delta) + \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t+\delta)$$

$$E = \frac{1}{2}m\omega^2 A^2$$

since the total energy (mechanical) at time t is independent of