MATH.APP.790 : Topics in Mathematics, Nonlinear time series analysis

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Lecture overview

- Recurrence networks.
- Complex network measures.
- Applications.
- Primary References
  1. Kantz, Holger, and Thomas Schreiber. Nonlinear time series analysis. Vol. 7. Cambridge university press, 2004.
  2. Puthanmadam Subramaniyam, Narayan. "Recurrence network analysis of EEG signals: A Geometric Approach." (2016).
  3. Recurrence quantification analysis. Theory and best practices.
  4. Zou et al. Complex network approaches to nonlinear time series. 2019 analysis.

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- These approaches can be broadly divided into three classes - 1) proximity networks, 2) visibility graphs and 3) transition networks.
- Such complex networks can be directed or undirected.
- The vertices are represented by state vectors and the edges between the vertices are defined based on some criteria such as mutual closeness or transition probabilities.

Types of time series networks

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- Recurrence networks

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- An edge is formed between the nodes if the state space distance between the corresponding cycles is lesser than a predetermined threshold

Correlation networks

In correlation networks, the Pearson correlation coefficient between two state vectors is used as a closeness measure to define the existence of an edge between two nodes (i.e., state vectors)

Recurrence networks

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- $\varepsilon-$recurrence network.

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- Such a network is asymmetric and directed because $i \in \mathcal{N}(j)$ does not imply $j \in \mathcal{N}(i)$.

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- If vertex $j \in \mathcal{N}(i)$, then vertex i is removed from $\mathcal{N}(j)$ to avoid the possibility of double-counting.
- This results in a symmetric adjacency matrix with an average degree of $2K_0$, where K_0 is the fixed number of neighbors initially assigned.

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The $k-$ nearest neighbor networks, Adaptive nearest neighbor networks and ε-recurrence networks are directly related with the concept of Poincaré recurrence in phase space.
They can be used to determine the dynamical invariants of the underlying system and provide an alternative framework for studying recurrences from a geometric point of view.

ε-recurrence networks

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3. Borrowing a rich set of measures from graph theory, we can characterize the topology of the RN.
4. Thus one can characterize the geometric skeleton of the attractor and gain insight into the dynamics.

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- The basic idea is to consider individual time series observations as vertices.
- Given a time series $\{x_i\}_{i=1}^{N}$ with $x_i = x(t_i)$.
- The VG approach will result in a $N \times N$ binary, adjacency matrix.

Visibility graphs

- Two points t_i and t_j are considered mutually visible in a VG setting if

$$\frac{x_i - x_k}{t_k - t_i} > \frac{x_i - x_j}{t_j - t_i}$$

is fulfilled for all time points t_k with $t_i < t_k < t_j$.

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- Advantages : No parameters to tune!

Horizontal visibility graphs

- The definition of VG was modified to obtain horizontal VGs (HVG), with the criterion

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- HVG is more restrictive (lesser connected graph) compared to VG.
- Algorithmically simple, compared to VG.
- HVG is not invariant under affine transformations.
- Many other definitions of VGs exist (See Zou and Donner 2019)

VG vs HVG

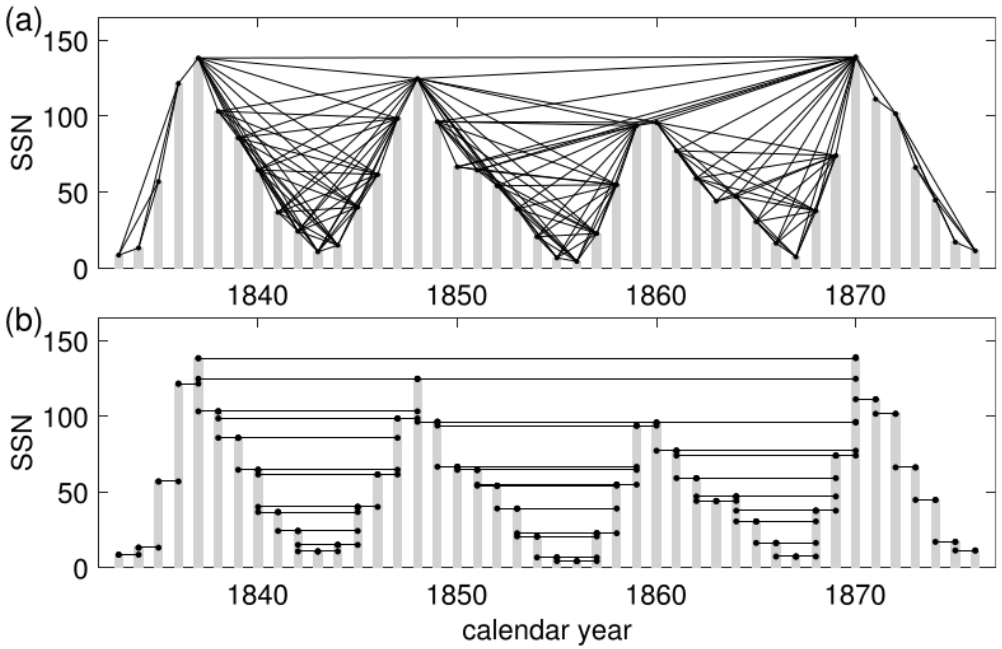

Figure: VG and HVG constructed from sunspot numbers.

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- The nodes of a transition network correspond to certain discrete states or patterns.
- Directed links are established if one of these nodes is followed by the other with non-zero (empirical) probability along the observed trajectory of the system under study.
- It is equivalent to a Markov chain with given transition probabilities between discrete states.

Ordinal Partition Transition networks

An OPTN is based on the ordinal symbolic encoding of a time series and consists of

- nodes, which represent the individual ordinal patterns and
- probability-weighted edges, which represent the transition frequencies between two successive ordinal patterns.

Ordinal Partition Transition networks

- Given a univariate time series $X = \{x_t\}_{t=1}^{T}$, following Takens' embedding theorem, we can qualitatively reconstruct the underlying trajectory.
- $$\mathbf{z}_t = [x_t, x_{t+\tau}, \ldots, x_{t+(k-1)\tau}],$$

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- $$\mathbf{z}_t = [x_t, x_{t+\tau}, \ldots, x_{t+(k-1)\tau}],$$
- Each embedding vector $\mathbf{z}_t$ is mapped to a sequence of integers $(s_0, s_1, \ldots, s_{k-1})$ that describes the rank order of its components.

Ordinal Partition Transition networks

The rank order is a unique permutation of the set $\{0, 1, \ldots, k-1\}$, thereby satisfying

$$x_{t+s_0\tau} \leq x_{t+s_1\tau} \leq x_{t+s_2\tau} \leq \ldots \leq x_{t+s_{k-1}\tau}$$

and

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Note that there exist $k!$ different possible ordinal patterns when a time series is embedded in k dimensions, and we denote these patterns by $\pi_1, \pi_2, \ldots, \pi_{k!}$.

Ordinal Partition Transition Networks

As an example, consider a 5-dimensional embedding of a time series yielding an embedding vector

$$\{x_t, x_{t+\tau}, x_{t+2\tau}, x_{t+3\tau}, x_{t+4\tau}\} = \{3, 9, 10, 1, 6\}.$$

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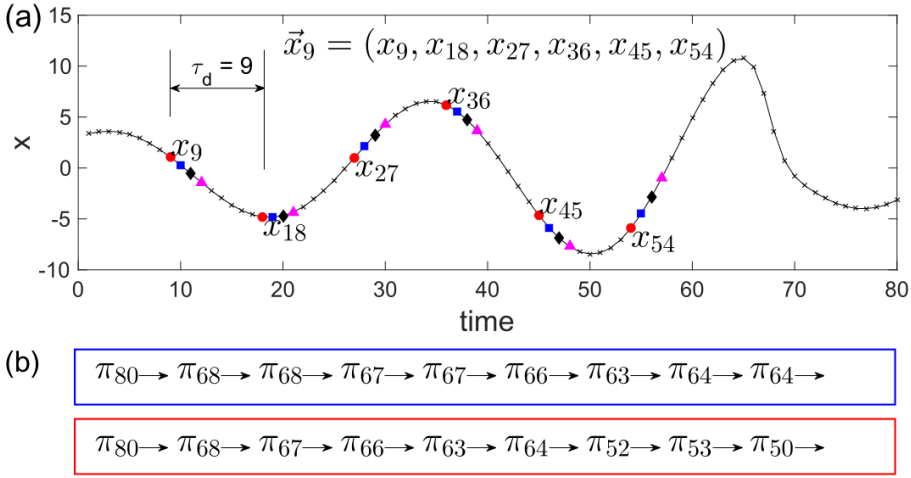
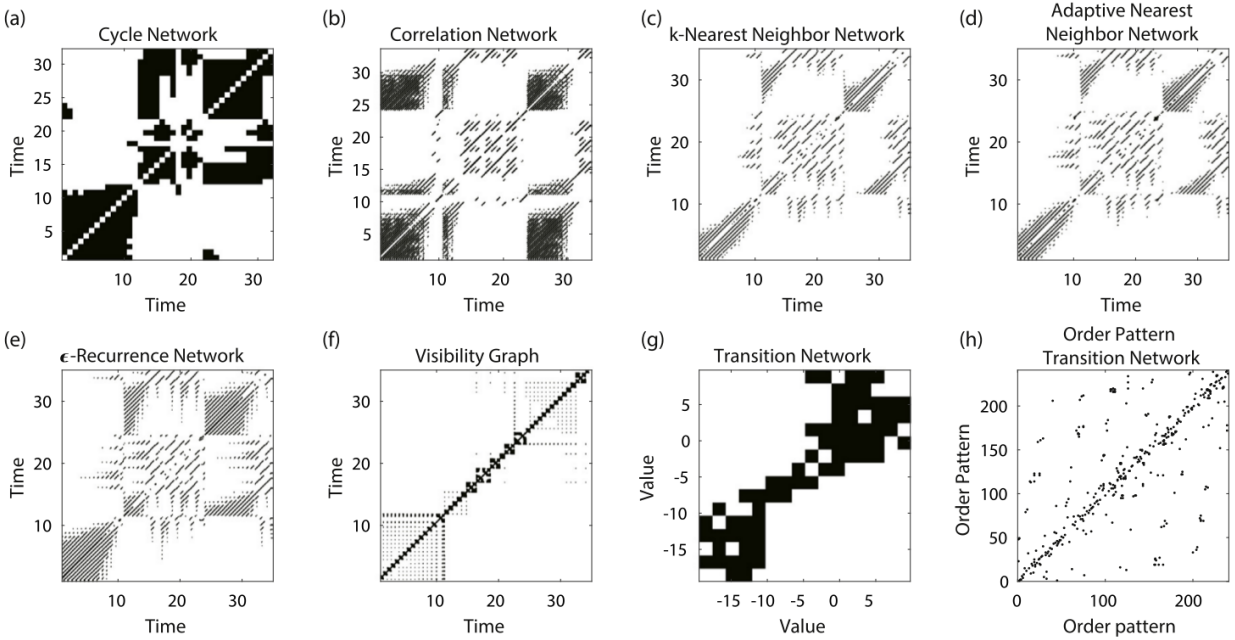
- This partition would be mapped to the ordinal pattern or symbol $\pi_k = \{3, 0, 4, 1, 2\}$.
- The exact numerical value of the resulting integer index $k \in \{1, \ldots, M!\}$ depends on the specific sorting of the permutations.
- For a univariate time series, we can then construct an (unweighted or weighted) OPTN with $k!$ nodes by repeating this encoding procedure for each embedding vector.

Ordinal Partition Transition networks

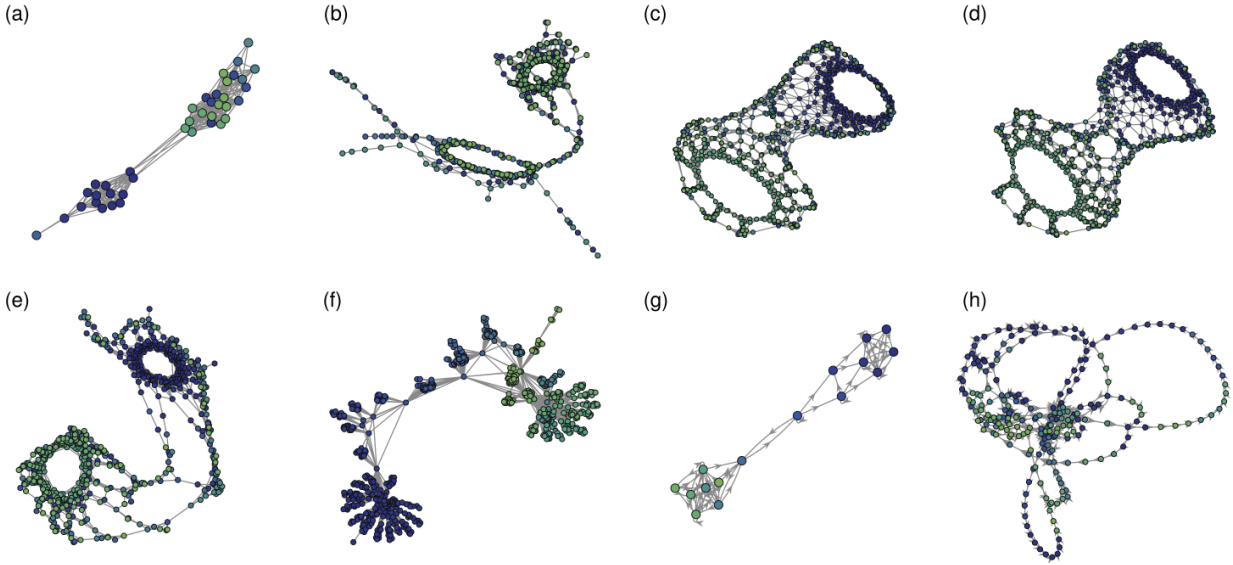
- A weighted OPTN is obtained by setting the weight of the edge between two nodes (permutations) to be equal to the empirical frequency of transitions.
- An unweighted OPTN simply contains a directed edge of unit weight between the corresponding nodes if this frequency is nonzero.



Adjacency matrices corresponding to different types of time series networks ($x-$component of Lorenz system)



Complex networks based on adjacency matrices corresponding to $x-$component of Lorenz system



$\varepsilon-$ recurrence networks

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- After constructing the RN, the next step is to investigate its basic statistical properties in terms local and global complex network measures such as degree distribution, average path length, local clustering coefficient etc

Vertex-based local measures : Degree centrality

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k_i can be normalized to describe the local phase space density, which gives the probability that a randomly chosen state vector in the phase space is within the ε-neighborhood of the state vector $\mathbf{x}_i$ represented by the vertex i in the complex network. It is given as

$$\rho_i = \frac{k_i}{N-1},$$

which also corresponds to the definition of the local recurrence rate (RR).

Example of Lorenz system

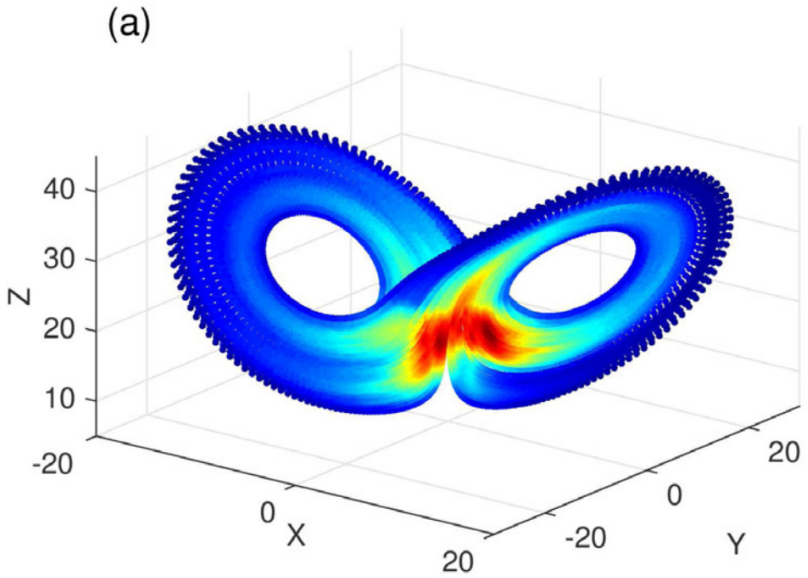

Figure: Spatial distribution of local recurrence rate for the chaotic Lorenz attractor.

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- Given the geodesic distance $d(i, j)$ between two vertices i and j in a network, the closeness centrality of the vertex i can be then defined as,

$$c_i = \frac{N-1}{\sum_{j=1}^{N} d(i,j)}$$

Example of Lorenz system

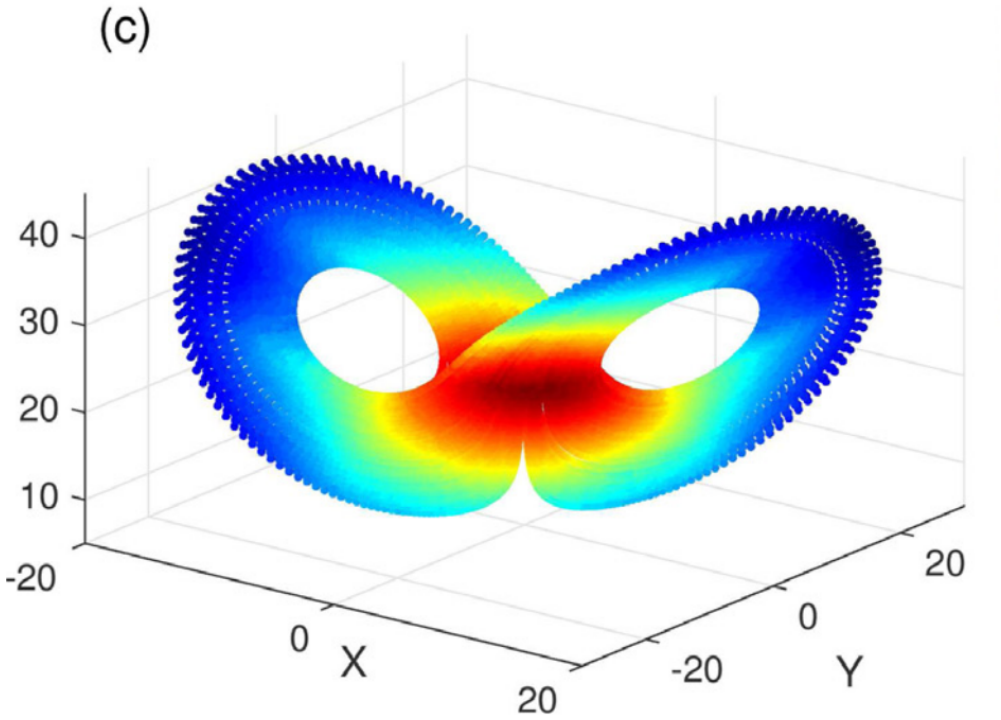

Figure: Spatial distribution of closeness centrality for the chaotic Lorenz attractor.

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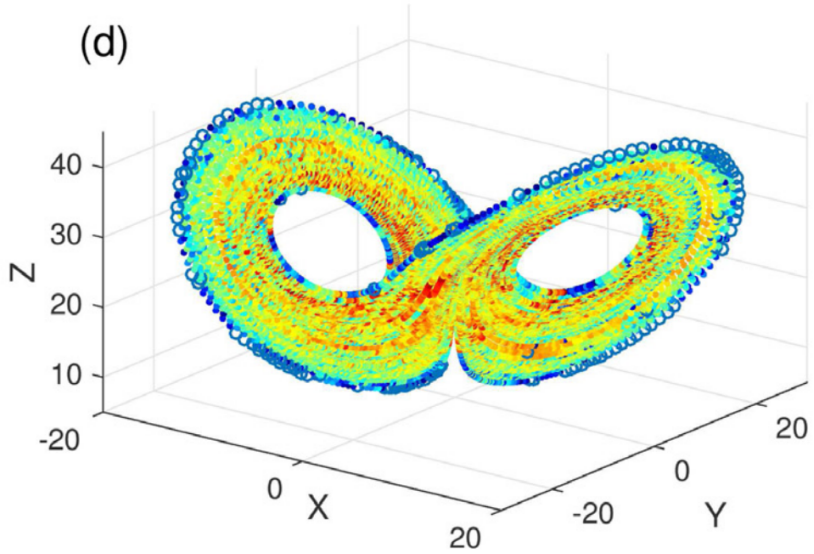
- The betweenness centrality for a vertex i can then be defined as,

$$b_i = \sum_{j \neq i \neq k} \frac{\sigma_{jk}(i)}{\sigma_{jk}}.$$

Example of Lorenz system



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- A triangle Δ can be considered as a subgraph of three vertices of undirected graph G representing a complex network and the number of triangles on vertex i can be defined as $\delta(i) = |\{\{j,q\} \in E : \{i,j\} \in E : \{i,q\} \in E\}|$.

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- A **triple** at a vertex i can be defined as a path of length two, for which i is the central vertex.
- The number of triples (for an undirected network) for a vertex i is given as $T_i = \binom{k_i}{2}$, where k_i is the node degree or degree centrality.

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The above expression can be given in terms of the adjacency matrix as

$$\mathcal{C}_i = \frac{\sum_{j,q} A(i,j)A(j,q)A(q,i)}{2T_i}$$

Example of Lorenz system

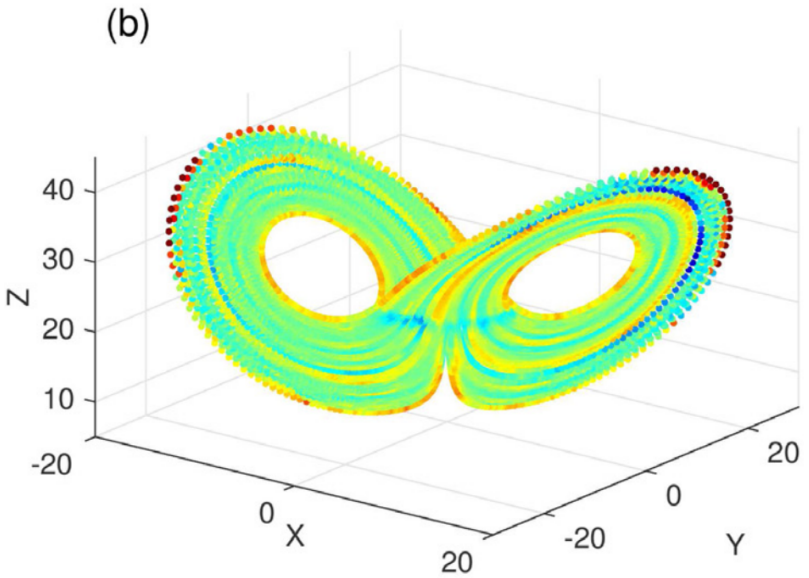

Figure: Spatial distribution of local clustering coefficient for the chaotic Lorenz attractor.

What do these local measures reveal ?

- State space regions with a high density of points are characterized by a high degree density.
- The local clustering coefficient in a RN is associated with the geometric alignment of state vectors.
- If the dynamics of the system is effectively lower-dimensional, it results in a locally enhanced fraction of closed paths of length 3 ("triangles") and, thus, a higher local clustering coefficient.
- Regions of unstable periodic orbits (UPOs) are typically associated with high local clustering coefficient.
- Closeness centrality is typically highest at the center of the RN.
- Betweeness centrality reveals the local degree of framentation of the attractor.

Global network measures

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- In terms of the matrix $\mathbf{A}$, $\mathcal{T}$ can be given as

$$\mathcal{T} = \frac{\sum_{i,j,q=1}^{N} A(i,j)A(j,q)A(q,i)}{\sum_{i,j,q=1}^{N} A(i,j)A(q,i)}.$$

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- **The transitivity dimension** at a single-scale ε can be given as [Donner et al. 2012]

$$\mathcal{D}_{\mathcal{T}_\varepsilon} = \frac{\log \mathcal{T}_\varepsilon}{\log(3/4)}$$

- Where does this idea come from ?

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- $f(x, y) = \Theta(\varepsilon - |x - y|)$.

Random geometric graphs

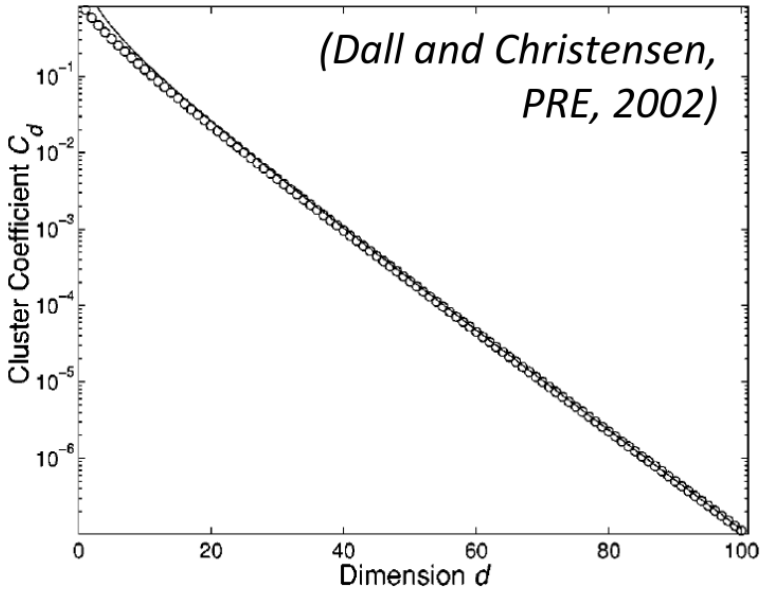


Figure: For a random geometric graph, network measures such as clustering coefficient are purely geometric quantity

Asymptotic solution $C_d \approx 3\sqrt{\frac{2}{\pi d}}(\frac{3}{4})^{\frac{d+1}{2}}$

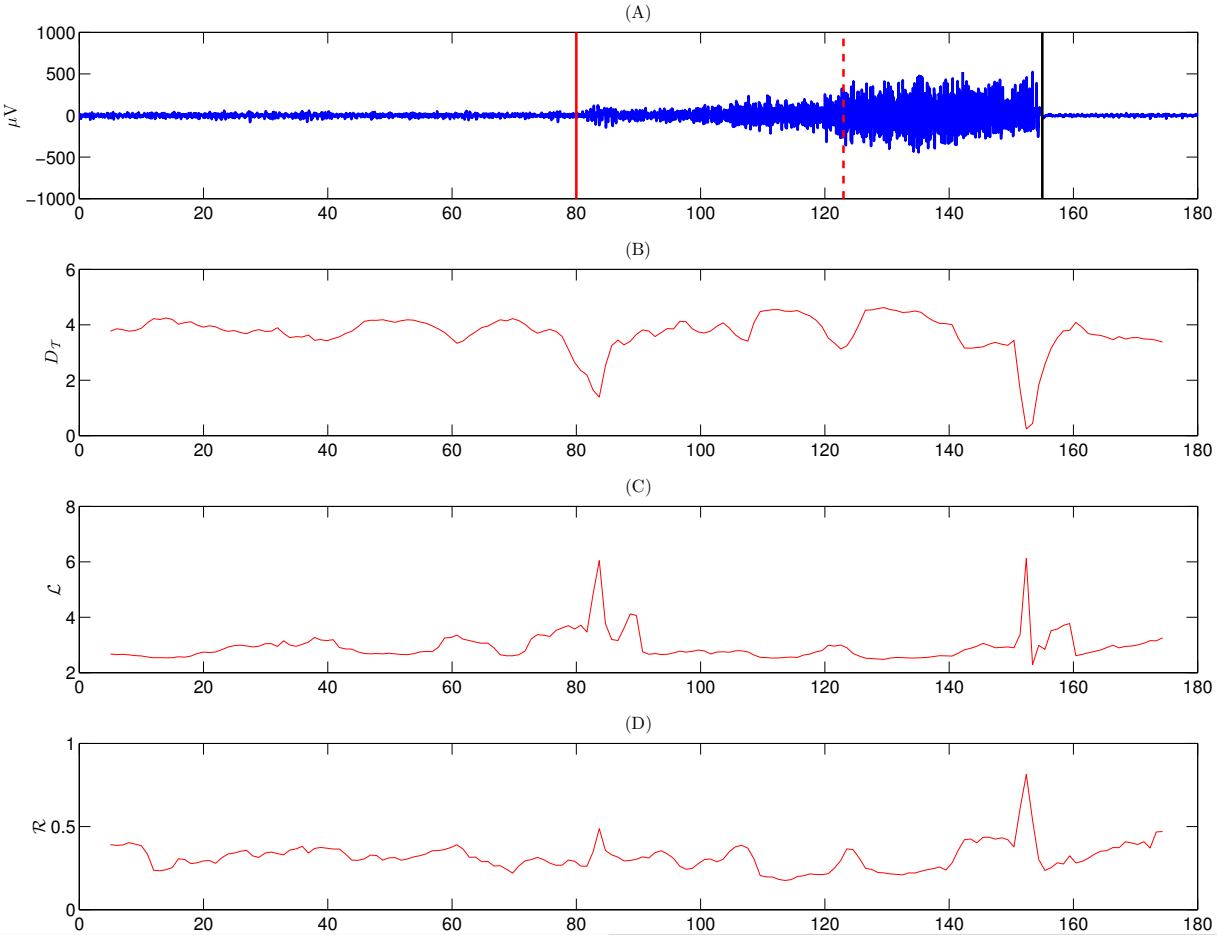
Dynamical systems and RGG

- We can think of $\mathcal{A}$ as the manifold describing a chaotic attractor and $p(x)$ can be some invariant density.
- The structure of the attractor can be approximated by a set of elements of $\mathcal{A}$ drawn randomly according to $p(x)$.
- The (network) properties of such RGGs reveal geometric characteristics of the attractor!

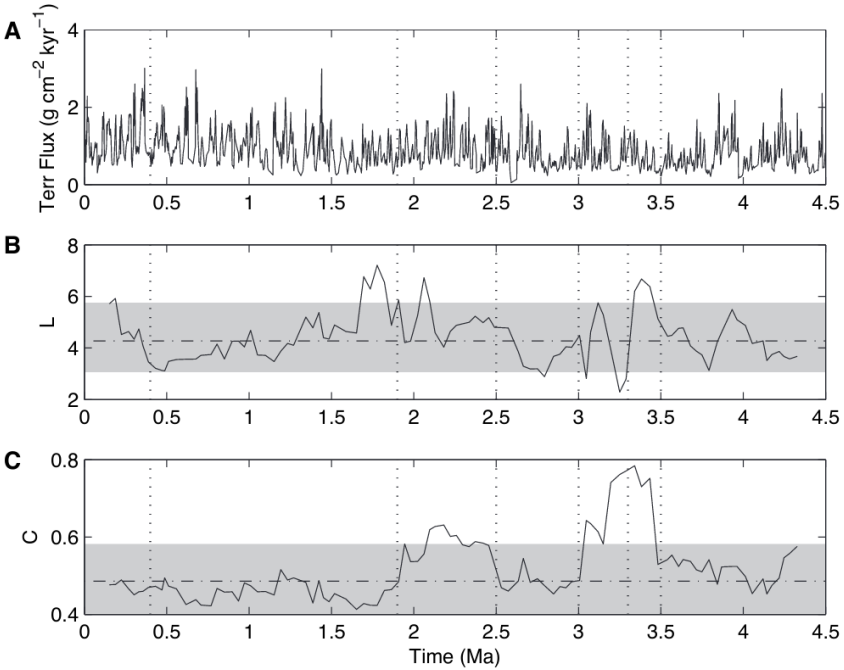
Summary of measures [Subramaniyam 2016]

Network measures	Definition	Interpretation	Key references
Degree centrality (k_i)	$\sum_{j=1}^{N} A(i,j)$.	Local recurrence in phase space, emergence of scale-free distributions	Donner *et al.* [16] ; Zout *et al.* [169]
Closeness centrality (c_i)	$\frac{N-1}{\sum_{j=1}^{N} d(i,j)}$.	Identification of the attractor's center of gravity	Donner *et al.* [16]
Betweenness centrality (b_i)	$\sum_{j\neq i\neq k} \frac{\sigma_{jk}(i)}{\sigma_{jk}}$.	Local fragmentation of attractor	Donner *et al.* [16, 154]
Clustering coefficient ($\mathcal{C}_i$)	$\frac{\sum_{j,q} A(i,j)A(j,q)A(q,i)}{2T_i}$, $T_i = \frac{k_i(k_i-1)}{2}$.	Presence of lower order UPOs	Donner *et al.* [16] ; Zou *et al.* [165]
Transitivity ($\mathcal{T}$)	$\frac{\sum_{i,j,q=1}^{N} A(i,j)A(j,q)A(q,i)}{\sum_{i,j,q=1}^{N} A(i,j)A(q,i)}$.	Attractor dimension, UPOs, regularity of dynamics	Donner *et al.* [163]; Donges *et al.* [170]
Assortativity ($\mathcal{R}$)	$\frac{\frac{1}{N}\sum_{j>i} k_i k_j A(i,j) - [\frac{1}{N}\sum_{j>i}\frac{1}{2}(k_i+k_j)A(i,j)]^2}{\frac{1}{N}\sum_{j>i}\frac{1}{2}(k_i^2+k_j^2)A(i,j) - [\frac{1}{N}\sum_{j>i}\frac{1}{2}(k_i+k_j)A(i,j)]^2}$.	Smoothness of phase space density	Donner *et al.* [16]
Average path length ($\mathcal{L}$)	$\frac{1}{N(N-1)}\sum_{i\neq j} d(i,j)$.	Average phase space separation, abrupt dynamical changes	Donner *et al.* [16]; Zou *et al.* [165]; Donges *et al.* [170]
Cross-clustering coefficient ($\mathcal{C}_{xy}$)	$\frac{\sum A_{xy}(i,j)A_{xy}(i,k)A_y(j,k)}{k_{i,xy}(k_{i,xy}-1)}$.	Mutual interaction between complex networks	Donges *et al.* [149]

Real-world examples : Detecting epileptic seizure



Real-world examples : Terrigenous dust flux



Take home messages

- RN measures are advantageous as they are conceptually simple and attractor topology can be quantified using various network measures.
- Clustering coefficient is sensitive to periodicity. (longer time-scales)
- Average path length, Transitivity dimension is sensitive to transitions. (short time-scales)
- Complementary to methods such as RQA and other linear/nonlinear methods.