MATH.APP.790 : Topics in Mathematics, Nonlinear time series analysis

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Lecture overview

- Estimation of embedding dimension
- Lyapunov exponents
- References

1. Kantz, Holger, and Thomas Schreiber. Nonlinear time series analysis. Vol. 7. Cambridge university press, 2004.
2. Puthanmadam Subramaniyam, Narayan. "Recurrence network analysis of EEG signals: A Geometric Approach." (2016).
3. Wolf, Alan, et al. "Determining Lyapunov exponents from a time series." Physica D: Nonlinear Phenomena 16.3 (1985): 285-317.

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- There are various methods in the literature on the selection of k and many of these methods are based on false-nearest neighbor (FNN) principle.
- The rationale : Examine if the points along a trajectory in dimension k are also neighbors in dimension $k+1$.

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- For every increase in k, the percentage of false neighbors is also computed.
- The value of k at which the percentage of false neighbors becomes zero (or arbitrarily small due to the effect of noise) is considered as an appropriate choice for k.

Embedding dimension

Given a scalar time series and τ, one reconstructs the phase space vector at an initial dimension k. Thus we have a state vector at time instance t_n,

$$x_n^k = (x_{t_n}, x_{t_n-\tau}, \cdots, x_{t_n-(k-1)\tau})$$

which has a nearest neighbor

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If the above reconstruction occurred in an insufficient dimension k, then this closeness could be a result of trajectories crossing.

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which can be normalized and computed for all the points as

$$\xi = \sum_{n=1}^{N-k-1} \Theta\left(\frac{D_n}{||x_n^k - x_{\mathcal{N}(n)}^k||} - r\right). \tag{2}$$

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- In other words, if ξ is greater than r then these two points can be called false neighbors.
- Thus, ξ is the amount of false neighbors that one would find in dimension k.
- The optimal embedding dimension is then defined as the dimension for which ξ becomes zero (or very small in case of noisy time series).

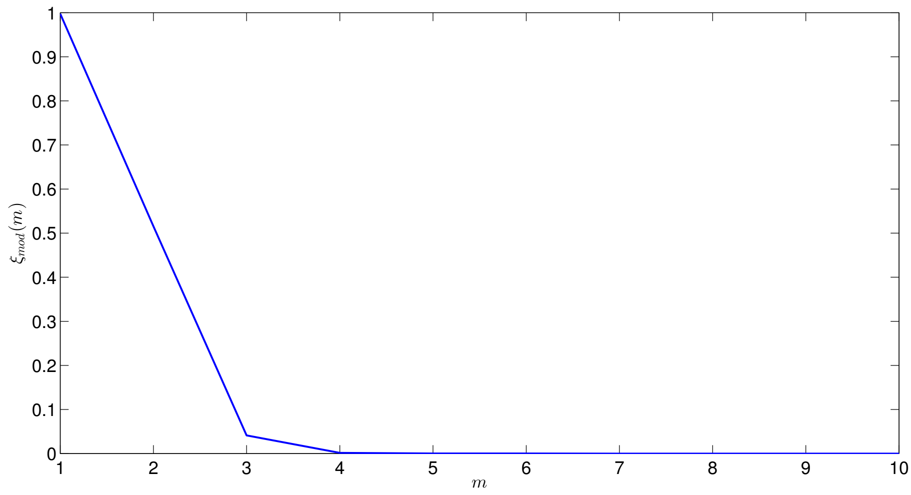
Embedding dimension

Hegger and Kantz proposed a modification to the FNN method by excluding points whose initial distance was already greater σ/r, where σ is the standard deviation of the data. The modified FNN method is given by

$$\xi_{mod} = \frac{\sum_{n=1}^{N-k-1} \Theta\left(\frac{||x_n^{k+1} - x_{\mathcal{N}(n)}^{k+1}||}{||x_n^k - x_{\mathcal{N}(n)}^k||} - r\right) \Theta\left(\frac{\sigma}{r} - ||x_n^k - x_{\mathcal{N}(n)}^k||\right)}{\sum_{n=1}^{N-k-1} \Theta\left(\frac{\sigma}{r} - ||x_n^k - x_{\mathcal{N}(n)}^k||\right)}.$$

Embedding dimension

Figure below shows the minimum embedding dimension using the x-component of the Lorenz system with the modified FNN method.



It can be seen at at $k = 3$, already the percentage of false nearest neighbors is very close to zero and for $k > 3$, this value falls to zero.

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- It does not depend on the coordinate system.
- Value of dynamic invariant obtained from original dynamic system is equal to the one obtained from time delay embedding!

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- Positive Lyapunov exponent $\rightarrow$ Bounded system with deterministic chaotic dynamics

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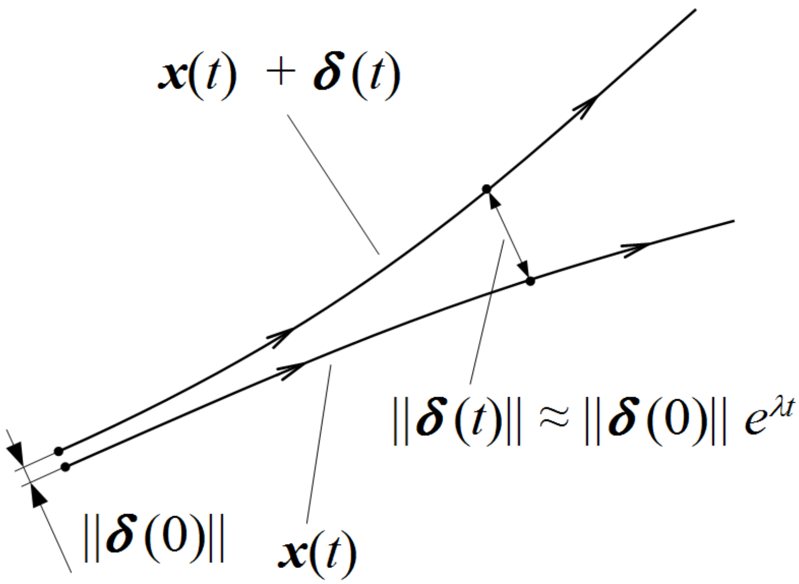
- The Lyapunov exponent can be used to measure this separation.
- Since chaotic orbits are bounded, this separation and hence the mutual distance between two diverging points cannot tend to infinity.
- Consider two neighboring points x_0 and $x_0 + \delta x_0$, with initial separation δx_0. The divergence of these two points occurs at a rate given by

$$\delta x(t) \approx e^{\lambda t} |\delta x_0|$$

The Lyapunov exponent



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- For dissipative systems, the sum of all the Lyapunov exponents must be negative.
- Flows have one zero exponent $\rightarrow$ for the occurrence of chaos at least three dimensions are required!

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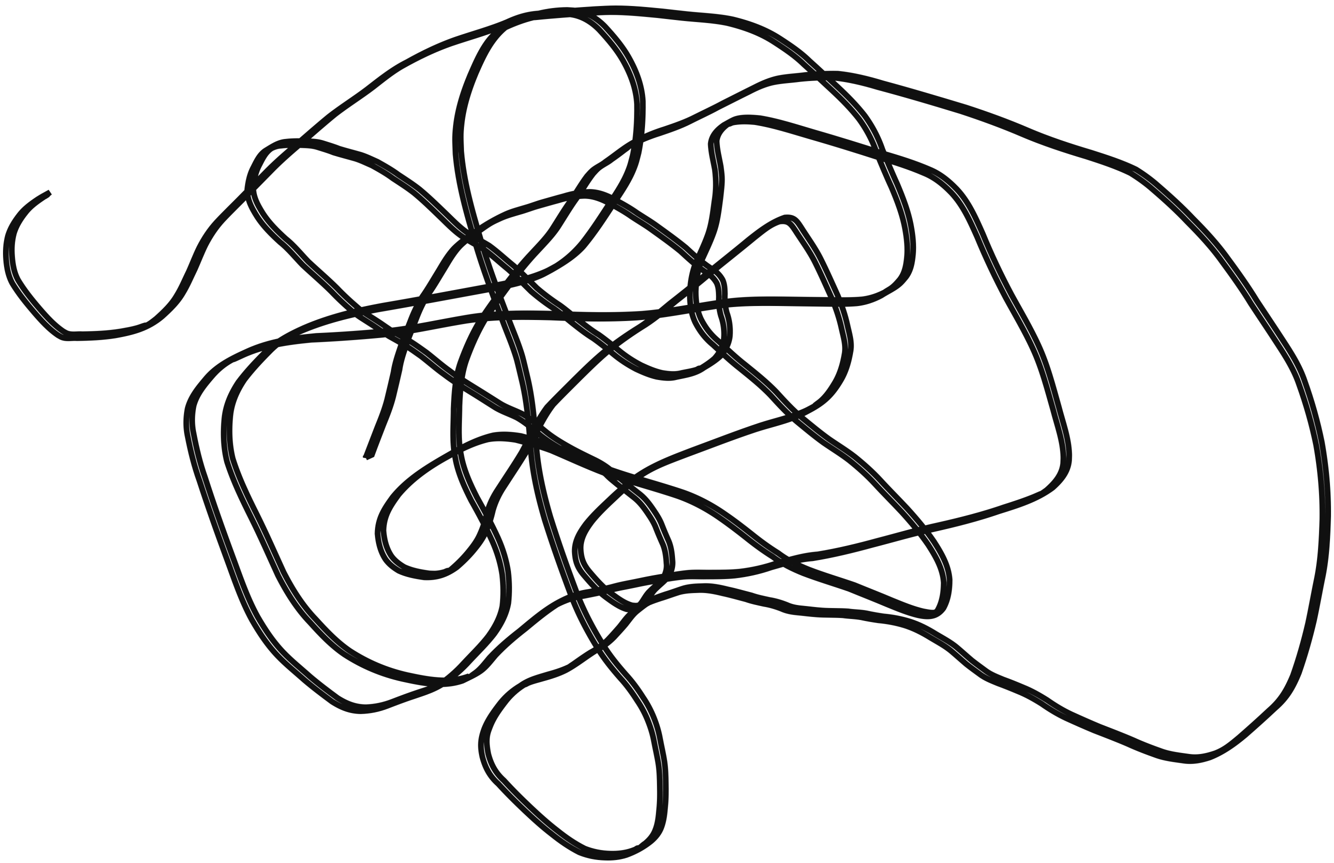
- A chaotic system has at least one positive Lyapunov exponent.
- Note : Lyapunov exponents are invariant under deformations of the attractor that preserve its topology.
- If the reconstruction is done correctly, λ obtained from reconstructed attractor is equal to that of the original system.

The Lyapunov exponent



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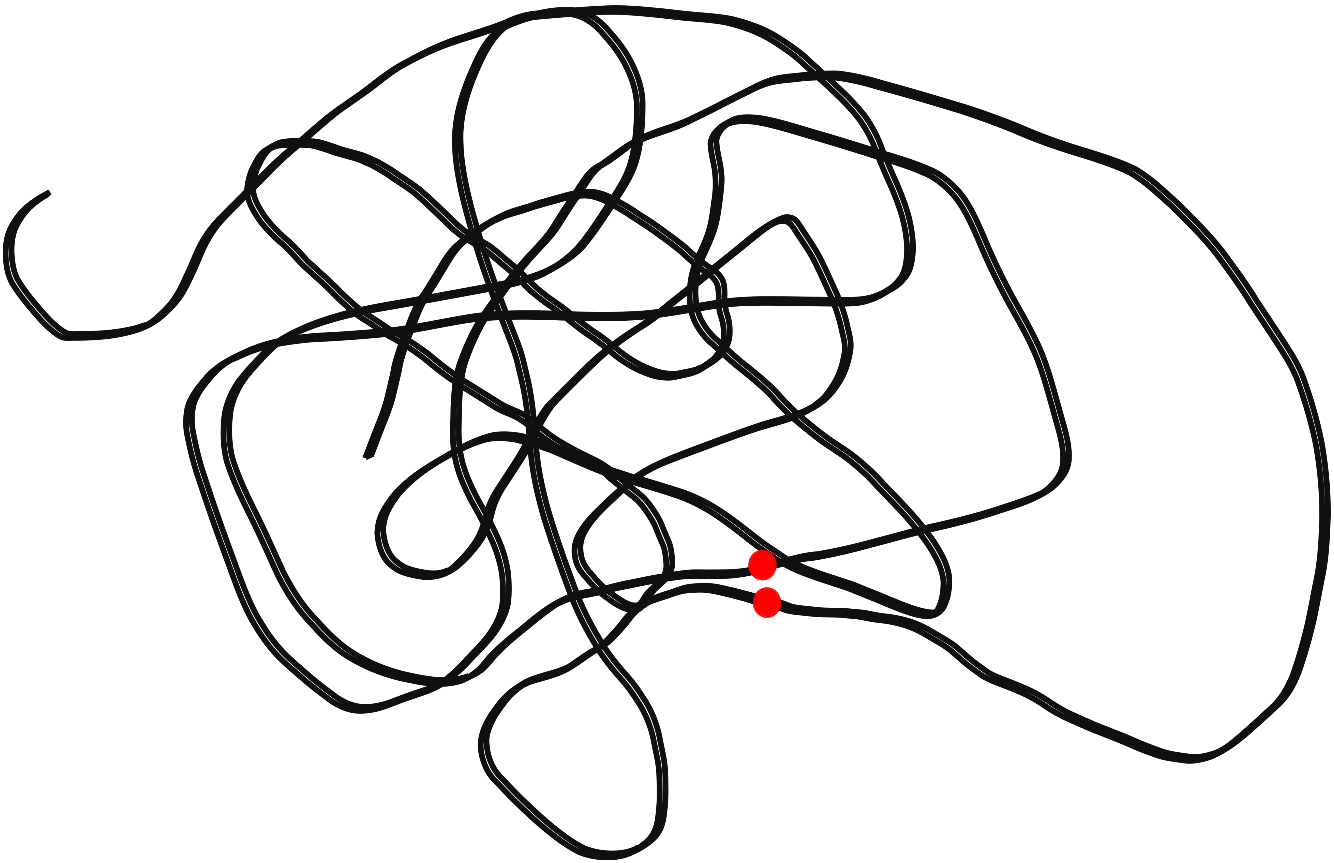

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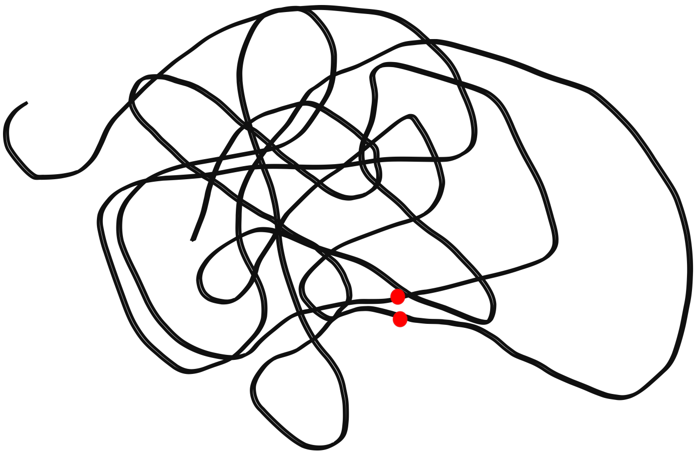
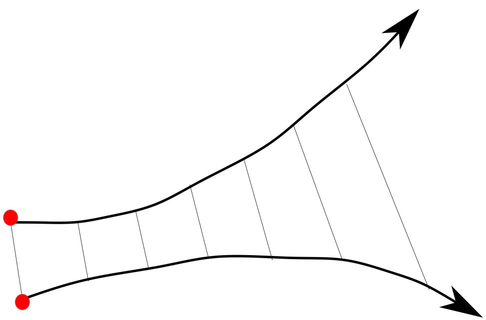
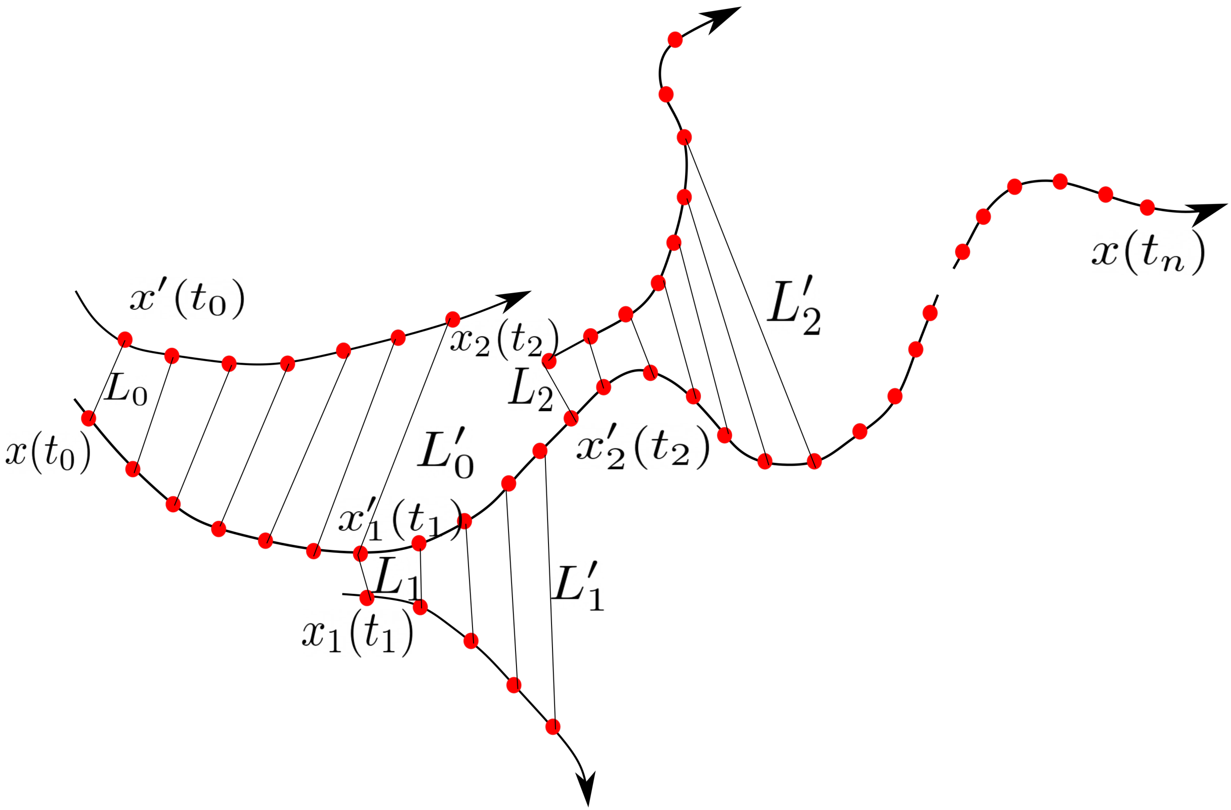




How to compute Lyapunov exponents from time series data ?



How to compute Lyapunov exponents from time series data ?

Wolf algorithm :

1. Perform embedding after selecting τ and k
2. Pick a point $x(t_0)$ and its nearest neighbor $x'(t_0)$.
3. Compute $L_0 = ||x'(t_0) - x(t_0)||$.
4. Increment i and compute $L_0(i) = ||x'(t_i) - x(t_i)||$, until $L_0(i) > \epsilon$.
5. $t \leftarrow t_1$ and $L_0(i) \leftarrow L'_0$.
6. Find $x'_1(t_1)$, the nearest neighbor of $x_1(t_1)$ and repeat step 3 to step 5, until $t = t_n$.

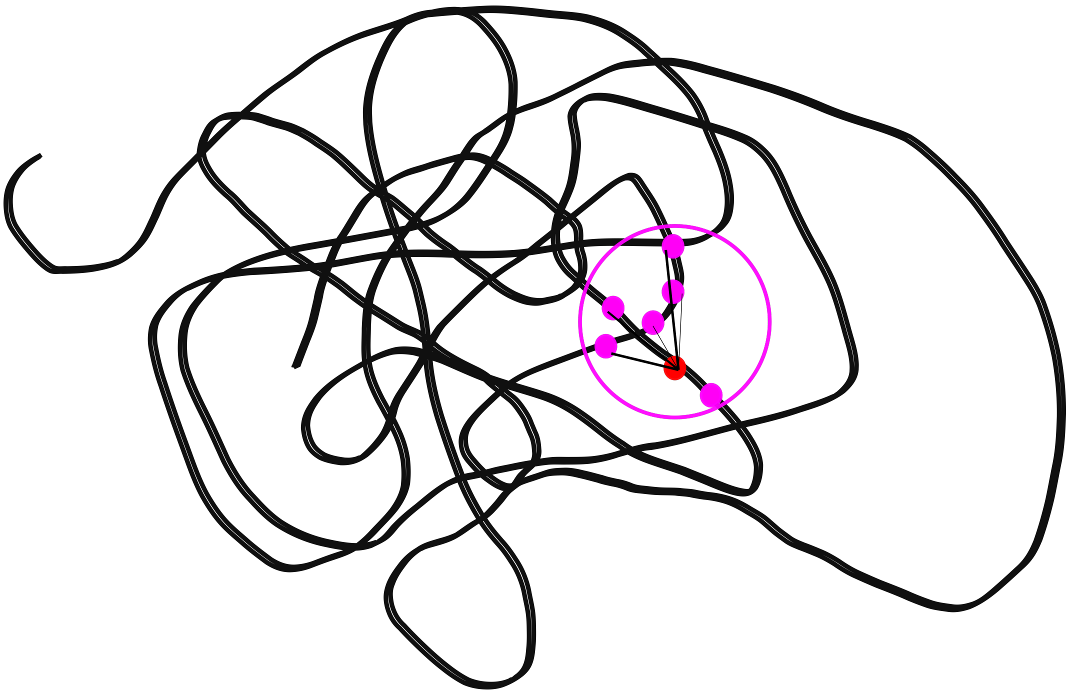
The biggest Lyapunov exponent is given by

$$\lambda_1 \approx \frac{1}{N\Delta t} \sum_{1}^{M-1} \log \frac{L'_i}{L_i}$$

Problem with Wolf algorithm

- Single point nature of the algorithm - At every step one starting point and one nearest neighbor is picked.
- This could be a major problem if data is noisy.
- The Kantz algorithm (and related methods) address this issue by instead picking multiple points and averaging their distance to some central point.

Kantz algorithm



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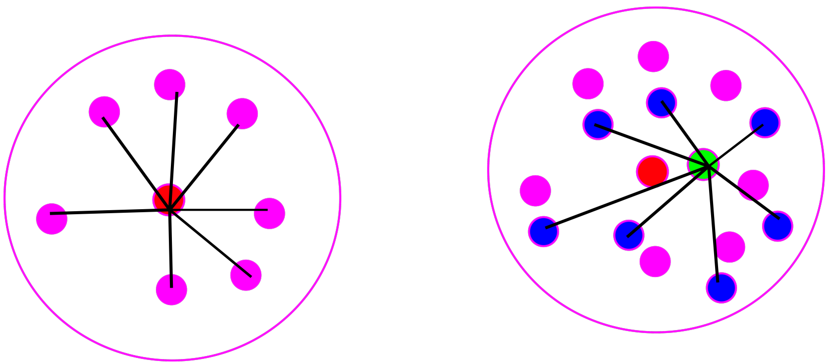

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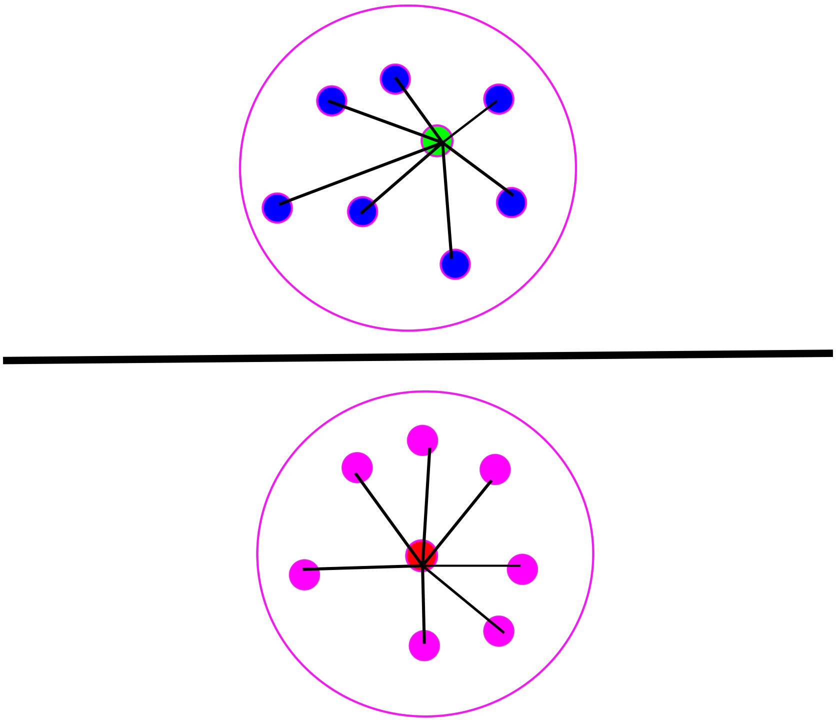

Kantz algorithm

$$S(\Delta n) = \frac{1}{N} \sum_{n_0=1}^{N} \log \left(\frac{1}{\mathcal{U}(s_{n_0})} \sum_{s_n \in \mathcal{U}(s_{n_0})} \left| s_{n_0+\Delta n} - s_{n+\Delta n} \right| \right)$$