MATH.APP.790 : Topics in Mathematics, Nonlinear time series analysis

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January 2021

Lecture overview

- Recurrence plots.
- Recurrence quantification analysis.
- Applications.
- Primary References
    1. Kantz, Holger, and Thomas Schreiber. Nonlinear time series analysis. Vol. 7. Cambridge university press, 2004.
    2. Puthanmadam Subramaniyam, Narayan. "Recurrence network analysis of EEG signals: A Geometric Approach." (2016).
    3. Recurrence quantification analysis. Theory and best practices.

Poincaré recurrences

- The metatheorem of dynamical systems theory states that, for an appropriately bounded phase space X, the trajectories of the motion will exhibit some form of recurrence.
- That is, they will return close to their initial position.
- In 1890, Poincaré formulated the first precise result and proved that whenever a dynamical system preserves volume, almost all the trajectories return arbitrarily close to their initial position, infinite number of times

Poincaré recurrence theorem

Formally, let $(X, \mathcal{B}, \mu)$ be a measure space of finite total measure. Here, $\mathcal{B}$ is a σ-algebra (i.e., $\mathcal{B} \subset 2^X$) μ is a probability measure. Let $f : X \to X$ be a function such that $f^{-1}\mathcal{A} \in \mathcal{B}$ for any $\mathcal{A} \in \mathcal{B}$ and $\mu(f^{-1}\mathcal{A}) = \mu(\mathcal{A})$. Such a measure μ is called f-invariant.

Theorem

Let $f : (X, \mathcal{B}) \to (X, \mathcal{B})$ and $(X, \mathcal{B}, \mu)$ be a f-invariant measure with $\mu(X) < \infty$. For $\mathcal{A} \in \mathcal{B}$ with $\mu(\mathcal{A}) > 0$ almost every $x \in \mathcal{A}$ has the property that :

$$\mu(\{x \in \mathcal{A} : \text{ there exists infinite positive integers } n \in \mathbb{Z}^+ \text{ such that } f^n x \in \mathcal{A}\}) = \mu(\mathcal{A}).$$

The set $\{f^n x \,|\, n \in \mathbb{Z}^+\}$ is called the orbit of x and $x \in \mathcal{A}$ is **recurrent** if the orbit of x intersects $\mathcal{A}$ infinitely many times.

Poincaré recurrence theorem

Thus, the Poincaré recurrence theorem states that if μ is f-invariant then almost every point of a measurable subset with positive measure is recurrent.

Recurrence plots

- Eckmann *et al.* introduced recurrence plots (RPs) as a method to visualize the recurrences of trajectories in the phase space of a dynamical system.
- An RP can be represented using a recurrence matrix $\mathbf{R}$ whose elements are given as

$$R_{i,j} = \Theta(\varepsilon - ||\mathbf{x}_i - \mathbf{x}_j||),$$

where Θ is the Heaviside step function, $||\cdot||$ is a distance norm, ε is a pre-determined distance threshold, $\mathbf{x}_i$ and $\mathbf{x}_j$ are state vectors at time instant t_i and t_j respectively.

Recurrence plots

We can write the elements of the matrix $\mathbf{R}$ as

$$R_{i,j} = \begin{cases} 1 & \text{when } \varepsilon < ||\mathbf{x}_i - \mathbf{x}_j||, \\ 0 & \text{when } \varepsilon > ||\mathbf{x}_i - \mathbf{x}_j||. \end{cases}$$

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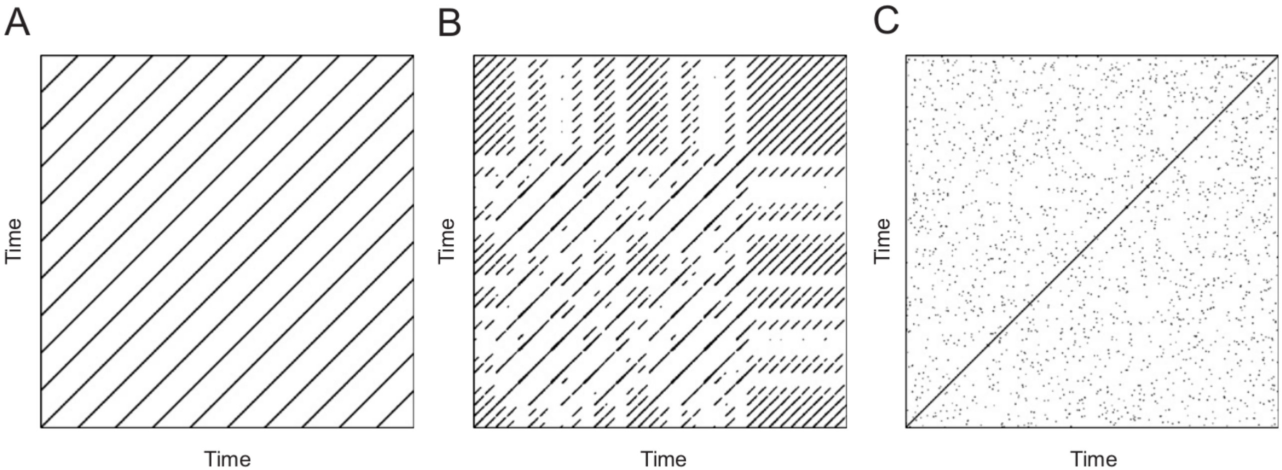
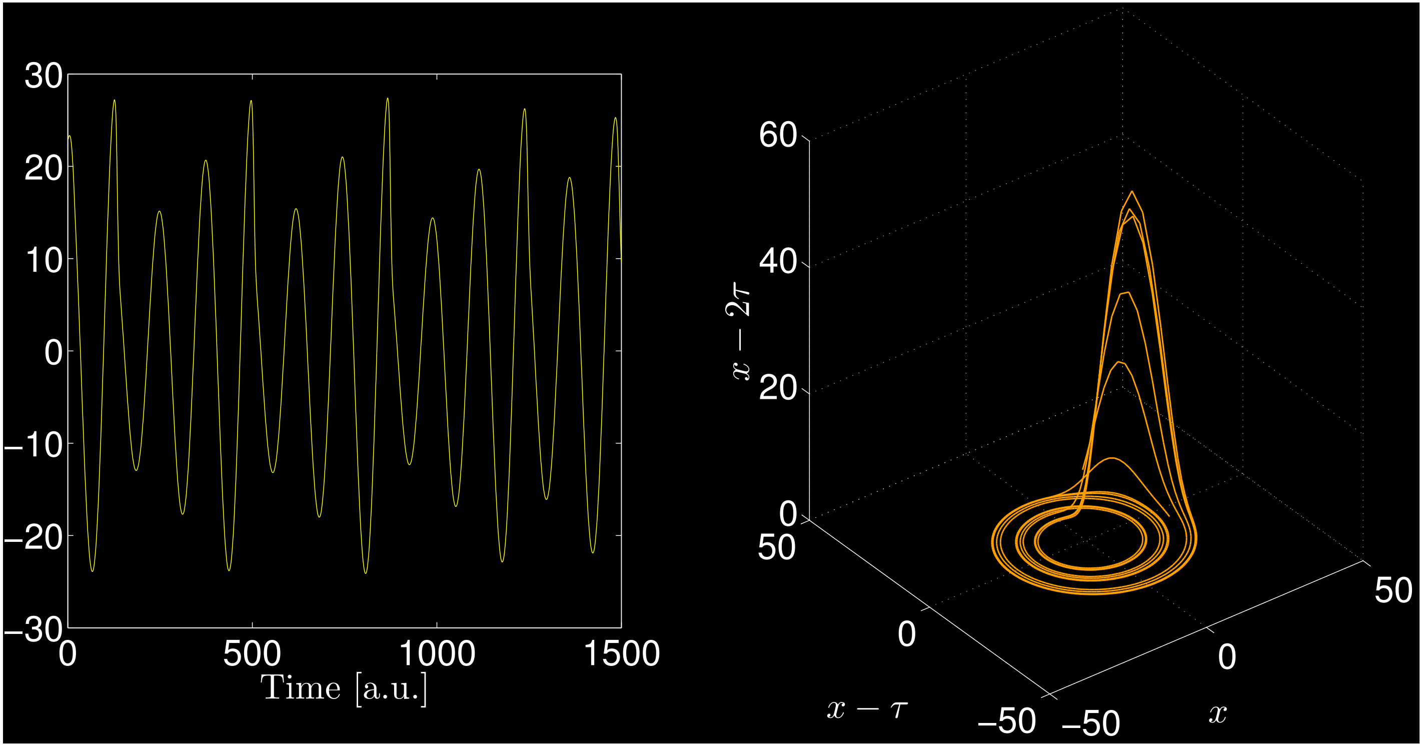
- If the trajectory in the phase space returns at time instance t_j into the ε-neighborhood of where it was at time t_i $(j > i)$, then the corresponding entry $R_{i,j}$ is 1 otherwise it is 0.
- The matrix $\mathbf{R}$ is a symmetric, binary matrix.
- The patterns in the RP can give information about the temporal evolution of the trajectories.
- For e.g. A periodic dynamics will contain long and non-interrupted diagonal lines.

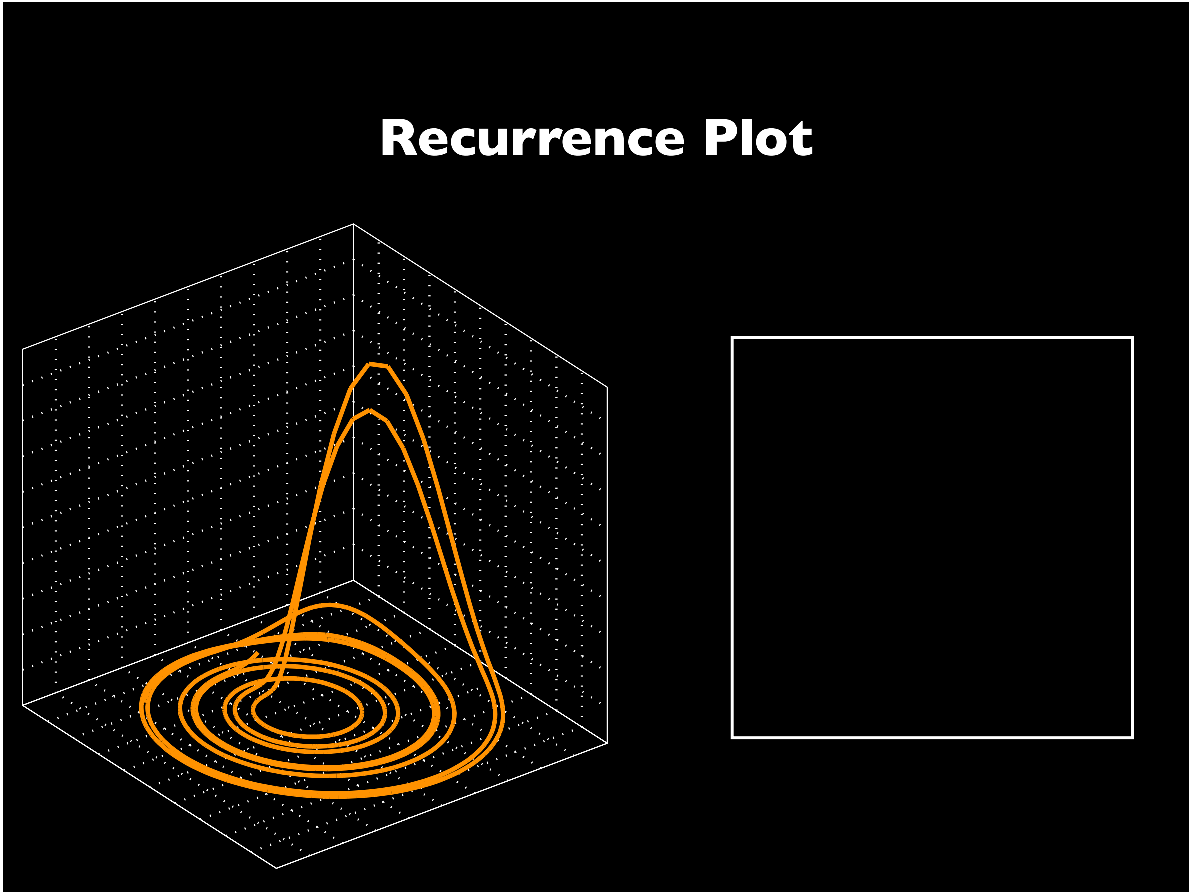
Example of a recurrence plots : Periodic, chaotic and stochastic dynamics



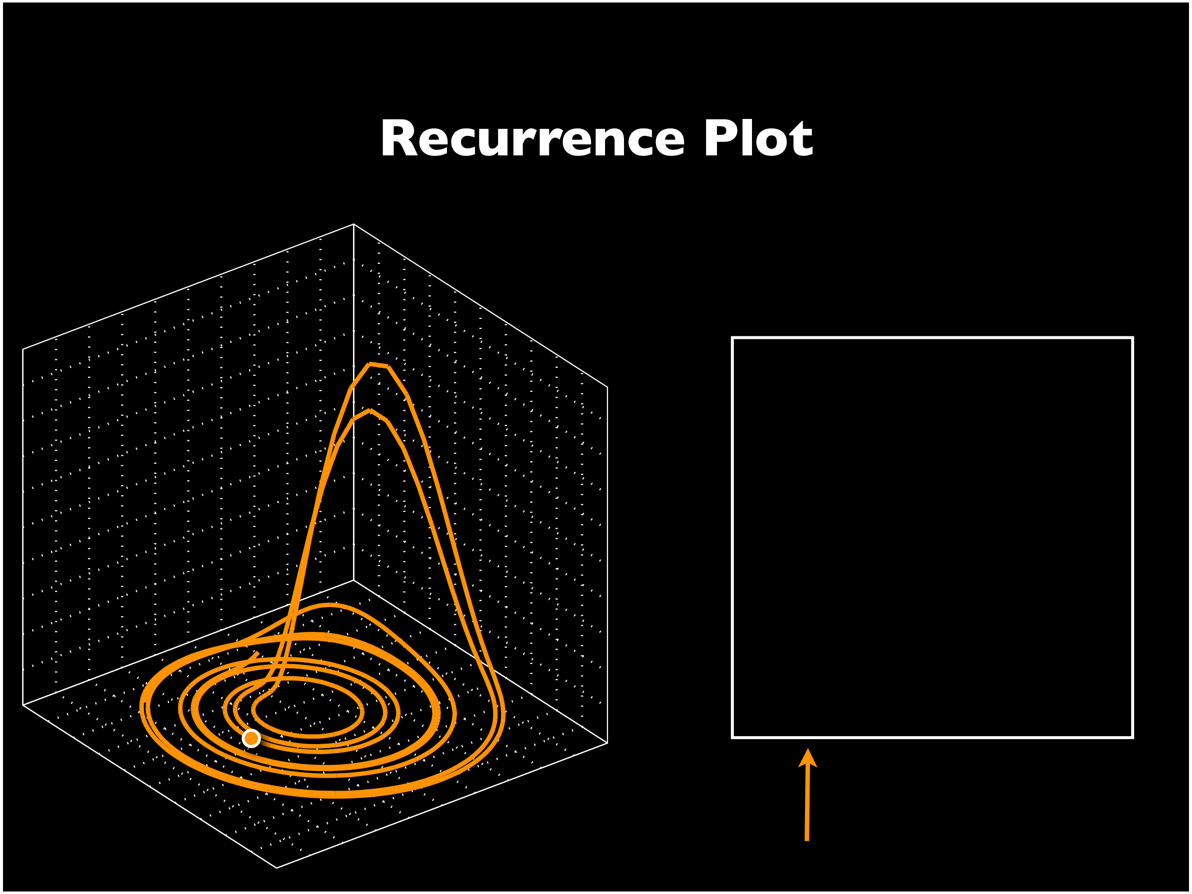
Time series to Phase Space



Recurrence plot



Recurrence plot



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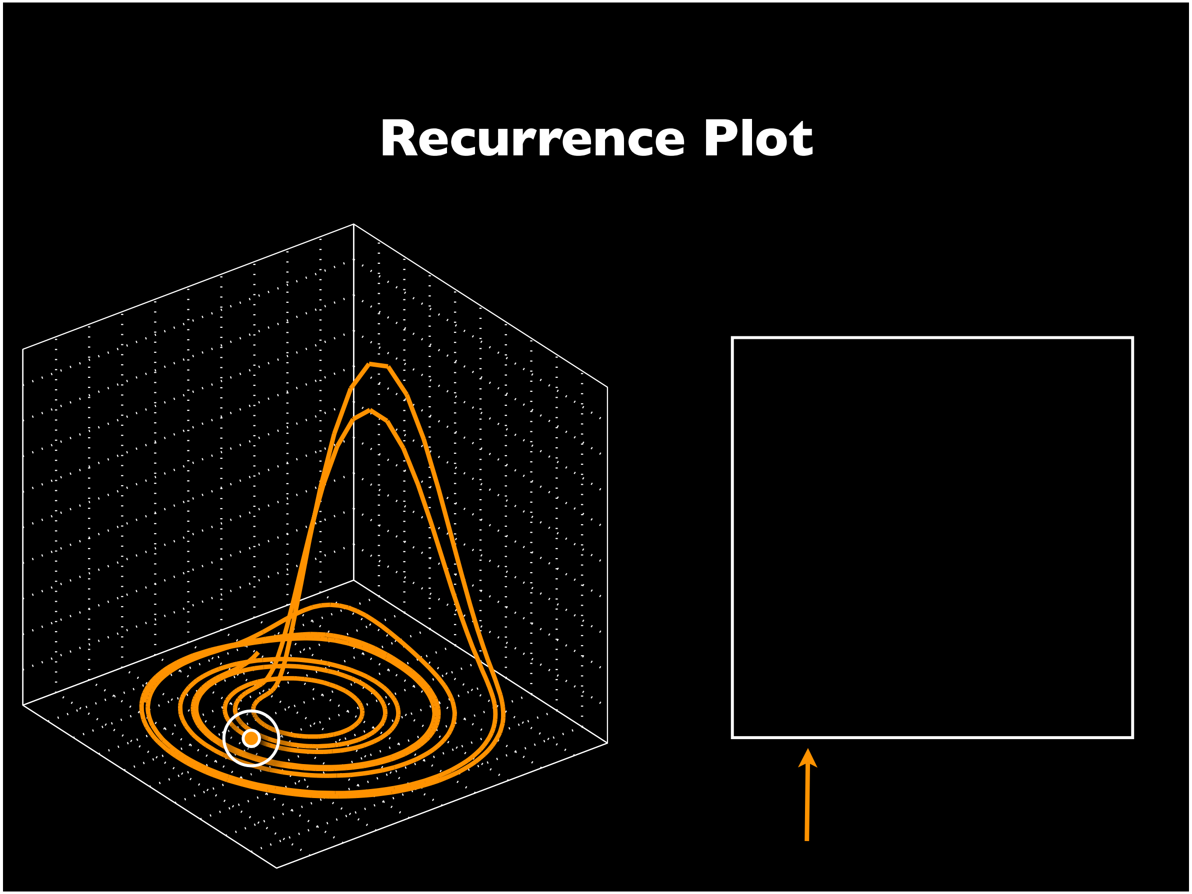

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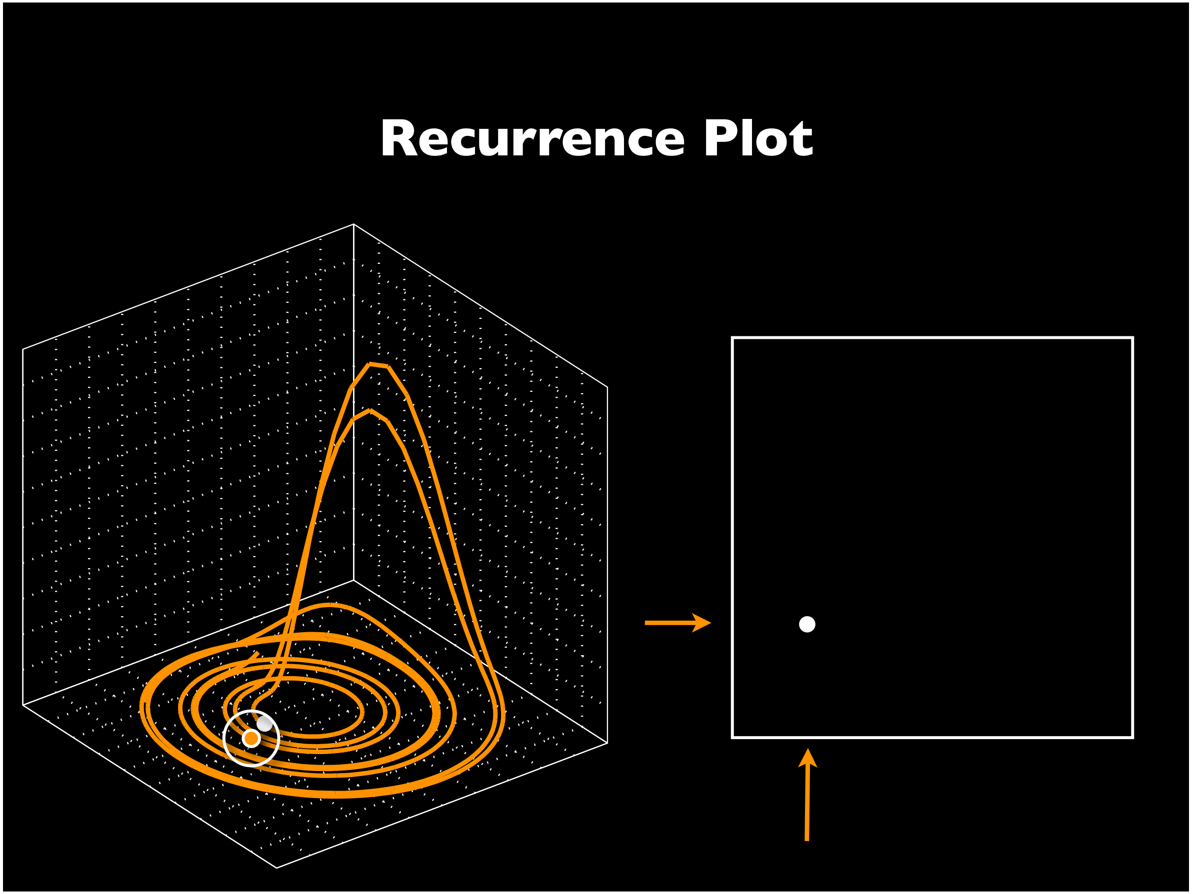

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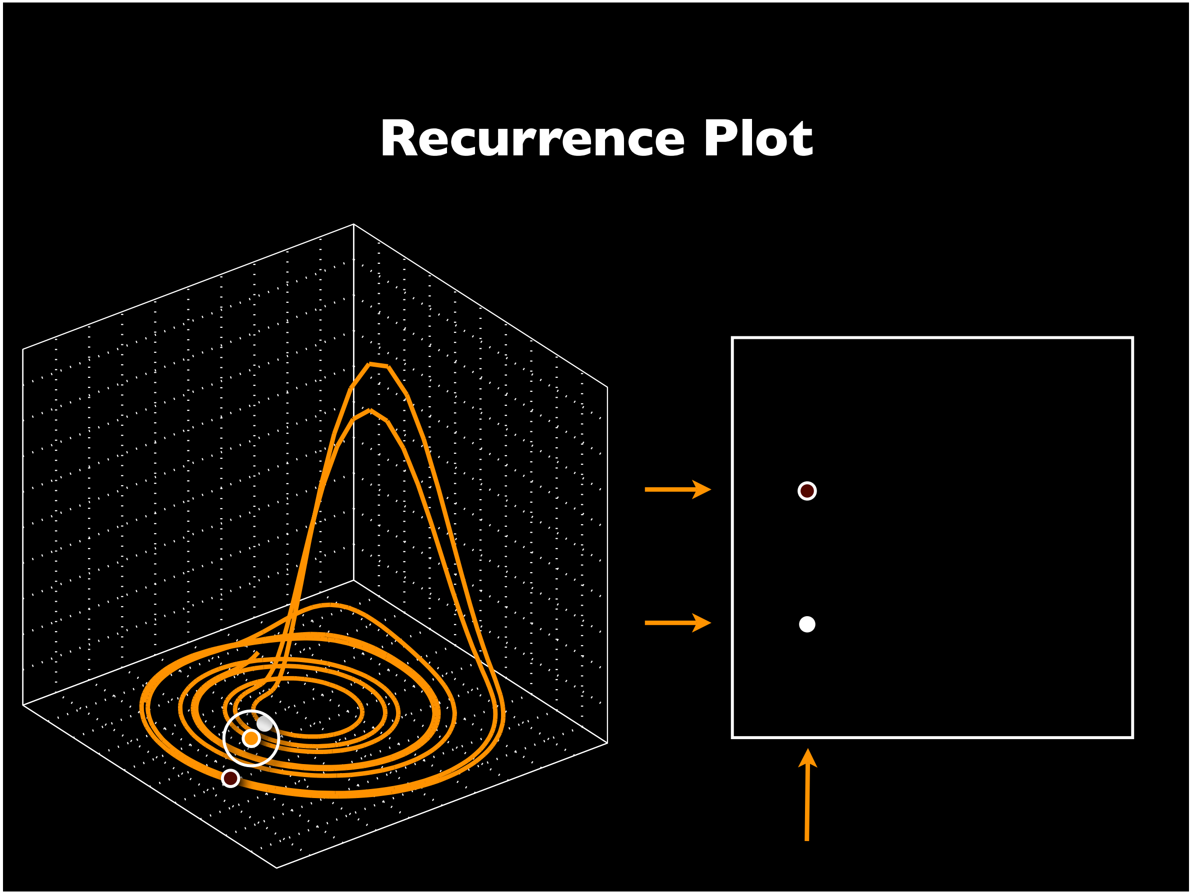

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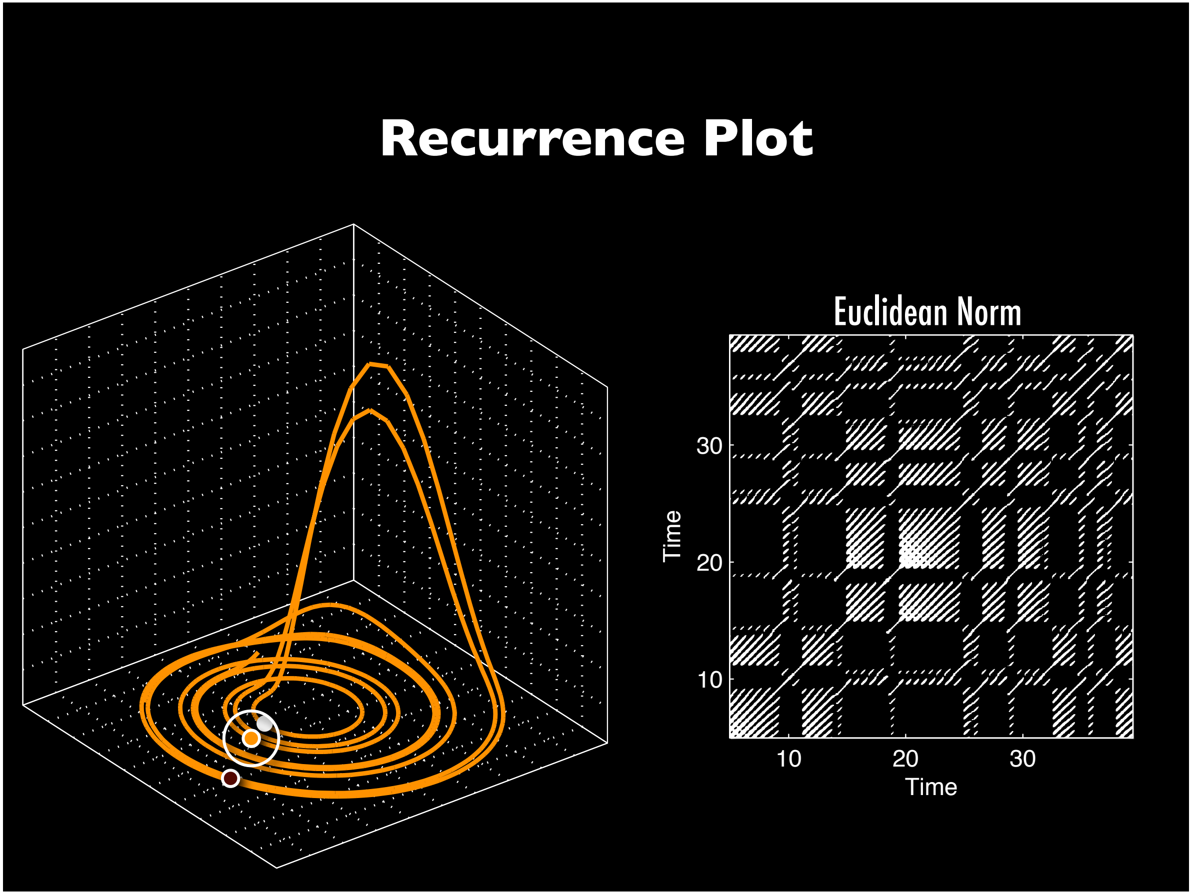

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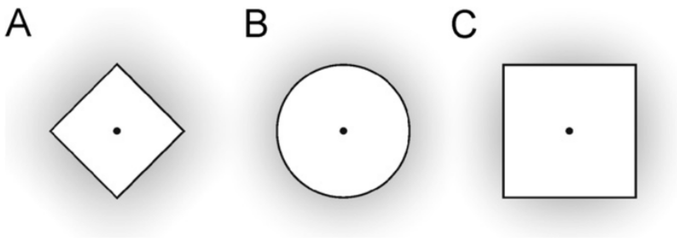
- A large ε can causes thick and longer structures in the RP.
- **Tangential motion** : A large ε includes also points into the neighbourhood which are simple consecutive points on the trajectory.
- If ε is chosen to be too small, true recurrence points might get excluded.
- An RP with very small ε might not reflect the true recurrence structure.

How to choose ε ?

- The amount of noise present also influences what ε should be set to.
- Noise would distort any existing structure in the RP and choosing higher ε, structure may be preserved.
- Rule of thumb : Some $\%$ of the maximum phase space diameter (typically not more than 5 to 10 $\%$).
- If SNR is known, then $\varepsilon > 5\sigma_{noise}$

Choice of distance norm.



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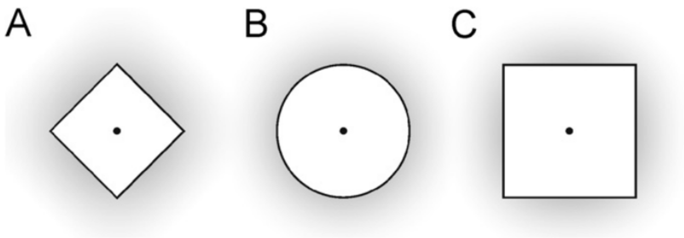
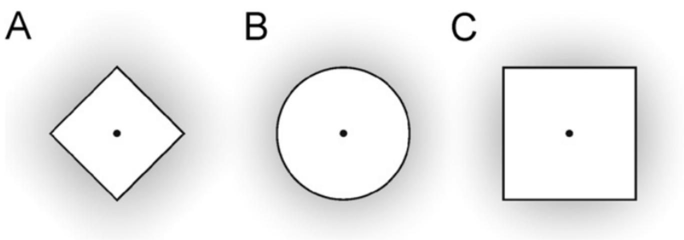


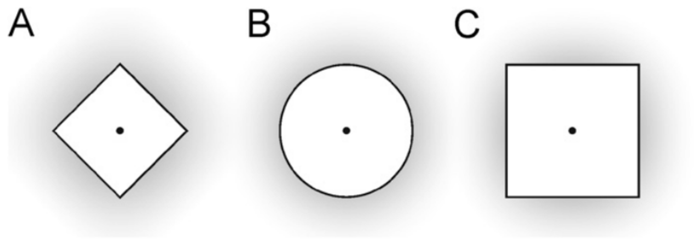
- Which distance norm to choose ?

Choice of distance norm.



- Which distance norm to choose ?
- Manhattan (A), Euclidean (B) or Maximumn (C) norm ?

Choice of distance norm.



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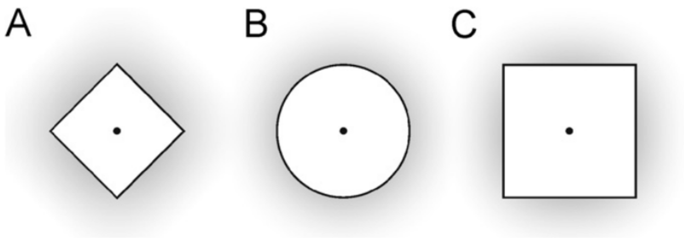
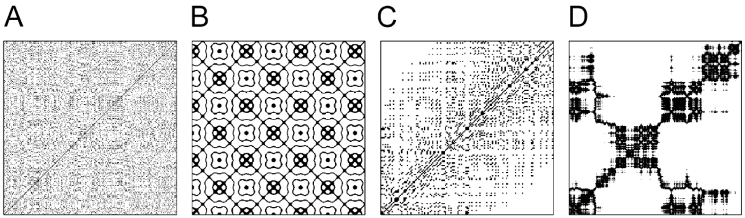


- Which distance norm to choose ?
- Manhattan (A), Euclidean (B) or Maximumn (C) norm ?
- Maximum norm has one obvious advantage : the distance computed is independent of the embedding dimension.
- This allows for comparing RPs obtained with different embedding dimensions.

Structures in RPs

The typology of RPs can be classified into (A) Homogeneous, (B) Periodic, (C) Drift and (D) Disrupted.



- Homogeneous RPs are exhibited by stationary and autonomous system (for e.g., stochastic systems).
- Periodic structures in RPs can be seen in oscillating systems (for e.g. harmonic oscillations)
- Drift structures are exhibited by system with slowly-varying parameter.
- Disrupted structures are caused by systems undergoing extreme changes or events.

Recurrence quantification analysis

- Recurrence quantification analysis (RQA) , Zbilut and Webber, is an objective approach to quantify the structure of a RP.
- Several RQA measures can be computed to interpret the dynamics of the system from which time series was generated.
- A convenient and easy way to measure the complexity of a dynamical system.

Recurrence rate

- **Recurrence rate** (RR) is a measure of the density of recurrences in an RP.
- It is closely related to the definition of correlation sum and is given as

$$RR = \frac{1}{N^2 - N} \sum_{i \neq 1, j \neq 1}^{N} R_{i,j}.$$

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- The RR is simply the sum of **ones** in a RP.

When number of state vectors (N) is large, RR is the probability that a state vector returns to its ε-neighborhood.

Percent determinism

- **Percent determinism** (DET) can be defined as the ratio of recurrence points that form the diagonal lines of at least length l_{min} to all the recurrence points in an RP.
- It is given as

$$DET = \frac{\sum_{l=l_{min}}^{N} l P_D(l)}{\sum_{i,j=1}^{N} R_{i,j}},$$

where $P_D(l)$ is the histogram of the lengths of the diagonal structures in an RP and is given as,

$$P_D(l) = \sum_{i,j=1}^{N} (1 - R_{i-1,j-1})(1 - R_{i+l,j+l}) \prod_{k=0}^{l-1} R_{i+k,j+k}.$$

Percent determinism

- RP of periodic dynamics contain long diagonals and that of chaotic dynamics contain many short lines.
- In case of stochastic signals, diagonal lines are absent!
- DET can be interpreted as a measure of predictability of a system.
- What happens if $l_{min} = 1$?

Percent determinism

- RP of periodic dynamics contain long diagonals and that of chaotic dynamics contain many short lines.
- In case of stochastic signals, diagonal lines are absent!
- DET can be interpreted as a measure of predictability of a system.
- What happens if $l_{min} = 1$?
- We get the definition of recurrence rate!

Maximum diagonal length

- **Maximum diagonal line length** (L_{max}) can be defined as

$$L_{max} = max\left(\{l_i; i = 1, 2, \cdots, N_l\}\right) = \arg\max_l P_D(l),$$

where N_l is the total number of diagonal lines.

- A related measure known as **Divergence** (DIV) is given as the inverse of L_{max}. These two measures are related to the exponential divergence of the trajectories
- The faster the trajectories diverge, shorter will be the diagonal lengths leading to a lower value of L_{max} (or conversely a higher value for DIV).
- Related to (largest) Lyapunov exponent!

Average diagonal length

Average diagonal line length (L_{mean}) can be defined as the average time for which two segments of the trajectories are in proximity of each other. This measure is related to the mean prediction time and it is given as

$$L_{mean} = \frac{\sum_{l=l_{min}}^{N} l P_D(l)}{\sum_{l=l_{min}}^{N} P_D(l)}.$$

Entropy

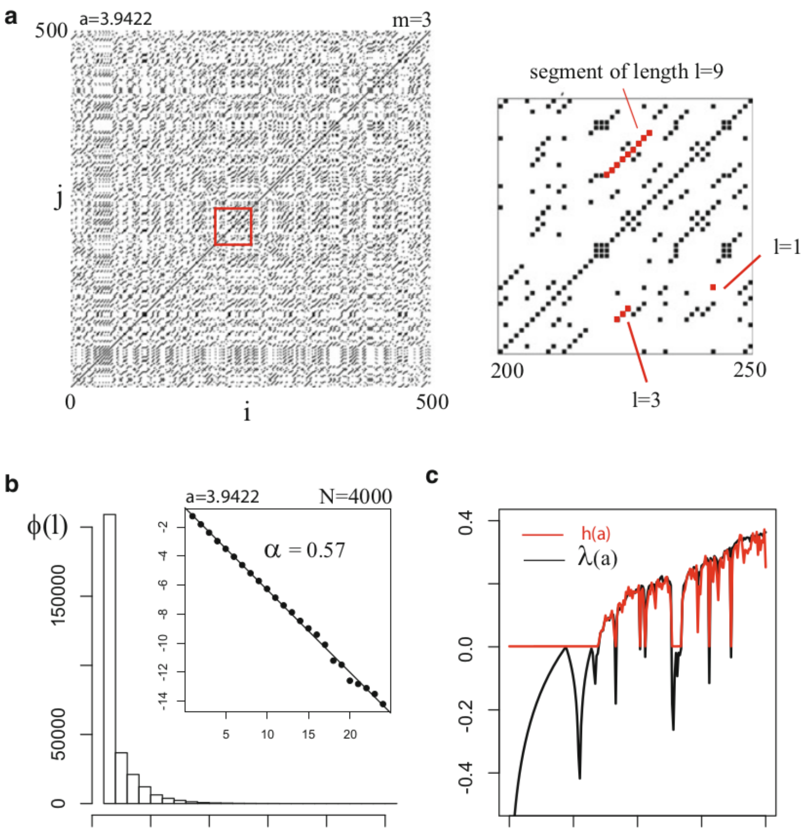
We can also compute the Shannon entropy of the frequency distribution of the diagonal line,

$$ENT = -\sum_{l=l_{min}}^{N} p(l)\log(p(l)),$$

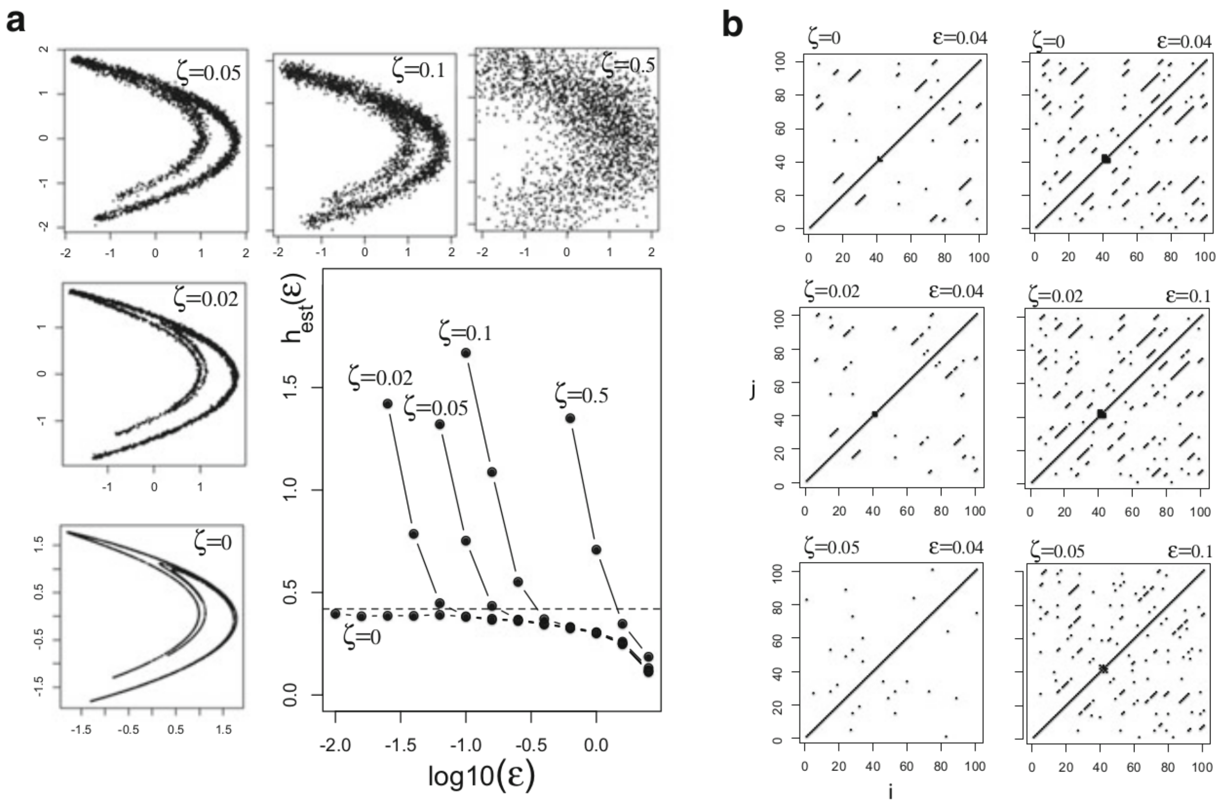
where $p(l) = P_D(l/\sum_{l=l_{min}}^{N} P_D(l))$.

- The parameter, ENT, can be used to characterize the complexity of a deterministic structure in an RP.
- For a stochastic or uncorrelated dynamics, ENT is rather small.
- High values of ENT reflect increased complexity.
- Sensitive to parameter choices such as number of bins, l_{min} and ε.

Example of computing ENT from RP for Logistic map [Faure and Lense 2015]



Effect of noise on computing ENT from RP for Henon map [Faure and Lense 2015]



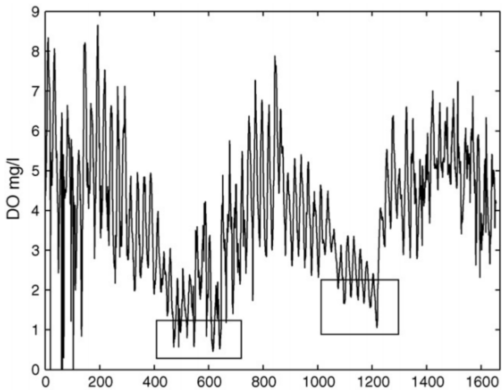
Theiler window

- Useful to exclude trivial recurrences such as line of identity (LOI) in RP, which refers to the recurrence of a state with itself.
- Also long diagonal lines can occur directly below and above the LOI for smooth or high resolution data (highly sampled data) due to finite ε.
- Theiler correction refers to exclusion of the diagonal lines in a small corridor around the LOI.
- Thus consider phase space points satisfying the constraint $|x_j - x_i| \geq w$.
- w can be set using autocorrelation value or $(k-1)\tau$ (Gao et al. 1994).

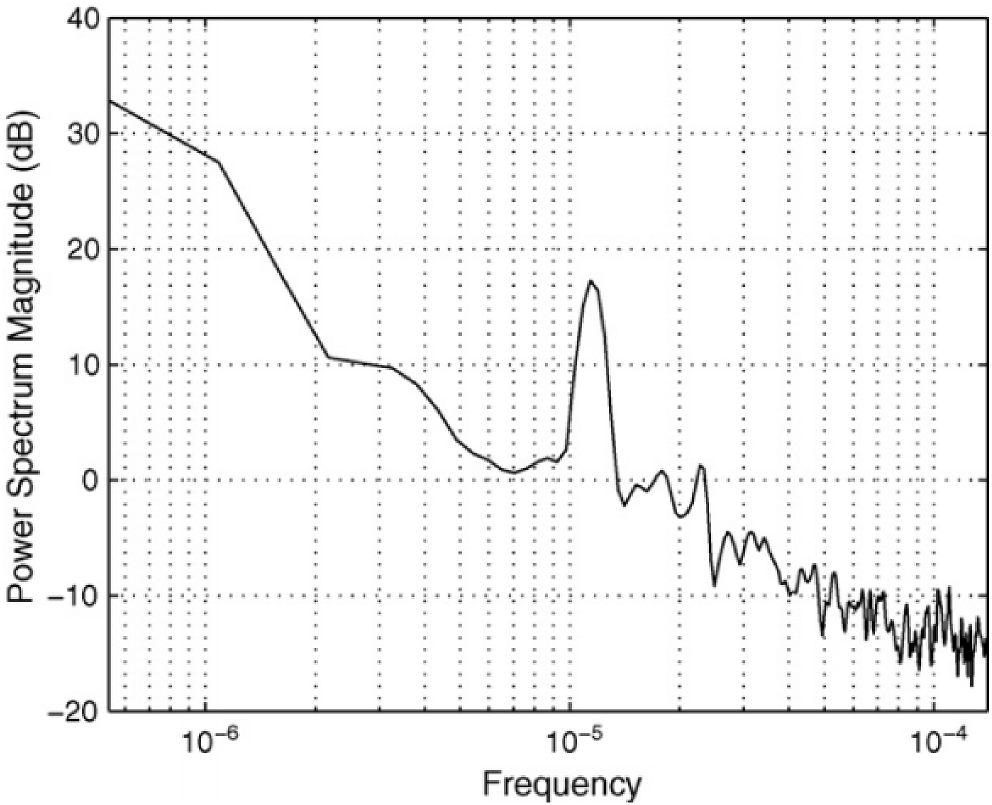
Applications [Facchini et al. 2006]

Detecting transition in the oscillations of dissolved oxygen just before the onset of an anoxic crisis.

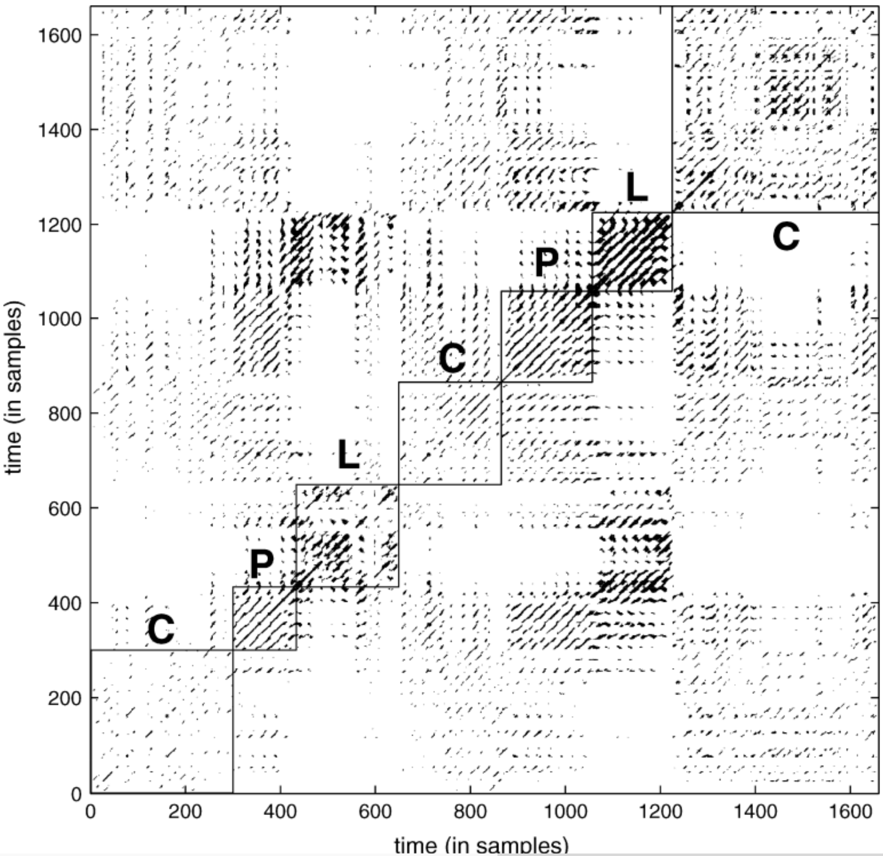
- Nonlinear time series analysis of data representing dissolved oxygen (DO) collected in the Lagoon of Orbetello (Grosseto, Italy) using RPs.
- DO is a highly informative variable which represents reliably important features of the ecosystem.
- DO oscillations can give insight into the functioning of aquatic ecosystems and help prevent anoxic crises.



PSD of DO oscillations



RP of DO oscillations



RQA of DO oscillations