MATH.APP.790 : Topics in Mathematics, Nonlinear time series analysis

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Lecture overview

- Diffeomorphism
- Estimation of embedding lag
- Estimation of embedding dimension
- References

1. F. Takens, Detecting Strange Attractors in Turbulence — Dynamical Systems and Turbulence, Lecture Notes in Mathematics 366, Springer (1981).
2. T. Sauer et al., J. Stat. Phys. 65, pp 579 (1991).
3. K. Judd and A. Mees, Physica D 120, pp 273 (1998)
4. Kantz, Holger, and Thomas Schreiber. Nonlinear time series analysis. Vol. 7. Cambridge university press, 2004.

Taken's embedding theorem

Theorem

Consider a compact, m-dimensional manifold M, with $\phi : M \to M$, a smooth diffeomorphism (at least in the sense of C^2). Consider a smooth (again in the sense of C^2) observation function, $f : M \to \mathbb{R}$. It is a generic property that,

$$Rec_k : M \to \mathbb{R}^{2m+1},$$

is an embedding and it is given as

$$Rec_k(x) = (f(x), f(\phi^1(x)), f(\phi^2 x)), \cdots, f(\phi^{2m}(x))$$

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The components of Rec_k correspond to the time lagged observations of the dynamics on M (or $\mathcal{A}$), as defined by the smooth observation function f.

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- To perform the embedding in practice, we need to know the embedding lag (τ) and dimension (k).
- In theory, the theorem will work for any arbitrary value of τ and the embedding theorem assumes $\tau = 1$.
- The embedding dimension is $k > 2m$, but we do not know what is m in practice!

In general, any value of $\tau \in \mathbb{R}_+$ can be chosen. The reconstructed embedding vector, for an arbitrary τ can be given as,

$$x_n \in M \Longleftrightarrow (y_n, y_{n+\tau}, y_{n+2\tau}, \cdots y_{n+(k-1)\tau}).$$

Taken's embedding theorem in practice : Embedding parameters

For the right τ and k, the reconstructed attractor is diffeomorphic to (have the same topology as) the original attractor.

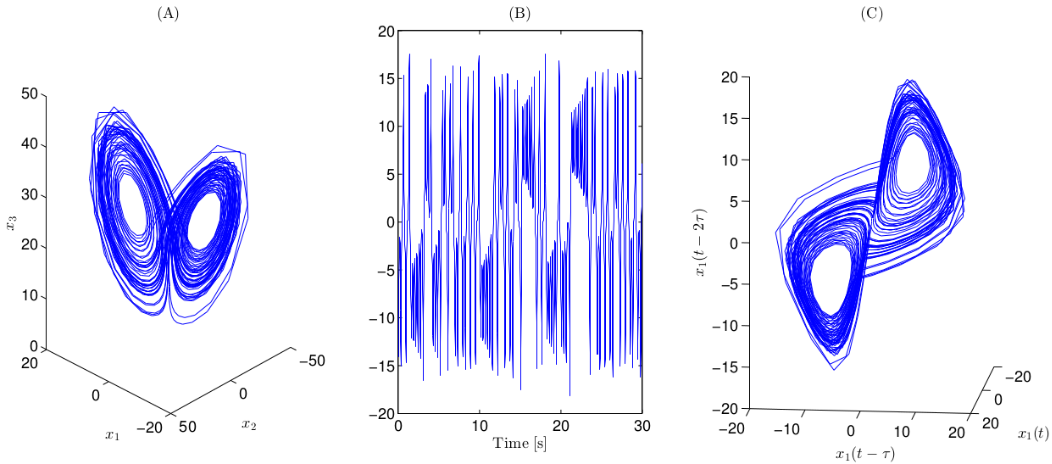
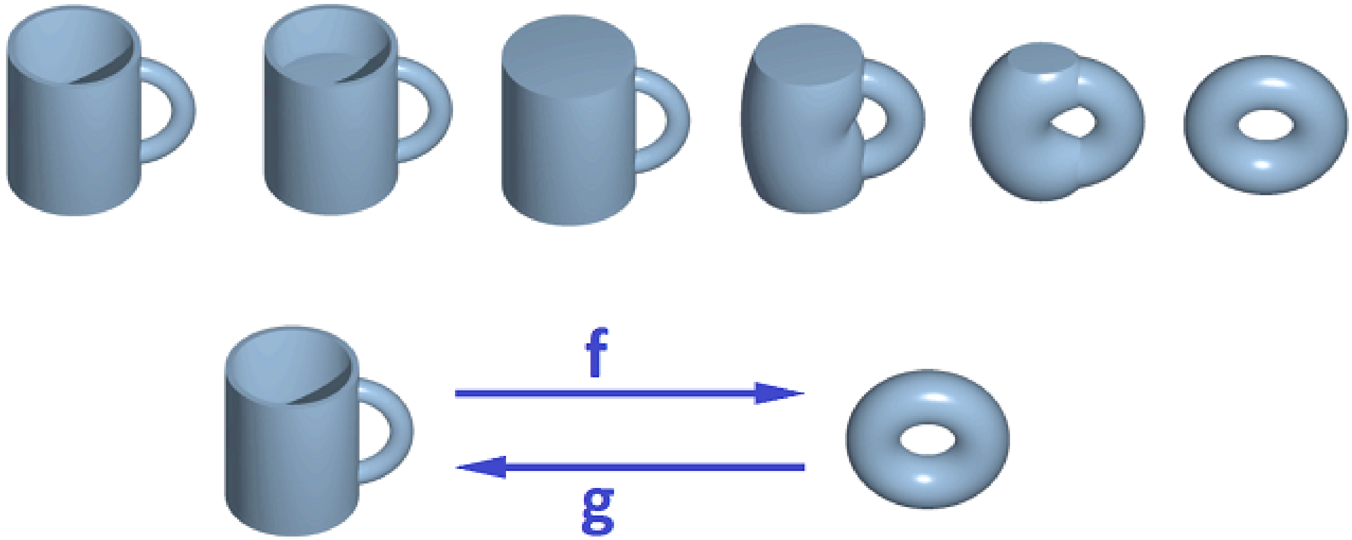


Figure: (A) The original Lorenz attractor. (B) The x_1-component of the Lorenz attractor, which is observed. (C) The Lorenz attractor in the reconstructed space after applying embed- ding theorem using only the x_1-component.

Topology and diffeomorphism



Topology and diffeomorphism

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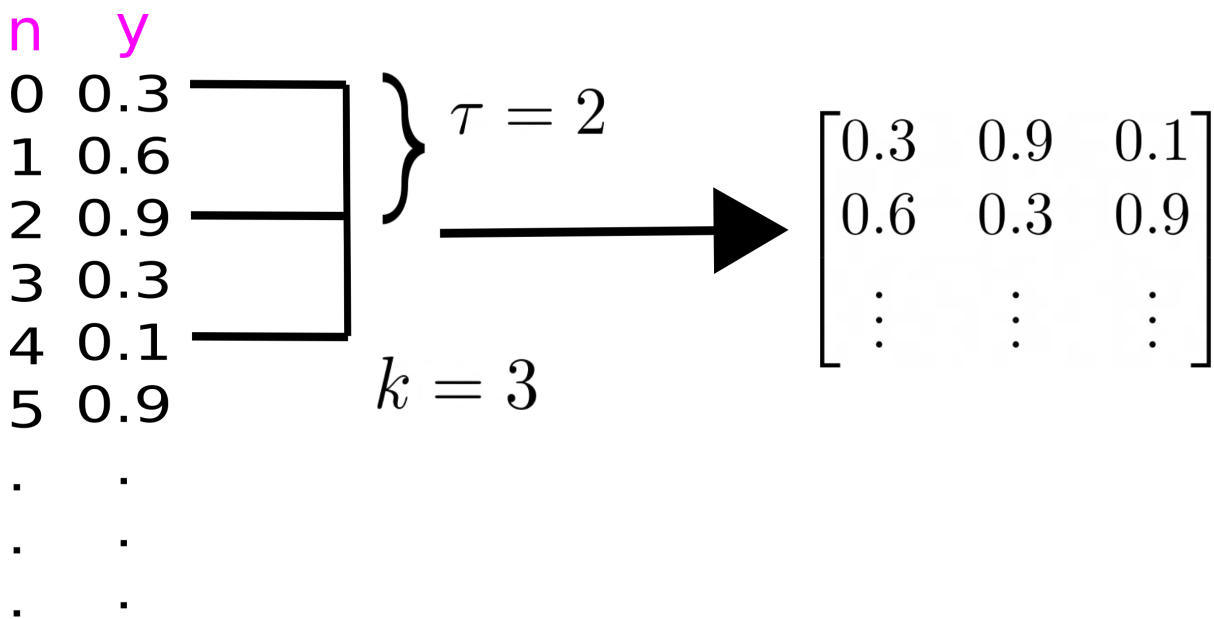
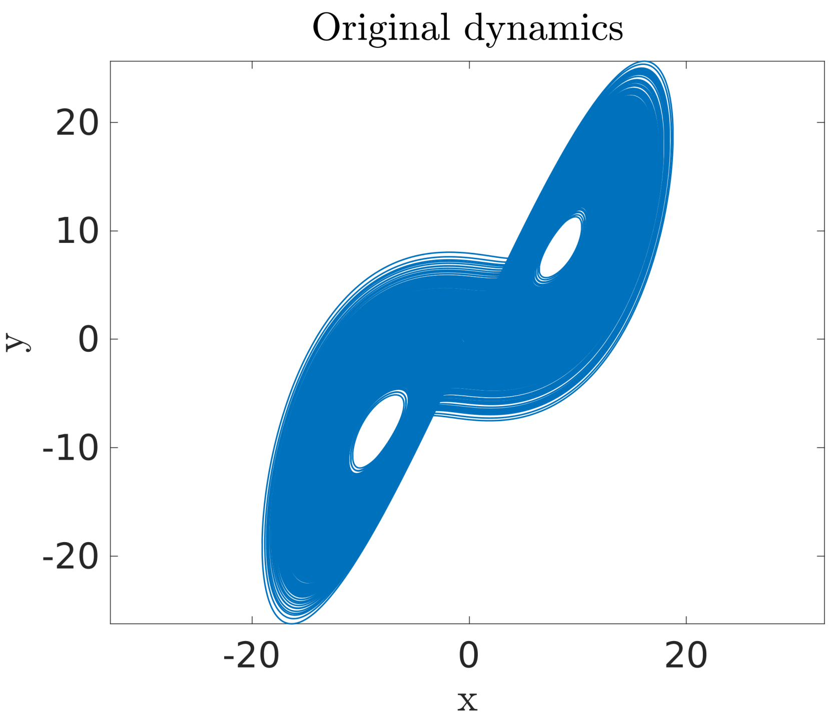
- A correct embedding is related to the original dynamics by diffeomorphism (provided conditions of the theorem are met).
- The reconstructed attractor and the original attractor have qualitatively the same shape.
- Many important properties of dynamical systems are invariant under such transformations (dynamical invariants). For e.g., the Lyapunov exponent.

Estimation of the embedding lag

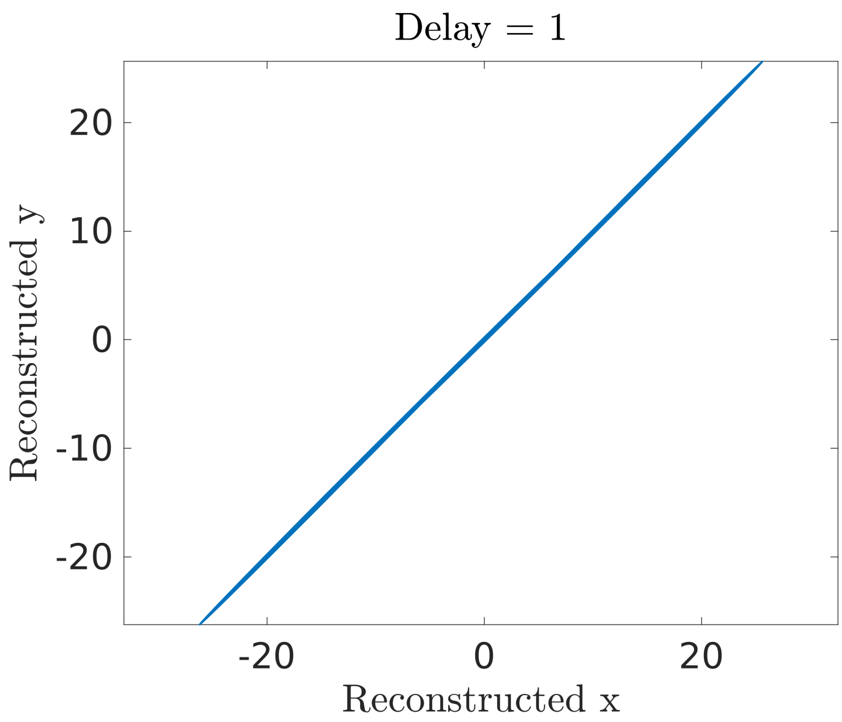


What is the effect of increasing τ, assuming k is fixed ?

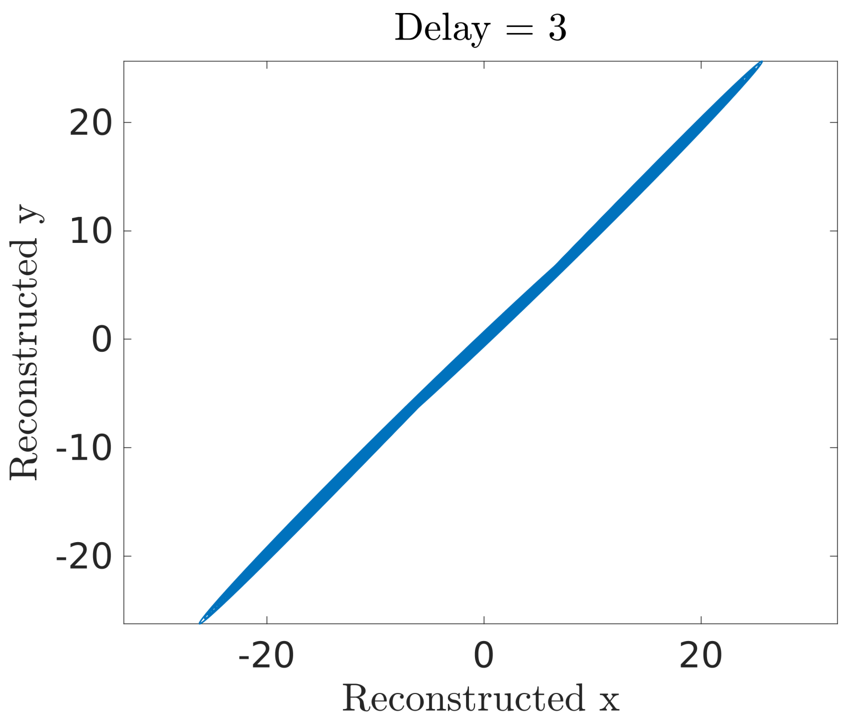
Original dynamics



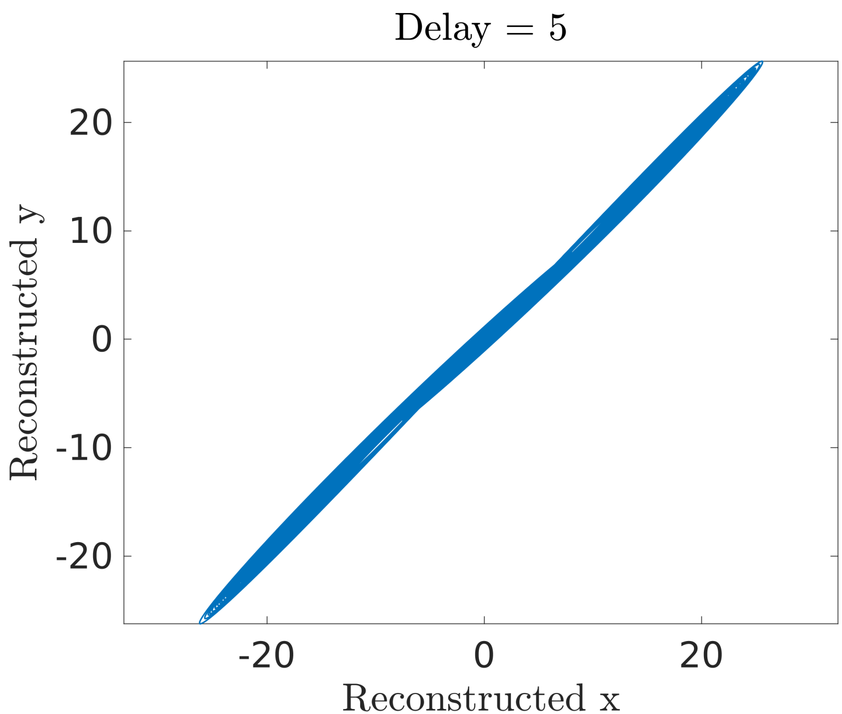
$\tau = 1$



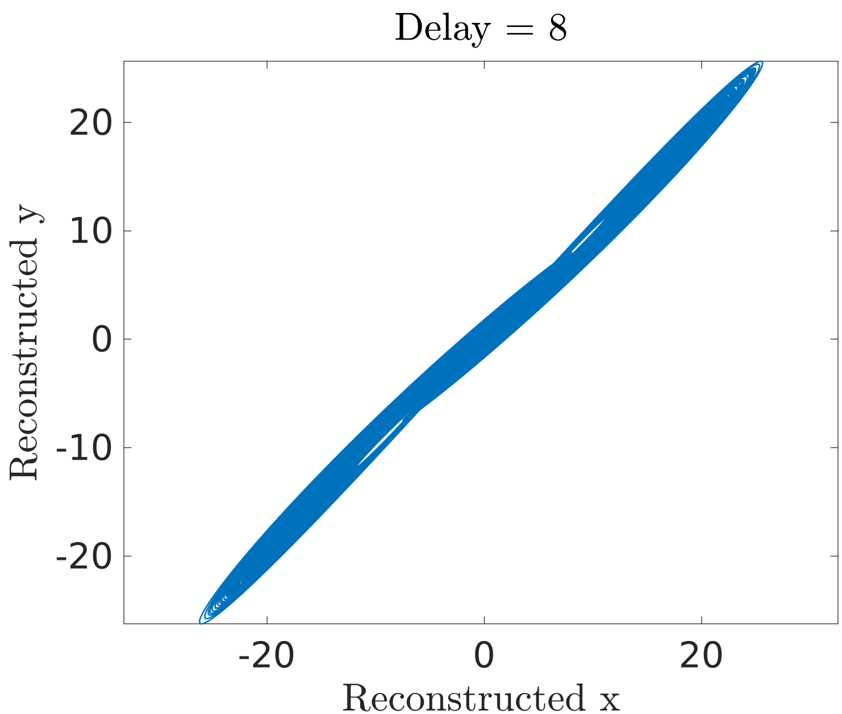
$\tau = 3$



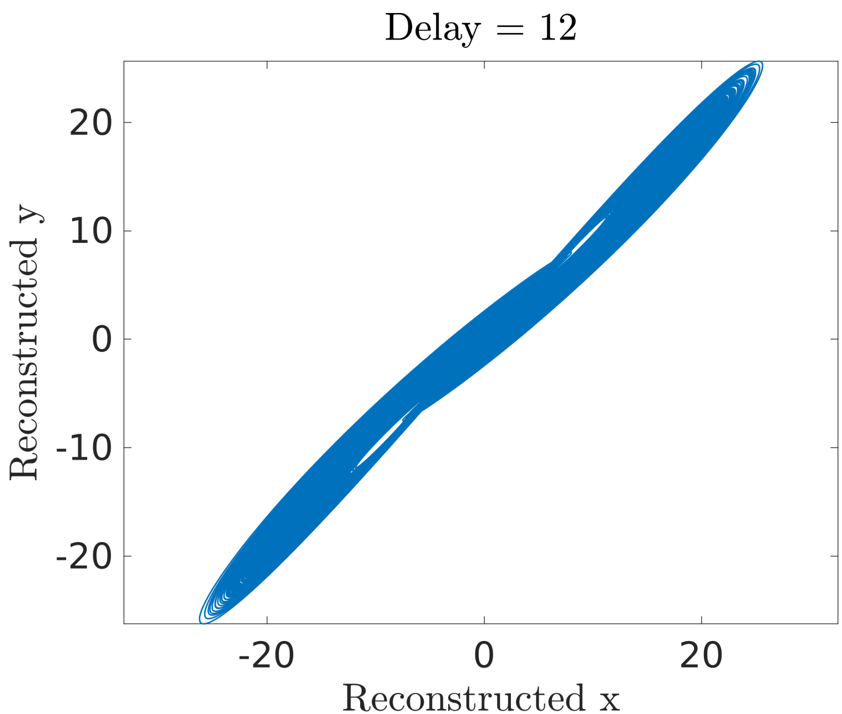
$\tau = 5$



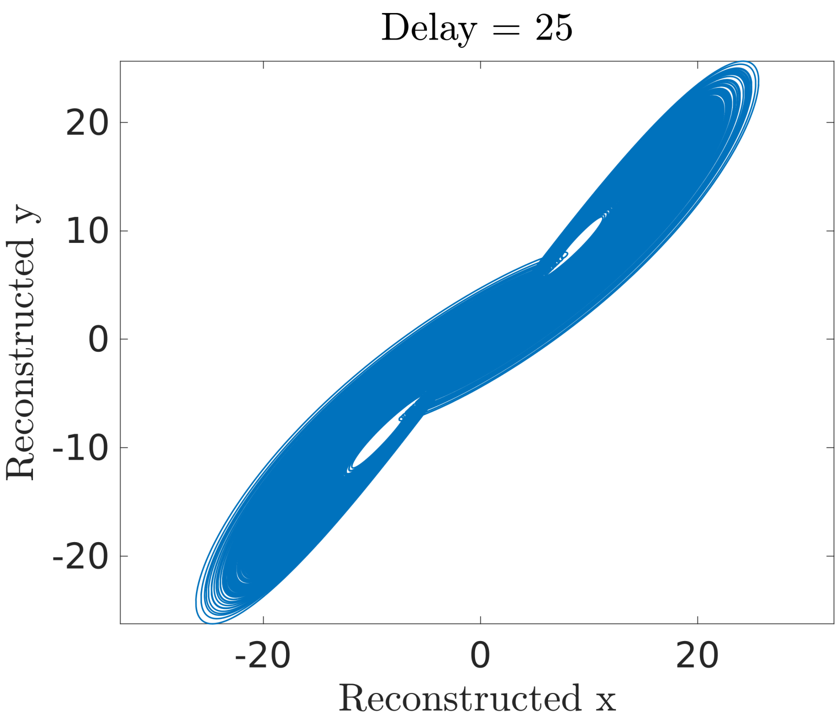
$\tau = 8$



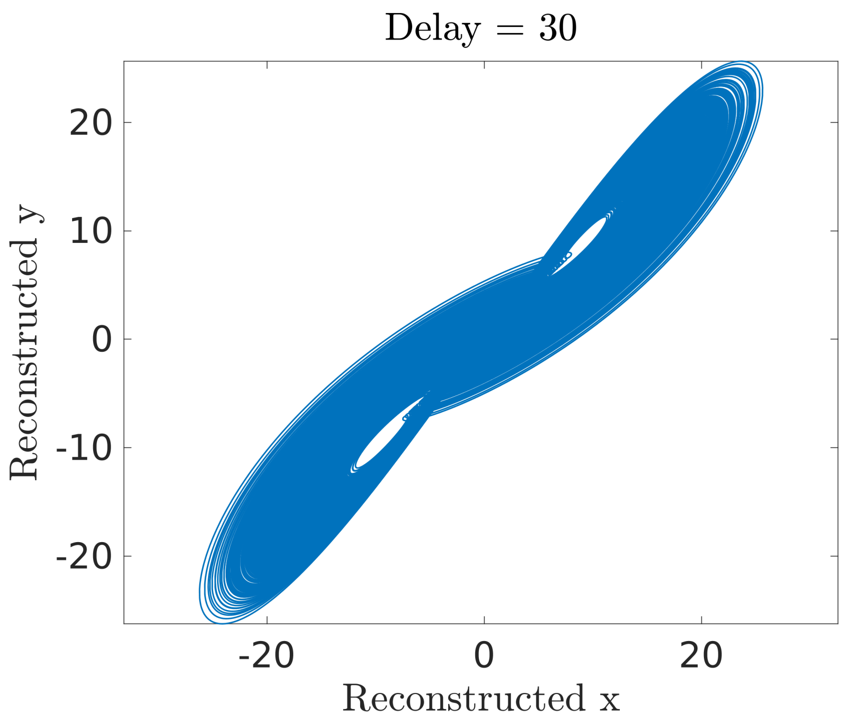
$\tau = 12$



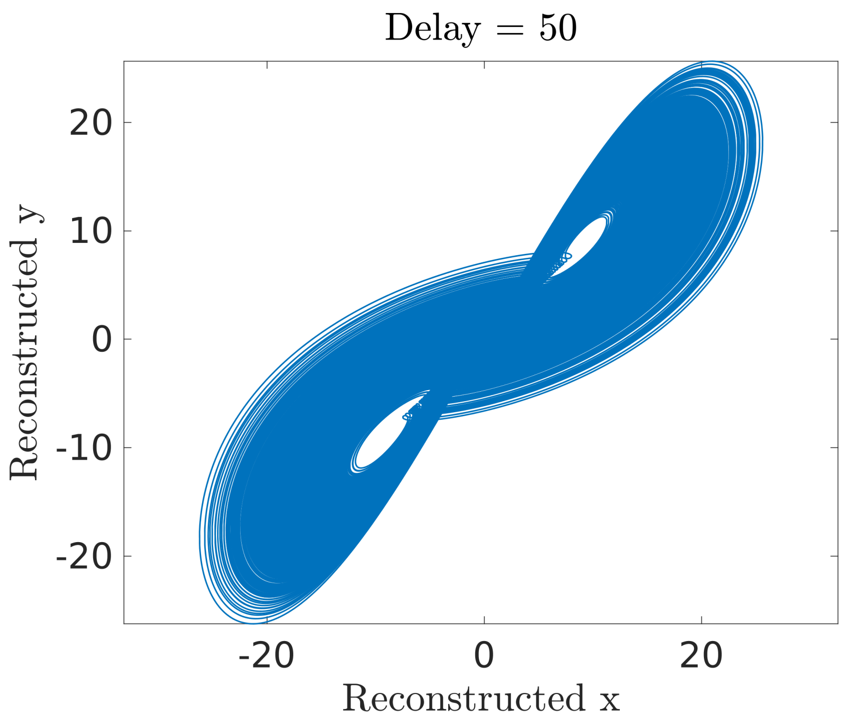
$\tau = 25$



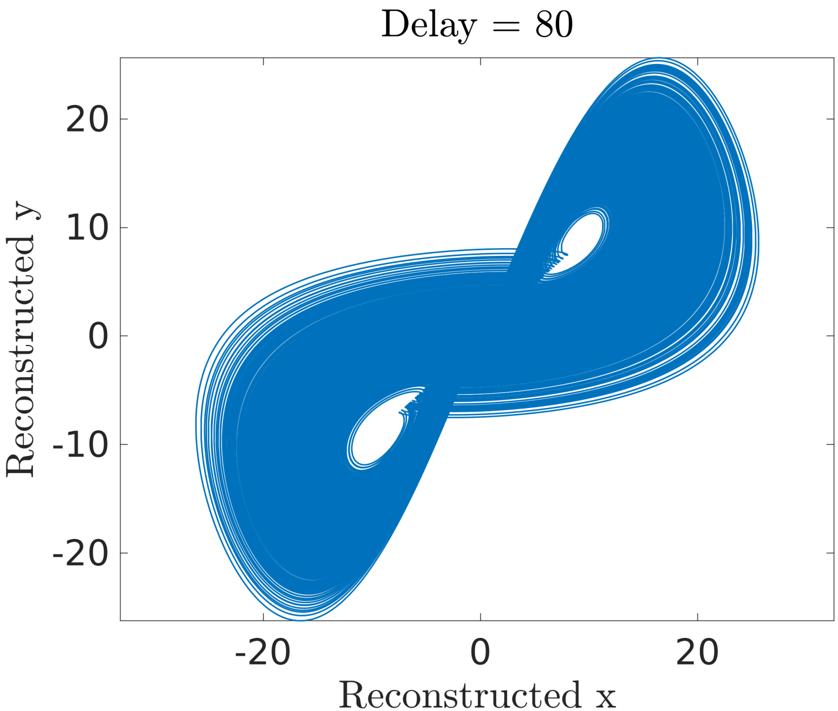
$\tau = 30$



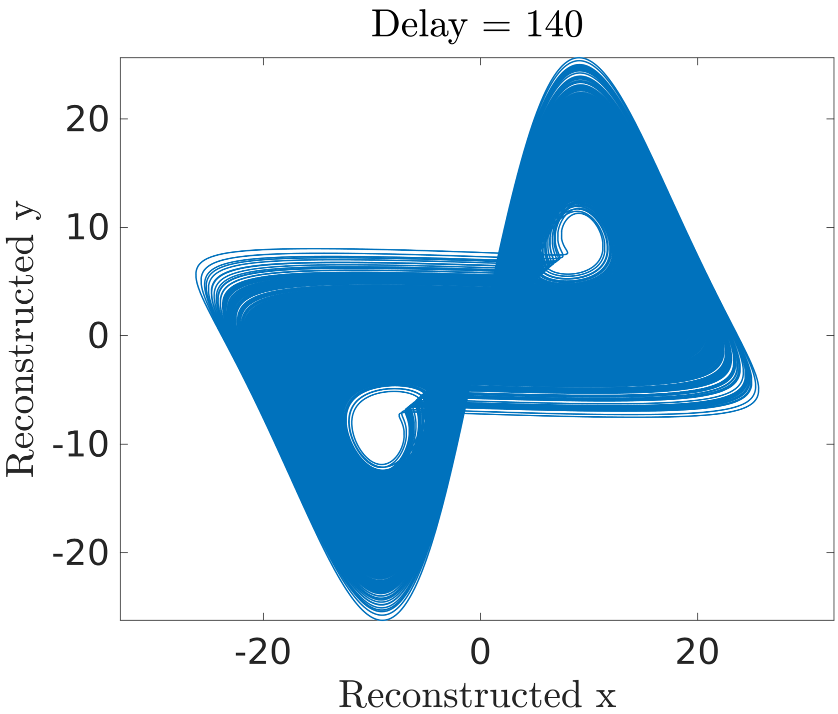
$\tau = 50$



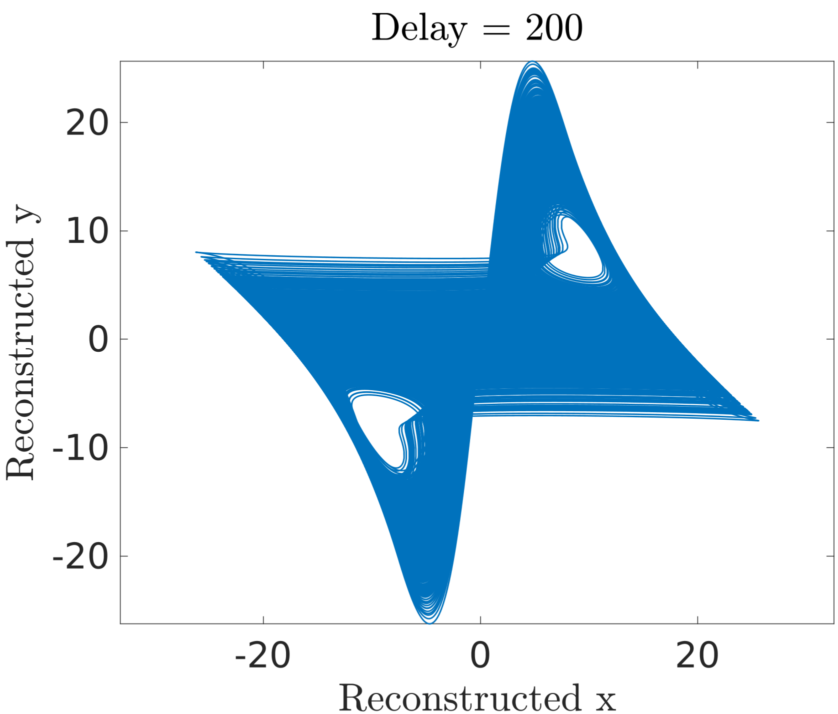
$\tau = 80$



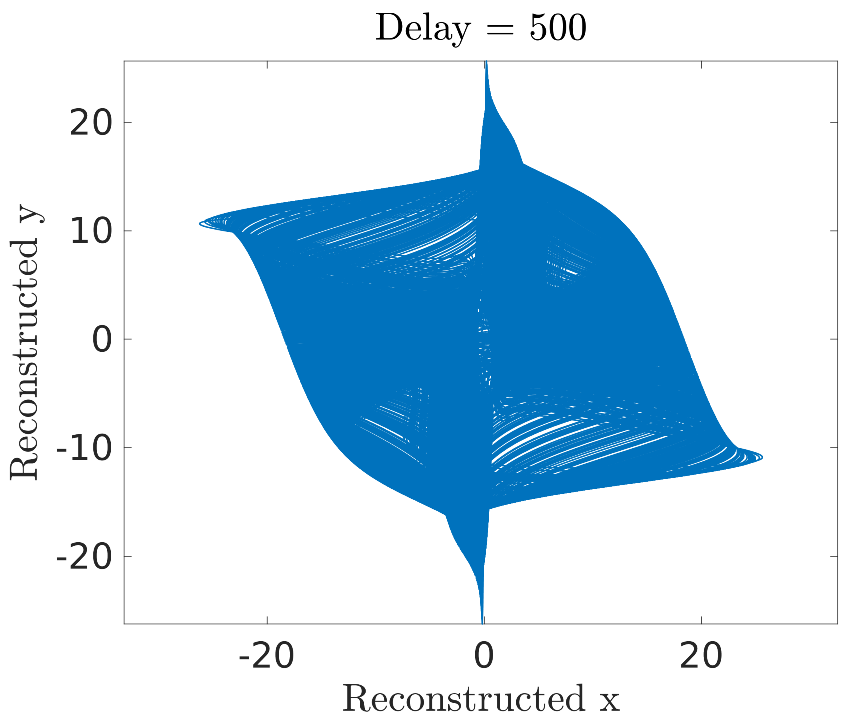
$\tau = 140$


Delay = 140
Reconstructed y
Reconstructed x
20
10
0
-10
-20
-20
0
20

$\tau = 200$


Delay = 200
20
10
0
-10
-20
-20
0
20
Reconstructed y
Reconstructed x

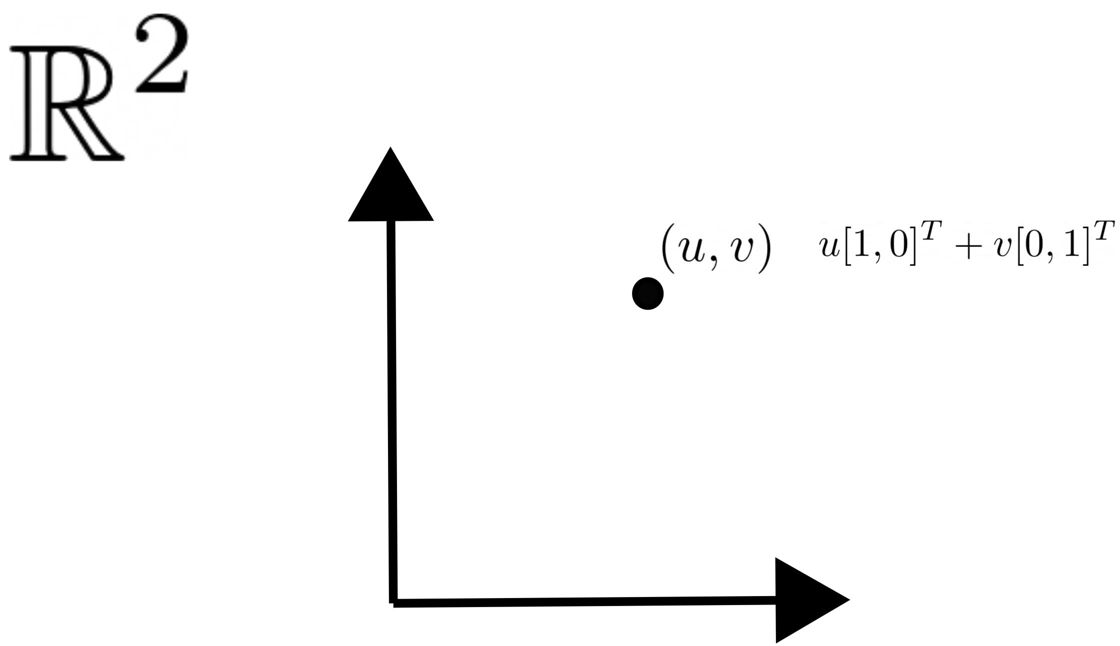
$\tau = 500$


Delay = 500
Reconstructed y
Reconstructed x
20
10
0
-10
-20
-20
0
20

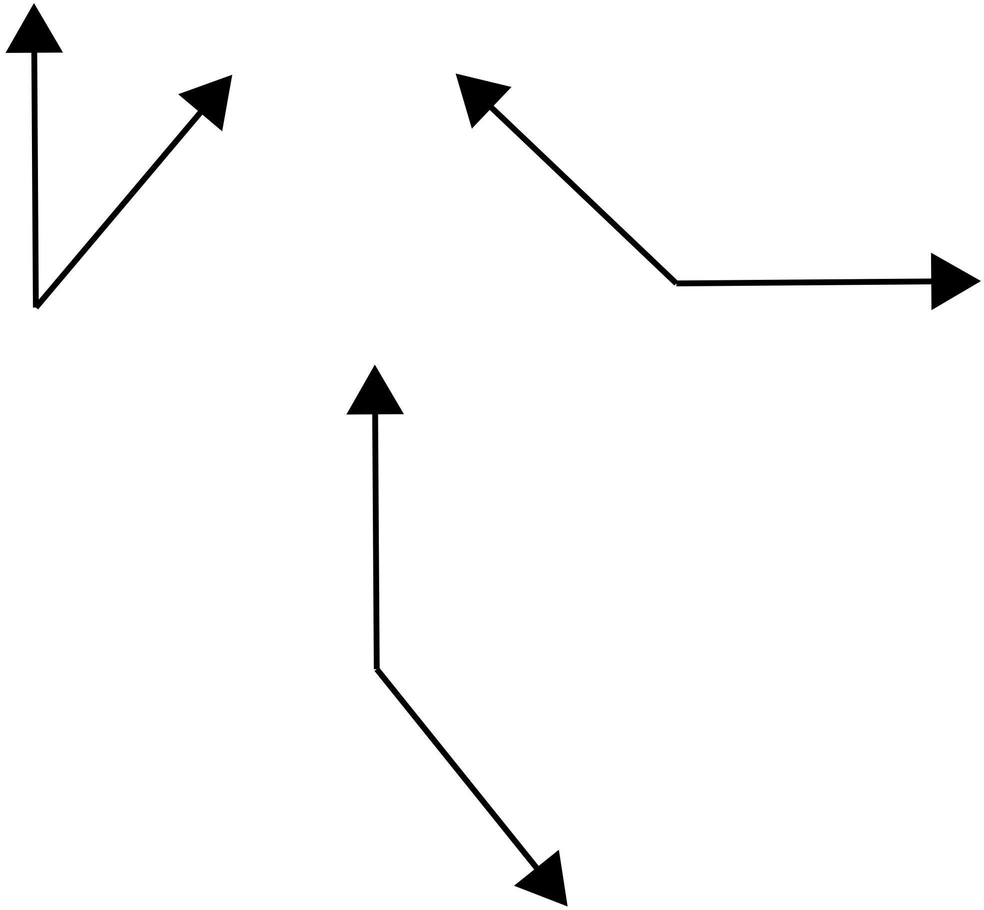
Effect of τ

- As τ is increased the reconstructed attractor folds onto itself.
- At each τ, the reconstructed Lorenz attractor is topologically equivalent to the original attractor (two holes!)
- However, how to we know at what τ the dynamics is fully unfolded and the reconstruction is not least complicated ? (Experimental data!)

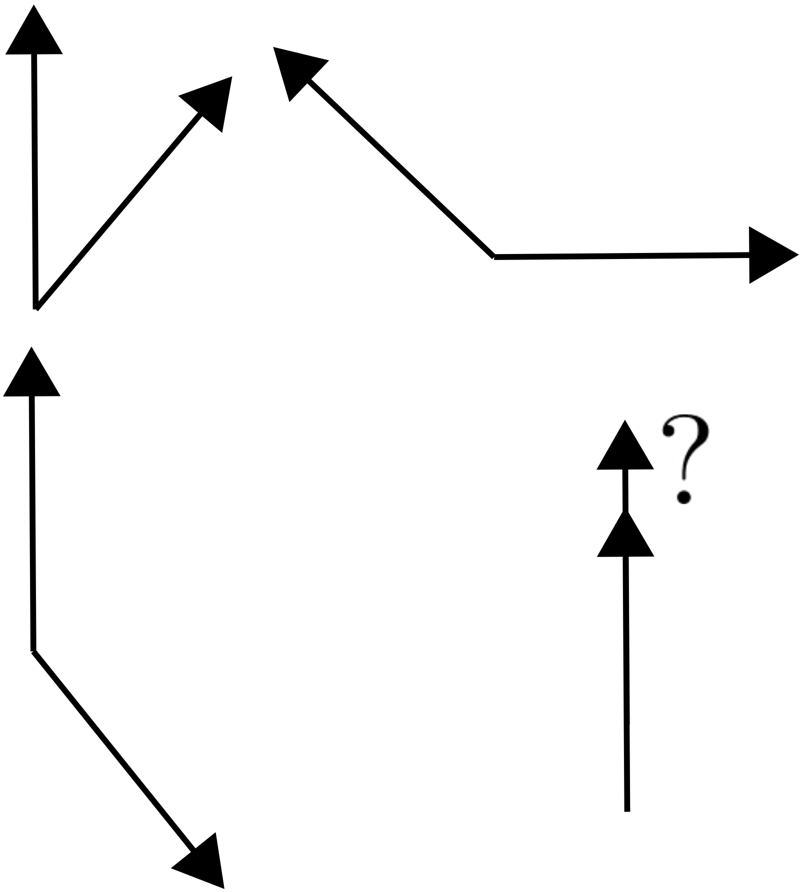
A solution from linear perspective!



A solution from linear perspective!



A solution from linear perspective!



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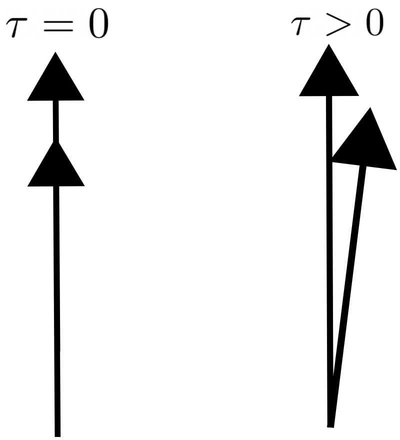


The autocorrelation function

The linear autocorrelation function for a scalar time series $X_t = (x_1, x_2, \cdots, x_N)$ is given as

$$\rho(\tau) = \frac{\sum_{t=1}^{N-\tau}[x_{t+\tau} - \bar{x}][x_t - \bar{x}]}{\sum_{t=1}^{N}[x_t - \bar{x}]^2},$$

where $\bar{x}$ is the sample mean.

- An autocorrelation function tells how similar is the shape of a signal at time $t + \tau$ to that at time t
- The above equation can be simplified by normalizing the time series to zero mean and unit variance

$$\rho(\tau) = \frac{1}{N - \tau} \sum_{t=1}^{N-\tau} x_{t+\tau} x_t,$$

The autocorrelation function

- Note : $\rho(\tau) = \rho(-\tau)$ for $\tau < 0$, i.e., the autocorrelation function is an even function of τ.
- One way of selecting τ for embedding is to choose the first minimum of the autocorrelation function. (Discuss why ?)
- Studies have shown that this choice (first minimum) leads to poor reconstruction.
- Instead, reasonable choice for the embedding lag can be given as the decorrelation time

$$\tau_{dec} = min\left\{\tau : \rho(\tau) < \frac{1}{e}\right\}.$$

Mutual information

- The field of information theory and dynamical systems are closely related.
- In a chaotic system, the distance between two near-by points typically increases in an exponential fashion
- $|x_1(t') - x_2(t)| \approx |x_1(t) - x_2(t)|e^{(\lambda|t'-t|)}$, where λ is the largest Lyapunov exponent.
- These two near-by points that were experimentally (or numerically) indistinguishable, can now be resolved.
- Thus, chaos can be thought of as an information source!
- The rate of the generated information and to quantify the complexity of a dynamical system can be given by Kolmogorov-Sinai entropy.

Mutual information

- Information theory : Message can be considered as a sequence of symbols.
- The state of a dynamical system can be sequence of symbols as well!
- The dynamics of the system can be given by the shift operator, which switches the message sequence one step.
- Using information-theoretic (entropy) measures, one can measure how well can the next sequence be predicted.

Mutual information

- Given a time series, one can learn how much information can be gained from the measurement at one time, from the measurement acquired at another time.
- In general, given two discrete random variables X and Y, the mutual information function can be defined as

$$I(X,Y) = \sum_{x,y} p(x,y) \log \left(\frac{p(x,y)}{p(x)\, p(y)} \right)$$

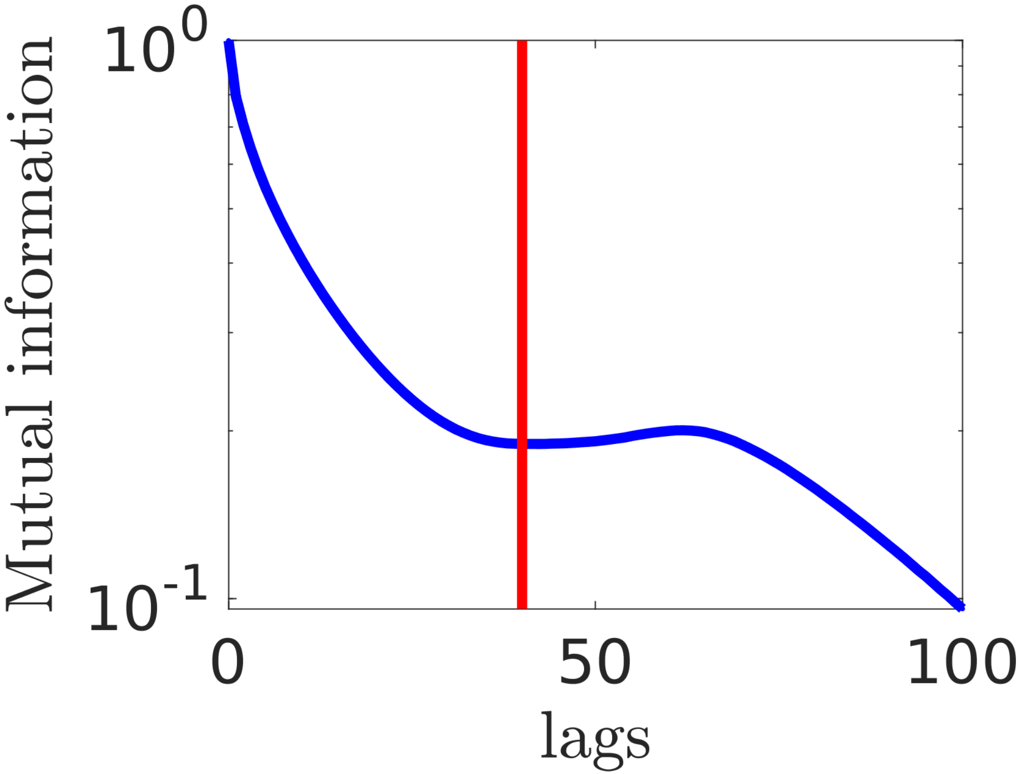
- In the context of a scalar time series, auto mutual information is given as

$$I(\tau) = \sum_{n=1}^{N} p(y_n, y_{n+\tau}) \log \left(\frac{p(y_n, y_{n+\tau})}{p(y_n) p(y_{n+\tau})} \right).$$

Mutual information

- The embedding lag can be given as first minimum of auto mutual information.
- This method has an obvious advantage over the autocorrelation method as it takes into account the nonlinear interrelations as well.

Mutual information



Next lecture : Estimation of embedding dimension