MATH.APP.790 : Topics in Mathematics, Nonlinear time series analysis

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Course Info

- This course will cover the fundamental aspects of nonlinear time series analysis, with the goal to build a foundation for analyzing observational data using dynamical systems theory.
- Seven lectures (Tuesdays 10.00-12.00, Zoom meeting) and exercise consultation sessions (Thursday 10.00-12.00).
- Exercises uploaded to moodle. If consultation session is required send a message or mail at least one day before (i.e., by Wednesday).
- Graded pass or fail based on submission of all exercise solutions and presentation (or research paper) on a chosen topic.

Lecture overview

- What is nonlinear time series analysis and why we need it?

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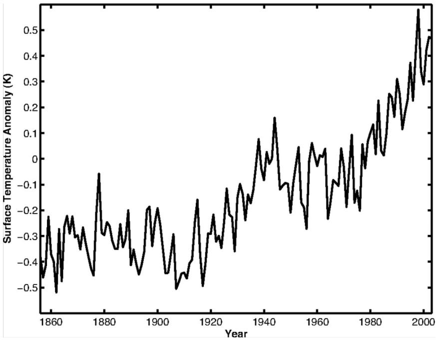
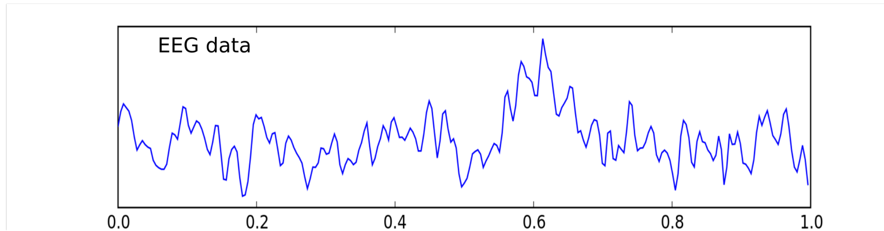
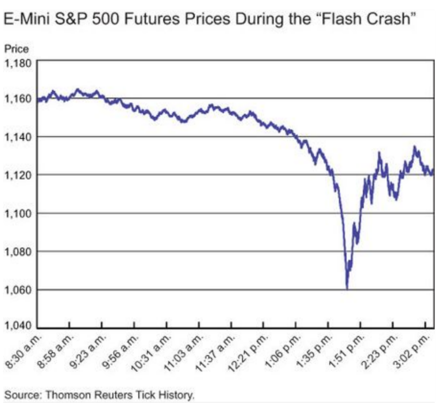
- Limitations of linear methods.
- Introduction to dynamical systems.
- Some more formal definitions and examples.
- What is chaos ?

Some examples of time series







Why to analyze time series ?

- We are surrounded by time series data (Age of big data!)
- Motivation to analyze time series
    - Characterization
    - Prediction
    - Modelling
- What are the tools available for analysis?

Linear time series analysis

- Linear stochastic models - autoregressive (AR) and moving average (MA) models.

$$y_t = a_t + \sum_{m=1}^{M} b_m y_{t-m} + \sum_{n=0}^{N} c_n x_{t-n}$$

- Taken together, we have an ARMA model.
- The model parameters (b_m and c_n) can be determined using the autocorrelation function.

$$c(\tau) = \frac{\langle (y_t - \mu_y)(y_{t-\tau} - \mu_y) \rangle}{\sigma_y^2}$$

Linear time series analysis

- From Weiner-Kinchin theorem, the autocorrelation function of a (wide-sense stationary) process is equal to its Fourier transform.
- Thus linear stochastic models can also be fully characterized by their power spectrum.
- Typically stochastic processes have exponentially decaying autocorrelation function, but so do data from deterministic (chaotic) system!
- Autocorrelations are not adequate to distinguish stochastic from deterministic (chaotic) signals.

Linear time series analysis

- Goal of prediction : Minimize the expectation value of squared prediction error, i.e., $\langle(\hat{y}_{T+1} - y_{T+1})^2\rangle$
- For linear time series models we can write,

$$\hat{y}_{t+1} = \sum_{j=1}^{m} a_j y_{t-m+j}$$

and minimize

$$\sum_{n=m}^{T-1} (\hat{y}_{t+1} - y_{t+1})^2$$

with respect to a_j.

Linear time series analysis

To summarize, the limitations of linear methods include,

- Assumption of Gaussian distribution of the data should hold.
- Model order should be somehow known (Number of coefficients to be estimated must be limited to prevent over-fitting!).
- Only AR part of model can be used for prediction as noise inputs are often unknown.
- Autocorrelation function is not adequate to distinguish stochastic from deterministic signals.

Why nonlinear time series analysis ?

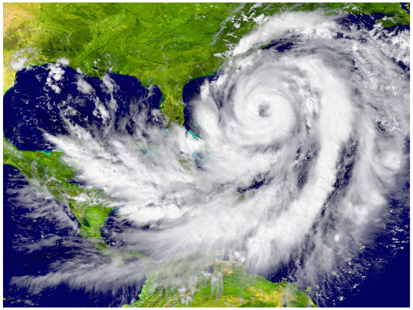


Figure: Tornado

Figure: Swirl of water in a glass container.

Figure: Ecosystems

Figure: Financial markets

Dynamical systems

- Systems change with time $\rightarrow$ dynamic behaviour.
- Dynamical systems can be thought of a study of long-term behavior of such evolving systems.
- Deterministic dynamical systems are governed by rules that map future state of the system from the current state.
- If the evolution rule is purely deterministic $\rightarrow$ deterministic dynamical systems
- Applications to physics, biology, meteorology, astronomy, and other areas.

A formal definition

Definition 1

A dynamical system consists is a 3-tuple $(\mathcal{T}, \mathcal{X}, \Phi)$, where $\mathcal{X}$ is the state space, $\mathcal{T}$ is the parameter space and $\Phi : (\mathcal{X} \times \mathcal{T}) \rightarrow \mathcal{X}$ is the evolution or mapping operator.

For $x \in \mathcal{X}$, the following must hold

1. $\Phi^0(x) = x$ (identity)
2. $\Phi^t(\Phi^s(x)) = \Phi^{t+s}(x)$, $\forall t, s \in \mathcal{T}$ (additivity)

A formal definition

- The collection of maps cannot be arbitrary and must have a group or semigroup structure.
- For a semigroup structure, the following must hold
  - If $\Phi^s, \Phi^t \in \{\Phi^t\}$, then $\Phi^{s+t} := \Phi^s \circ \Phi^t \in \{\Phi^t\}$.
  - Φ^0 is an identity map

  For a group structure, in addition to the above two properties, the inverse property must hold, i.e., every Φ^s has an inverse denoted by Φ^{-s}.

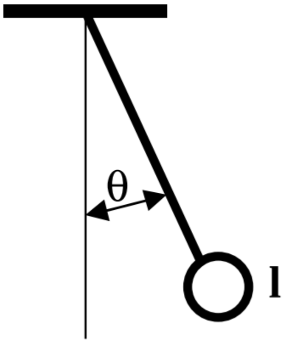
A formal definition

In general, $(\mathcal{X}, \Phi)$ can be,

- A measure space and a measure preserving map
- A topological space and a continuous map
- A metric space and isometry
- A smooth manifold and a differentiable map

Phase space

- A state of a dynamical system is minimum number of variables needed to fully represent the system.
- Example : A simple pendulum is a dynamical system and its state can represented by θ and $\dot{\theta}$.



- Phase space (or state space) is really an abstraction to represent the space of such possible states a dynamical system can be in.

Classification of dynamical systems

Based on the group structure (i.e.,properties of $\mathcal{T}$ and Φ), dynamical systems can be classified into,

- **Continuous-time dynamical systems** : $\mathcal{T} \in \mathbb{R}$ (or $\mathbb{R}_0^+$) and Φ is a continuous and differentiable function. Also known as **Flows**
- **Discrete-time dynamical systems** : $\mathcal{T} \in \mathbb{Z}$ and $\Phi^t(x)$, represents an iterated map, i.e., a single element Φ can generate $\{\Phi^t\}$. Also known as **Maps**

Some definitions

Let $x \in \mathcal{X}$, then the **trajectory** or **orbit** of x is the set

$$\mathcal{O}(x) = \{\Phi^t(x) \colon t \in \mathcal{T}\}$$

For example, if $\mathcal{T} \in \mathbb{Z}$, then

$$\mathcal{O}(x) = \{x, \Phi(x), \Phi^2(x), \Phi^3(x), \cdots,\} = \bigcup_t \Phi^t(x)$$

is the orbit of x and $\Phi^n = \Phi \circ \cdots \circ \Phi$

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- **Positive semiorbit** $\mathcal{O}^+(x) = \bigcup_{t \geq 0} \Phi^t(x)$.
- **Negative semiorbit** $\mathcal{O}^-(x) = \bigcup_{t \leq 0} \Phi^t(x)$
- The **orbit** of $x \in \mathcal{X}$ is $\mathcal{O}^+(x) \bigcup \mathcal{O}^-(x) = \bigcup_t \Phi^t(x)$
- If $\Phi^T(x) = x$, then x is a **periodic point** of T.
- **Periodic orbit** is the orbits of a periodic point
- A subset $\mathcal{A} \subset \mathcal{X}$ is **Φ-invariant** if $\Phi^t(\mathcal{A}) \subset \mathcal{A}$.

Continuous dynamical systems

Consider a system of first-order ordinary differential equations (ODEs) of the form

$$\dot{x} = \Phi(x)$$

where $x(t) \in \mathbb{R}^d$ is a d-dimensional vector, $\Phi \colon \mathbb{R}^d \to \mathbb{R}^d$ is a **vector field** and $\dot{x}$ is the time derivative.

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Example Lorenz system (Nonlinear dynamics) The Lorenz system for $(x, y, z) \in \mathbb{R}^3$ is given by,

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

Example Simple pendulum (Linear dynamics)

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{Tc}{ml^2}$$

Discrete-time dynamical system

A discrete-time system or map is given as

$$x_{n+1} = \Phi(x_n)$$

where $\Phi(x_n)$ defines the mathematical rule governing the evolution.

Example : Logistic Map (Nonlinear dynamics)

$$x_{n+1} = r x_n (1 - x_n)$$

Example : Matrix difference equation (Linear dynamics)

$$x_{n+1} = A x_n + b$$

Some signatures of nonlinear systems

- Exponential sensitivity to initial conditions
- Discontinuous changes (example bifurcations)
- Self-similarity
- Chaos

Sensitivity to initial conditions

The Lorenz system for $(x, y, z) \in \mathbb{R}^3$ is given by,

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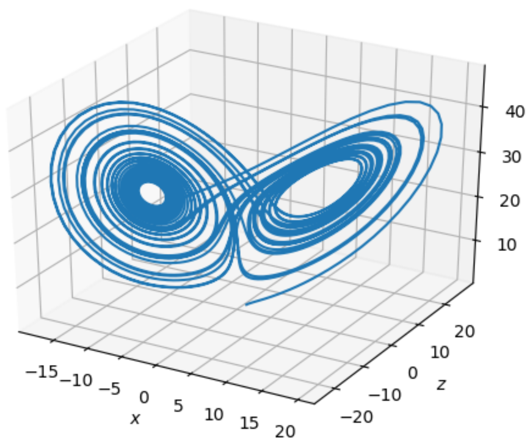
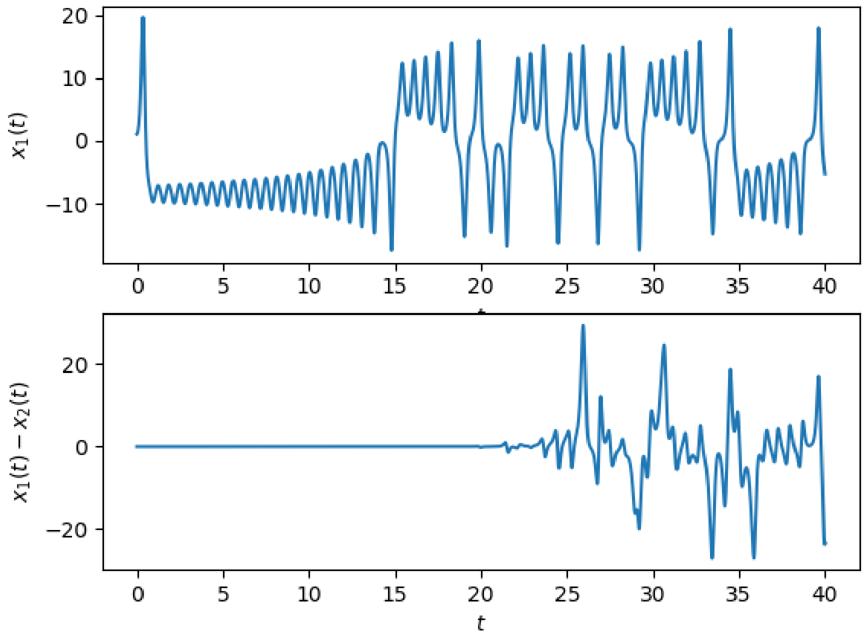


Sensitivity to initial conditions

$x_1(t)$ is with initial conditions $[1, 1, 1]$ and $x_2(t)$ with $[1, 1, 1.0001]$



Bifurcations

- As parameters of the dynamical system are varied the topology of the phase portraits changes.
- This could include changes in number or stability of fixed points, closed orbits, or saddle connections.
- Such qualitative changes in the phase portrait are called bifurcations.
- The corresponding parameters are often called bifurcation parameters.

Bifurcations : Phase portrait

Consider the system

$$\dot{x} = f(x) = \lambda x - x^3$$

The fixed points (i.e., solution for $\dot{x} = 0$) are

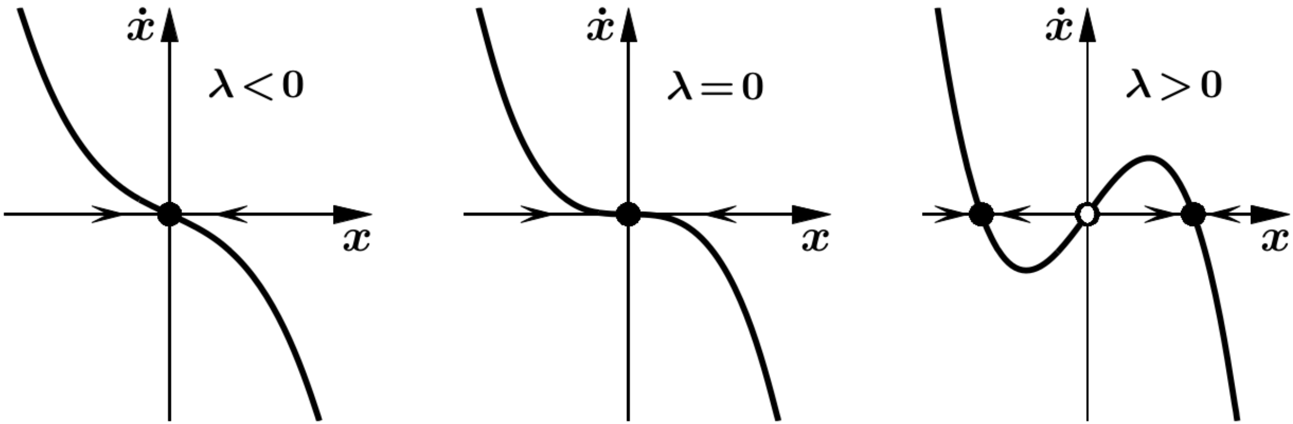
$$\tilde{x}_1 = 0$$
$$\tilde{x}_{2,3} = \pm\sqrt{\lambda}$$

$$f'(x) = \lambda - 3x^2$$

Thus, $f'(\tilde{x}_1 = 0) = \lambda$ and $f'(\tilde{x}_{2,3} = \pm\sqrt{\lambda}) = -2\lambda$

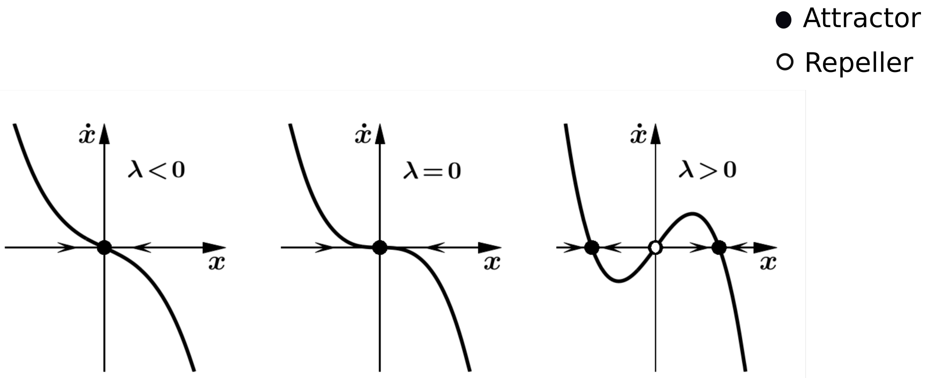
Bifurcations

Phase portraits of $\dot{x} = \lambda x - x^3$ for different values of λ



Bifurcations

Phase portraits of $\dot{x} = \lambda x - x^3$ for different values of λ



- For $\lambda < 0$ and $\lambda = 0$ single fixed point with negative slope. Thus, it is a stable point (**attractor**).
- For $\lambda > 0$, two fixed points, both of them stable appear at $x = \pm\sqrt{\lambda}$.
- In addition, the previously fixed point at $x = 0$ becomes unstable (repeller).
- The system exhibits bistability (two stable points) for $\lambda > 0$.

A even more simple (linear) system

Consider the one-dimensional linear system $\dot{x} = \lambda x$ with general solution $x = c \exp \lambda t$.
Phase portrait for $\lambda < 0$ (all trajectories evolve towards origin), $\lambda = 0$ (No trajectory, system sits at $x(t) = c$) and $\lambda > 0$ (all trajectories evolve away origin)

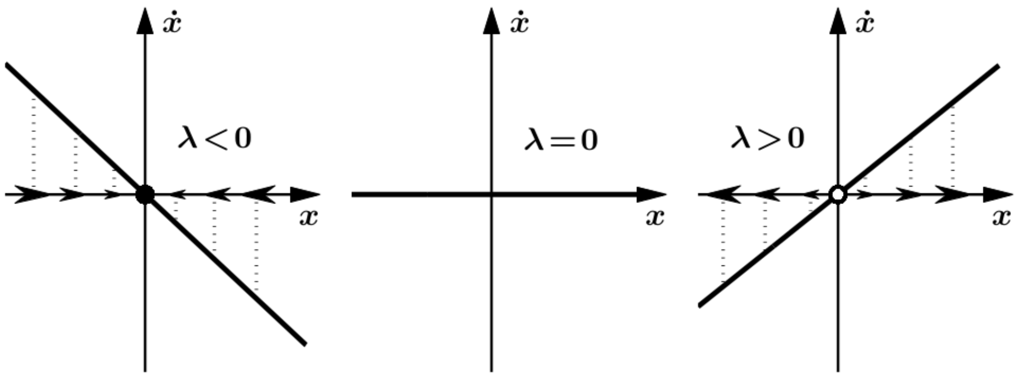


Figure: For $\lambda = 0$ There are infinitely many fixed points, i.e. every point on the real axis. However, they are all neutrally stable, neither attractors nor repellers

Attractor

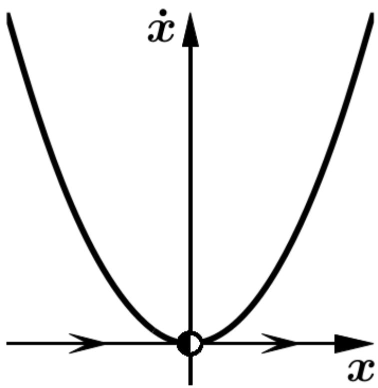
Attractor of a dynamical system can be defined as a subset of the phase space to which the trajectories emerging from typical initial conditions accumulate to, as time increases
More formally, a set $\mathcal{A} \subset \mathcal{X}$ is known is known as an attractor of a dynamical system if the following conditions are met

- $\mathcal{A}$ is an invariant set. (Any trajectory that starts in $\mathcal{A}$ stays in $\mathcal{A}$ for all time)
- $\mathcal{A}$ attracts open set of initial conditions. Thus, if $x(0) \in \mathcal{U}$, then distance of $x(t)$ from $\mathcal{A}$ tends to zero as $t \to \infty$.
- This means, $\mathcal{A}$ attracts all trajectories that start sufficiently close to it.
- The basin of attraction $B(\mathcal{A})$ is the largest such set.
- $\mathcal{A}$ cannot be decomposed into further trivial sets.

Bifurcations

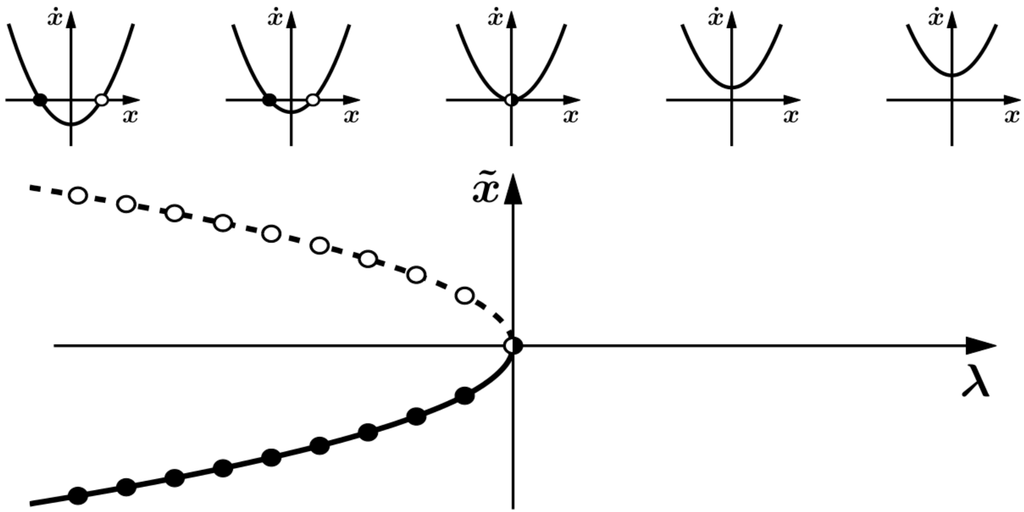
- We see that as control parameter is varied, the change in the system's dynamical behavior is not gradual but qualitative.
- A monostable system with a single attractor at the origin, changes to a bistable system with three fixed points, two attractor and one repeller!

In addition to stable and unstable fixed points, another type of fixed point arises in the nonlinear world - saddle point or half-stable point.



Bifurcation diagram

- In addition to using phase portraits, bifurcation diagram which displays the locations and stability of fixed points as a function of the parameter, can be used.
- Consider $\dot{x} = \lambda + x^2$ with $\tilde{x}_{1,2} = \pm\sqrt{\lambda}$ (Saddle-node bifurcation)

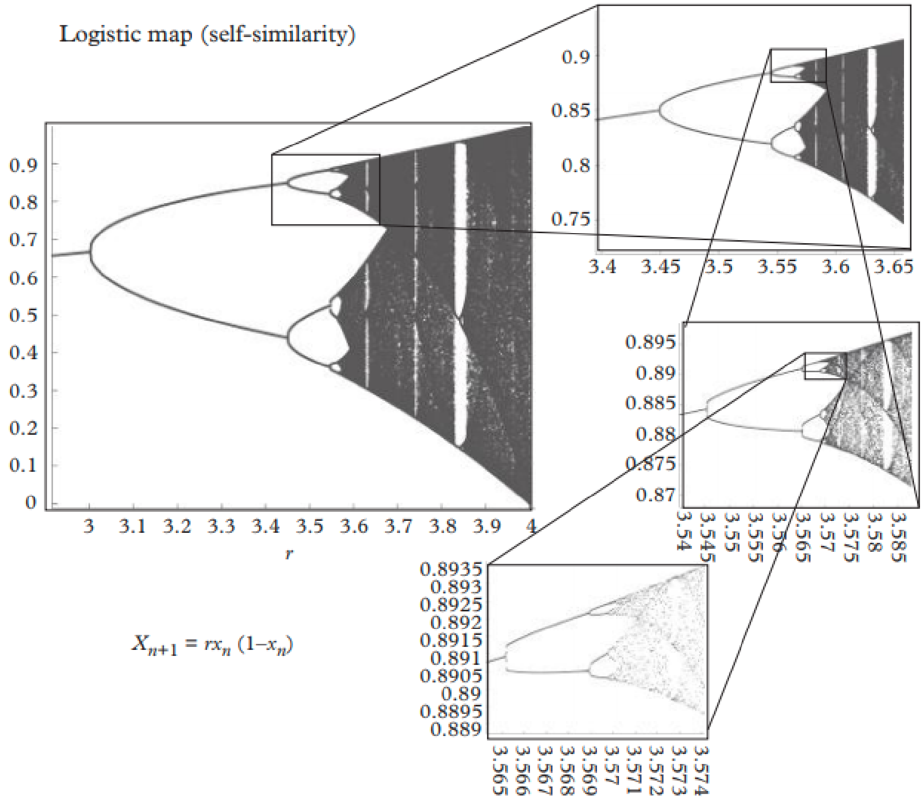


Bifurcation types

For a one-dimensional system, there are only four types of bifurcations possible

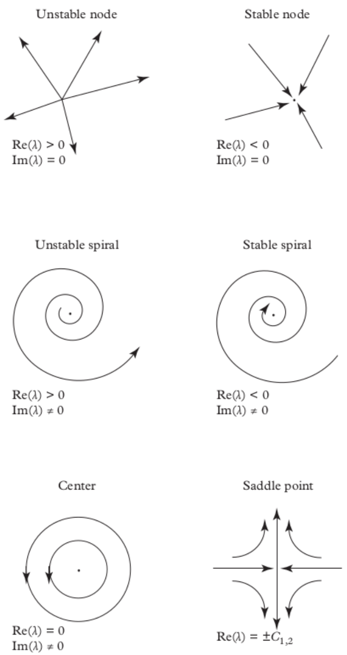
- Saddle-node
- Transcritical
- Super pitchfork
- Subcritical pitchfork

Self-similarity



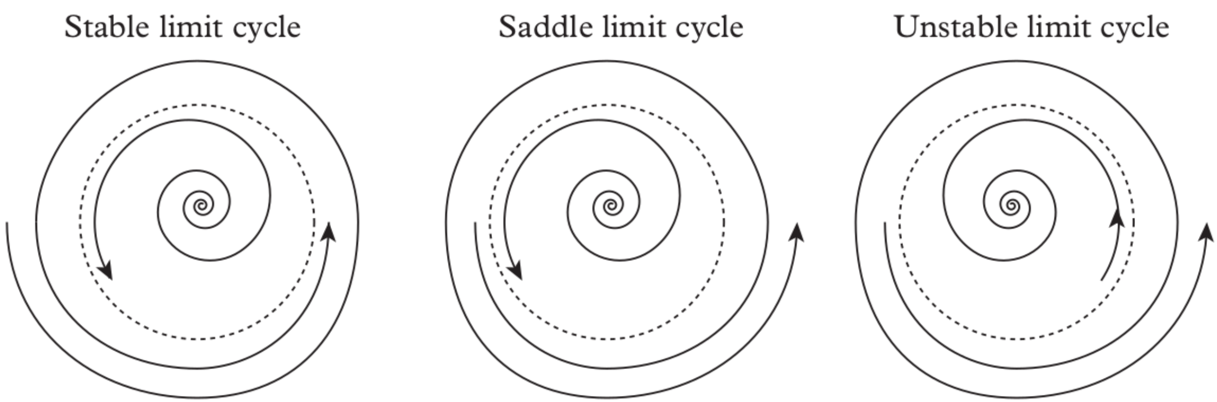
Two-dimensional systems

Given a two-dimensional system, there can be numerous possible types of fixed points.



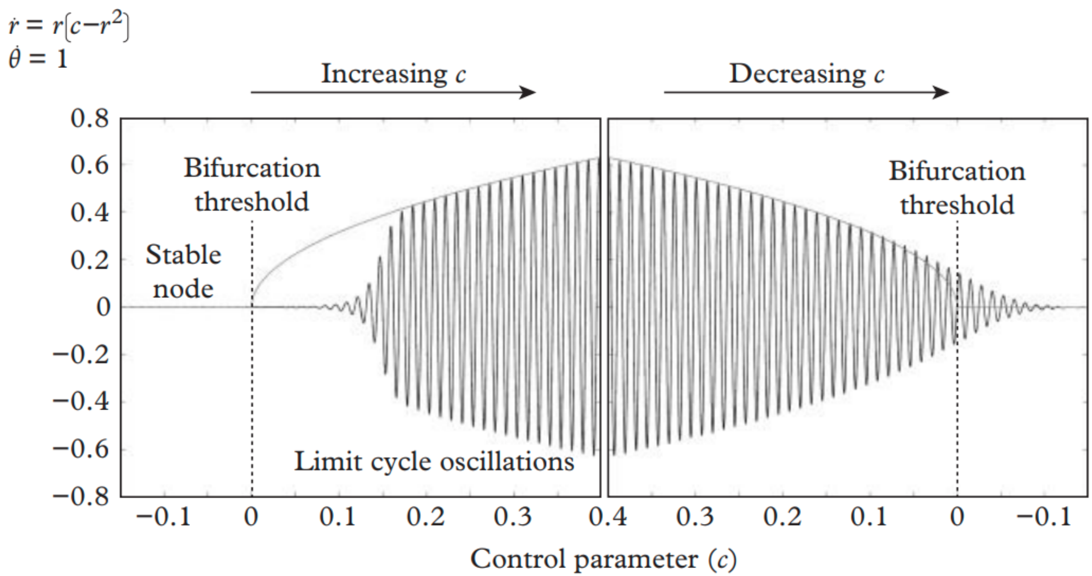
Limit cycles

A steady state solution for dynamical system also includes limit cycles, in addition to fixed points.



The van der Pol oscillator is an example of limit cycle that produces self-sustained oscillations!

One more example of bifurcation diagram

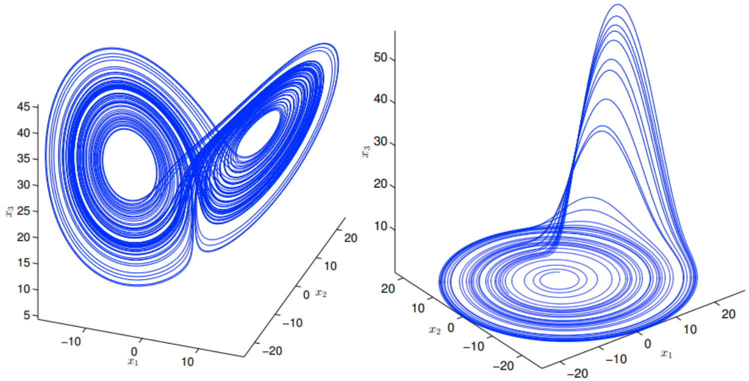


From 2D to 3D - Strange things happen!

- No-crossing rule in deterministic systems: No orbits or trajectories can cross itself (why?)
- In 3D or higher dimenions, trajectories tend to move above, below and around other trajectories without crossing.
- This gives rise to some interesting behaviour as we transition from 2D to 3D - **chaotic attractor**.
- A chaotic dynamical system, although deterministic, is sensitive to initial conditions and hence trajectories diverge.
- As the system is bounded, the separation between the trajectories cannot tend to infinity and hence the trajectories start to converge.

From 2D to 3D - Strange things happen!

- This repeated divergence and convergence (i.e., stretching and folding) of trajectories on the attractor, produces very complicated structures in phase space.
- Also, the trajectories are confined to a bounded set of zero volume.
- It gives the attractor its strange and fractal (zero volume but infinite surface area) nature.



Summary of terminologies

- **State space or phase space** Space of possible states of a dynamical system.
- **Orbit or trajectory** The path taken by state vector in a state space as time evolves. Trajectories do not intersect!
- **Attractor** A set (finite or infinite) in state space towards which the trajectories evolve (get attracted to!) asymptotically.
- **Basin of attraction** Set of initial conditions that evolve towards a attractor.
- **Fixed points** Points in state space for which $\dot{x} = 0$.
- **Stability** Fixed points can be stable (attractor), unstable (repeller), half-stable(saddle) or neutrally stable (neither attractor or repeller $\rightarrow$ No changes in the system!)
- **Chaotic dynamical systems** Deterministic dynamical systems whose trajectories appear to evolve randomly due to exponential sensitivity to initial conditions.