MATH.APP.790 : Topics in Mathematics, Nonlinear time series analysis

Narayan P. Subramaniyam

Tampere University (MET Faculty)

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Lecture overview

- Estimation of embedding dimension
- Fractal dimensions
- Lyapunov exponents
- Entropy measures

Embedding dimension

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- There are various methods in the literature on the selection of k and many of these methods are based on false-nearest neighbor (FNN) principle.
- The rationale : Examine if the points along a trajectory in dimension k are also neighbors in dimension $k+1$.

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- For every increase in k, the percentage of false neighbors is also computed.
- The value of k at which the percentage of false neighbors becomes zero (or arbitrarily small due to the effect of noise) is considered as an appropriate choice for k.

Embedding dimension

Given a scalar time series and τ, one reconstructs the phase space vector at an initial dimension k. Thus we have a state vector at time instance t_n,

$$x_n^k = (x_{t_n}, x_{t_n-\tau}, \cdots, x_{t_n-(k-1)\tau})$$

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If the above reconstruction occurred in an insufficient dimension k, then this closeness could be a result of trajectories crossing.

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which can be normalized and computed for all the points as

$$\xi = \sum_{n=1}^{N-k-1} \Theta\left(\frac{D_n}{||x_n^k - x_{\mathcal{N}(n)}^k||} - r\right). \tag{2}$$

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- In other words, if ξ is greater than r then these two points can be called false neighbors.
- Thus, ξ is the amount of false neighbors that one would find in dimension k.
- The optimal embedding dimension is then defined as the dimension for which ξ becomes zero (or very small in case of noisy time series).

Embedding dimension

Hegger and Kantz proposed a modification to the FNN method by excluding points whose initial distance was already greater σ/r, where σ is the standard deviation of the data. The modified FNN method is given by

$$\xi_{mod} = \frac{\sum_{n=1}^{N-k-1} \Theta\left(\frac{||x_n^{k+1} - x_{\mathcal{N}(n)}^{k+1}||}{||x_n^k - x_{\mathcal{N}(n)}^k||} - r\right)\Theta(\frac{\sigma}{r} - ||x_n^k - x_{\mathcal{N}(n)}^k||)}{\sum_{n=1}^{N-k-1} \Theta(\frac{\sigma}{r} - ||x_n^k - x_{\mathcal{N}(n)}^k||)}.$$

Embedding dimension

Figure below shows the minimum embedding dimension using the x-component of the Lorenz system with the modified FNN method. It can be seen at at $k = 3$, already the percentage of false nearest neighbors is very close to zero and for $k > 3$, this value falls to zero.

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- Value of dynamic invariant obtained from original dynamic system is equal to the one obtained from time delay embedding!

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- Positive Lyapunov exponent $\rightarrow$ Bounded system with deterministic chaotic dynamics

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- Since chaotic orbits are bounded, this separation and hence the mutual distance between two diverging points cannot tend to infinity.
- Consider two neighboring points x_0 and $x_0 + \delta x_0$, with initial separation δx_0. The divergence of these two points occurs at a rate given by

$$\delta x(t) \approx e^{\lambda t} |\delta x_0|$$

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- Flows have one zero exponent $\rightarrow$ for the occurrence of chaos at least three dimensions are required!

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- A chaotic system has at least one positive Lyapunov exponent.
- Note : Lyapunov exponents are invariant under deformations of the attractor that preserve its topology.
- If the reconstruction is done correctly, λ obtained from reconstructed attractor is equal to that of the original system.

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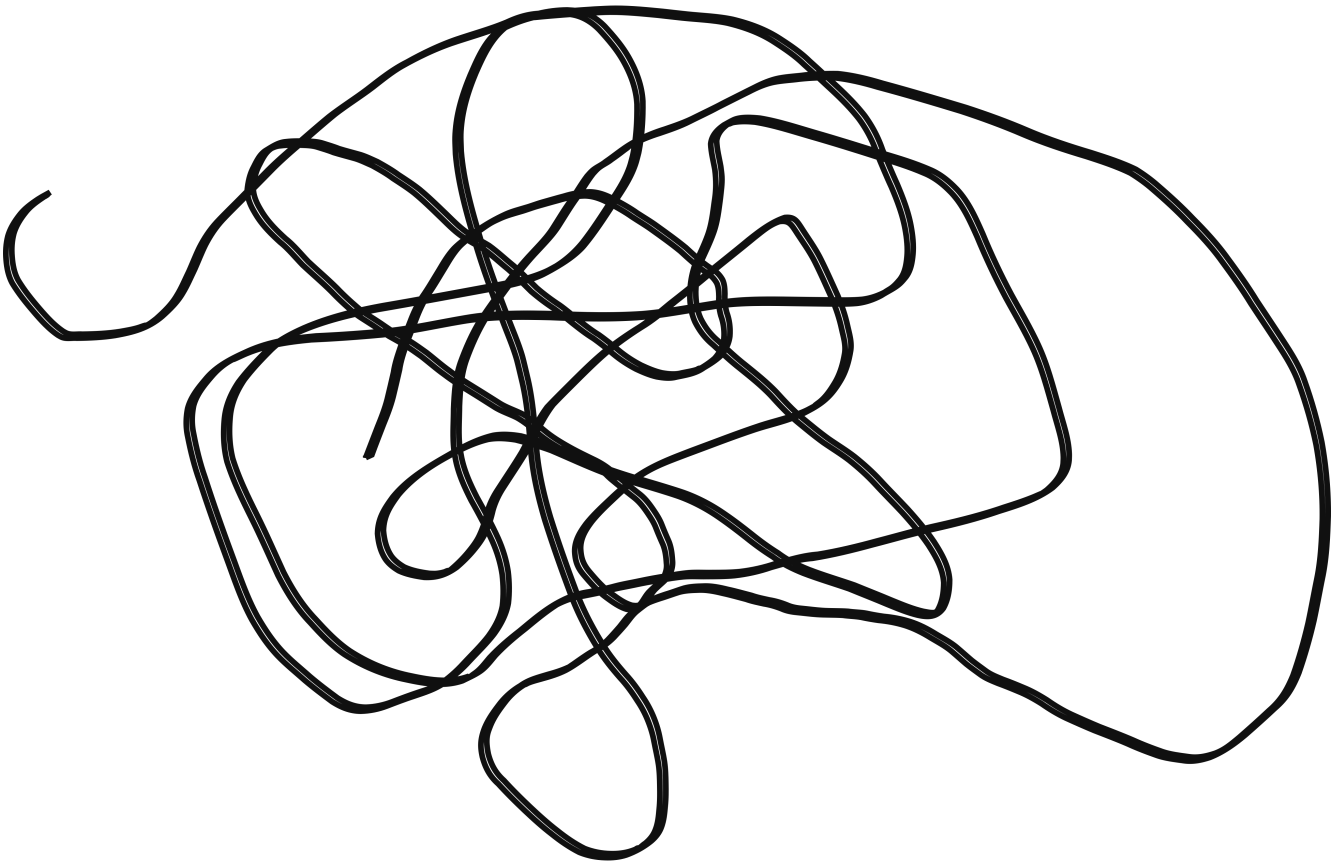

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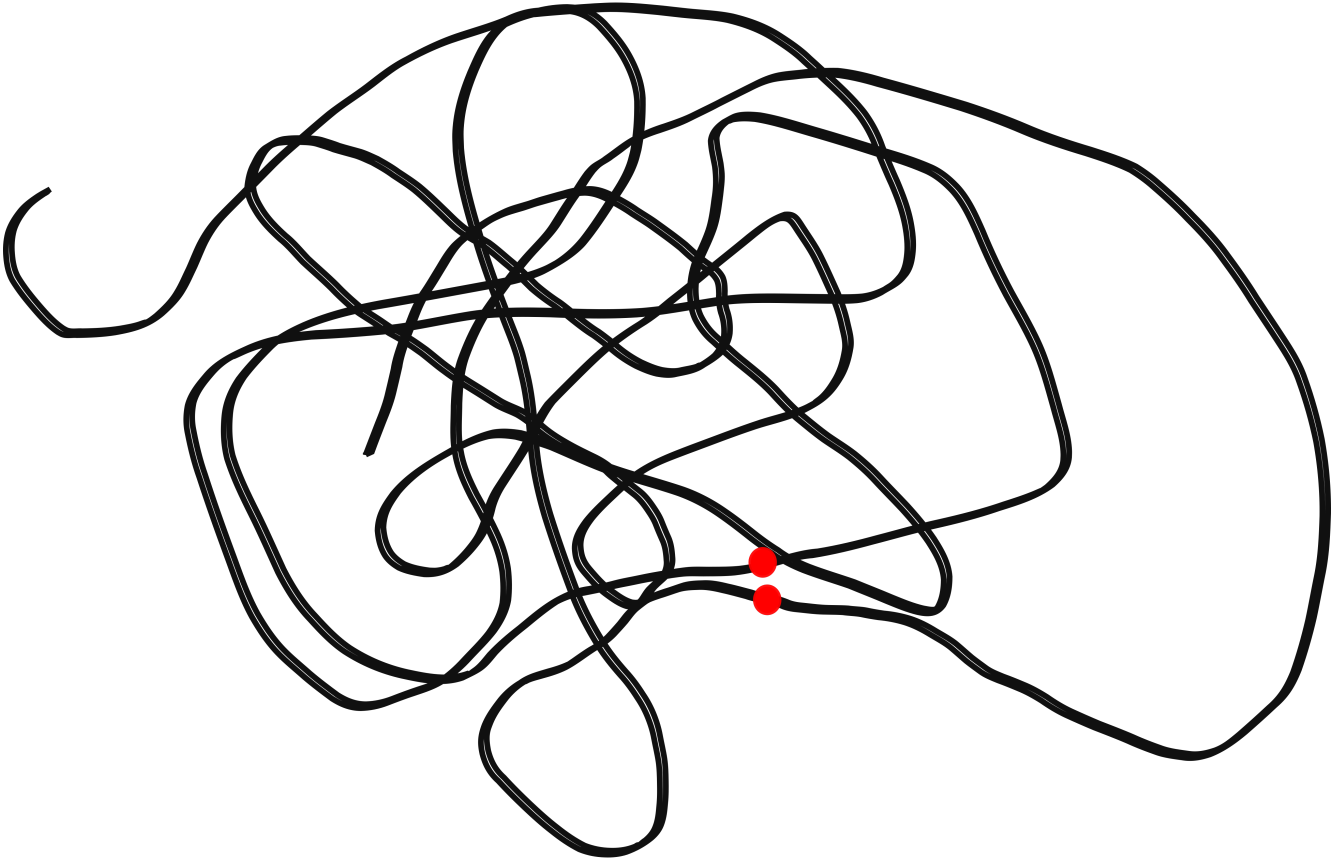

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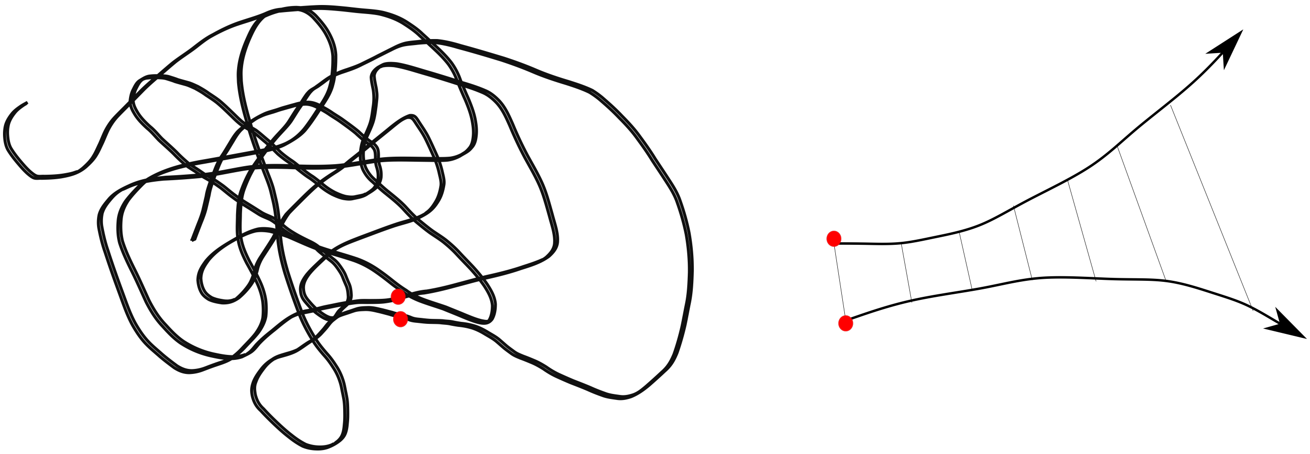

How to compute Lyapunov exponents from time series data ?

- Perform embedding after selecting τ and k

Entropy measures