Knowledge Compilation und #SAT

Narek Bojikian

Humboldt University of Berlin

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Definitions

- The SAT Problem (**SAT**).

SAT

– Given a Boolean formula φ of n variables.

? Find an assignment that satisfies φ.

Definitions

- The SAT Problem (**SAT**).
- Counting SAT Problem (**#SAT**).

#SAT

– Given a Boolean formula φ of n variables.

? How many assignments in $2^{\mathrm{Var}(\varphi)}$ satisfy φ?

Definitions

- The SAT Problem (**SAT**).
- Counting SAT Problem (**#SAT**).

Notation

Let $\mathrm{SAT}(\chi) \subseteq 2^{\mathrm{VAR}(\chi)}$ be the set of all satisfying assignments of χ

$$\mathrm{SAT}(\chi) = \{\rho : \mathrm{VAR}(\chi) \to \{0,1\} : \rho(\chi) = 1\}.$$

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SAT: Is $\mathrm{SAT}(\varphi) = \emptyset$. #SAT: Find $|\mathrm{SAT}(\varphi)|$.

Definitions

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Example

$$\varphi = X_1 \wedge (X_2 \vee \neg X_3)$$

Clearly, $\#\mathrm{SAT}(\varphi) = 3$.

Definitions

- The SAT Problem (**SAT**).
- Counting SAT Problem (**#SAT**).
- Negation Normal Form (**NNF**).

Negation Normal Form

A Boolean formula φ is in NNF form, if it contains only disjunctions and conjunctions over a set of positive and negative literals.
Example. $\varphi = X_1 \vee \neg X_2$.

Definitions

- The SAT Problem (**SAT**).
- Counting SAT Problem (**#SAT**).
- Negation Normal Form (**NNF**).
- Conjunctive Normal Form (**CNF**).

Conjunctive Normal Form

A Boolean formula φ is in CNF, if it is a conjunction of one or more clauses, where each clauses is a disjunction of one or more literals. Note that each CNF formula is an NNF formula as well.

Definitions

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- Counting SAT Problem (**#SAT**).
- Negation Normal Form (**NNF**).
- Conjunctive Normal Form (**CNF**).
- Decomposable Negation Normal Form (**DNNF**).

Decomposable Negation Normal Form

A Boolean formula φ is in DNNF, if it is in NNF and for each conjunction subformula *phi'* $:= \psi_1 \wedge \psi_2$ we have $\mathrm{VAR}(\psi_1) \cap \mathrm{VAR}(\psi_2) = \emptyset$.

- Satisfying each subformula is independent.

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deterministic Decomposable Negation Normal Form

A Boolean formula φ is in d-DNNF, if it is in DNNF and for each disjunction subformula $\varphi' = \psi_1 \vee \psi_2$ we have $\mathrm{SAT}(\psi_1) \cap \mathrm{SAT}(\psi_2) = \emptyset$.

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- decision Decomposable Negation Normal Form (**dec-DNNF**).

decision Decomposable Negation Normal Form

A Boolean formula φ is in dec-DNNF, if it is in DNNF and each disjunction subformula φ' is of the form $\varphi' = (X \wedge \psi_1) \vee (\neg X \wedge \psi_2)$ for some variable $X \in \mathrm{VAR}(\varphi)$.

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Note. Each dec-DNNF is a d-DNNF.

Examples

[On the blackboard..]

Goals for today

Build a dec-DNNF of polynomial size for *β-acyclic* formulas.

Show a subclass of β-acyclic formulas, where each *structured* d-DNNF has exponential size.

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Example

$$\varphi := (v_1 \vee \neg v_2 \vee v_3) \wedge (v_1 \vee v_2) \wedge (\neg v_2 \vee \neg v_3)$$

For $V' = \{v_1, v_2\}, \tau_{|V'}(v_1) = 1, \tau_{|V'}(v_2) = 0$, the partial assignment $\tau_{|V'}$ satisfies φ.

Hypergraphs

- Hypergraph $\mathcal{H}$.
  - A set of vertices $V(\mathcal{H})$.
  - Edges $E(\mathcal{H})$, defined as subsets over $V(\mathcal{H})$.

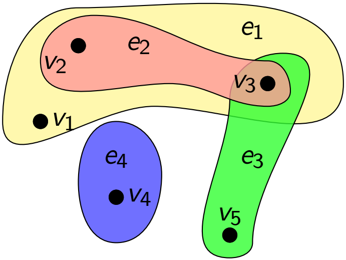

Figure: https://tex.stackexchange.com/a/1195/163902

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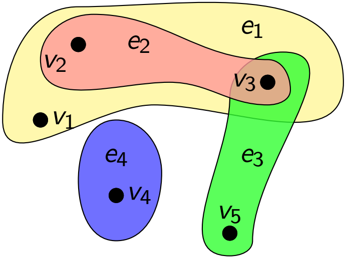

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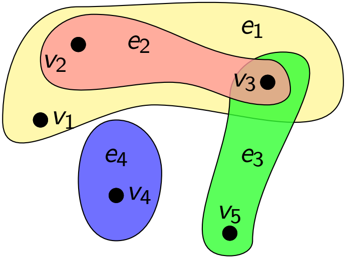

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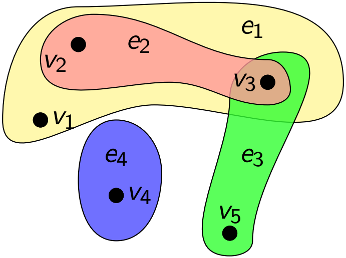
- Different ways to translate

  acyclicity to hypergraphs.



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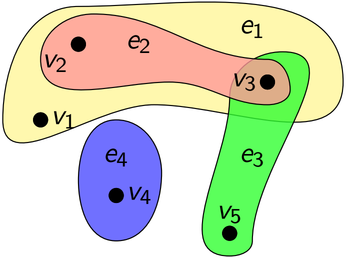
β-acyclic graphs

- Let $\rho := v_1, \dots v_n$ be an enumeration of the vertices.



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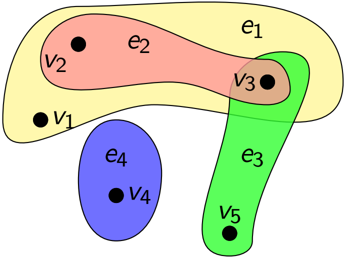

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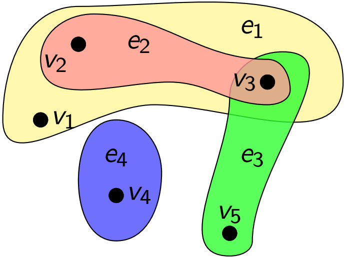
$$e_1, e_2 \in E(\mathcal{H}) \text{ and } v_i \in e_1 \cap e_2,$$
$$e_{1|\geq i} \subseteq e_2 \text{ or } e_{2|\geq i} \subseteq e_1.$$



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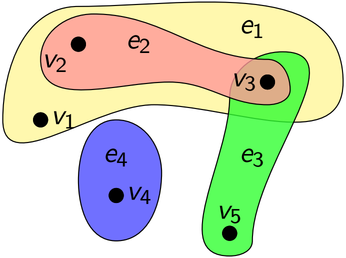

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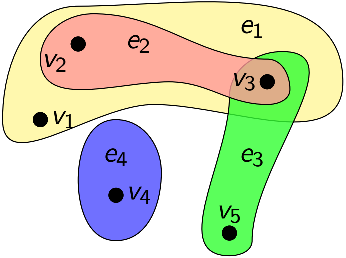
- We define $V_{\leq v_i} := \{v_j ; j \leq i\}$.



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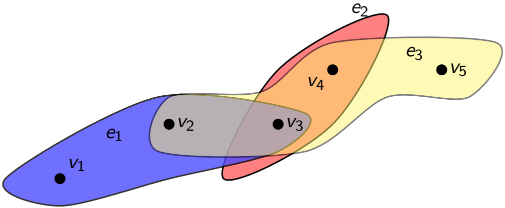
Polynomial upper-bound on the practical method

Lemmas on β-acyclic graphs

- Let $\mathcal{H}$ be a β-acyclic graph and $v_1, \ldots v_n$ a β-elimination.



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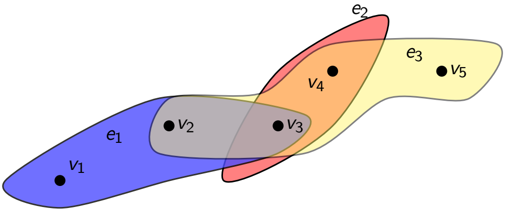
- Let $\mathcal{H}$ be a β-acyclic graph and $v_1, \dots v_n$ a β-elimination.
- For two edge $e, f \in \mathcal{H}$, $e < f$, if and only if $\max\{e \Delta f\} \in f$



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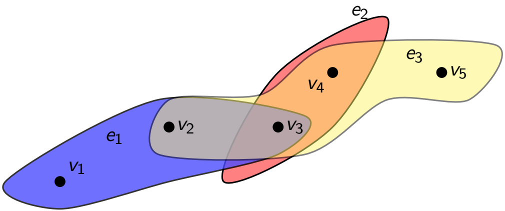

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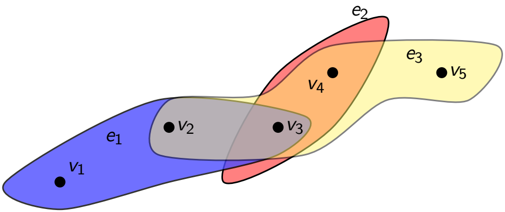

Figure: Note that $e_2 \notin \mathcal{H}_{e_3}^{v_2}$ meanwhile $e_2 \in \mathcal{H}_{e_3}^{v_3}$

Lemmas on β-acyclic graphs

Lemma (lemma 2)

For $x, y \in V(\mathcal{H}), x \leq y$ and for $e, f \in \mathcal{H}, e \leq f$,

$$\text{if } V(\mathcal{H}_e^x) \cap V(\mathcal{H}_f^y) \cap V_{\leq x} \neq \emptyset, \text{ then } \mathcal{H}_e^x \subseteq \mathcal{H}_f^y.$$

In particular, for all $y \in V(\mathcal{H})$,

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Proof sketch. For $g \in \mathcal{H}_e^x$, there is a path from g to e using edges smaller than e and vertices smaller than x.
There is also a path from e to f. Concatenate both paths to get a path from g to f.

Lemmas on β-acyclic graphs

Lemma (lemma 4)

For $e, f \in \mathcal{H}, e \leq f$, If there exists a vertex $x \in V(\mathcal{H})$, such that $x \in e \cap f$, then $e \cap V_{\geq x} \subseteq f$.

Lemmas on β-acyclic graphs

Lemma (lemma 4)

For $e, f \in \mathcal{H}, e \leq f$, If there exists a vertex $x \in V(\mathcal{H})$, such that $x \in e \cap f$, then $e \cap V_{\geq x} \subseteq f$.

Proof sketch. If $y \in e \setminus f$ such that $y > x$, then $\mathcal{H}$ is not β-acyclic.

Lemmas on β-acyclic graphs

A path $(e_1, x_1, \dots e_{n+1})$ is called decreasing, if $e_i > e_{i+1}$ and $x_i > x_{i+1}$ for all i.

Lemma (lemma 5)

For $x \in V(\mathcal{H}), e \in \mathcal{H}$ and $f \in \mathcal{H}_e^x$, there exists a decreasing path from e to f going through vertices smaller than x.

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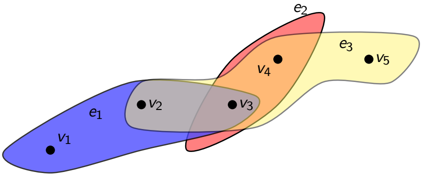
Proof sketch. Any shortest path from e to f is decreasing. A path exists by definition.

Lemmas on β-acyclic graphs

Theorem (theorem 3)

For every $x \in V(\mathcal{H})$ and $e \in \mathcal{H}$, $V(\mathcal{H}_e^x) \cap V_{\geq x} \subseteq e$



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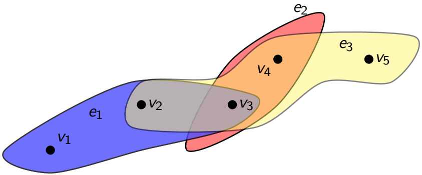

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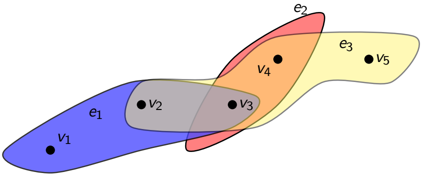

Proof sketch. Prove that all edges of a decreasing path are subsets of the first edge by induction over the length of the path.
Intuitively, this allows us to use dynamic programming, since all variables in $\mathcal{H}_e^x$ not contained in e are smaller than x.

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  $F[\tau_C^x]$ results from F by removing all variables greater than x from each clause.

Solving #SAT in β-acyclic graphs

Lemma (lemma 6)

Let $x \neq x_1 \in \mathrm{VAR}(F)$ and let y be the predecessor of x for $<$. Let $e \in \mathcal{H}$ and $\tau : (e \cap V_{\geq x}) \to \{0,1\}$. Then either $F_e^x[\tau] \equiv 1$ or there exists $U \subseteq \mathcal{H}_e^x$ such that

$$F_e^x[\tau] \equiv \bigwedge_{g \in U} F_g^y[\tau_{C_g}^y],$$

where C_g is some clause in F_e^x such that $\mathrm{VAR}(C_g) = g$.
Moreover, all and-gates are decomposable and U can be computed in polynomial time.

$$F_e^x[\tau] \equiv \bigwedge_{g \in U} F_g^y[\tau_{C_g}^y],$$

Proof sketch.

- Let A be the set edges g, where $\tau \not\models C_g$ for some *corresponding* C_g.

Solving #SAT in β-acyclic graphs

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- U can be computed in polynomial time.
- The edges in U are pairwise disjoint. Hence, the and-gate is decomposable.

Solving #SAT in β-acyclic graphs

Corollary (corollary 7)

Let $x \neq x_1 \in \mathrm{VAR}(F)$ and let y be the predecessor of x for $<$. For every $C \in \mathcal{H}$, there exist $U_0, U_1 \subseteq \mathcal{H}^x_{\mathrm{VAR}(C)}$ such that

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Proof sketch. Let $\tau_1 := \tau^x_C \cup \{x \mapsto 1\}$ and $\tau_0 := \tau^x_C \cup \{x \mapsto 0\}$.

$$F^x_{\mathrm{VAR}(C)}[\tau^x_C] = (x \wedge F^x_{\mathrm{VAR}(C)}[\tau_1]) \vee (\neg x \wedge F^x_{\mathrm{VAR}(C)}[\tau_0])$$

Apply lemma 6 on each of the terms.

Solving #SAT in β-acyclic graphs

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Let F be a β-acyclic CNF-formula. One can construct in polynomial time in $\mathrm{size}(F)$ a dec-DNNF D of size $O((size(F))$ and fanin at most $|\mathcal{H}|$ computing F.

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- For $e = \max \mathcal{H}$ and a clause C such that $\mathrm{VAR}(C) = e$, we have $\mathcal{H}_e^{v_n} = \mathcal{H}$ and $\tau_{\mathrm{VAR}(C)}^{x_n} = \emptyset$, hence there is a gate in D_n computing $F_e^{x_n}[\tau_C] = F$.

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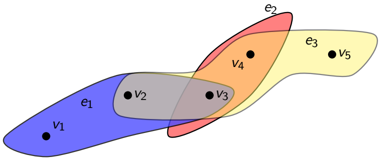
- We add at most 7 gates per edges per vertex.

Example



$$F = \{\{\overline{v_1}, v_2, v_3\}, \{\overline{v_3}, v_4\}, \{v_2, v_3, \overline{v_4}, \overline{v_5}\}\}$$

The rest on the blackboard..

concluding the practical method

- Exhaustive DPLL is a very-well used in practice method.

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conclusion

Exhaustive DPLL can yield efficient algorithms "theoretically", if we can find a good order to choose the variable (such an ordering must be computable in polynomial time) and a good method of cashing.

Lower-bound on the theoretical method

Branch decomposition and MIM-width

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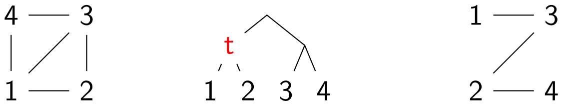

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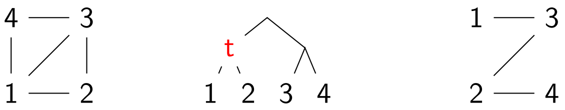

MIM-width$(t) = 2$.

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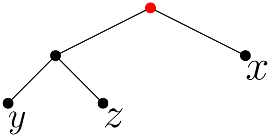
$(x \wedge (y \vee z)) \vee (z {\color{red}\wedge} \neg x)$



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- A formula φ is structured, if there is a *v*tree T over the vertices of φ, such that φ respects T.

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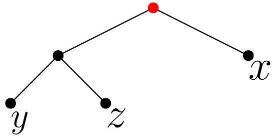

Incidence graphs and structure of formulas

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- The MIM-width of a CNF-formula is the MIM-width of its incidence graph.

Results on the structured d-DNNF

Theorem (theorem 9)

There exists an infinite family $\mathcal{F}$ of β-acyclic CNF-formulas such that for every $F \in \mathcal{F}$ having n variables, there is no structured DNNF of size less than $2^{\Omega(\sqrt{n})}$ computing F.

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Theorem (theorem 1)[1]

There exists an infinite family of β-acyclic hypergraphs of incidence MIM-width $\Omega(n)$ where n is the number of vertices of the hypergraph.

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Results on the structured d-DNNF

- Let r be a boolean function over X and let (Y, Z) be a partition of X. We call r a (Y, Z)-rectangle if and only if for every $\tau, \tau' \in \{0,1\}^X$ such that $\tau \models r$ and $\tau' \models r$, we have $\tau|Y \cup \tau'|Z) \models r$.
- A (Y, Z)-rectangle cover of a boolean function f is a set $R = \{r_1, \ldots, r_q\}$ of (Y, Z)-rectangles such that $\mathrm{sat}(f) = \bigcup_{i=1}^{q} \mathrm{sat}(r_i)$.

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Theorem (theorem 11)[2,3]

Let D be a DNNF on variables X respecting the vtree T. For every vertex t of T, there exists a $(X_t, X \setminus X_t)$-rectangle cover of D of size at most $|D|$, where $X_t = \mathrm{VAR}(T_t)$.

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Let F be a CNF-formula. Let $\hat{F} := \{K \cup \{c_K\} | K \in F\}$ where we add a fresh variable to each clause.

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Proof sketch (lemma 12). Find an assignment τ of $\hat{F}$ such that

$$\hat{F}[\tau] \equiv \bigwedge_{e \in N} (x_e \vee c_e).$$

Conclusion

Takeaway

- Building a structured d-DNNF is not always the best choice we have.

Takeaway

- Building a structured d-DNNF is not always the best choice we have.
- If the structure implies a good elimination ordering, exhaustive DPLL might be a better shot.