Maximum Cardinality Matching Problem

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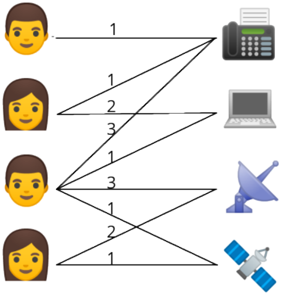
June 10, 2019

Assignment Problem

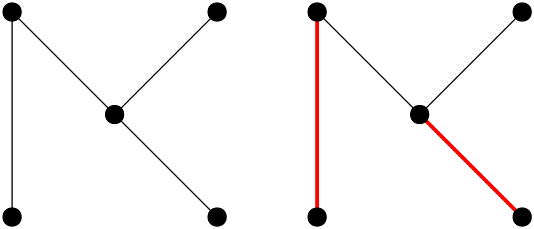


Definitions

Matching

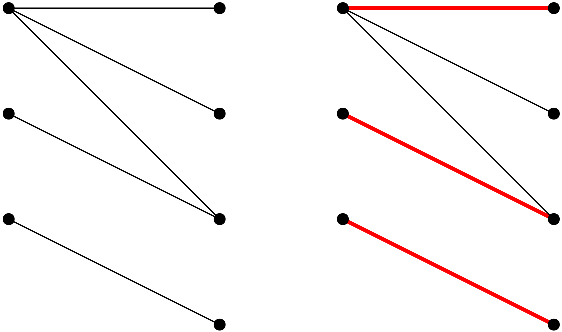
Definition

Set of non overlapping edges.



Definition

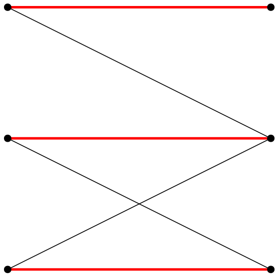
Matching on a bipartite graph.



Perfect Matching

Definition

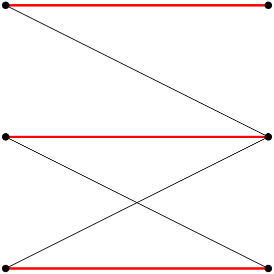
Matching of size $\frac{|V(G)|}{2}$.



Maximum Matching

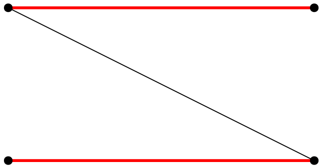
Definition

Matching of the maximum size.

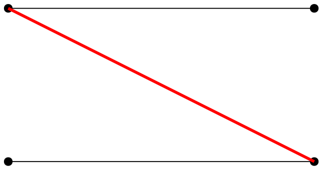


Maximum Matching vs Maximal Matching

Maximum: matching of the maximum size:



Maximal: no more edges can be added:



Bipartite graphs

Hall's Theorem

Definition

A bipartite graph G consisting of sets U and W, has a matching satisfying $|u|$ if and only if $|N(x)| \geq |x|$ for every nonempty subset X of U.

Examples - Hall's Theorem

- Examples on the board –

Königs Theorem

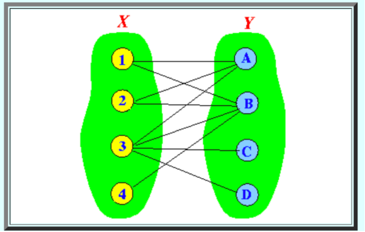
Definition

The maximum matching for a bipartite graph equals its minimum vertex cover

Königs Theorem - Example

– Example on the Board –

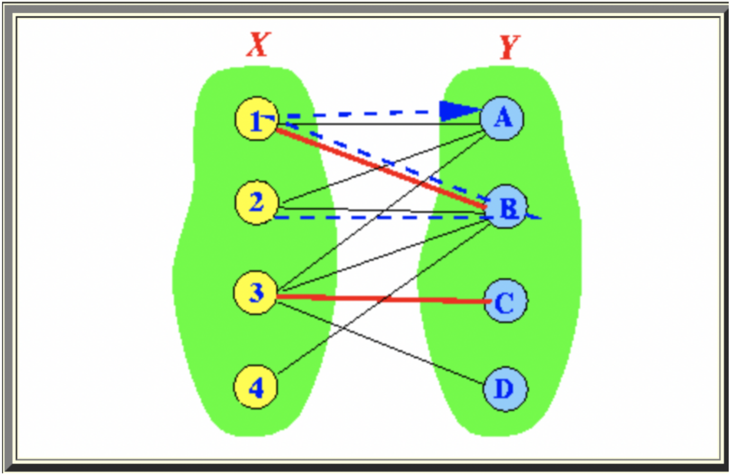
Alternating Path



Definition

Let $G = (X, Y, E)$ be a bi-partite graph where the vertices are divided into the sets X and Y and E the edges.

Alternating Path



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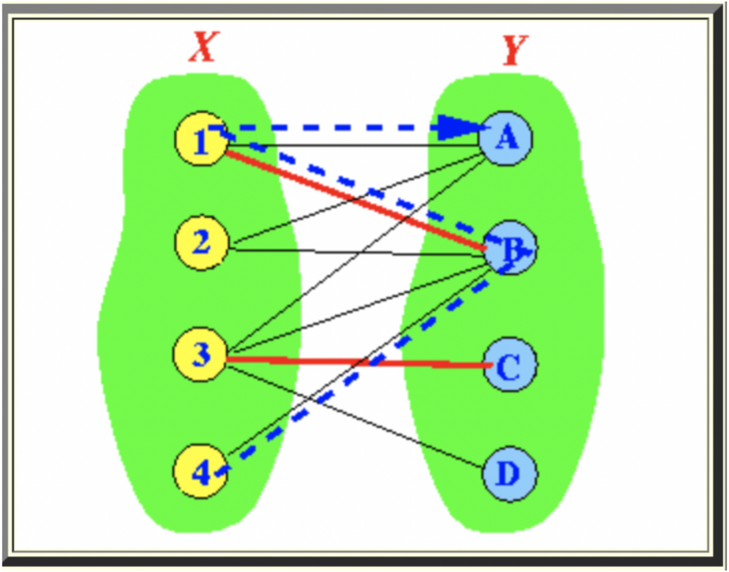


Alternating Path - Summary

- starts in a vertex element of X and ends in vertex element of Y
- must have an odd-number of edges
- will visit nodes in X und Y alternatedly
- And it starts and ends in free/unmatched vertices

$\rightarrow$ To go forward, use an edge that is not part of the matching
$\rightarrow$ To go backward, use an edge that is part of the matching

Augmenting Path - Definition

Definition

An augmenting path is an alternating path where the first and last vertex are unmatched.

Augmenting Path - example

– Example on the Board –

Breadth First Search - repetition

– Example on the Board –

Berge Theorem

A matching is a maximum matching if it contains no augmenting path.

Hungarian Method

Search augmenting paths in the graph until no augmenting path can be found

$\rightarrow$ **Time complexity:** $O(|V|^3)$

Precise information can be found here:
https://brilliant.org/wiki/hungarian-matching/

Note: It is not an algorithm, so it does not specify a implementation

Maximum Flow Reduction

– Example on the Board –

Hopcraft-Carp Algorithm

Input: **A bipartite Graph** Initialize Matching

1. Repeat

$\rightarrow$ Build alternating level graph rooted at unmatched vertices using bfs

$\rightarrow$ Augment M via maximal set of vertex disjoint shortest-length paths

$\rightarrow$ until no augmenting paths exists

2. Return M

Time complexity: $O(|E| * \sqrt{|V|})$

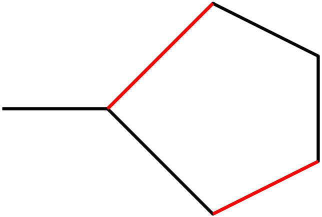
Hopcraft-Carp Algorithm - bipartite Graph

– Example on the Board –

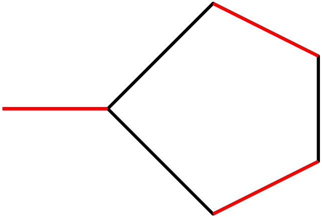
General graphs

General graphs

Problem: odd-length cycles



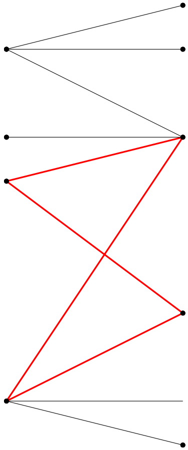
Maximal: matching of size 2



Optimum: matching of size 3

General graphs

Bipartite graphs can have cycles, but always only of even length:



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Input: Graph $\mathcal{G}$, initial matching $\mathcal{M}$ on $\mathcal{G}$
Output: maximum matching $\mathcal{M}^*$ on $\mathcal{G}$

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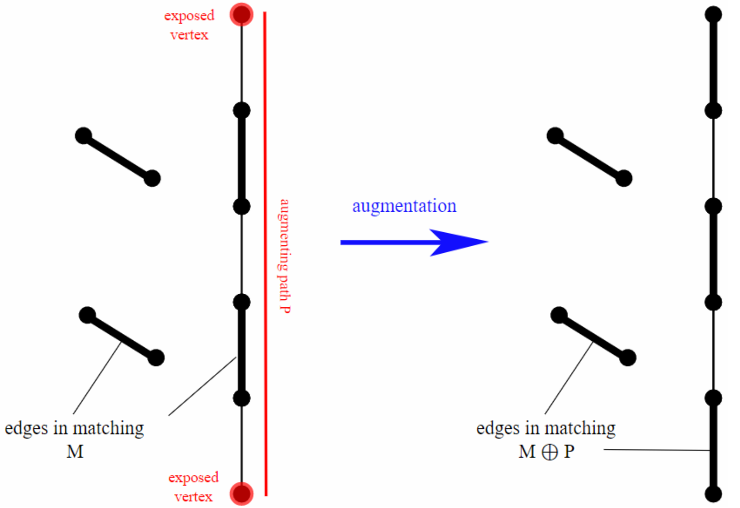
In other words, Blossom algorithm improves existing matching $\mathcal{M}$ in $\mathcal{G}$ as long as augmenting paths exist, then returns.

Edmonds' Blossom algorithm (1965)



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Definition: Exposed vertex

Vertex v is exposed iff no edge of M is incident with v.

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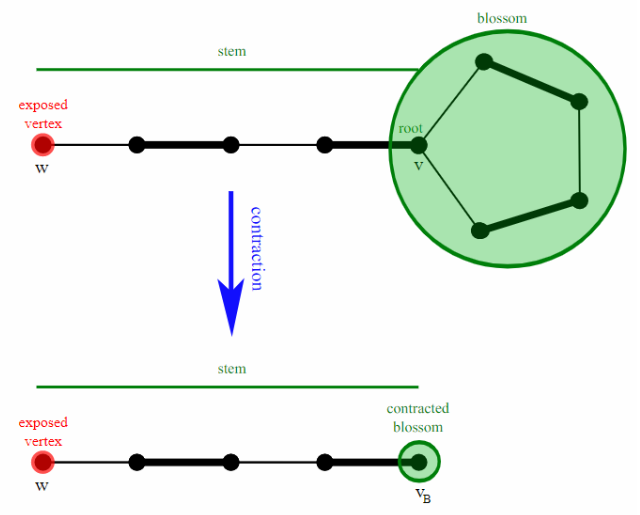
Vertex v is exposed iff no edge of M is incident with v.

Blossom algorithm examines all exposed vertices v and

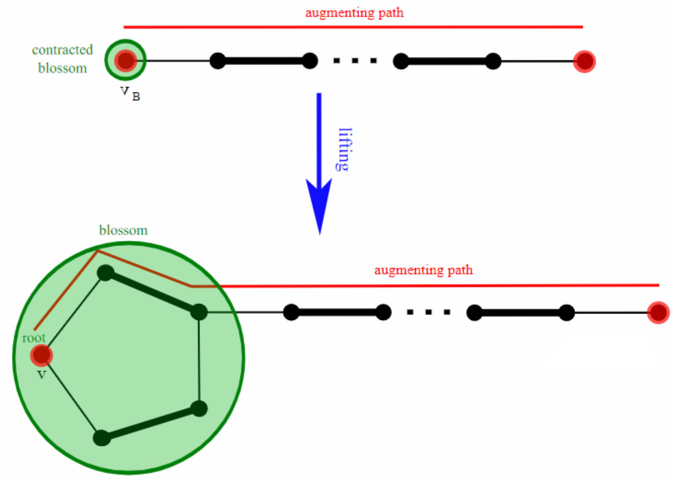
- if an augmenting path is found, improve matching
- if a "blossom" (odd cycle*) is found, temporarily remove cycle and execute algorithm on a modified graph
- (delete vertex v from exposed vertices)

Edmonds' Blossom algorithm (1965)

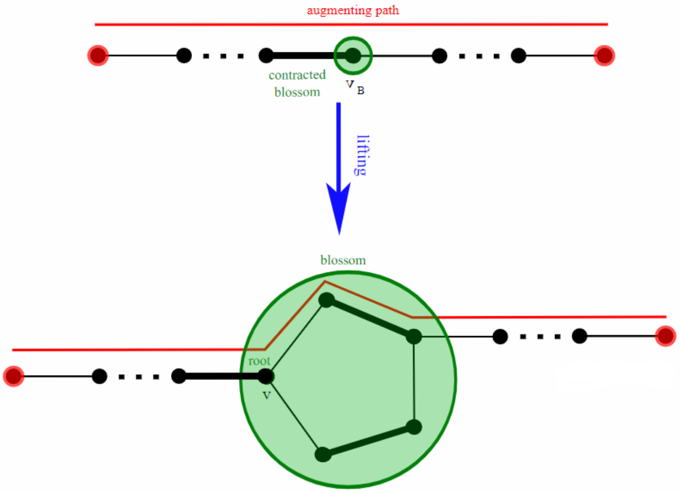
Blossom contraction



Blossom lift



Blossom lift (other variant)



General graphs

Complexity: For general graphs a straightforward implementation of the maximum matching algorithm of Edmonds (1965) runs in $O(n^4)$ time (Papadimitriou and Steiglitz, 1982). More efficient general matching algorithms have been designed with the following running times:

- $O(n^3)$ - Gabow, 1976,
- $O(n * m)$ - Kameda and Munro, 1974,
- $O(n^{2,5)}$ - Even and Kariv, 1975,
- $O(n^{l/2} * m)$ - Micali and Vazirani, 1980

Randomized Algorithms

Determinant and Permanent

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Permutation

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i	1	2	3	4	5
$\delta(i)$	3	2	4	1	5

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Determinant of a matrix

$$det(A) = \sum_{\pi \in \mathcal{S}} sign(\pi) \prod_{i \in [n]} A_{i,\pi(i)}$$

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Laplace expansion (on board)

Efficient Gaussian Elimination

$O(n^\omega), \omega = 2.373$

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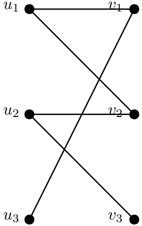
NP-Hard problem.
Presumably no polynomial time algorithm to compute.

Randomized Perfect Matching

How to check if a bipartite graph admits a perfect matching?



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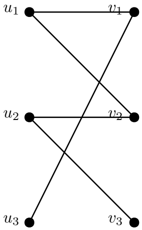

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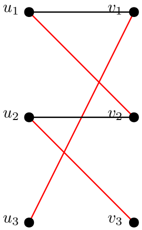

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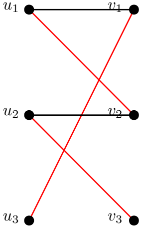

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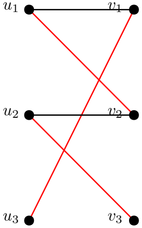

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u_i	1	2	3
$\pi(u_i)$	2	3	1

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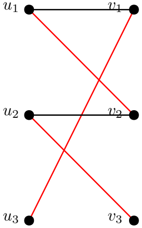

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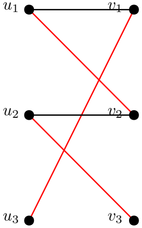

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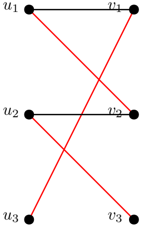

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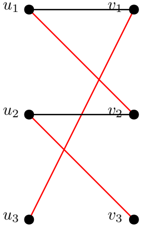

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$$\underbrace{\sum_{\pi \in \mathcal{S}} \prod_{i \in [n]} A^G_{i,\pi(i)}}_{perm(A^G)} > 0$$

Permanent and Determinant

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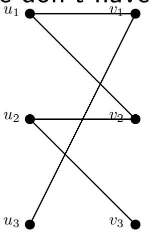

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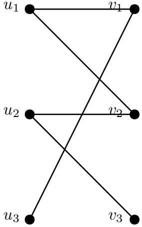

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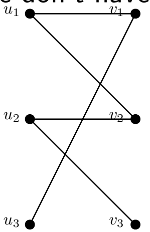

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$det(A'^G)$ is a polynomial of degree n.
The graph admits a perfect matching $\iff$ the polynomial $det(A'^G)$ is not identical zero.

Schwartz-Zippel lemma

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Let ρ be a non-zero polynomial of n variables and degree d over a field $\mathbb{F}$. Let $\mathcal{S} \subseteq \mathbb{F}^n$, then for $x_0 \underset{u.a.r}{\in} \mathcal{S}$

$$Pr[\rho(x_0) = 0] \leq \frac{d}{|\mathcal{S}|}$$

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Choosing $|\mathcal{S}| = 2d$, we get a method that tells with probability at least $1/2$, if the polynomial is identical zero.
Since one non-zero answer is enough to know the polynomial is a non-zero, we can repeat the operation a couple of times magnifying the probability.