Proofs from Assignment 2.1.

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1 Proof from subsection 1.2.4.

Show that the volume of the tetrahedron with vertices $(\vec{X_1}, \vec{X_2}, \vec{X_3}, \vec{X_4})$ is given by $\frac{1}{6}|det(\mathbf{D}_m)|$.

We know that for a matrix $\mathbf{D}_m \in \mathbb{R}^{3\times3}$ the following holds

$$det(\mathbf{D}_m) = \vec{a_1}^T.(\vec{a_2} \times \vec{a_3}) \tag{1}$$

In our case, $\vec{a_1} = \vec{X_1} - \vec{X_4}$, $\vec{a_2} = \vec{X_2} - \vec{X_4}$, $\vec{a_3} = \vec{X_3} - \vec{X_4}$.

Base area of the parallelepiped, defined by three vectors $(\vec{a_1}, \vec{a_2}, \vec{a_2})$ can be calculated the following way:

$$\mathbf{A_p} = \vec{a_2} \times \vec{a_3}$$

and vector $\vec{a_1}^T$ will be the height $\mathbf{h}$ of the according parallelepiped.

Using $\mathbf{A_p}$ and $\mathbf{h}$ we can rewrite Equation 1 in the following form

$$det(\mathbf{D}_m) = \mathbf{hA_p} \tag{2}$$

We know, that the base area of the tetrahedron, defined by the same vectors $(\vec{a_1}, \vec{a_2}, \vec{a_2})$ will be the half of the base area of an according parallelepiped.

$$\mathbf{A_t} = \frac{1}{2}\mathbf{A_p}$$

Volume of the tetrahedron $T = (\vec{X_1}, \vec{X_2}, \vec{X_3}, \vec{X_4})$ is given us by

$$Vol(T) = \frac{1}{3}\mathbf{hA_t} \tag{3}$$

This way we get

$$Vol(T) = \frac{1}{3}\mathbf{hA_t} = \frac{1}{6}\mathbf{hA_p} = \frac{1}{6}|det(\mathbf{D}_m)| \tag{4}$$

$\blacksquare$

2 Proof from subsection 1.2.7.

Show that $\frac{Vol(\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4})}{Vol(\vec{X_1}, \vec{X_2}, \vec{X_3}, \vec{X_4})} = |det(\mathbf{F})|$

$$\frac{Vol(\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4})}{Vol(\vec{X_1}, \vec{X_2}, \vec{X_3}, \vec{X_4})} = |\frac{det(\mathbf{D_s})}{det(\mathbf{D_m})}| = |det(\mathbf{D_s}\mathbf{D_m^{-1}})| = |det(\mathbf{F})|$$

$\blacksquare$