Heap

- The *heap property*:

  each node is more extreme (greater or less) than each of its children

- + Shape property

Binary heap (…) (by default for us)

Binomial heap

a forest of binomial trees satisfying the heap property

Fibonacci heap

a collection of trees satisfying the heap property

Binary heap

A binary tree with two additional constraints:

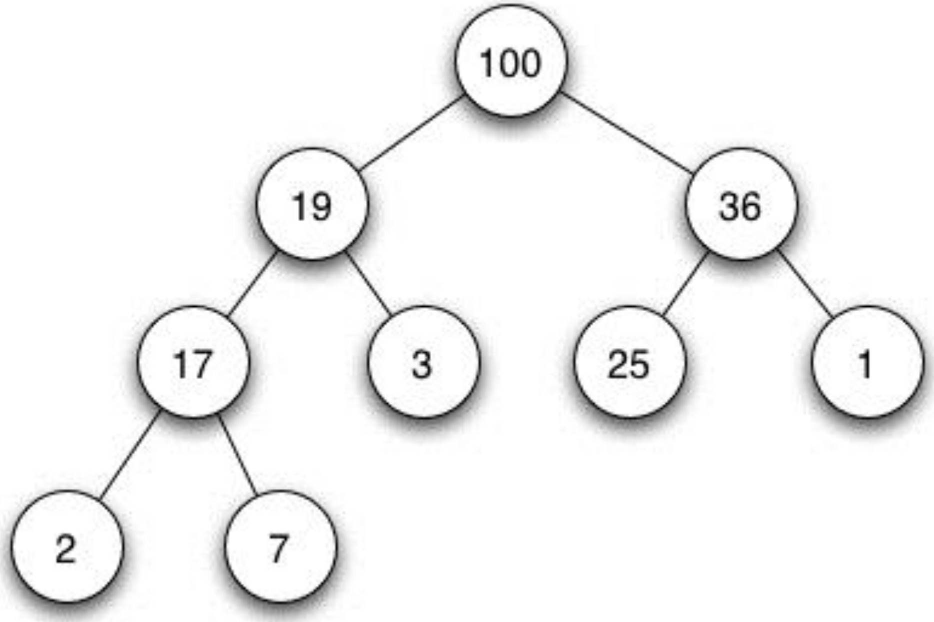
- The *shape property*
  (almost) complete
- The *heap property*:
  each node is more extreme (greater or less) than each of its children

or equal

NO ordering of siblings

Convention

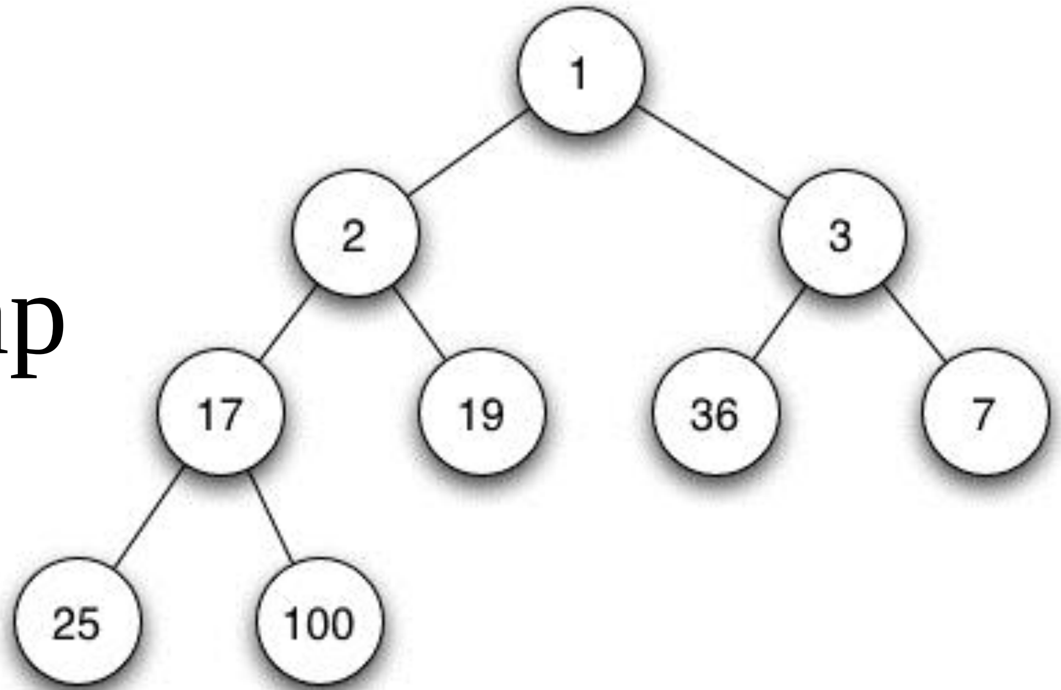
During these classes, the term heap will refer to max binary heap, when not explicitly specified otherwise.



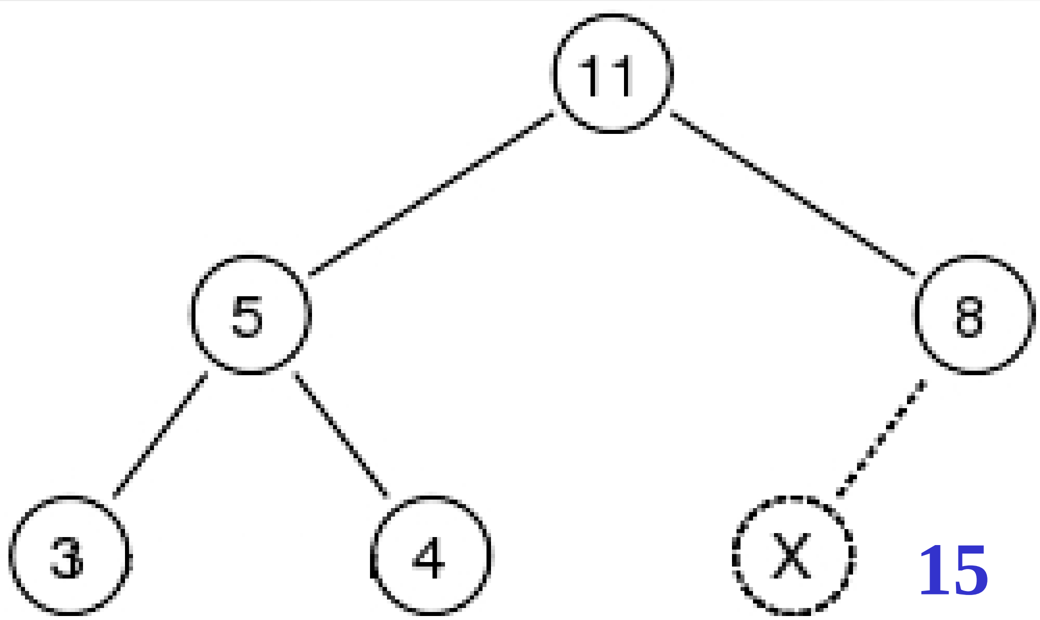
Max binary heap

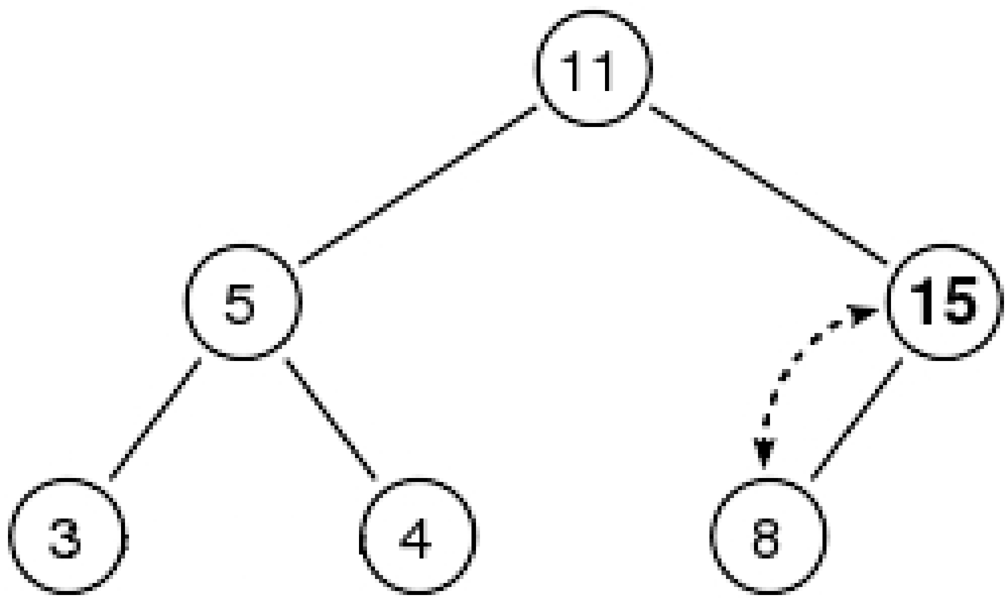
by default, for us

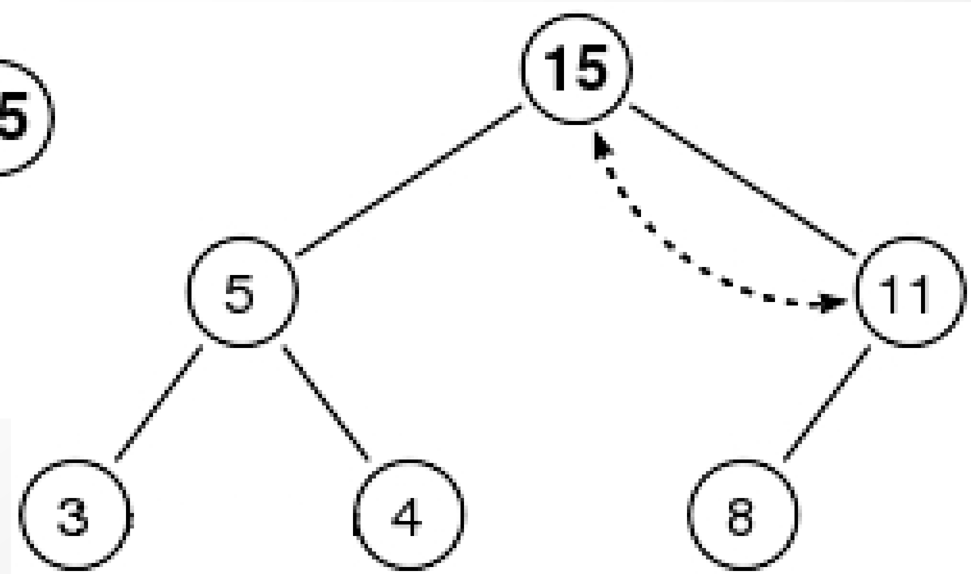
Min binary heap



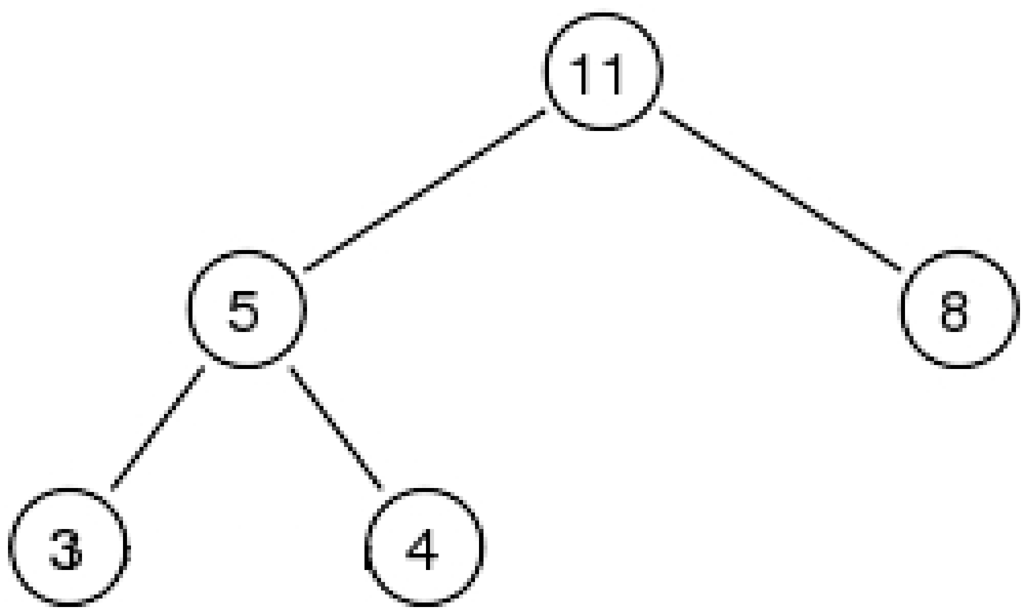
Heap - insertion

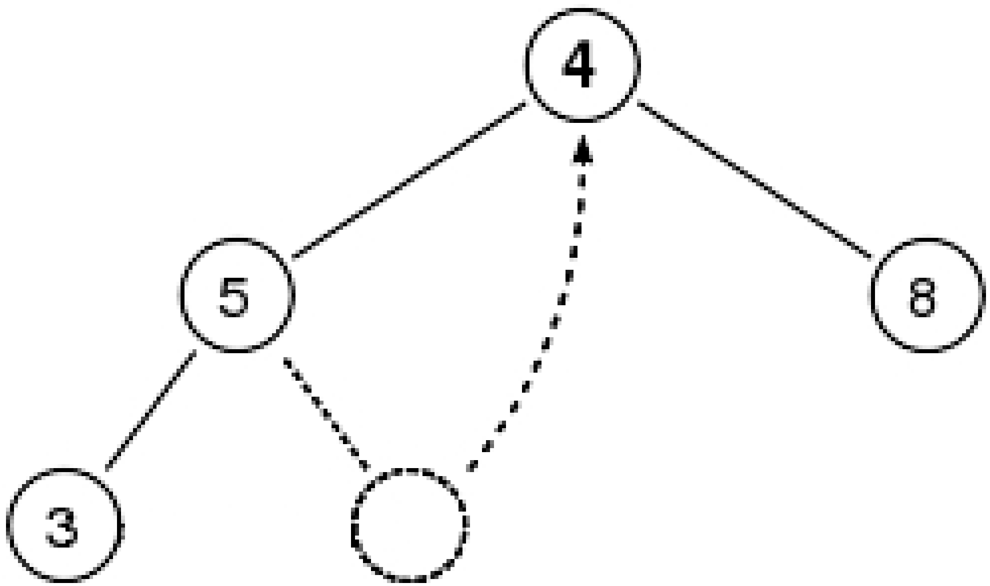


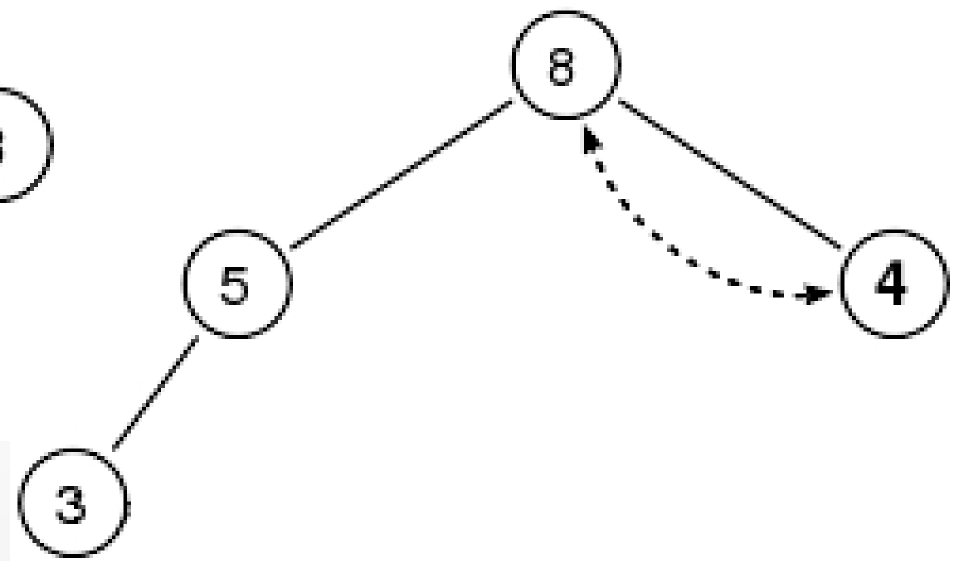




Heap – delete root







Binary heap

Max binary heap

getMax	O(1)
insert	O(log n)
deleteMax	O(log n)

The height of binary heap: O(log n)

Heap – stored in array

tree root item has index 1

n tree elements: $a[1]$ .. $a[n]$

element $a[i]$

children: $a[2i]$ and $a[2i+1]$

parent $a[\text{ floor }(i/2)]$

tree root item has index 0

n tree elements: $a[0]$ .. $a[n-1]$

element $a[i]$

children: $a[2i+1]$ and $a[2i+2]$

parent $a[\text{ floor }((i-1)/2)]$

```
Heap: record
   n: Integer
   els: array [1..MAX] of TComparable
end
```

Extract maximum (root)

```
Funct. extractMax (H)        //if size(H)>=1
   extractMax :=H.els [1]
   H. els[1]:=H. els[H.n]
   H.n := H.n -1
   downHeap(H,1)
end_extractMax
```

subalg. downHeap(H,poz)

```
el:=H.Element[poz];
p:=poz; ch:=2*poz
while ch<=H.n do
    if ch<H.n then
                    if H. els[ch]<H. els[ch+1] then
                                ch:=ch+1
    endif           endif
    if H. els[ch]< el then break;
    else            H. els[p]:=H. els[ch]
                    p:=ch; ch:=2*ch
    endif
endwhile
H. els[p]:=el
```

end_downHeap

add

```
subalg. add (H,el)
H.n := H.n +1
H. els[H.n] := el
upHeap (H, H.n)
end_add
```

```
subalg. upHeap (H, i)
el := H. els[i]
ch:=i
p:=ch div 2
while (p>=1) and (H. els[p]<el) do
    H. els[ch] := H. els[p]
    ch:=p
    p:=p div 2
endwhile
H. els[ch]:=el
end_upHeap
```

build heap - complexity

- A heap could be built by successive insertions. $O(n \log_2 n)$
- optimal method:
  - starts by randomly putting the elements
  - then: build the *heap property*

// build the heap property

```
Subalg. buildHeapProp(H)
    for i:=[H.n / 2] , 1 , step = -1 do
        downHeap (H,i)
    endfor
endbuildHeapProp
```

$$nrNodes_h \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$$

build heap - complexity

- obvious: complexity $\in O(n*\log_2(n))$
- not obvious, but proved: complexity $\in O(n)$

proof ideas

- nr. nodes of height h $\quad nrNodes_h \le \left\lceil \frac{n}{2^{h+1}} \right\rceil$
- complexity (*nr. of oper.*)

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}\right)$$

$$\le O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right)$$

$$= O(n)$$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

HeapSort

- build a heap => O(n)
- repeatedly extract maximum => n*O(log(n))

=> O(n*log(n)) (even in the worse case)

Heap - usage

- used in the sorting algorithm

  heapsort

  one of the best sorting methods

  with no quadratic worst case scenarios

- used to implement priority queues

  Java util: Priority Queue

  based on a priority heap

  head of this queue is the ***least*** *element*

C++ STL .

Standard Template Library: Algorithms

Heap:

push_heap

pop_heap

make_heap …

(uses RandomAccessIterator)

sort_heap …

C++ STL

priority queue

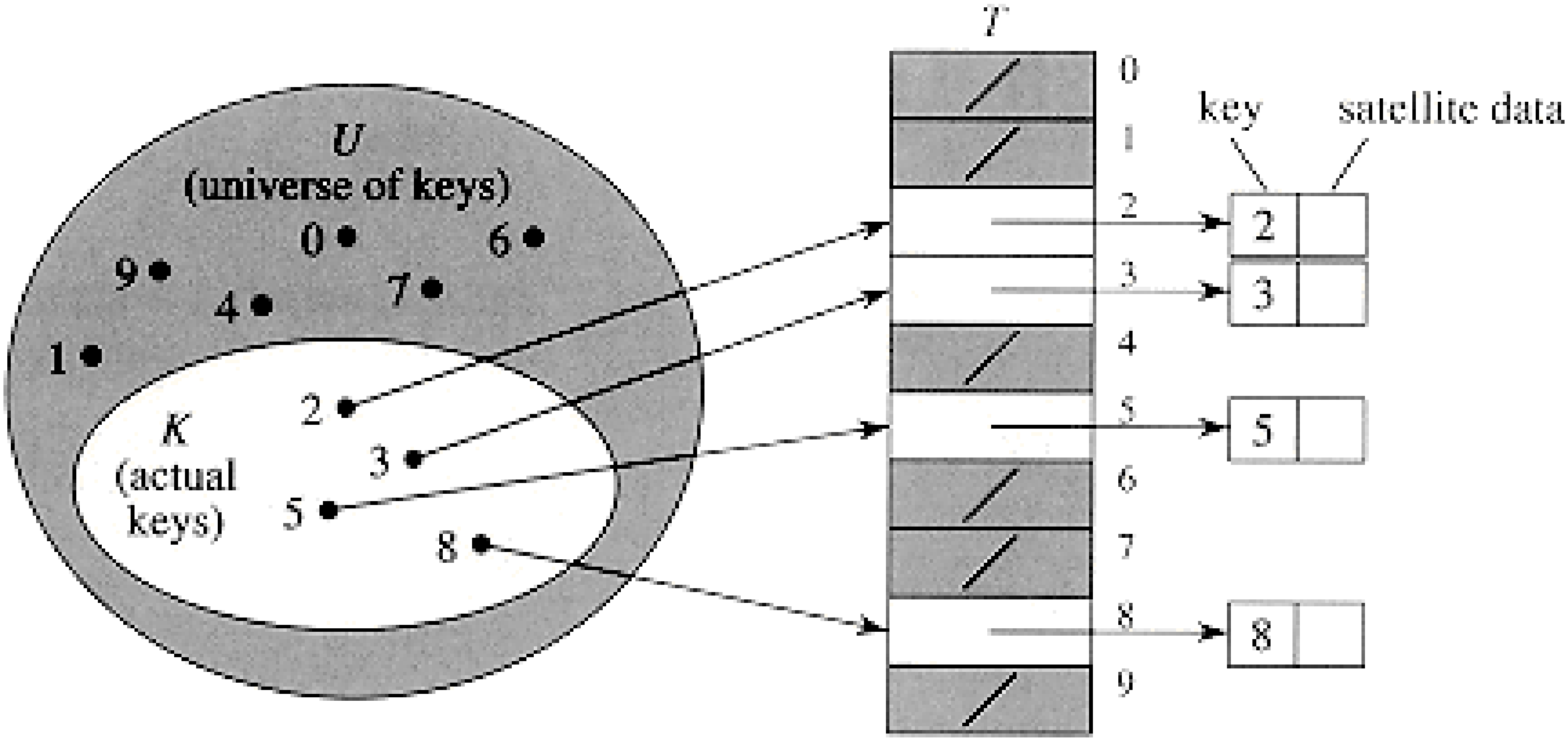
Priority queues are implemented as container adaptors

The underlying container

- accessible through random access iterators
- operations:
  - front()
  - push_back()
  - pop_back()

- random access iterators is required to keep a heap structure internally
- container adaptor call make_heap, push_heap and pop_heap

Direct address table



Direct address table

idea:

- allocate an array that has one position for every possible key

applicable: *when we can afford to …*

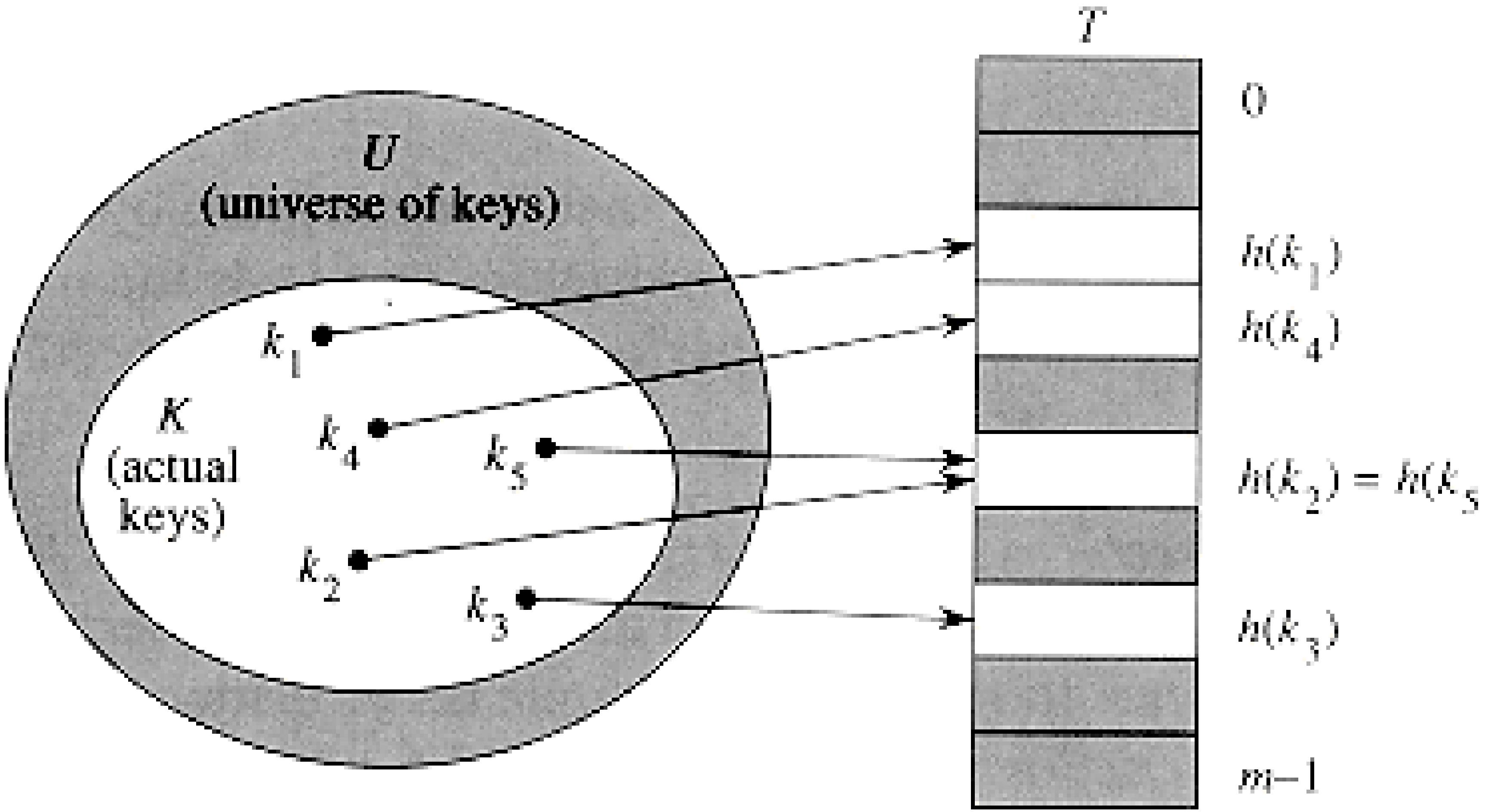
the universe U of keys is reasonably small

  - each element has a key
  - drawn from the universe $U = \{0,1, . . . , m - 1\}$, where m is not too large.
  - no two elements have the same key.

Possible ways to store elements:

1. satellite data - object external to the direct-address table with a pointer from a slot in the table to the object
2. the elements can be stored in the direct-address table itself.

Hash table


T
U
(universe of keys)
K
(actual
keys)
k_1
k_4
k_5
k_2
k_3
0
$h(k_1)$
$h(k_4)$
$h(k_2) = h(k_5)$
$h(k_3)$
$m-1$

Collision problem

- ideal solution - avoid collisions

a well-designed hash function

- minimize collisions
- deterministic:

  a given input *k* should always produce the same output *h(k)*

If $|U| > m$

- there must be two keys that have the same hash value
- avoiding collisions altogether is therefore impossible

(sometimes?)

Collision resolution by chaining