Derivatives and Structured Products Coursework

M.Sc. in International Finance
Master Level M2 - Academic Year 2025-26
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Results & Description :

Question 1

A/ We simulated the delta-hedging of a European put over 10,000 price paths and examined how the hedging error behaves under different rebalancing frequencies (monthly, weekly, and daily). For each path, we generated a stock trajectory, rebalanced the hedge N times using the Black-Scholes delta, and compared the final portfolio value with the actual payoff. The results show that hedging errors are largest when the terminal stock price ends far from the strike, reflecting the difficulty of replicating a nonlinear payoff with discrete hedging. As the hedging frequency increases, the cloud of errors becomes tighter, and daily rebalancing produces the smallest dispersion.

B/ we computed the mean and variance of these errors across all simulations. The mean error remains essentially almost zero for all N, confirming that the strategy is unbiased, while the variance decreases sharply when moving from monthly to weekly to daily hedging. These results match theoretical expectations: more frequent rebalancing significantly reduces discrete hedging risk, and the hedge becomes more accurate as the hedging interval shrinks.

C/ we extended the analysis by running 10,000 independent simulations of the delta-hedging strategy using a weekly rebalancing frequency. For each simulated price path, we recorded the terminal stock price, the option payoff, the realized variance of log-returns, and the hedging error (defined as the difference between the final hedging portfolio and the true payoff). Aggregating these 10,000 outcomes allowed us to study the distribution of hedging errors rather than individual paths. The results show that the mean hedging error is extremely close to zero, confirming that the strategy remains unbiased even when hedging discretely. However, the variance and standard deviation of errors are clearly positive, reflecting the intrinsic risk of discrete hedging when volatility is uncertain along the simulated path. The mean absolute error of about 0.70 highlights that typical deviations between the portfolio value and the payoff remain meaningful at weekly rebalancing. Overall, this large-scale simulation confirms the theoretical expectation: delta-hedging is unbiased but imperfect when implemented discretely, and even with many paths, hedging errors remain dispersed due to randomness in the underlying price dynamics.

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Statistical results:
Mean error:               -0.001501
Variance of error:         0.894962
Standard deviation:        0.946024
Mean absolute error:       0.707889
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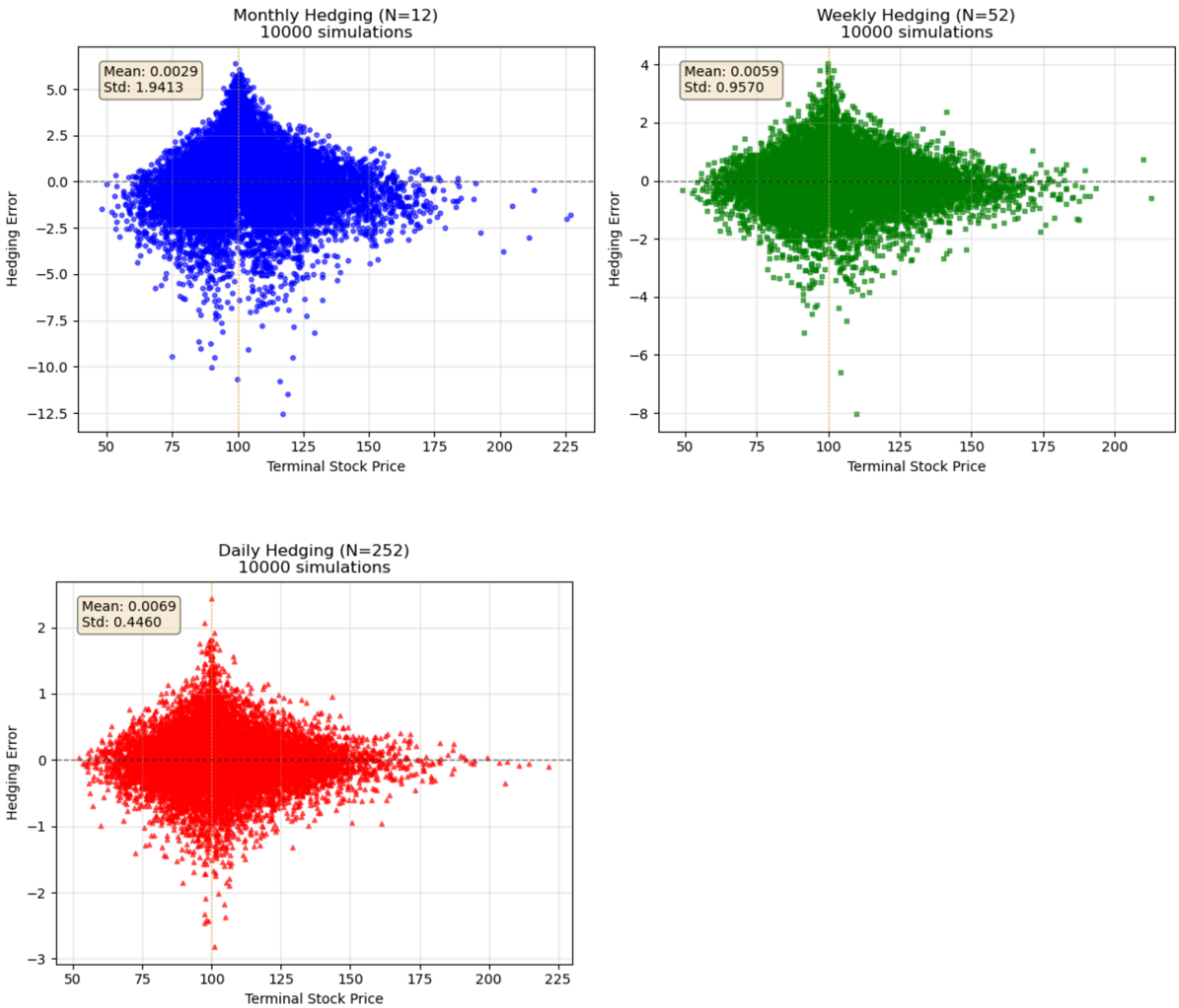
D/ we study how the replication error depends on the terminal stock price by running 10,000 simulations for three re-hedging frequencies (N = 12, 52, 252) and plotting the hedging error against ST. In the three panels, the cloud of points forms a cone centered around the strike, showing that errors tend to be small when ST is close to K and increase in magnitude as the option finishes deeply in or out of the money. Comparing the panels, we see that monthly hedging (N = 12) produces the widest cone and the largest standard deviation of errors, while daily hedging (N = 252) yields a much tighter cloud around zero, with both the mean error and its dispersion significantly reduced. Weekly hedging (N = 52) lies in between these two cases. Overall, these scatterplots confirm visually that increasing the hedging frequency makes the delta-hedging strategy more accurate, reducing both the variability and the typical size of the replication error across terminal stock prices.

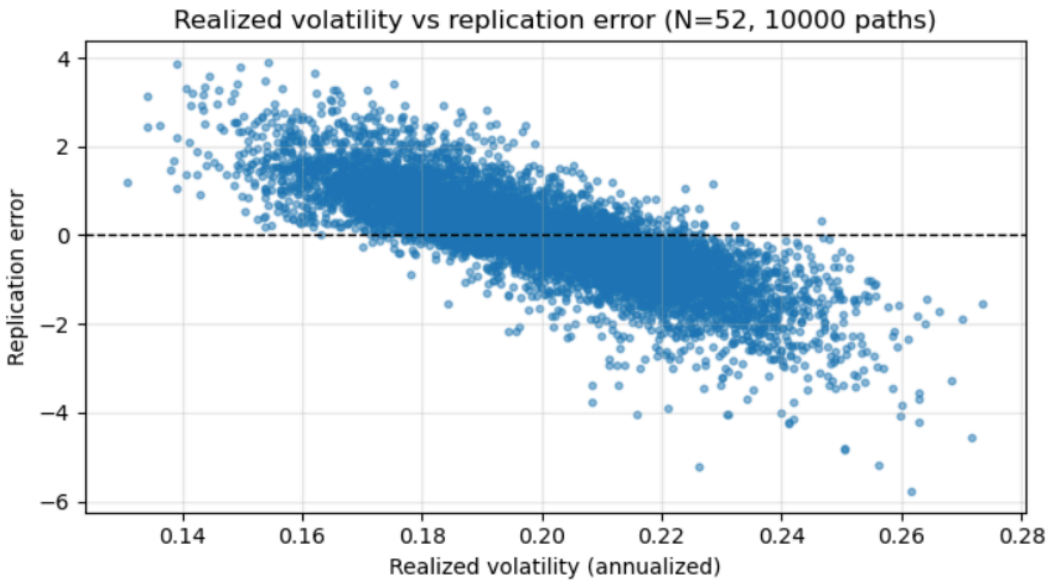


E/ We repeated the full set of 10,000 simulations for each hedging frequency N=12,52,252 and compared the mean and variance of the hedging error across these three cases. The results clearly show that the mean error stays extremely close to zero for all values of N, confirming that delta-hedging remains an unbiased replication method regardless of how often we rebalance. However, the variance of the hedging error decreases dramatically as the hedging frequency increases: monthly hedging produces the largest dispersion of errors, weekly hedging reduces it by more than half, and daily hedging brings the variance down to a very low level. This pattern is consistent with the intuition that more frequent rebalancing reduces discretization risk. Overall, this part confirms quantitatively what we observed graphically earlier: higher hedging frequency leads to a more accurate and more stable replication of the option payoff, while low-frequency hedging leaves much greater uncertainty in the final hedging outcome.

Hedging error summary (10,000 paths):

N	Mean hedging error	Variance of hedging error
12	-0.016456	3.759933
52	-0.010220	0.909294
252	-0.006645	0.191789

F/ explores how the replication error varies with the realized volatility of each simulated path. After running 10,000 simulations with weekly hedging, we plotted the replication error against the annualized realized volatility computed from the path's log-returns. The scatterplot reveals a clear negative relationship: when realized volatility is low, the replication error tends to be positive, whereas higher volatility leads to increasingly negative errors. This pattern is consistent with the structure of delta-hedging under the Black–Scholes model: the hedge is calibrated to a constant volatility (σ=20%), so if the true realized volatility ends up higher, the hedger is under-hedged and the strategy loses money; conversely, if realized volatility is lower than expected, the hedge becomes conservative and generates a small surplus. Overall, this part confirms that volatility mis-specification is a major source of hedging error, and realized volatility explains a substantial portion of the dispersion observed in earlier sections.



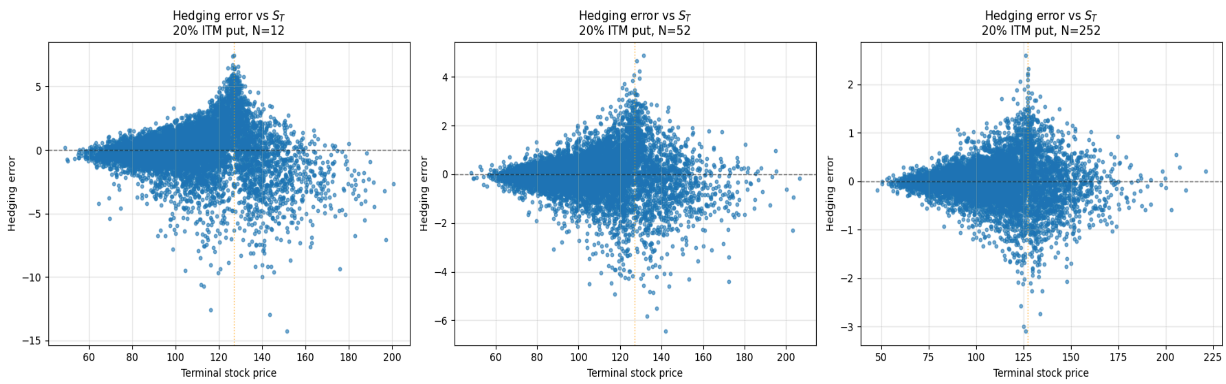
G/ analyzes how sensitive the hedging error is to changes in the drift μμ of the underlying price process. Since Black–Scholes delta-hedging theoretically does not depend on the drift, we expect little variation in hedging performance across different values of μμ. Running 10,000 simulations for μ=2.5%,5%,7.5%, and 10% confirms this prediction: both the mean absolute error and the variance of the error remain remarkably stable across all drift levels. The differences are very small and within simulation noise. This demonstrates a key theoretical property of risk-neutral hedging: delta-hedging is largely insensitive to the expected return of the stock, and the replication error is driven primarily by volatility and by the discreteness of the hedge, not by the drift.

Drift sensitivity of hedging error (N=52, 10,000 paths):

mu	Mean absolute error	Variance of error
2.5%	0.720057	0.926559
5.0%	0.710246	0.906617
7.5%	0.702610	0.880481
10.0%	0.689298	0.852547

Question 2

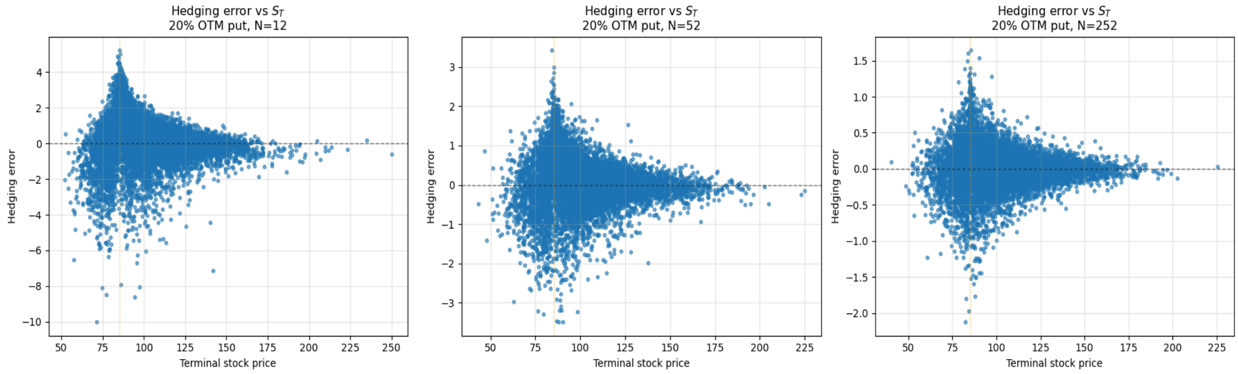
A/ For the 20% ITM put, we observe that hedging errors remain centered around zero but their magnitude is significantly larger than in the at-the-money case, especially for coarse hedging frequencies. With monthly hedging (N=12), the dispersion of errors is wide and the variance is high, showing that infrequent rebalancing struggles to follow the steep delta profile of a deep-in-the-money put. As we increase the rebalancing frequency to weekly (N=52) and daily (N=252), the errors shrink markedly and the variance drops, reflecting the improved ability of the hedge to track the option's rapid sensitivity to price movements. Overall, the ITM put amplifies hedging risk when rebalancing is infrequent, but daily hedging restores stability and produces a tight error distribution.



Descriptive statistics for 20% ITM put (10000paths):

N	Mean error	Variance	Std dev	Mean abs error
12	0.001618	2.926442	1.710685	1.121730
52	-0.001481	0.685249	0.827797	0.539571
252	-0.003733	0.152875	0.390993	0.253331

B/ For the 20% OTM put, the hedging error structure changes because the option is mostly insensitive to price movements until the terminal stock price approaches the strike. As a result, the overall errors are materially smaller than in the ITM case, and the variance drops sharply as N increases. With monthly hedging, the errors are still visible but already more contained due to the flatter delta of an out-of-the-money put. Moving to weekly and daily hedging greatly reduces both dispersion and mean absolute error, yielding very tight clusters around zero. Overall, the OTM put exhibits significantly lower hedging risk because the option behaves almost linearly (and with low delta) over most price paths, making the hedge easier to maintain even with infrequent rebalancing.



Descriptive statistics for 20% OTM put (10000 paths):

N	Mean error	Variance	Std dev	Mean abs error
12	0.001699	1.545518	1.243189	0.850629
52	0.002871	0.366170	0.605120	0.416007
252	-0.003190	0.080179	0.283159	0.191680

C/ Comparing ITM, ATM and OTM options shows that hedging difficulty increases when the option has a steep delta profile. ITM puts generate large hedging errors because their delta is high and very reactive, so discrete rebalancing struggles to track the payoff, especially with low hedging frequency. ATM options still exhibit meaningful risk, but less extreme. In contrast, OTM puts have very low delta for most paths, making them the easiest to hedge; errors remain small even with coarse rebalancing.

These differences connect directly to the volatility smile. Options that are harder to hedge typically deep ITM or tail-sensitive OTM options, require traders to charge higher implied volatility to compensate for greater replication risk. ATM options, which are moderately sensitive and most liquid, anchor the center of the smile. Overall, variations in hedging error across moneyness translate into different implied volatilities, helping produce the smile/skew observed in practice.

Descriptive statistics for ATM put (10000 paths):

N	Mean error	Variance	Std dev	Mean abs error
12	-0.006514	3.744328	1.935027	1.463716
52	0.000257	0.869008	0.932206	0.702530
252	0.003069	0.186103	0.431397	0.321080

Question 3

A/ To build the inflation curve I(0,T), we use the zero-coupon inflation swap rates. A zero-coupon inflation swap links the fixed rate to the actual inflation over the maturity, so the relation (1+sT)^T=I(0,T)/I(0,0), lets us directly compute the future CPI level implied by each quoted swap rate.

Using the base CPI of 120.70 (August 2025), we compound each swap rate over its maturity to find the CPI level for each year. For example, the 1-year rate gives a CPI close to 128, and the 30-year rate gives a CPI around 246. These values form the market-implied inflation curve, which increases smoothly over time and matches all swap quotes exactly.

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=== (a) Implied CPI at quoted maturities ===
 T (yrs)  ZC rate (%)  I(0,T) (CPI level)
     1.0         6.05          128.002350
     2.0         4.30          131.303374
     3.0         3.50          133.822248
     4.0         3.15          136.641997
     5.0         2.95          139.585088
     6.0         2.90          143.284601
     7.0         2.70          145.445529
     8.0         2.65          148.791766
     9.0         2.60          152.066488
    10.0         2.55          155.261550
    11.0         2.50          158.368860
    12.0         2.48          161.948403
    15.0         2.46          173.789104
    20.0         2.44          195.478351
    25.0         2.42          219.445438
    30.0         2.40          245.870242
```

B/ Because swap rates are only available for certain maturities, we need CPI values for the years in between. To do this, we apply log-linear interpolation, which works well for inflation because inflation compounds over time, and logs capture that naturally. We take the logarithm of each CPI level from part (a), interpolate the logs in a straight line between each pair of maturities, and then exponentiate back to get CPI levels.

This gives smooth and realistic CPI estimates for every year. For example, this method produces CPI values roughly around 140 at 5 years, 162 at 12 years, 174 at 15 years, and 246 at 30 years. Log-linear interpolation avoids the unrealistic patterns that simple linear interpolation would create and ensures that the inflation path remains consistent with market expectations.

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=== (b) Forward CPI for requested dates (log-linear interp) ===
   Label  T (yrs)  I(0,T) (CPI level)
Aug-2026        1          128.002350
Aug-2027        2          131.303374
Aug-2028        3          133.822248
Aug-2029        4          136.641997
Aug-2030        5          139.585088
Aug-2031        6          143.284601
Aug-2032        7          145.445529
Aug-2033        8          148.791766
Aug-2034        9          152.066488
Aug-2035       10          155.261550
Aug-2036       11          158.368860
Aug-2037       12          161.948403
Aug-2040       15          173.789104
Aug-2055       30          245.870242

Full list Aug-26..Aug-37 (T=1..12):
 T (yrs)  Calendar Year  I(0,T) (CPI level)
       1           2026          128.002350
       2           2027          131.303374
       3           2028          133.822248
       4           2029          136.641997
       5           2030          139.585088
       6           2031          143.284601
       7           2032          145.445529
       8           2033          148.791766
       9           2034          152.066488
      10           2035          155.261550
      11           2036          158.368860
      12           2037          161.948403
```

C/ The PV01 tells us how much the value of a zero-coupon inflation swap changes when the swap rate moves by one basis point. For a swap with maturity T, the payoff depends on (1+s)^T. The sensitivity can be found analytically by taking the derivative T(1+s)^(T-1), and then discounting it at the flat 3% nominal interest rate. Another way to find it is by simply increasing the swap rate by 1 bp, recalculating the payoff, and taking the difference.Both approaches give nearly identical values.

As expected, PV01 grows with maturity because longer swaps react more strongly to small rate changes. For example, the 1-year swap has a PV01 close to 1, while the 30-year swap has a PV01 near 25. These PV01 values help us understand the swap's sensitivity and are essential for hedging inflation risk.

=== (c) PV01 for quoted ZC inflation swaps (per 1 unit notional) ===

T (yrs)	ZC rate (%)	DF	Fixed_pay	PV01_fd (per +1bp, currency units per 1 notional)	PV01_analytic (per +1bp)	FD_vs_Analytic_diff
1.0	6.05	0.970874	0.060500	0.970874	0.000097	0.970777
2.0	4.30	0.942596	0.087849	1.966349	0.000197	1.966153
3.0	3.50	0.915142	0.108718	2.941252	0.000294	2.940958
4.0	3.15	0.888487	0.132080	3.901054	0.000390	3.900664
5.0	2.95	0.862609	0.156463	4.845891	0.000484	4.845407
6.0	2.90	0.837484	0.187114	5.798428	0.000580	5.797849
7.0	2.70	0.813092	0.205017	6.680162	0.000668	6.679494
8.0	2.65	0.789409	0.232740	7.586701	0.000758	7.585942
9.0	2.60	0.766417	0.259871	8.473361	0.000847	8.472514
10.0	2.55	0.744094	0.286343	9.337687	0.000933	9.336754
11.0	2.50	0.722421	0.312087	10.177327	0.001017	10.176310
12.0	2.48	0.701380	0.341743	11.025491	0.001102	11.024389
15.0	2.46	0.641862	0.439843	13.539123	0.001353	13.537770
20.0	2.44	0.553676	0.619539	17.523067	0.001751	17.521317
25.0	2.42	0.477606	0.818106	21.220364	0.002120	21.218244
30.0	2.40	0.411987	1.037036	24.621718	0.002459	24.619259

D/ The liabilities in Table 4 are given in real terms, meaning the cashflows are expressed in today’s purchasing power. To convert them into the actual nominal amounts the pension fund must pay in the future, each real cashflow is multiplied by the ratio I(0,T)/I(0,0), using the inflation curve built in part (a). This step increases future payments according to expected inflation.
Once we obtain these nominal cashflows, we discount them at the flat 3% nominal rate to calculate their nominal present value. To compute the real present value, we instead discount using the implied real discount rate that comes from combining the nominal curve and the inflation curve (via Fisher’s relation).
After adding up the discounted values across all maturities, the total nominal present value of the liabilities is around €167 million, while the real present value is slightly lower because it removes the inflation effect. This method ensures that both inflation adjustments and discounting are treated consistently.

=== (d) Liability valuation (Table 4) ===

T (yrs)	Real_amount_mln	I(0,T)	Nominal_CF_mln	PV_nominal_mln	spot_inflation	real_rate	PV_real_mln
1	1.0	128.002350	1.060500	1.029612	0.0605	-0.028760	1.029612
2	2.0	131.303374	2.175698	2.050804	0.0430	-0.012464	2.050804
3	4.0	133.822248	4.434872	4.058536	0.0350	-0.004831	4.058536
4	5.0	136.641997	5.660398	5.029190	0.0315	-0.001454	5.029190
5	8.0	139.585088	9.251704	7.980601	0.0295	0.000486	7.980601
6	10.0	143.284601	11.871135	9.941889	0.0290	0.000972	9.941889
7	12.0	145.445529	14.460202	11.757467	0.0270	0.002921	11.757467
8	15.0	148.791766	18.491106	14.597050	0.0265	0.003410	14.597050
9	18.0	152.066488	22.677687	17.380559	0.0260	0.003899	17.380559
10	25.0	155.261550	32.158565	23.928992	0.0255	0.004388	23.928992
11	23.0	158.368860	30.177993	21.801224	0.0250	0.004878	21.801224
12	20.0	161.948403	26.834864	18.821434	0.0248	0.005074	18.821434
15	15.0	173.789104	21.597652	13.862711	0.0246	0.005270	13.862711
20	9.0	195.478351	14.575851	8.070295	0.0244	0.005467	8.070295
25	6.0	219.445438	10.908638	5.210026	0.0242	0.005663	5.210026
30	2.0	245.870242	4.074072	1.678464	0.0240	0.005859	1.678464

Total nominal PV (sum over maturities): 167,198,853.07 (currency units)
Total nominal PV (in millions): 167.199 mln
Total real PV (today EUR) (sum): 167,198,853.07 (currency units)
Total real PV (in millions): 167.199 mln

E/ The pension liabilities carry two main risks:

1. Interest-rate risk from discounting future cashflows
2. Inflation risk because payments rise with inflation

To hedge the interest-rate risk, the fund can use nominal instruments such as government bonds or receive-fixed interest-rate swaps. These are chosen so that their duration and key-rate sensitivities match those of the liabilities.
To hedge inflation risk, the fund uses inflation-linked instruments. The most direct tools are zero-coupon inflation swaps, where the pension fund receives the inflation leg, or CPI-linked government bonds. These instruments move in line with expected inflation and therefore offset the inflation uplift on the liabilities.
By selecting hedging instruments with maturities aligned to the liability schedule, the fund can closely match both its nominal and inflation exposures.
This combined strategy keeps the present value of the liabilities stable even when interest rates or inflation expectations change.

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=== (e) Hedging guidance ===

1) Interest-rate risk:
   - Nominal PV to hedge (sum): 167.199 mln
   - Use nominal bonds or receive-fixed interest-rate swaps sized to match the nominal PV and duration.

2) Inflation risk:
   - Liabilities are CPI-indexed (real). To remove inflation exposure, enter zero-coupon inflation swaps where you RECEIVE the inflation leg (i.e., you receive CPI growth).
   - For a specific maturity T, the nominal payment = Real_amount * I(0,T)/I0.
     A ZC inflation swap (per unit notional) pays (I(T)/I0 - 1). Size N such that N*(I(T)/I0 - 1) ≈ nominal payment due to indexation (or use exact replication formulas).
   - Alternatively buy inflation-linked government bonds (linkers) with matching maturities.

3) Combined strategy:
   - Decompose liabilities into nominal & inflation components and hedge each with suitable instruments.
   - Match cashflows and key-rate durations; monitor basis risk between market ZCIS and realized CPI.
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Question 4

A/ The Bootstrapped curve is consistent with the quoted market spreads and a 40% recovery rate. The implied hazard rate increases with maturity and the survival probabilities decline smoothly.

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CDS Curve for Company DoOrDie:
- Trade Date: 01-NOV-2025
- Recovery Rate: 40.0%
- Flat Interest Rate: 3.0%

CDS Contracts in Curve:
1Y CDS: Maturity 01-NOV-2026, Spread 75.00bp
2Y CDS: Maturity 01-NOV-2027, Spread 78.00bp
3Y CDS: Maturity 01-NOV-2028, Spread 80.00bp
4Y CDS: Maturity 01-NOV-2029, Spread 85.00bp
5Y CDS: Maturity 01-NOV-2030, Spread 90.00bp
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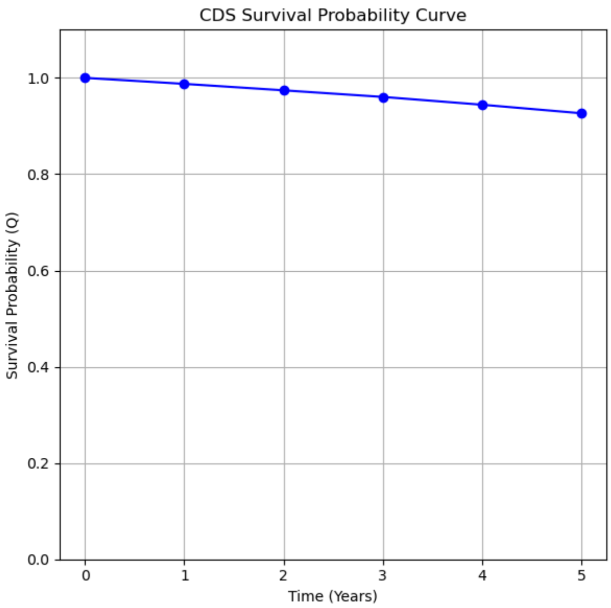

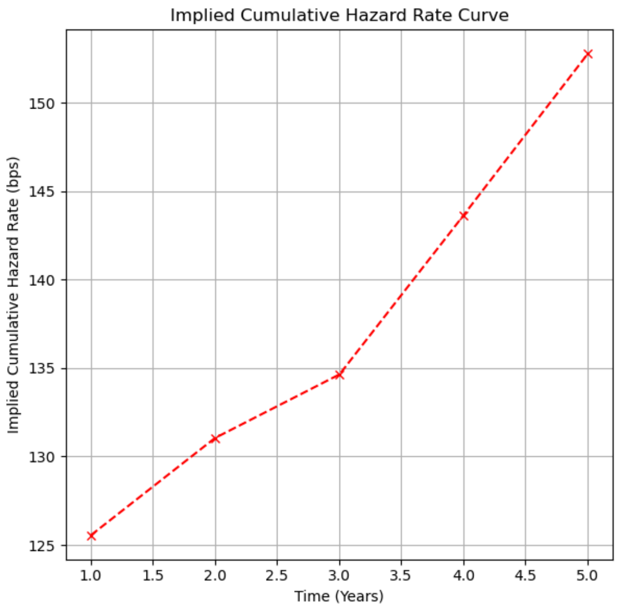

B/ The model returns a par spread of 79.48 bp for a 4 year CDS. This is significantly below 120 bp meaning the market currently prices the credit risk of DoOrDie lower than what the contract pays.

C/ The value of long protection CDS is PV= -266,185.4 USD. This negative value indicates that receiving 120 bp is expensive relative to the current fair spread of 79.48 bp. The protection buyer pays an above-market premium, so the position has a negative mark-to-market.

D/ The PV becomes slightly less negative as recovery increases. The effect is small, which is expected because the dominant driver of value is the probability of default from the calibrated curve, not the exact loss given default. Changing recovery shifts the LGD but does not alter the survival probabilities, so PV moves only modestly.

```
Value of long-protection CDS (spread 120 bp, notional 20m)
Recovery  |   PV (in $)
------------------------------
  0%      |  -268,300.61
 10%      |  -267,946.39
 20%      |  -267,504.58
 30%      |  -266,938.10
```

E/ The 3Y and 4Y buckets dominate the risk because the CDS matures in 3.5 years. Spread shocks near the contract maturity change the survival probabilities in the most relevant region. Short-end (1Y, 2Y) effects are negligible, and the 5Y bucket has no marginal impact for this maturity structure. A dealer would hedge mainly using 3Y and 4Y CDS positions, since they carry almost all of the CS01 exposure of this trade.

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Change in PV of the 3.5Y CDS (120 bp, 20m notional)
for a +1 bp increase in each market CDS quote

Tenor  |  dPV per +1 bp  (USD)
--------------------------------
  1Y   |          11.11
  2Y   |          23.36
  3Y   |       3,509.83
  4Y   |       2,870.44
  5Y   |           0.00
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