Online algorithms for two problems in high-dimensional convex geometry

Naren Sarayu Manoj (https://nsmanoj.com)

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2023 Nov 06

Modern input concerns for algorithmic data science

My goal – design and analyze algorithms for more realistic input scenarios.

- Research statement – https://narenmanoj.github.io/nsm_statement.pdf
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Streaming Algorithms for Ellipsoidal Approximation of Convex Polytopes

Conference of Learning Theory (COLT) 2022

Yury Makarychev, *NSM*, Max Ovsiankin

Symmetric ellipsoidal approximations

Basic problem

Given a convex body X, find an ellipsoid $\mathcal{E}$ such that $\mathcal{E}/\alpha \subseteq X \subseteq \mathcal{E}$ for a small α.

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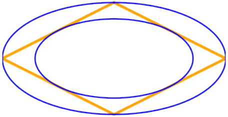

Figure: $\mathcal{E}/\sqrt{2} \subseteq X \subseteq \mathcal{E}$

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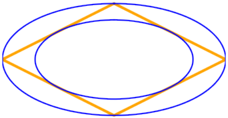

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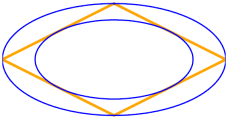

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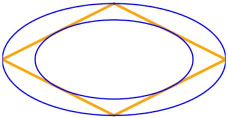

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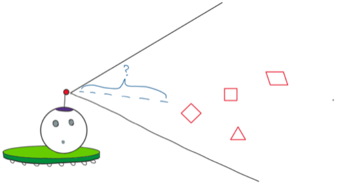

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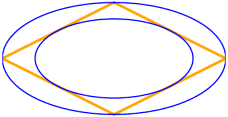

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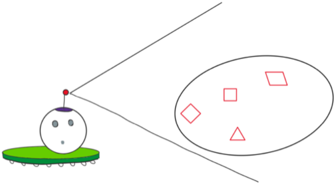

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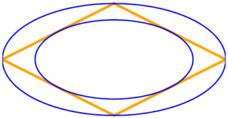

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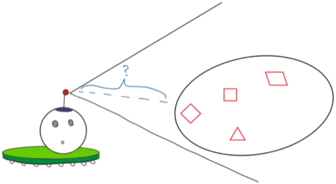

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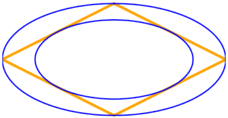

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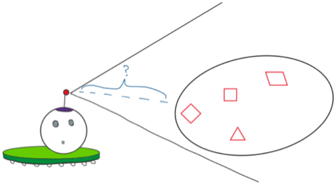

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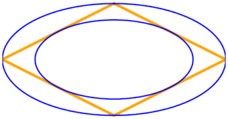

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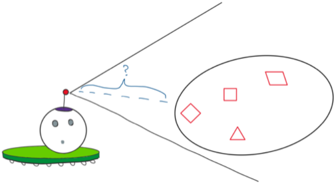
- Succinctly representing a convex body (d^2 floats)



Offline solution

Symmetric ellipsoidal approximation – offline

Given a symmetric convex body $X \subseteq \mathbb{R}^d$, compute an ellipsoid $\mathcal{E}$ with $\mathcal{E}/\alpha \subseteq X \subseteq \mathcal{E}$ that minimizes α. Let α be $\mathcal{E}$'s *approximation factor*.

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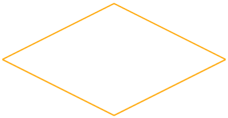

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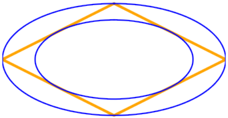

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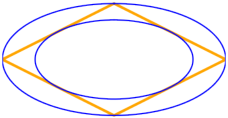

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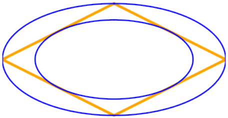

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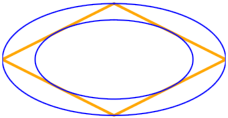

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There exists X for which the approximation factor $\sqrt{d}$ is tight for *any* ellipsoidal approximation for X (e.g. cross polytope, hypercube). For symmetric polytopes with n linear constraints, John's Ellipsoid can be approximated in time $\widetilde{O}\left(nd^2\right)$ [CCLY19].

Streaming/online ellipsoidal approximations

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Given a symmetric convex body $X = \mathrm{conv}(\pm \boldsymbol{x}_1, \ldots, \pm \boldsymbol{x}_n)$ as a stream of points, find an ellipsoid $\mathcal{E}$ with ${}^{\mathcal{E}}/_{\alpha} \subseteq X \subseteq \mathcal{E}$ that minimizes α.

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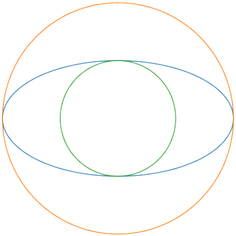

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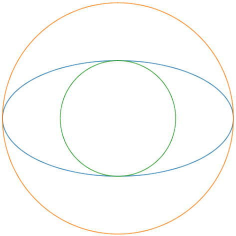

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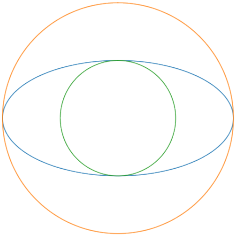

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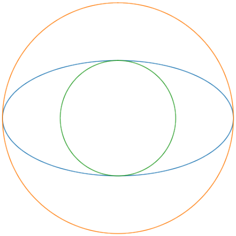

Figure: $1 \cdot B_2^d \subseteq \mathcal{E} \subseteq 2 \cdot B_2^d \Rightarrow \kappa(\mathcal{E}) = {}^{2}/_{1}$

Example

- For an ellipsoid $\mathcal{E}$ with major axis $\lambda_{\max}$ and minor axis $\lambda_{\min}$, we have $\kappa(\mathcal{E}) = {}^{\lambda_{\max}}/_{\lambda_{\min}}$;
- $\kappa(B_1^d) = \sqrt{d}$;
- $\kappa(B_2^d) = 1$

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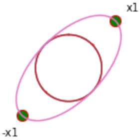

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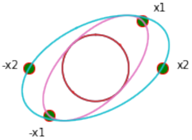

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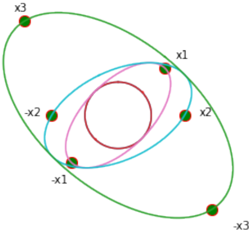

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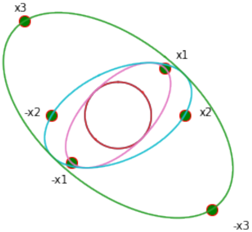

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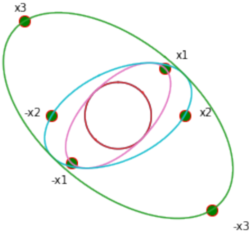

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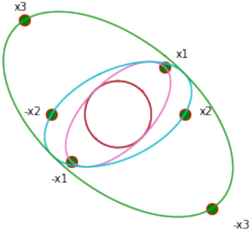

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- Compared to the optimal offline solution, we lose only an excess $O(\sqrt{\log\left({}^{R}/_{r} + 1\right)})$ factor.
- We have strong evidence to suggest that the $\sqrt{\log\left({}^{R}/_{r} + 1\right)}$ extra loss is necessary.

Proof strategy

$\mathcal{E}$ is an α-approximation if ${}^{\mathcal{E}}/_{\alpha} \subseteq \operatorname{conv}(\pm \boldsymbol{x}_1, \ldots, \pm \boldsymbol{x}_n) \subseteq \mathcal{E}$.

Define:

$$E_{R/r} := \{\mathcal{E} \supseteq X \ : \ \mathcal{E} \text{ has aspect ratio at most } {}^{R}/_{r}\}$$

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Part 1 – Identifying an ellipsoid to track

Consider an ellipsoid $\mathcal{E}^\star \supseteq X$ such that $\kappa(\mathcal{E}^\star) \leq {}^{R}\!/\!_{r}$. Pick a matrix $\mathbf{J}^\star$ such that $\mathcal{E}^\star = \left\{x \; : \; \|\mathbf{J}^\star \boldsymbol{x}\|_2 \leq 1\right\}$.

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Observation

If $\sigma_{\max}\left(\mathbf{J}^\star \cdot \mathbf{A}_n^{-1}\right) \leq \alpha$, and if for a point $\boldsymbol{x} \in \mathbb{R}^d$ we have $\|\mathbf{A}_n \boldsymbol{x}\|_2 \leq 1$, then we have $\left\|\left({}^{\mathbf{J}^\star}/_\alpha\right) x\right\|_2 \leq 1$.

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Interpretation – $\mathbf{J}^\star \cdot \mathbf{A}_n^{-1}$ describes the image of $\mathcal{E}_n$ where we transform the space such that $\mathcal{E}^\star$ corresponds to the unit ball.

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Upshot – it's sufficient to show that $\sigma_{\max}\left(\mathbf{J}^\star \cdot \mathbf{A}_n^{-1}\right) \lesssim \sqrt{d \log\left(R/r + 1\right)}$.

Part 2 – Potential functions

Goal – Identify some useful $\Phi(\mathcal{E}_t)$ and claim that
$\sigma_{\max}\left(\mathbf{J}^\star \cdot A_n^{-1}\right)^2 \leq \Phi(\mathcal{E}_n) \leq \Phi(\mathcal{E}_{n-1}) \leq \cdots \leq \Phi(\mathcal{E}_0) \lesssim d \log\left({}^{R}\!/_{r} + 1\right).$

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Interpretation – the singular values of $\mathbf{J}^\star \cdot \mathbf{A}_t^{-1}$ correspond to the lengths of the axes of $\mathbf{J}^\star \cdot \mathcal{E}_t$. Thus, $\Phi_{\mathbf{J}^\star}(\mathcal{E}_n) \gtrsim \sigma_{\max}\left(\mathbf{J}^\star \cdot \mathbf{A}_n^{-1}\right)^2$.

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$$\Phi_{\mathbf{J}^\star}(\mathcal{E}_t) := \left\| \mathbf{J}^\star \cdot \mathbf{A}_t^{-1} \right\|_F^2 - 2 \log\det\left(\mathbf{J}^\star \cdot \mathbf{A}_t^{-1}\right) = \sum_{i=1}^{d} \sigma_{i,t}^2 - \log\left(\sigma_{i,t}^2\right)$$

Interpretation – the singular values of $\mathbf{J}^\star \cdot \mathbf{A}_t^{-1}$ correspond to the lengths of the axes of $\mathbf{J}^\star \cdot \mathcal{E}_t$. Thus, $\Phi_{\mathbf{J}^\star}(\mathcal{E}_n) \gtrsim \sigma_{\max}\left(\mathbf{J}^\star \cdot \mathbf{A}_n^{-1}\right)^2$.

Next, we show:

$$\frac{\left\|\mathbf{A}_t^{-1}\right\|_F^2 - \left\|\mathbf{A}_{t-1}^{-1}\right\|_F^2}{2\log\det\left(\mathbf{A}_t^{-1}\right) - 2\log\det\left(\mathbf{A}_{t-1}^{-1}\right)} = \frac{1 - \frac{1}{\left\|\mathbf{A}_{t-1}\boldsymbol{x}_t\right\|_2^2}}{2\log\left\|\mathbf{A}_{t-1}\boldsymbol{x}_t\right\|_2} \cdot \left\|\mathbf{J}^\star \boldsymbol{x}_t\right\|^2$$

$$\leq 1$$

$$\Rightarrow \Phi_{\mathbf{J}^\star}(\mathcal{E}_t) \leq \Phi_{\mathbf{J}^\star}(\mathcal{E}_{t-1})$$

Part 2 – Potential functions

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Suffices to control the RHS.

Part 3 – Evaluating initial potential function value

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This is enough!

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Let J be the minimum volume outer ellipsoid covering X.
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Upshot – No "monotone" one-pass algorithm can have an output close to the true minimum-volume outer ellipsoid for every sequence of inputs.

Hard Instance – Outline

Hadamard basis

For a dimension d, we say there exists a Hadamard basis for $\mathbb{R}^d$ if we can find a collection of d vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ such that $\boldsymbol{v}_i \in \{\pm 1\}^d$ and for all $i \neq j$, we have $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0$.

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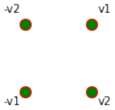

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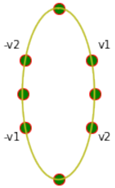
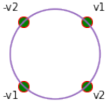
Phase 2

The instance selects $i \in [d]$ and $\varepsilon \in (0, d-1)$. Let $\boldsymbol{w}_i = \boldsymbol{e}_i \cdot {}^1/\sqrt{d-\varepsilon}$ and $\boldsymbol{w}_j = \boldsymbol{e}_j \cdot \sqrt{{}^{d-1}/\varepsilon}$, for all $j \neq i$. Give the points $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d$.



Hard Instance – Analysis Outline

- At the end of Phase 1, the minimum volume outer ellipsoid is B_2^d.



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- At the end of Phase 1, the minimum volume outer ellipsoid is B_2^d.
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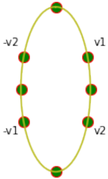
$$\left\{ \boldsymbol{x} \ : \ 1 \geq \frac{\boldsymbol{x}_i^2}{\left(1/\sqrt{d-\varepsilon}\right)^2} + \sum_{j \neq i} \frac{\boldsymbol{x}_j^2}{\left(\sqrt{d-1/\varepsilon}\right)^2} \right\}$$



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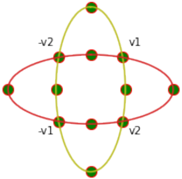

Key observation – it is not possible to output an ellipsoid at the end of Phase 1 such that all outcomes of Phase 2 yield approximation factor $< \sqrt{d-\varepsilon}$.

Conclusion – Streaming Algorithms for Ellipsoidal Approximation of Convex Polytopes

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3. Paper – https://arxiv.org/abs/2206.07250.

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Dueling convex optimization with a monotone adversary

under conference review

Avrim Blum, Meghal Gupta, Gene Li, *NSM*, Aadirupa Saha, Chloe Yang

Preference-based feedback

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Item $m+1$

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Original problem (without the monotone adversary) was studied by Jamieson, Nowak, and Recht [JNR12] and Saha, Koren, and Mansour [SKM22].

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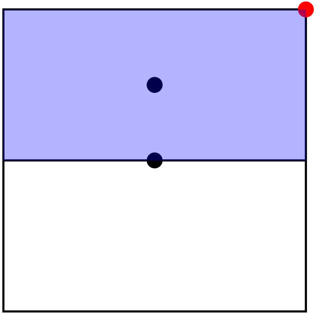

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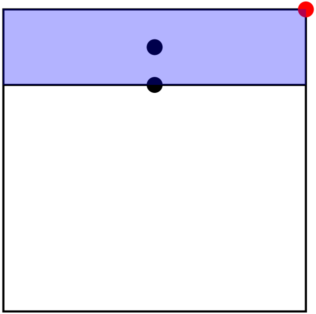

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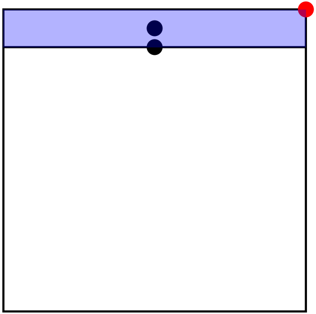

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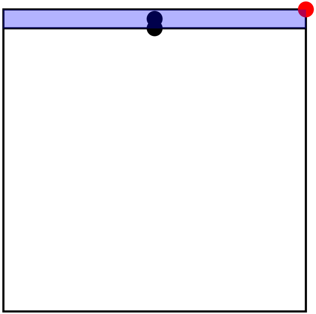

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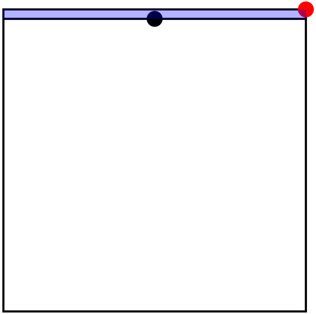

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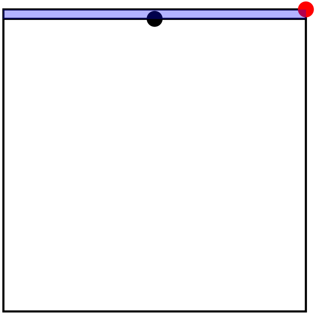

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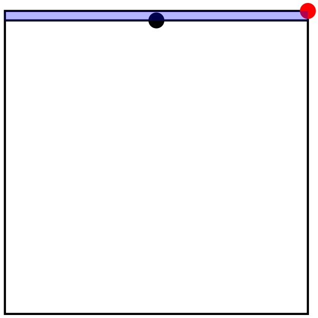

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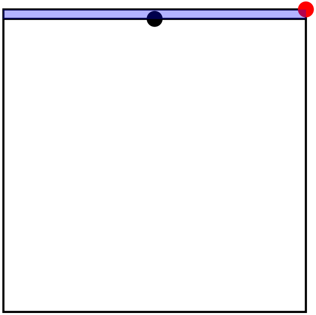

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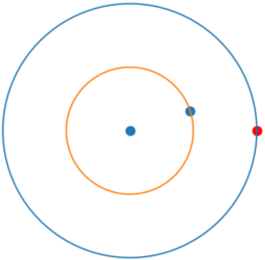

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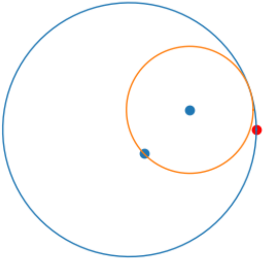

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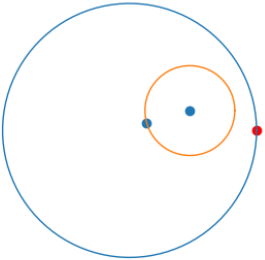

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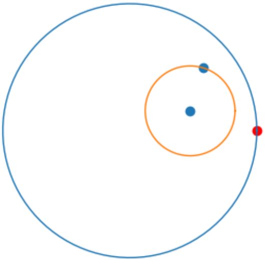

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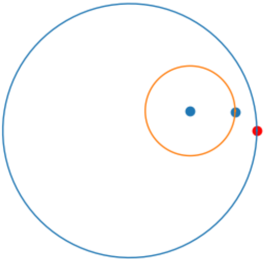

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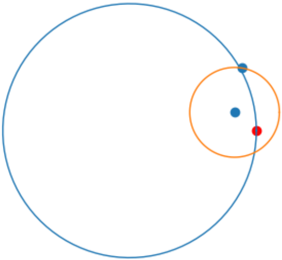

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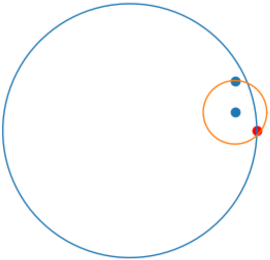

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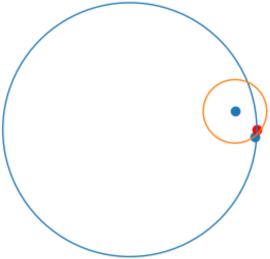

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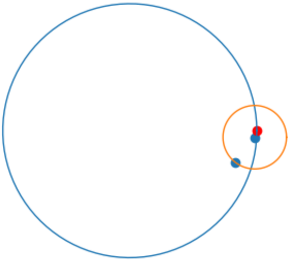

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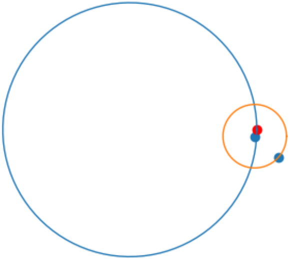

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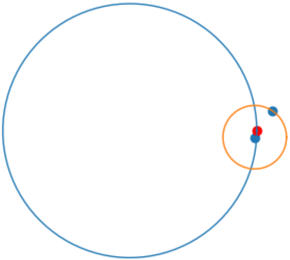

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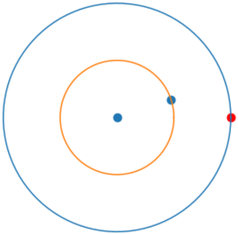

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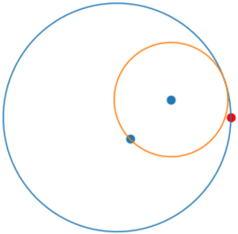

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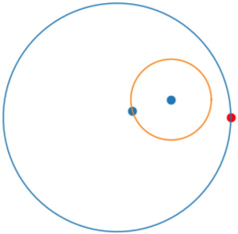

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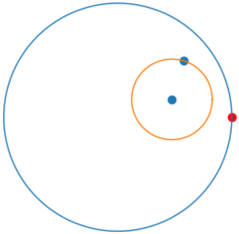

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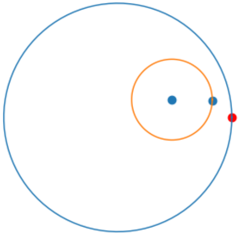

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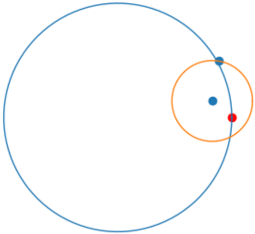

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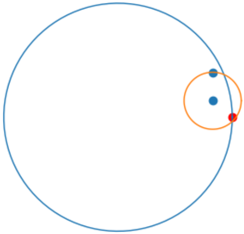

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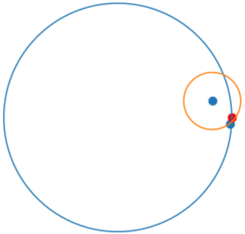

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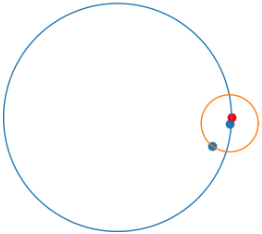

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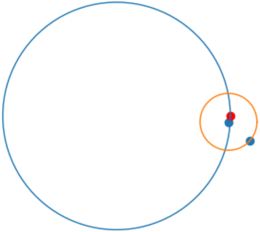

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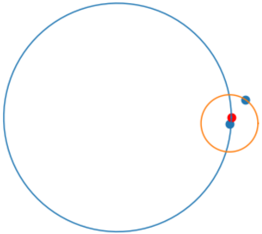

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If $f(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{x}^\star\|_2^2$, then any randomized algorithm for m-ary convex optimization that is allowed to suggest up to m points in each round must take at least $\sim \frac{d}{\log m}$ rounds to identify a 0.1^2-optimal solution.

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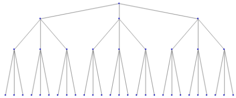

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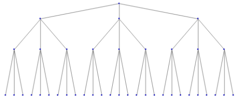

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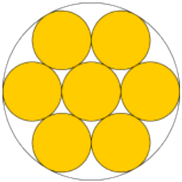

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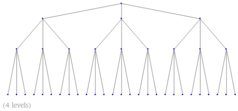

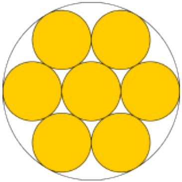

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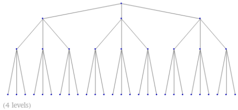

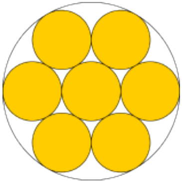

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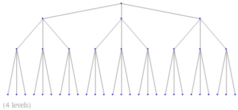

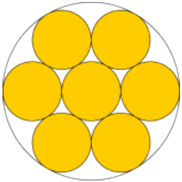

Upshot – our algorithm is **optimal**.

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- arXiv forthcoming! (or email me for the manuscript)

Modern input concerns for algorithmic data science

My goal – design and analyze algorithms for more realistic input scenarios.

Unwieldy input

- Approximating convex polytopes in a stream (Makarychev, Manoj, and Ovsiankin [MMO22; MMO23], COLT 2022/under submission).
- Approximating large convex objective functions (Manoj and Ovsiankin [MO23], under submission).
- Explaining classifier predictions on large inputs (Gupta and Manoj [GM23], SOSA 2023).

Unexpected input

- Robust machine learning under backdoor poisoning attacks (Manoj and Blum [MB21], NeurIPS 2021).
- Learning from out-of-list feedback (Blum, Gupta, Li, Manoj, Saha, and Yang [BGLMSY23], under submission).
- Generalization of short-program interpolators (Manoj and Srebro [MS23], COLT 2023).

- Research statement – https://narenmanoj.github.io/nsm_statement.pdf
- Publication list – https://narenmanoj.github.io/nsm_publist.pdf

Questions?

Thank you!!

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