Online algorithms for two problems in high-dimensional convex geometry

Naren Sarayu Manoj (https://nsmanoj.com)

TTIC

2023 Nov 06

Modern input concerns for algorithmic data science

- Research statement – https://narenmanoj.github.io/nsm_statement.pdf
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Table of contents

Streaming Algorithms for Ellipsoidal Approximation of Convex Polytopes

Conference of Learning Theory (COLT) 2022

Yury Makarychev, *NSM*, Max Ovsiankin

Symmetric ellipsoidal approximations

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Given a convex body X, find an ellipsoid $\mathcal{E}$ such that ${}^{\mathcal{E}}/_{\alpha} \subseteq X \subseteq \mathcal{E}$ for a small α.

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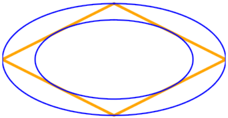

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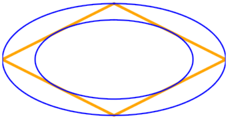

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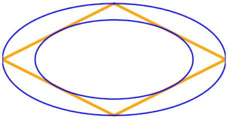

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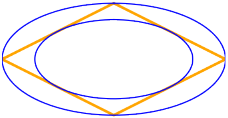

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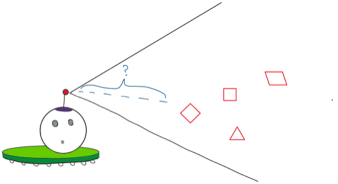

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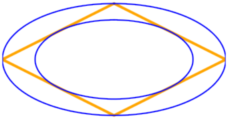

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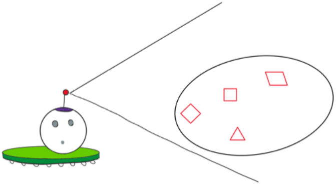

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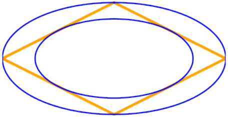

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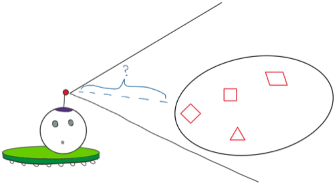

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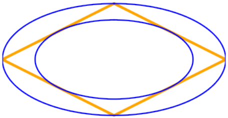

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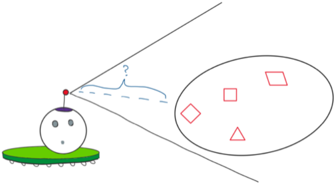

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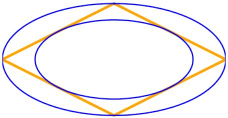

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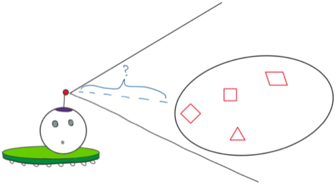
- Succinctly representing a convex body (d^2 floats)



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Given a symmetric convex body $X \subseteq \mathbb{R}^d$, compute an ellipsoid $\mathcal{E}$ with $\mathcal{E}/\alpha \subseteq X \subseteq \mathcal{E}$ that minimizes α. Let α be $\mathcal{E}$'s *approximation factor*.

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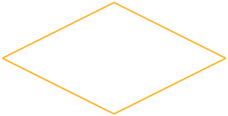

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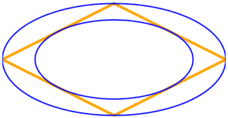

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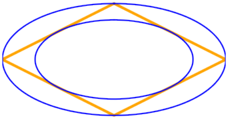

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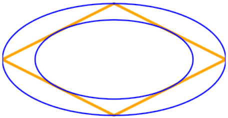

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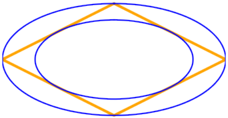

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There exists X for which the approximation factor $\sqrt{d}$ is tight for *any* ellipsoidal approximation for X (e.g. cross polytope, hypercube). For symmetric polytopes with n linear constraints, John's Ellipsoid can be approximated in time $\widetilde{O}\left(nd^2\right)$ [CCLY19].

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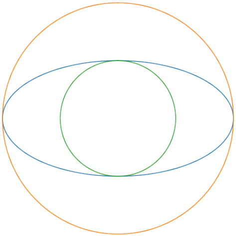

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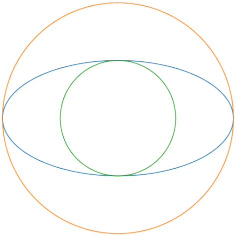

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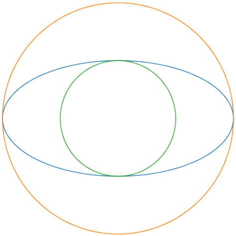

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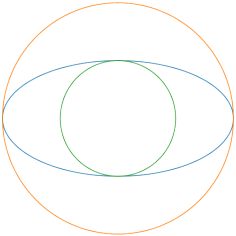

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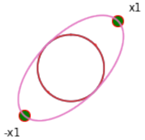

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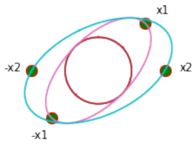

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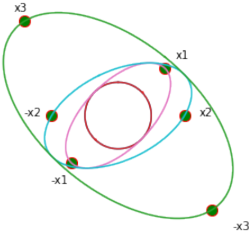

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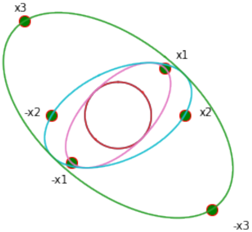

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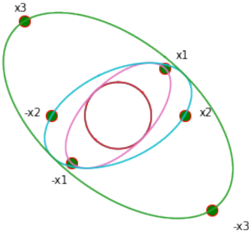

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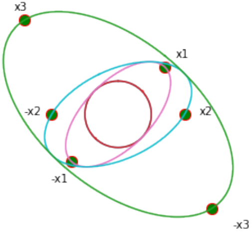

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Optimality:

- Compared to the optimal offline solution, we lose only an excess $O(\sqrt{\log\left({}^{R}/_{r} + 1\right)})$ factor.
- We have strong evidence to suggest that the $\sqrt{\log\left({}^{R}/_{r} + 1\right)}$ extra loss is necessary.

Proof strategy

$\mathcal{E}$ is an α-approximation if $\mathcal{E}/\alpha \subseteq \operatorname{conv}(\pm \boldsymbol{x}_1, \ldots, \pm \boldsymbol{x}_n) \subseteq \mathcal{E}$.

Define:

$$E_{R/r} := \{\mathcal{E} \supseteq X \ : \ \mathcal{E} \text{ has aspect ratio at most } R/r\}$$

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1. (1) follows from a potential function argument.
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Part 1 – Identifying an ellipsoid to track

Consider an ellipsoid $\mathcal{E}^\star \supseteq X$ such that $\kappa(\mathcal{E}^\star) \leq R/r$. Pick a matrix $\mathbf{J}^\star$ such that $\mathcal{E}^\star = \left\{x \ : \ \|\mathbf{J}^\star \boldsymbol{x}\|_2 \leq 1\right\}$.

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Upshot – it's sufficient to show that $\sigma_{\max}\left(\mathbf{J}^\star \mathbf{A}_n^{-1}\right) \lesssim \sqrt{d \log\left({}^R\!/_r + 1\right)}$.

Goal – Identify some useful $\Phi(\mathcal{E}_t)$ and claim that
$\sigma_{\max}\left(\mathbf{J}^\star \cdot A_n^{-1}\right)^2 \leq \Phi(\mathcal{E}_n) \leq \Phi(\mathcal{E}_{n-1}) \leq \cdots \leq \Phi(\mathcal{E}_0) \lesssim d \log\left({}^{R}/_{r} + 1\right)$.

Defining the potential

Part 2 – Potential functions

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Interpretation – the singular values of $\mathbf{J}^\star\mathbf{A}_t^{-1}$ correspond to the lengths of the axes of $\mathbf{J}^\star \cdot \mathcal{E}_t$. Thus, $\Phi_{\mathbf{J}^\star}(\mathcal{E}_n) \gtrsim \sigma_{\max}\left(\mathbf{J}^\star\mathbf{A}_n^{-1}\right)^2$.

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$$\frac{\left\|\mathbf{J}^\star \mathbf{A}_t^{-1}\right\|_F^2 - \left\|\mathbf{J}^\star \mathbf{A}_{t-1}^{-1}\right\|_F^2}{2\log\det\left(\mathbf{J}^\star \mathbf{A}_t^{-1}\right) - 2\log\det\left(\mathbf{J}^\star \mathbf{A}_{t-1}^{-1}\right)} = \frac{1 - \frac{1}{\left\|\mathbf{A}_{t-1}\boldsymbol{x}_t\right\|_2^2}}{2\log\left\|\mathbf{A}_{t-1}\boldsymbol{x}_t\right\|_2} \cdot \left\|\mathbf{J}^\star \boldsymbol{x}_t\right\|^2 \leq 1$$

Part 2 – Potential functions

Goal – Identify some useful $\Phi(\mathcal{E}_t)$ and claim that
$\sigma_{\max}\left(\mathbf{J}^\star \cdot A_n^{-1}\right)^2 \le \Phi(\mathcal{E}_n) \le \Phi(\mathcal{E}_{n-1}) \le \cdots \le \Phi(\mathcal{E}_0) \lesssim d \log\left({}^R/_r + 1\right)$.

Defining the potential

Let $\mathbf{A}_t$ be an invertible transformation such that $\mathcal{E}_t = \left\{\boldsymbol{x} : \|\mathbf{A}_t\boldsymbol{x}\|_2 \le 1\right\}$. Then:

$$\Phi_{\mathbf{J}^\star}(\mathcal{E}_t) := \left\|\mathbf{J}^\star\mathbf{A}_t^{-1}\right\|_F^2 - 2\log\det\left(\mathbf{J}^\star\mathbf{A}_t^{-1}\right) = \sum_{i=1}^{d} \sigma_{i,t}^2 - \log\left(\sigma_{i,t}^2\right)$$

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Suffices to control the RHS.

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This is enough!

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Let J be the minimum volume outer ellipsoid covering X.
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Upshot – No "monotone" one-pass algorithm can have an output close to the true minimum-volume outer ellipsoid for every sequence of inputs.

Hadamard basis

For a dimension d, we say there exists a Hadamard basis for $\mathbb{R}^d$ if we can find a collection of d vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ such that $\boldsymbol{v}_i \in \{\pm 1\}^d$ and for all $i \neq j$, we have $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0$.

Hard Instance – Outline

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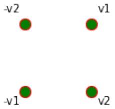
Give the points $\boldsymbol{v}_1/\sqrt{d}, \ldots, \boldsymbol{v}_d/\sqrt{d}$.



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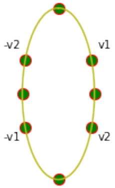
Phase 1

Give the points ${}^{\boldsymbol{v}_1}/_{\sqrt{d}}, \ldots, {}^{\boldsymbol{v}_d}/_{\sqrt{d}}$.

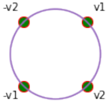
Phase 2

The instance selects $i \in [d]$ and $\varepsilon \in (0, d-1)$. Let $\boldsymbol{w}_i = \boldsymbol{e}_i \cdot {}^1/_{\sqrt{d-\varepsilon}}$ and $\boldsymbol{w}_j = \boldsymbol{e}_j \cdot \sqrt{{}^{d-1}/_{\varepsilon}}$, for all $j \neq i$. Give the points $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d$.



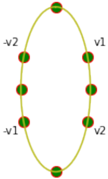
Hard Instance – Analysis Outline

- At the end of Phase 1, the minimum volume outer ellipsoid is B_2^d.



- At the end of Phase 1, the minimum volume outer ellipsoid is B_2^d.
- At the end of Phase 2, the final minimum volume outer ellipsoid is:

$$\left\{ \boldsymbol{x} \ : \ 1 \geq \frac{\boldsymbol{x}_i^2}{\left({}^1\!/\!\sqrt{d-\varepsilon}\right)^2} + \sum_{j \neq i} \frac{\boldsymbol{x}_j^2}{\left(\sqrt{d - {}^1\!/\!\varepsilon}\right)^2} \right\}$$



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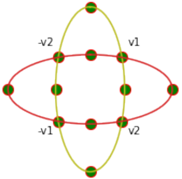

Key observation – it is not possible to output an ellipsoid at the end of Phase 1 such that all outcomes of Phase 2 yield approximation factor $< \sqrt{d - \varepsilon}$.

Conclusion – Streaming Algorithms for Ellipsoidal Approximation of Convex Polytopes

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3. Paper – `https://arxiv.org/abs/2206.07250`.

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Dueling convex optimization with a monotone adversary

under conference review

Avrim Blum, Meghal Gupta, Gene Li, *NSM*, Aadirupa Saha, Chloe Yang

Preference-based feedback

Consider a *recommendation* problem – we give a user some items and ask them to choose their favorite. Over time, we want to learn their preferences.

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2. minimize the total cost $\sum_{t=1}^{\infty}\left(\max\left\{f\left(\boldsymbol{x}_t^{(1)}\right), f\left(\boldsymbol{x}_t^{(2)}\right)\right\} - f(\boldsymbol{x}^\star)\right)$.

Original problem (without the monotone adversary) was studied by Jamieson, Nowak, and Recht [JNR12] and Saha, Koren, and Mansour [SKM22].

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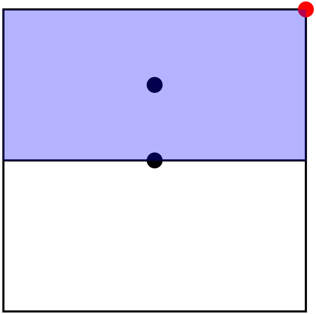
Choose query points so that we eliminate a constant fraction of the search space in each round.



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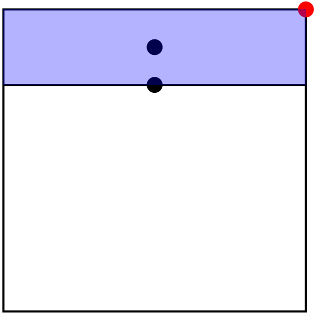

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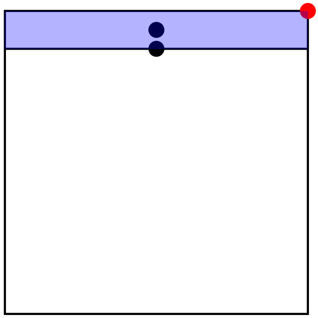

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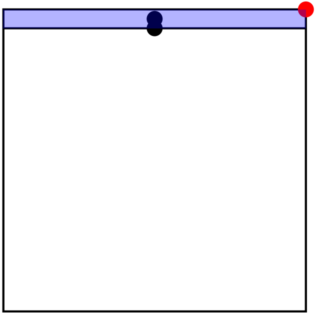

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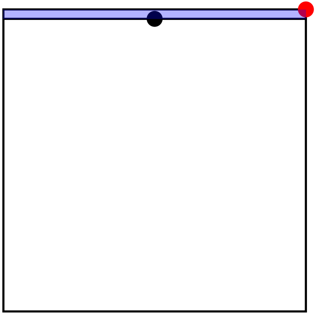

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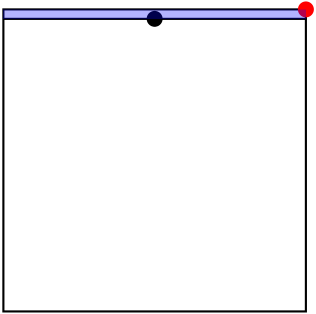

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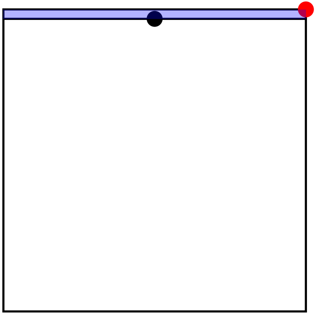

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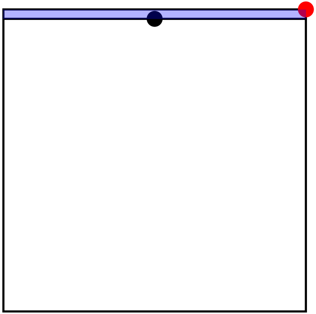

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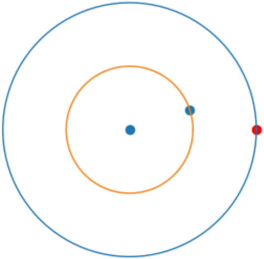

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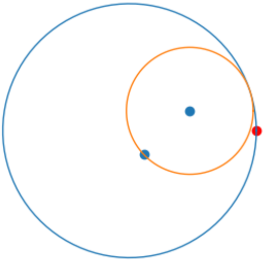

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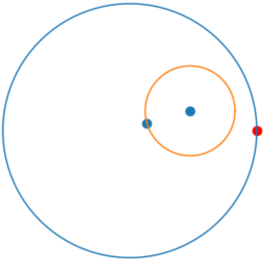

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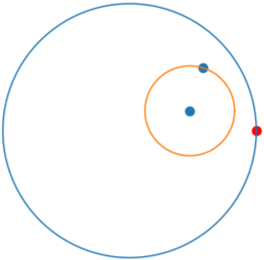

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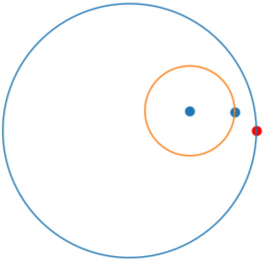

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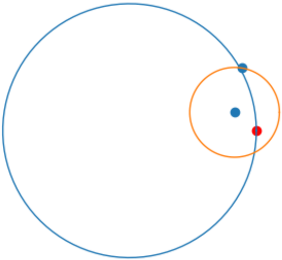

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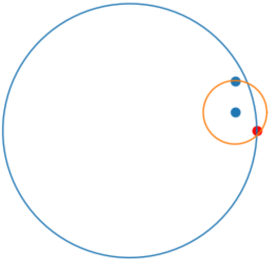

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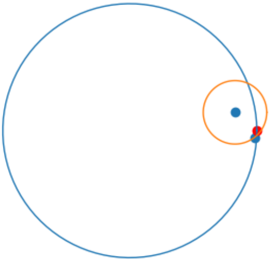

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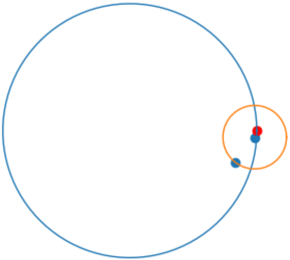

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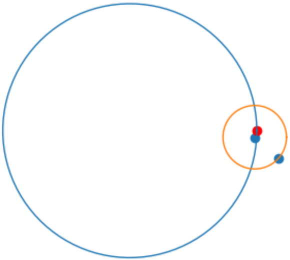

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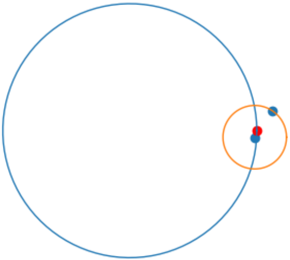

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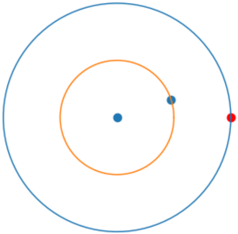

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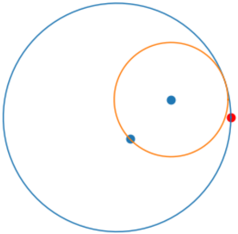

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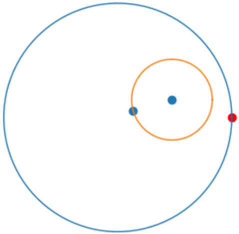

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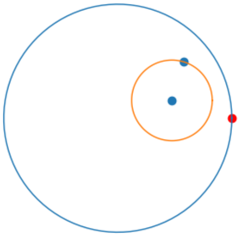

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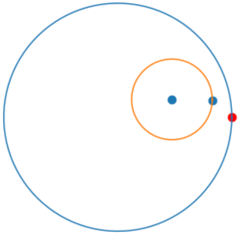

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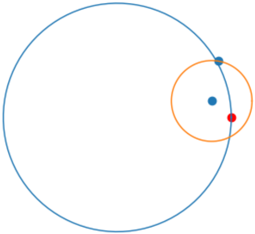

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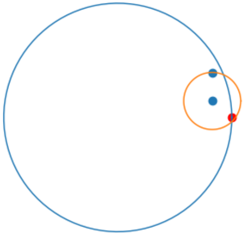

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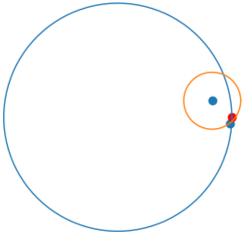

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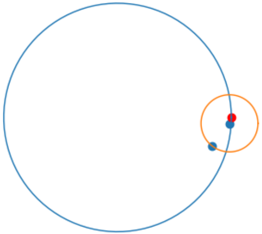

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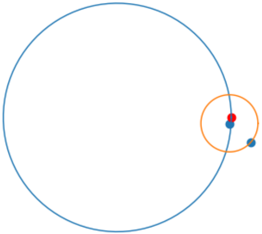

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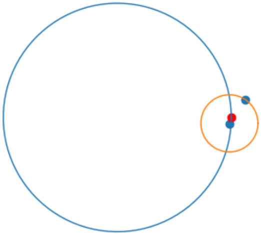

Table of contents

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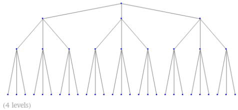

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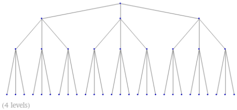

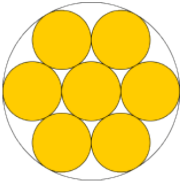

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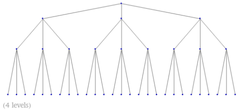

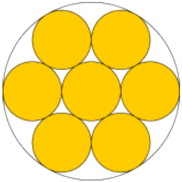

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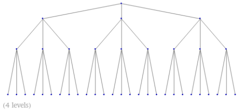

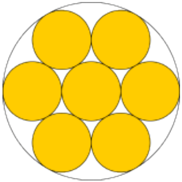

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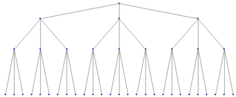

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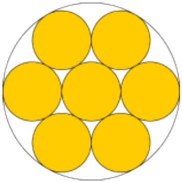

Upshot – our algorithm is **optimal**.

Conclusion – Dueling convex optimization with a monotone adversary

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- arXiv forthcoming! (or email me for the manuscript)

Modern input concerns for algorithmic data science

My goal – design and analyze algorithms for more realistic input scenarios.

Unwieldy input

- Approximating convex polytopes in a stream (Makarychev, Manoj, and Ovsiankin [MMO22; MMO23], COLT 2022/under submission).
- Approximating large convex objective functions (Manoj and Ovsiankin [MO23], under submission).
- Explaining classifier predictions on large inputs (Gupta and Manoj [GM23], SOSA 2023).

Unexpected input

- Robust machine learning under backdoor poisoning attacks (Manoj and Blum [MB21], NeurIPS 2021).
- Learning from out-of-list feedback (Blum, Gupta, Li, Manoj, Saha, and Yang [BGLMSY23], under submission).
- Generalization of short-program interpolators (Manoj and Srebro [MS23], COLT 2023).

- Research statement – https://narenmanoj.github.io/nsm_statement.pdf
- Publication list – https://narenmanoj.github.io/nsm_publist.pdf

Questions?

Thank you!!

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Aadirupa Saha, Tomer Koren, and Yishay Mansour. Dueling convex optimization with general preferences. *arXiv preprint arXiv:2210.02562*, 2022 (cited on pages 112–120).