1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES YOU THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. Submit your answers through this Google spreadsheet:

   `https://docs.google.com/forms/d/1PBq3u_fkktG746mG17EQNrs3LdqZGLT1pUNl6T8QAhs/viewform`

5. The solution packet will be released on March 24, 2015.

1. How many ordered pairs (x, y) for real numbers x and y simultaneously satisfy $x^2 + (y-2)^2 = 4$ and $\frac{x^2}{9} + \frac{y^2}{16} = 1$?

2. Circumscribe an equilateral triangle about a circle of area 100. Circumscribe a circle about that triangle. What is the area of the new circle?

3. Find the remainder when $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - ... + 2015 \cdot 2016$ is divided by 1000.

4. Consider the four digit number $S33D$. Jimmy and Timmy are playing a game where they randomly select digits S and D, respectively (note that S must be nonzero). Jimmy scores a point if the number $S33D$ is a multiple of 9 and Timmy gets a point if the number $S33D$ is a multiple of 11. Given that Jimmy scored a point, the probability that Timmy scored a point as well can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

5. Consider rectangle $AIME$ inscribed in circle O. The tangent to the circle through I intersects the tangent to the circle through M at P. If $PI = 5$ and $IM = 8$, then the area of figure $PIOM$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

6. Consider all strings containing only A's and B's of length 10. How many contain an AA or BB substring, but not an AAA or BBB substring?

7. Let S be the sum of ab over all distinct ordered pairs (a, b) such that $1 \leq a \leq b \leq 8$ and a and b are integers. Find S.

8. Let the roots of the polynomial $x^6 - x + 1$ be $a_1 ... a_6$. Find $a_1^5 + ... + a_6^5$.

9. Define $f(x) = [x]\{x\}^2$ where $[x]$ denotes the greatest integer less than or equal to x and $\{x\}$ denotes the fractional part of x. Let S be the set of all $1 \leq x \leq 2015$ such that $f(x)$ is an integer. The sum of the squares of the fractional parts of all elements of S can be expressed as $\frac{m}{n}$ for relatively prime integers m and n. Find the remainder when $m + n$ is divided by 1000.

10. In $\triangle ABC$, $AB = 5$, $BC = 8$, and $\angle ABC = 60°$. Construct point F on BC such that $BF = FC$. Construct E on AC such that BE bisects $\angle ABC$. Let AF and BE intersect at G. The ratio of the area of $\triangle AEG$ to that of $\triangle BGF$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

11. Consider the function $f(x) = x^4 + x^3 + x^2 + x + 1$. What's the maximum number of distinct prime factors under 100 that a given $f(x)$ can have?

12. Consider a triangle $\triangle ABC$. Let its circumcenter be O. Drop the perpendicular from O to BC such that this perpendicular intersects BC at O_{BC}. Define O_{AB} and O_{AC} similarly. Let OO_{BC} intersect AC at P and suppose that $O_{AC}P = O_{AC}O$. If $AB = 25$ and $BC = 31$, then the length of $O_{AB}P$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

13. For all positive integers x, we have $f(x) = (x + \sqrt{x^2 - 1})^{\frac{4}{3}} + (x + \sqrt{x^2 - 1})^{\frac{-4}{3}}$. Let the largest integer less than 1000 that can be expressed as $f(m)$ for some integer m be N. Determine $m + N$.

14. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC and let M and N be points on AB and AC such that $\angle ADM = \angle ADN = 30°$. Let MN and BC intersect at X. The length of XD can be written as a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

15. Let f be a function from $\{1...16\} \to \{1...16\}$ such that $f(16) = 16$, and for every other x, $f(f(x)) \equiv x^2 \pmod{17}$. Find the number of possible functions $f(x)$.

Answer Key

1. 001	6. 176	11. 006
2. 400	7. 750	12. 027
3. 128	8. 005	13. 582
4. 011	9. 107	14. 047
5. 103	10. 077	15. 064

1. Graphing both equations, it becomes apparent that one of the conics lies wholly within the other except for the one tangency point. Hence, the answer is $\boxed{001}$.

2. In a right triangle, the circumradius is exactly twice the inradius (proof is trivial). Thus, areas are scaled by 4 and we answer $\boxed{400}$.

3. Notice that we are asked to determine:

$$\sum_{n=1}^{2015} n^2 + \sum_{n=1}^{2015} n$$

This is equal to

$$\frac{2015 \cdot 2016 \cdot 4031}{6} + \frac{2015 \cdot 2016}{2}$$

Eliminating the denominators gives us

$$2015 \cdot 336 \cdot 4031 + 1008 \cdot 2015$$

Taking this modulo 1000 gives us $\boxed{360}$.

4. We break into cases based on the sum of the digits. Note that for $S33D$ to be a multiple of 9, $S + D$ is either 3 or 12.

Case 1: $S + D = 3$. Then, we can have $(S, D) = (1, 2), (2, 1), (3, 0)$. Notice that none of these possibilities can result in Timmy winning the game.

Case 2: $S + D = 12$. Then, we can have $(S, D) = (9, 3), (8, 4), (7, 5), (6, 6), (5, 7), (4, 8), (3, 9)$. Notice that only one of these - namely $(6, 6)$ can result in Timmy winning the game.

Our final probability is then $\frac{1}{10}$, and we answer $\boxed{011}$.

5. The radius of this circle, and therefore the length of OI, is 5. Let OP intersect IM at X. Since $IX = 4$ and $IX \perp OP$, $OX = 3$. By similar triangles, we have $OX \cdot OP = OI^2$. Plugging in our known information gives us $OP = \frac{25}{3}$. Our area is now given by $OP \cdot IX$ which is equal to $\frac{100}{3}$. Our final answer is therefore $\boxed{103}$.

6.

7. Notice that our answer is

$$\frac{1}{2}((1+2+3+4+5+6+7+8)^2 - (1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2)) + (1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2) = \boxed{750}$$

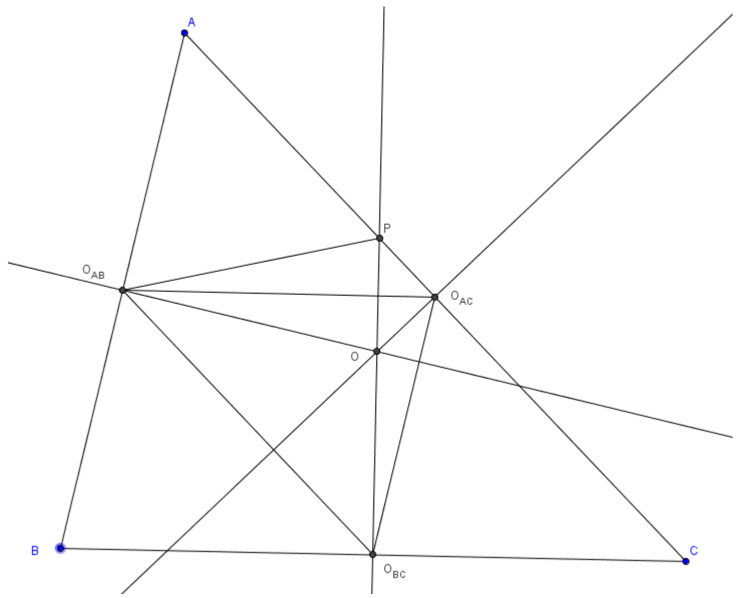
8. We have $x^6 = x - 1$, which implies $x^5 = 1 - \frac{1}{x}$. Summing over every possible x_i^5, we realize we want
$$6 - \sum_{i=1}^{6} \frac{1}{x_i}$$
. The summation is equal to the ratio of the coefficient of the linear term to the constant negated (the proof of this is from common denominators and Vieta's formulas). It thus follows that our answer is $6 - 1 = \boxed{005}$.

9. Let $x = i + f$ where i is an integer and $0 < f < 1$. We then have $[x] = i$ and $\{x\} = f$. Then, we have
$$f(i+f) = if^2 = n$$
where n is an integer. We then have
$$f^2 = \frac{n}{i}$$
In particular, note that the range of n is $1 \leq n \leq i - 1$.
We can now reinterpret the problem as:
$$\sum_{i=2}^{2015} S(i)$$
For a given i, we have $S(i)$ as follows:
$$S(i) = \frac{1}{i} + \frac{2}{i} + ... + \frac{i-1}{i} = \frac{\frac{i(i-1)}{2}}{i} = \frac{i-1}{2}$$
It then follows that the desired sum is simply:
$$\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + ... + \frac{2014}{2} = \frac{\frac{2014(2015)}{2}}{2} = \frac{1007 \cdot 2015}{2}$$
Taking the numerator modulo 1000 gives us 105. Adding the denominator gives us our final answer of $\boxed{107}$.

10. Let line CG intersect AB at D. The law of cosines tells us that $AC = 7$. By the angle bisector theorem, we get $EC = \frac{56}{13}$ and $AE = \frac{35}{13}$. Notice that the area of $\triangle AGE$ (denoted $[AEG]$) can be expressed as
$$AG \cdot GE \cdot \frac{\sin \angle AGE}{2}$$
, and similarly,
$$[BGF] = BG \cdot GF \cdot \frac{\sin \angle BGF}{2}$$
. The ratio between these is therefore
$$\frac{AG \cdot GE}{BG \cdot GF}$$
. We now bash a bit with mass points (not too bad) and find $\frac{AG}{GF} = \frac{5}{4}$ and $\frac{GE}{BG} = \frac{5}{13}$. Multiplying gives us our desired ratio of $\frac{25}{52}$, and our final answer is $\boxed{077}$.

11. (please contribute a solution as we don't have one written up)

12. We use complementary counting to count the first quantity. Notice that the total number of substrings is $2^10 = 1024$. Since only two strings (one alternating As and Bs and one alternating Bs and As) which do not contain a AA or BB substring. Hence, there are $1024 - 2 = 1022$ possible strings for the first quantity.



Notice that $O_{AB}O_{AC} \parallel BC$. From similar conclusions, it immediately follows that O is the orthocenter of $\triangle O_{AC}O_{AB}O_{BC}$. Since $OP \perp O_{AB}O_{AC}$ and since $OO_{AC} = PO_{AC}$, it follows that P is the reflection of O across $O_{AB}O_{AC}$. Hence, $PO_{AC}O_{AB}O_{BC}$ is a cyclic quadrilateral. Since $PO_{AC} \parallel O_{AB}O_{BC}$, $PO_{AC}O_{AB}O_{BC}$ must be an isosceles trapezoid. Therefore, $O_{AB}P = O_{AC}O_{BC}$. Since $O_{AC}O_{BC} = \frac{1}{2}AB = \frac{25}{2}$, we have that $O_{AB}P = \frac{25}{2}$. Our final answer is then $\boxed{027}$.

13. Let $y = x + \sqrt{x^2 - 1}$. Suppose we wish to rewrite $f(x)$ as some cleaner expression a. We then have

$$y^{\frac{4}{3}} + y^{\frac{-4}{3}} = a$$

Completing the square gives us

$$y^{\frac{4}{3}} + y^{\frac{-4}{3}} + 2 = (y^{\frac{2}{3}} + y^{\frac{-2}{3}})^2 = a + 2$$

We repeat this process, and we get

$$y^{\frac{1}{3}} + y^{\frac{-1}{3}} = \sqrt{\sqrt{a+2} + 2} = k$$

Cubing both sides gives us

$$y + \frac{1}{y} + 3(y^{\frac{1}{3}} + y^{\frac{-1}{3}}) = y + \frac{1}{y} + 3k = k^3$$

Now, we resubstitute x, which gives us

$$x + \sqrt{x^2 - 1} + \frac{1}{x + \sqrt{x^2 - 1}} = x + \sqrt{x^2 - 1} + x - \sqrt{x^2 - 1} = k^3 - 3k$$

$$2x = k^3 - 3k$$

Since we have $\sqrt{\sqrt{a+2}+2} = k$ and $a = N$, we have $\sqrt{\sqrt{N+2}+2} = k$. Notice that N must be less than 1000, indicating that $k < 6$. Testing $k = 5$ gives us:

$$\sqrt{\sqrt{N+2}+2} = 5$$
$$\sqrt{N+2} = 23$$
$$N = 527$$

Then, plugging this into the equation for x gives us:

$$2x = 5^3 - 3(5)$$
$$x = 55$$

Hence, we have $m = 55$. The correct answer is therefore $55 + 527 = \boxed{582}$.

14. Let MN and AD intersect at Y. Because $\angle YDM = \angle YDN$ and $\angle YDX = 90^\circ$, $\{X, Y : M, N\}$ form a harmonic bundle. When we project through A onto line BC, we see that $\{X, D : B, C\}$ forms a harmonic bundle as well. Then $\frac{XB}{XC} = \frac{DB}{DC} = \frac{5}{9}$. If we let $XB = x$, we have that $\frac{x}{x+14} = \frac{5}{9} \implies x = \frac{35}{2}$. Then $XD = XB + 5 = \frac{35}{2} + 5 = \frac{45}{2}$. Hence $m + n = 45 + 2 = \boxed{047}$.

15. (please contribute a solution as we don't have one written up)