Topic of the Week (0908): Cyclic Quadrilaterals

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1 Introduction

A *cyclic quadrilateral* is a quadrilateral which can be inscribed in a circle. In the following pages, you will discover some of the unique properties of the cyclic quadrliateral and use these properties to solve problems. Note that in your exploration, you may want to recall the following facts:

- If there exists an inscribed angle $\angle ABC$, then the measure of arc ABC is $2\angle ABC$.
- If two inscribed angles subtend the same arc, then the inscribed angles are equal.

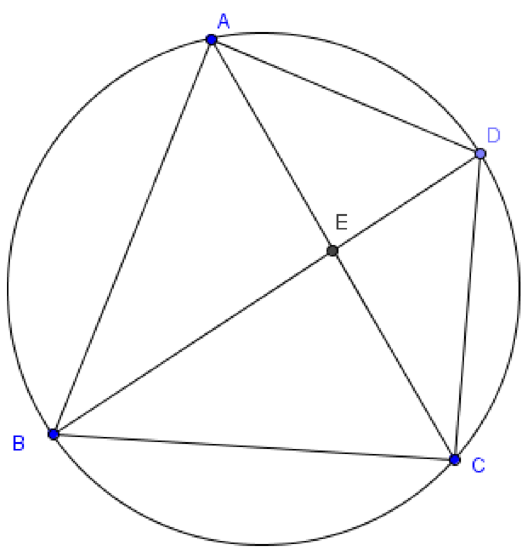


Figure 1.1: A cyclic quadrilateral ABCD

2 Properties

All problems refer to figure 1.1 unless otherwise stated.

1. Show that $\angle ABC + \angle ADC = 180^\circ$.
2. Show that $\angle BAC = \angle BDC$.
3. Show that $AE(EC) = DE(EB)$.
4. Show that $AB(DC) + AD(BC) = AC(BD)$ (this is known as Ptolemy's Theorem).
5. Show that given a quadrilateral $ABCD$, if any of the above properties hold, then $ABCD$ is cyclic.

3 Easy Problems

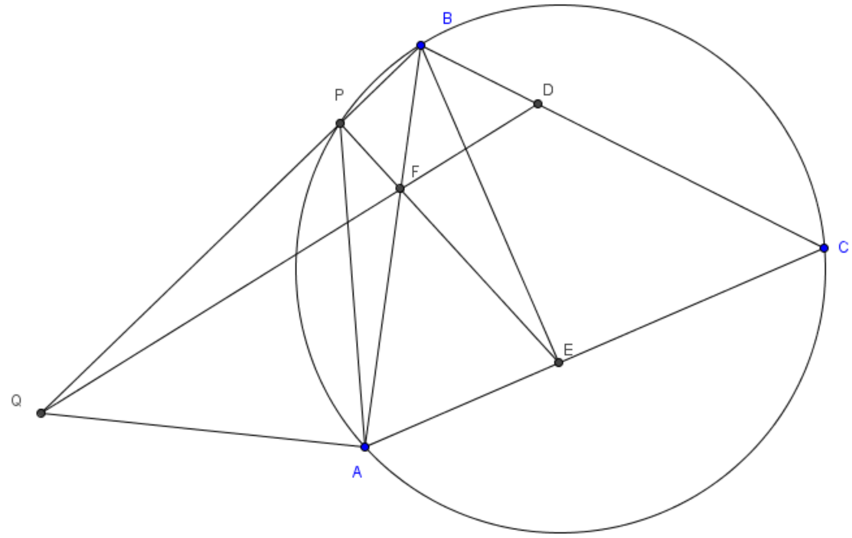
1. Let $\triangle ABC$ have altitudes AD and BE. Prove that $ABDE$ is cyclic.

4 Medium Problems

1. Suppose we have a triangle $\triangle ABC$. Let points X, Y, Z be located on sides BC, AC, AB, respectively. Prove that the circumcircles of triangles $\triangle CYX, \triangle BXZ, \triangle AYZ$ all intersect at one common point (Miquel's Theorem).

5 Hard Problems

1. Draw a triangle $\triangle ABC$. Let D, E, F be the feet of the altitudes from A, B, C, respectively. Suppose all these altitudes intersect at H (this point is known as the *orthocenter*). Let D', E', F' be the midpoints of AH, BH, CH, respectively. Let X, Y, Z be the midpoints of BC, AC, AB, respectively. Prove that $D, E, F, D', E', F', X, Y, Z$ all lie on the same circle (this circle is known as the nine-point circle).
2. Let $\triangle ABC$ be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB, respectively. One of the intersection points of the line EF and the circumcircle of ABC is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$ (IMO Shortlist 2010).



(a) We will be using many of the properties of cyclic quadrilaterals to attack this problem. Take careful note of the diagram above.

(b) We're trying to prove that $AP = AQ$. However, seeing that we aren't given any information about lengths, it may be challenging to attack this problem using lengths. Instead, we'll focus on angles. What two angles, if we could prove them equal, would prove $AP = AQ$?

(c) First, let's focus on $\angle QPA$. Prove that $\angle QPA = \angle BCA$.

(d) Prove $\angle BFD = \angle BCA$. The significance of this will come to light shortly.

(e) It's pretty trivial that $\angle QFA = \angle BFD$. However, we just now showed that $\angle BFD = \angle BCA = \angle QPA$. Hence, $\angle QFA = \angle QPA$. What conclusion can we draw about quadrilateral $QPFA$?

(f) Now, we can direct our attention to $\angle PQA$. Based on our previous conclusion, what can we say about $\angle PFA$?

(g) Notice that we also have $\angle PFA + \angle BFP = 180°$. However, it is trivial to note that $\angle BFP = \angle AFE$. Based on the conclusions from this step and the last step, what two angles must we prove equal?

(h) Prove $\angle AFE = \angle BCA$. Why does proving this immediately finish the problem? Convince yourself that it does, and once you do that, you're finished!

6 Solution to ISL Problem (along with my thoughts)

This is my approximate thought process when I first looked at/solved this problem.

"Hmm so if I could show $\angle PQA = \angle QPA$, I'd be finished. Let's see how to do this.

Well $APBC$ is cyclic so I have $\angle BCA = 180° - \angle BPA$, and I see that $\angle CPA + \angle APB = 180°$. Now I have $\angle QPA = \angle BCA$ Showing $\angle PQA = \angle BCA$ would finish the problem, so that is my new goal. "

Then I ended up staring at $\angle PQA$ and I realized there was absolutely nothing I can do with this angle. Perhaps I was missing something... if only I had more information about this diagram that would tell me more about angles...

"Wait. From this diagram, it looks as though $\angle QPA = \angle QFA$. This would prove $QPFA$ cyclic. Let's play this hunch. Will this help me in any way? Let's see... if I had that, then I could say $\angle PQA = 180° - \angle PFA = \angle PFB$. But, $\angle PFB$ looks a bit annoying still."

Then, I saw the vertical angle, and then I stated that if $QPFA$ were cyclic, then $\angle PQA = \angle AFE$, and $\angle AFE$ is not particularly hard to compute; after all, E and F are just feet of altitudes, and right angles make for fairly simple angle chasing.

"Let's calculate $\angle AFE$. Because $\angle BEC = 90°$, I have $\angle EBC = 90° - \angle C$. Similarly, $\angle FCB = 90° - \angle B$. Also, I know for a fact that $BFEC$ is cyclic because $\angle BEC = \angle CFB = 90°$. Then, I say $\angle BEF = 90° - \angle B$, and $\angle AEF = \angle B$ and $\angle AFE = \angle C$... what. Okay so if $QPFA$ is cyclic, then $\angle PQA = \angle AFE = \angle BCA = \angle QPA$. This would make me finished... now, my new goal is to prove $QPFA$ cyclic."

But, I had already done the heavy lifting. Using a similar process, I showed $\angle BFD = \angle BCA$, then I already had $\angle BFD = \angle QFA$, and $\angle QFA = \angle BCA = \angle QPA$, which I proved earlier without making any assumptions. Thus, I had $\angle QPA = \angle QFA$, which automatically meant $QPFA$ was definitely cyclic, and then I was done.