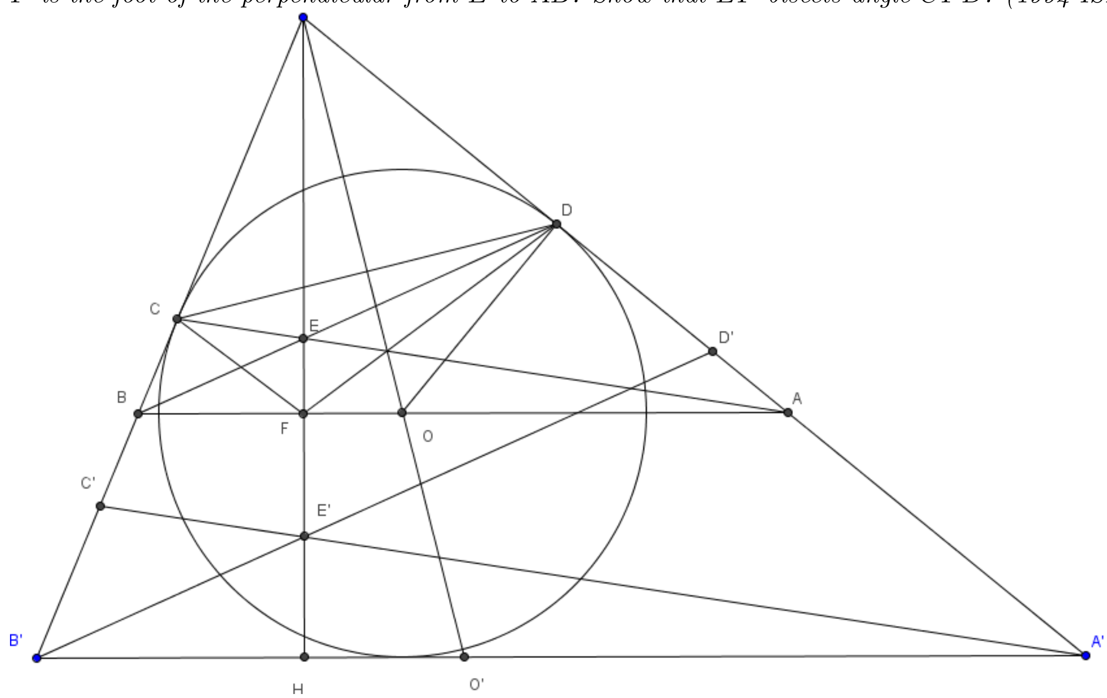
Contents

1 2014-11-28

1.1 (1994 ISL G1)

C and D are points on a semicircle. The tangent at C meets the extended diameter of the semicircle at B, and the tangent at D meets it at A, so that A and B are on opposite sides of the center. The lines AC and BD meet at E. F is the foot of the perpendicular from E to AB. Show that EF bisects angle CFD. (1994 ISL G1)



Let BC intersect DA at P. Then, because PC and PD are two tangents emanating from the same point, we have $PC \cong PD$. Furthermore, since $\angle OCP = \angle ODP = 90$ (because they are radii to tangents), quadrliateral $PDOC$ must be cyclic.

Now, consider the homothety taking B to B' and A to A' such that $A'B'||AB$ and $A'B'$ is tangent to the circle with center O and radius OC. Then, O is the incenter of $\triangle PB'A'$.

Additionally, note that P, E, F are collinear. (How?)

Therefore, $\angle PFO = 90$. Since $\angle ODP = 90$ as well, we have that $PFOD$ is cyclic. Hence, C and F lie on the circumcircle of $\triangle POD$, and we have $PCFD$ cyclic. Thus, because inscribed angles are equal, we have $\angle PCD = \angle PFD$ and $\angle PDC = \angle PFC$. However, since $PC \cong PD$, we have $\angle PCD = \angle PDC$, and as a result, we have $\angle PFD = \angle PFC$. Therefore, EF must bisect $\angle CFD$, as desired.

1.2 (1996 ISL A1)

Suppose that $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ac}{ac + a^5 + c^5} \leq 1$$

(1996 ISL A1)
By the rearrangement inequality, we have

$$a^5 + b^5 \geq a^2b^3 + b^3a^2$$
$$a^5 + b^5 \geq (a^2b^2)(a + b)$$

Then, we have:

$$\frac{ab}{ab + a^5 + b^5} \leq \frac{ab}{ab + (a^2b^2)(a + b)}$$
$$\frac{ab}{ab + a^5 + b^5} \leq \frac{1}{1 + (ab)(a + b)}$$
$$\frac{ab}{ab + a^5 + b^5} \leq \frac{1}{abc + (ab)(a + b)}$$
$$\frac{ab}{ab + a^5 + b^5} \leq \frac{1}{ab(a + b + c)}$$

Similarly,

$$\frac{bc}{bc + b^5 + c^5} \leq \frac{1}{bc(a + b + c)}$$
$$\frac{ac}{ac + a^5 + c^5} \leq \frac{1}{ac(a + b + c)}$$

Summing gives us

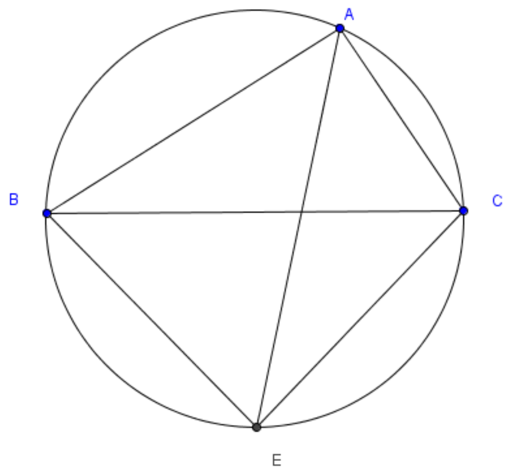
$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ac}{ac + a^5 + c^5} \leq \frac{1}{a + b + c}(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac})$$
$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ac}{ac + a^5 + c^5} \leq \frac{1}{a + b + c}(c + a + b)$$
$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ac}{ac + a^5 + c^5} \leq 1$$

as desired.

2 2014-11-29

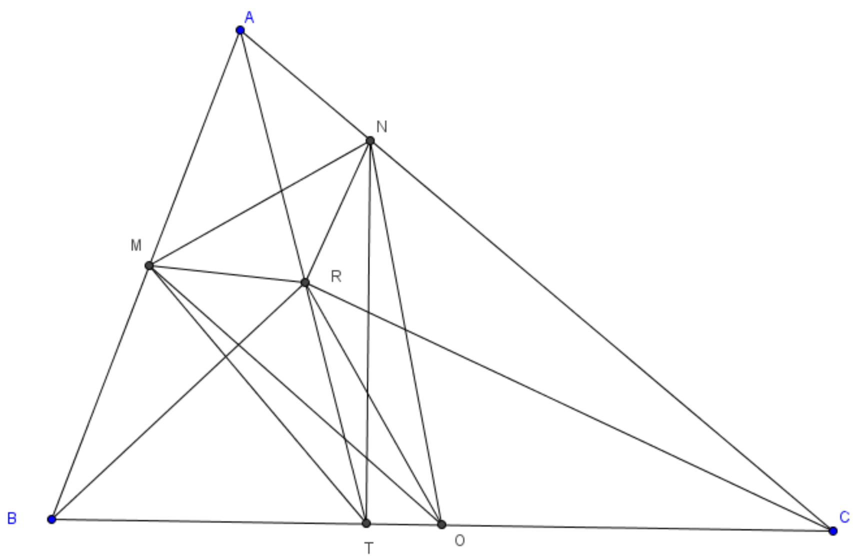
2.1 2004 ISL G1

Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.



Lemma 1: Suppose $\triangle ABC$ has angle bisector AE, and $EB = EC$. Then, $ABEC$ is cyclic.
Proof: Suppose there exists another point E' such that it lies on AE and $E'B = E'C$. Since E' must lie on the intersection of the perpendicular bisector of BC and AE, there is only one possible location for E'; namely, E itself. Now, let AE impact the circumcircle at F. Since $\angle BAF = \angle FAC$ and by inscribed angles, $\angle BAF = \angle BCF$ and $\angle FAC = \angle FBC$, we have $\angle FBC = \angle FCB$. Hence, $FB = FC$. However, we previously proved that there exists only one possible point P such that $PB = PC$ and P lies on AE (namely, this point is E). Hence, $E = F$, and we are finished.

Now we proceed to the actual problem. The diagram is below.



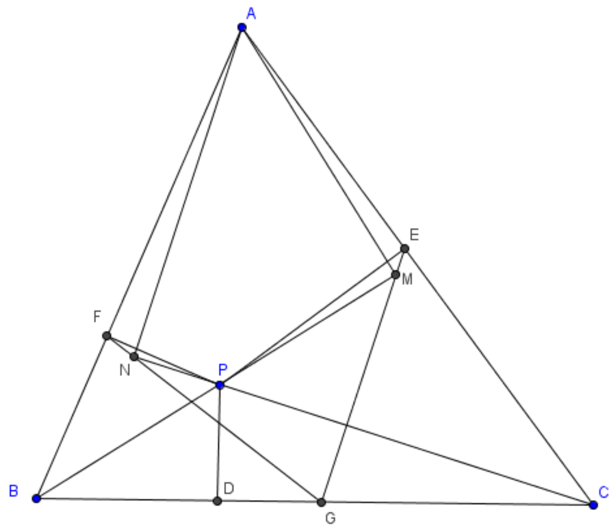
Because OM and ON are radii, we have $OM = ON$. Since OR is an angle bisector of isosceles triangle $\triangle MON$, OR must be the perpendicular bisector of MN. Hence, $MR = NR$. Now, by lemma 1, quadrilateral $AMRN$ is cyclic. Since $BMNC$ is also cyclic, we have $180 - \angle BMN = \angle ACB = \angle AMN$. Similarly, $\angle ABC = \angle ANM$. Additionally, by inscribed angles, $\frac{\angle BAC}{2} = \angle NMR = \angle MNR$. Hence, $\angle AMR = \angle ACB + \frac{\angle BAC}{2}$.

Also, note that $\angle ATB = \frac{\angle BAC}{2} + \angle ACB$. Then, $\angle ATB = \angle AMR$, and thus, $BMRT$ is cyclic. Similarly, $TRNC$ is cyclic, and hence, the circumcircles of BMR and CNR must intersect each other at point T on BC, as desired.

3 2014-12-01

3.1 1991 ISL G1

Given a point P inside a triangle $\triangle ABC$. Let D, E, F be the orthogonal projections of the point P on the sides BC, CA, AB, respectively. Let the orthogonal projections of the point A on the lines BP and CP be M and N, *respectively. Prove that the lines ME, NF, BC are concurrent.*



Notice that we have $\angle ANP = \angle AMP = 90$; hence, $ANPM$ is cyclic. Furthermore, note that $\angle AFP = \angle ANP = 90$; therefore, F lies on the circumcircle of $ANPM$. Similarly, so does E.

Now, let the intersection of FN and EM be G, which is not necessarily on BC. By power-of-a-point, we have $GN \cdot GF = GM \cdot GE$. Furthermore, let the intersection of FN and BC be G'. Then, by Miquel's Theorem, the circumcircles of $AFNPME$, $\triangle BFP'$, and $\triangle CEP'$ must concur at a point.

Also, we have $\angle BFP = \angle BDP = 90$. Then, $BFPD$ is cyclic, and similarly, $CEPD$ is cyclic. [UNFINISHED]

4 2014-12-21

4.1 1997/5 USAMO

Prove that, for all positive reals a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (a^3 + c^3 + abc)^{-1} \leq (abc)^{-1}$$

By the rearrangement inequality, we have:

$$a^3 + b^3 + abc \geq a^2b + b^2a + abc$$

$$a^3 + b^3 + abc \geq (a + b + c)(ab)$$

$$\frac{1}{a^3 + b^3 + abc} \leq \frac{1}{a + b + c} \cdot \frac{1}{ab}$$

We can do the same for the remaining terms in the summation, and then we have:

$$\sum_{cyc} \frac{1}{a^3 + b^3 + abc} \leq \frac{1}{a + b + c} \cdot (\sum_{cyc} \frac{1}{ab})$$

With common denominators, the right hand side becomes:

$$\frac{1}{a + b + c} \cdot (\sum_{cyc} \frac{1}{ab}) = \frac{1}{a + b + c}(\frac{a + b + c}{abc}) = \frac{1}{abc}$$

as desired.

4.2 1995 ISL A1

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3\left(b+c\right)}+\frac{1}{b^3\left(c+a\right)}+\frac{1}{c^3\left(a+b\right)}\geq\frac{3}{2}$$

Note that

$$\frac{1}{a^3(b+c)}=\frac{abc}{a^3(b+c)}=\frac{bc}{a^2(b+c)}=\frac{1}{a^2}\cdot\frac{1}{\frac{1}{b}+\frac{1}{c}}$$

This warrants the following substitutions:

$$\frac{1}{a}=x$$
$$\frac{1}{b}=y$$
$$\frac{1}{c}=z$$

Notice we have $xyz = 1$.
Then,

$$\frac{1}{a^3(b+c)}=\frac{x^2}{y+z}$$

Using similar substitutions, we realize that it suffices to prove

$$\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\geq\frac{3}{2}$$

By the Cauchy-Schwarz inequality, we have:

$$(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y})(x+y+y+z+z+x)\geq(x+y+z)^2$$

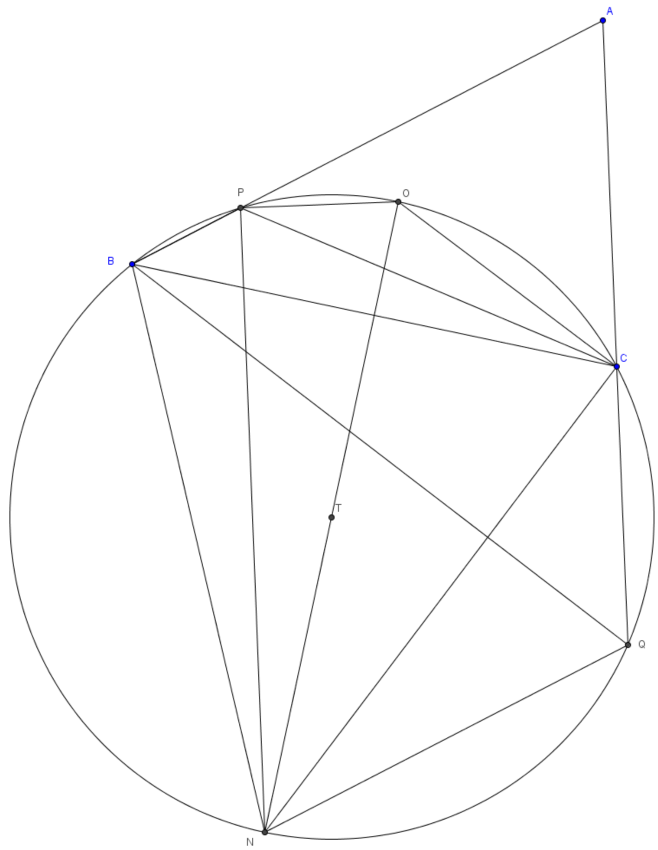
$$2(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y})(x+y+z)\geq(x+y+z)^2$$

$$2(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y})\geq x+y+z\geq 3(xyz)^{\frac{1}{3}}$$

where the last inequality follows from AM-GM. Substituting $xyz = 1$ into the above gives the desired.

5 2014-12-22

5.1 2010 APMO 1



First, we will make a list of observations:

- $NB = NC$. This is because OT is a perpendicular bisector of BC. Point T must lie on this perpendicular bisector as does O by the definition of the circumcenter.
-

From here, we realize $\angle BCN = \angle NBC$. Then, using inscribed angles and other properties of cyclic quadrilaterals, we obtain the following:

$$\angle BCN = \angle NBC = \angle NPC$$
$$180 - 2\angle BCN + \angle NPC = 180 - \angle BCN$$
$$180 - 2\angle BCN + \angle NPC = 180 - 2\angle BCN + \angle BCN$$
$$180 - 2\angle BCN + \angle NPC = \angle BNC + \angle BCN$$
$$180 - 2\angle BCN + \angle NPC = \angle AQB + \angle BQN$$
$$180 - 2\angle BCN + \angle NPC = \angle AQN$$
$$\angle NPC + \angle BNC = \angle AQN$$
$$\angle NPC + \angle BQA = \angle AQN$$
$$\angle NPC + \angle CPA = \angle AQN$$
$$\angle NPA = \angle AQN$$

6 2014-12-23

6.1 1997 Ireland, unknown

Suppose that $a+b+c+d=1$ and $a,b,c,d>0$. Show that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{a+d}\geq\frac{1}{2}$$

By Cauchy, we have:

$$(a+b+b+c+c+d+d+a)(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{a+d})\geq(a+b+c+d)^2$$
$$2(a+b+c+d)(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{a+d})\geq(a+b+c+d)^2$$
$$2(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{a+d})\geq 1$$

as desired.

6.2 1999 Belarus, unknown

Let a,b,c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\geq\frac{3}{2}$$

By Cauchy, we have

$$(ab+ac+bc+3)(\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac})\geq 3^2=9$$

Because $a^2+b^2+c^2=3\geq ab+ac+bc$, we have:

$$6(\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac})\geq 9$$

as desired.

6.3 2002 Canada NMO 3

Prove that for all $a,b,c>0$,

$$\frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}\geq a+b+c$$

and determine when equality occurs.
This inequality is homogenous; hence, without loss of generality, let $abc=1$. Then, we wish to prove

$$a^4+b^4+c^4\geq a+b+c$$

However, we know that, for the same conditions,

$$a^4+b^4+c^4\geq a^2+b^2+c^2\geq a+b+c$$

and we're done. Equality is achieved when $a=b=c$.

6.4 2004 Moldova, unknown

Prove that for all $a,b,c\geq 0$,

$$a^3+b^3+c^3\geq a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}$$

Consider the sequences $(3,0,0)$ and $(2,\frac{1}{2},\frac{1}{2})$. Since the former majorizes the latter, by Muirhead, we obtain the desired.

7 2014-12-24

7.1 2004 Estonia, unknown

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ac} \geq 1$$

By Titu's Lemma, we have:

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ac} \geq \frac{3^2}{3+2(ab+ac+bc)} = \frac{9}{a^2+b^2+c^2+2(ab+ac+bc)} = \frac{9}{(a+b+c)^2}$$

By Cauchy, we have:

$$(a^2+b^2+c^2)(1^2+1^2+1^2) \geq (a+b+c)^2$$

$$9 \geq (a+b+c)^2$$

Then, we have:

$$\frac{9}{(a+b+c)^2} \geq 1$$

as desired.

8 2014-12-28

8.1 2011/1 USAMO

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3$$

Expand the condition and rearrange it as follows:

$$(a+b)^2 + (b+c)^2 + (a+c)^2 \leq 4$$

Now, let $x = a+b, y = b+c, z = a+c$. Then, we have:

$$ab + 1 = \frac{x^2 - y^2 - z^2 + 2xy + 4}{4}$$

$$ab + 1 \geq \frac{x^2 - y^2 - z^2 + 2xy + x^2 + y^2 + z^2}{4} = \frac{x^2 + yz}{2}$$

$$\frac{ab+1}{(a+b)^2} \geq \frac{x^2+yz}{2x^2} = \frac{1}{2} + \frac{yz}{2x^2}$$

We repeat for $bc+1, ac+1$. Hence, it suffices to prove

$$\frac{3}{2} + \frac{yz}{2x^2} + \frac{xz}{2y^2} + \frac{xy}{2z^2} \geq 3$$

or

$$\frac{yz}{2x^2} + \frac{xz}{2y^2} + \frac{xy}{2z^2} \geq \frac{3}{2}$$

which is true by AM-GM.

8.2 2012/3 USAJMO

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a} \geq \frac{2}{3}(a^2+b^2+c^2)$$

INSERT A/A SOLUTION HERE

9 2014-12-31

9.1 1998 ISL A3

Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

Consider the following manipulation:

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} = \frac{x^4}{x(1+y)(1+z)}+\frac{y^4}{y(1+z)(1+x)}+\frac{z^4}{z(1+x)(1+y)}$$

By Cauchy, we have

$$(x(1+y)(1+z)+y(1+z)(1+x)+z(1+x)(1+y))(\frac{x^4}{x(1+y)(1+z)}+\frac{y^4}{y(1+z)(1+x)}+\frac{z^4}{z(1+x)(1+y)}) \geq (x^2+y^2+z^2)^2$$

$$(3xyz+2(xy+yz+xz)+x+y+z)(\frac{x^4}{x(1+y)(1+z)}+\frac{y^4}{y(1+z)(1+x)}+\frac{z^4}{z(1+x)(1+y)}) \geq (x^2+y^2+z^2)^2$$

Now, we note the following well-known inequalities given the $xyz = 1$ constraint:

$$x^2+y^2+z^2 \geq x+y+z$$
$$x^2+y^2+z^2 \geq xy+yz+xz$$

Also, by AM-GM, we have:

$$x^2+y^2+z^2 \geq 3(xyz)^{\frac{2}{3}} = 3 = 3xyz$$

We then substitute these into our inequality:

$$4(x^2+y^2+z^2)(\frac{x^4}{x(1+y)(1+z)}+\frac{y^4}{y(1+z)(1+x)}+\frac{z^4}{z(1+x)(1+y)}) \geq (x^2+y^2+z^2)^2$$

$$4(\frac{x^4}{x(1+y)(1+z)}+\frac{y^4}{y(1+z)(1+x)}+\frac{z^4}{z(1+x)(1+y)}) \geq x^2+y^2+z^2 \geq 3$$

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

as desired.

9.2 2000/2 IMO

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

We use the well-known substitution:

$$a = \frac{x}{y}$$
$$b = \frac{y}{z}$$
$$c = \frac{z}{x}$$

Substituting everything in, it remains to prove

$$(x + z - y)(x + y - z)(y + z - x) \leq xyz$$

Expanding everything and rearranging, it would suffice to prove

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2$$

However, this is trivial by Schur's Inequality, and as a result, we are done.

10 2014-01-02

10.1 Own

Let x, y, z be positive reals such that $xyz = 1$. Prove that

$$\frac{x^{14} - x^8 + 1}{(x + y)(x + z)} + \frac{y^{14} - y^8 + 1}{(y + z)(y + x)} + \frac{z^{14} - z^8 + 1}{(z + x)(z + y)} \geq \frac{3}{x^2 + y^2 + z^2 + 1}$$

Begin by noting that $x^{14} + 1 \geq x^8 + x^6$. This is because $(14, 0)$ majorizes $(8, 6)$. Then, simply manipulate the above to obtain:

$$x^{14} - x^8 + 1 \geq x^6$$

Using a similar procedure for y, z, we realize it suffices to prove:

$$\frac{x^6}{(x + y)(x + z)} + \frac{y^6}{(y + z)(y + x)} + \frac{z^6}{(z + x)(z + y)} \geq \frac{3}{x^2 + y^2 + z^2 + 1}$$

Consider the following manipulation:

$$\frac{x^6}{(x + y)(x + z)} = \frac{x^8}{x^2(x + y)(x + z)}$$

Repeat for the rest of the terms. Expand the denominator; we get

$$x^4 + x^2(xy + yz + xz)$$

The other two denominators are:

$$y^4 + y^2(xy + yz + xz)$$

$$z^4 + z^2(xy + yz + xz)$$

Adding these three expressions and defining $f(x, y, z)$ to be this sum gives us

$$f(x,y,z) = x^4 + y^4 + z^4 + (xy + yz + xz)(x^2 + y^2 + z^2)$$

Then, by Cauchy, we have:

$$f(x,y,z)(\frac{x^6}{(x+y)(x+z)} + \frac{y^6}{(y+z)(y+x)} + \frac{z^6}{(z+x)(z+y)}) \geq (x^4 + y^4 + z^4)^2$$

Notice that $x^2 + y^2 + z^2 \geq xy + yz + xz$ and, for the given condition, $x^4 + y^4 + z^4 \geq x^2 + y^2 + z^2$. Then, we have:

$$x^4 + y^4 + z^4 + (x^4 + y^4 + z^4)(x^2 + y^2 + z^2) = (x^4 + y^4 + z^4)(x^2 + y^2 + z^2 + 1) \geq f(x,y,z)$$

Therefore, we conclude

$$(x^4 + y^4 + z^4)(x^2 + y^2 + z^2 + 1)(\frac{x^6}{(x+y)(x+z)} + \frac{y^6}{(y+z)(y+x)} + \frac{z^6}{(z+x)(z+y)}) \geq (x^4 + y^4 + z^4)^2$$

$$(x^2 + y^2 + z^2 + 1)(\frac{x^6}{(x+y)(x+z)} + \frac{y^6}{(y+z)(y+x)} + \frac{z^6}{(z+x)(z+y)}) \geq x^4 + y^4 + z^4$$

By AM-GM, $x^4 + y^4 + z^4 \geq 3(xyz)^{\frac{4}{3}} = 3$. Then, we have

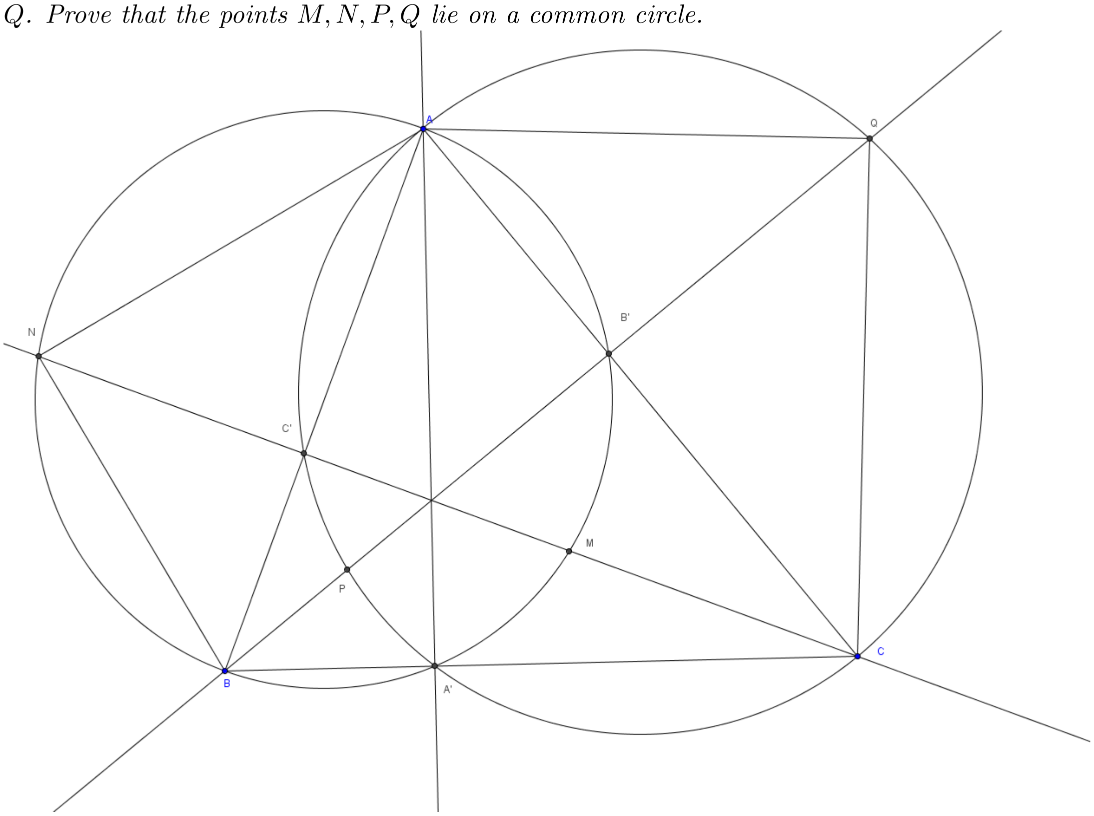
$$(x^2 + y^2 + z^2 + 1)(\frac{x^6}{(x+y)(x+z)} + \frac{y^6}{(y+z)(y+x)} + \frac{z^6}{(z+x)(z+y)}) \geq 3$$

as desired.

11 2015-01-08

11.1 1990/5 USAMO

An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

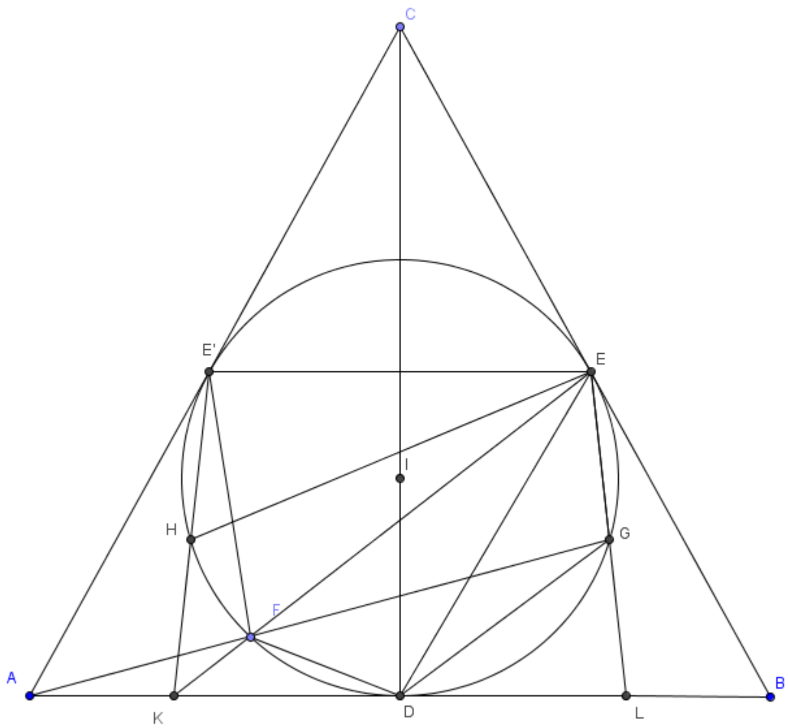


Since MN is perpendicular to diameter AB, AB must be the perpendicular bisector of MN. It follows that $AN = AM$. With similar logic, we have $AP = AQ$.
Using the fact that $\triangle AC'N \sim \triangle ANB$, we conclude $AN^2 = AC' \cdot AB$. Similarly, we have $AQ^2 = AB' \cdot AC$. Also, we have that $\angle BC'C = \angle CB'B = 90$; hence, $BC'B'C$ is cyclic. Then, by power of a point, we have $AC' \cdot AB = AB' \cdot AC$. Hence, $AN = AQ$. From our previous equality, we conclude $AN = AP = AM = AQ$, and thus, A is the circumcenter of the circle circumscribed about the cyclic quadrilateral $NPMQ$.

12 2015-01-09

12.1 WOOT POTD Wednesday, Jan 7

Let ABC be an isosceles triangle with $AC = BC$. Its incircle touches AB in D and BC in E. A line distinct of AE goes through A and intersects the incircle in F and G. Line AB intersects line EF and EG in K and L, respectively. Prove that $DK = DL$.



Notice that $EE'||AB$. Then, $\angle E'EK = \angle EKL$. Also, because AC is a tangent to the incircle at E', we have $\angle AE'F = \angle EE'K$. Therefore, we have $\angle EKD = \angle AE'F$, and from this, we conclude that $AE'FK$ is cyclic. By Miquel's Theorem, we know that the circumcircles of $\triangle AFK, \triangle EFG, \triangle EKL$ must intersect at a single point. Since we know that this point cannot be F, it must be E'. Then, we have $E'ELK$ cyclic, and since $EE'||AB$, and since CD is the perpendicular bisector of EE', it follows that CD is the perpendicular bisector of KL, as desired.

13 2015-01-10

13.1 WOOT POTD Wednesday Oct 15

If positive integers x and y are such that $3x+4y$ and $4x+3y$ are perfect squares, prove that both x and y are divisible by 7.
Let

$$3x + 4y = a^2$$

$$4x + 3y = b^2$$

for positive integers a, b. We then obtain $7(x+y) = a^2 + b^2$ by adding the above equations. Notice that any perfect square modulo 7 is either $0, 1, 2, 4$; hence, it follows that both a and b are $0(mod 7)$. Let $a = 7c, b = 7d$. Then, we have $7(x+y) = 49(c^2 + d^2)$, or $x + y \equiv 0(mod 7)$. Write the first equation as $3(x+y) + y \equiv 0(mod 7)$. It then follows that $y = \equiv 0(mod 7)$, as desired.

14 2015-01-11

14.1 1996 ISL G1

Let ABC be a triangle, and H its orthocenter. Let P be a point on the circumcircle of triangle ABC (distinct from the vertices A, B, C), and let E be the foot of the altitude of triangle ABC from the vertex B. Let the parallel to the line BP through the point A meet the parallel to the line AP through the point B at a point Q. Let the parallel to the line CP through the point A meet the parallel to the line AP through the point C at a point R. The lines HR and AQ intersect at some point X. Prove that the lines EX and AP are parallel.

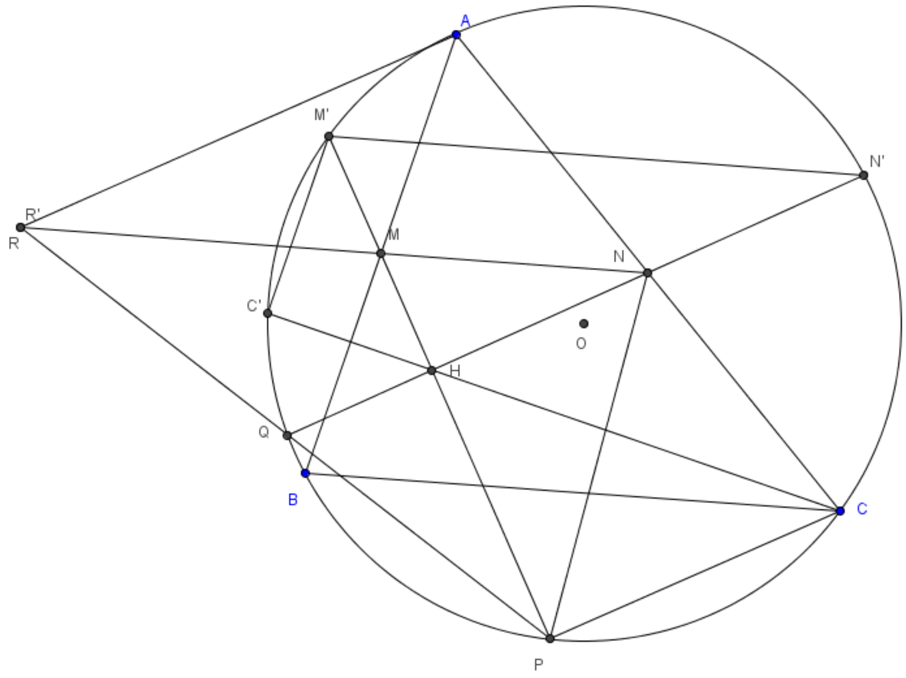
OUTLINE:

Show $AHCR$ cyclic. Since $\angle HRC + \angle ARH = \angle ARC = \angle APC$, with a lot of other angle chasing, you get $\angle AXR = 90$. Then, $AXHE$ is cyclic, and you get $\angle XAH = \angle HBQ = \angle XEH$. Then, we get $XE||BQ||AP$ as desired.

15 2015-01-12

15.1 Geometric Transformations Problem 7

Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.



Reflect H over M, N to obtain M', N', as shown. These must lie on the circumcircle. Since $HM = MM'$ and $HN = NN'$, we know that HN is the midline of $\triangle HM'N'$, and therefore, $N'M'||NM$. This immediately gives $MNPQ$ cyclic. By power of a point, we have $RQ \cdot RP = RM \cdot RN$. Hence, R has the same power with respsect to the circumcircle of $\triangle AMN$, ω, and the circumcircle of $MNPQ$. This means that R must lie on the radical axis of the circumcircle of $\triangle AMN$ and ω.

Now, note that $AMON$ is cyclic; this is because $\angle AMO = \angle ANO = 90$. Since the center of this circle must lie on AO (this is because both $\angle AMO$ and $\angle ANO$ are right), the circumcircle of $AMON$ and circle ω are internally tangent at A. Hence, the radical axis of these two circles is a line tangent to ω at A. Since we previously concluded R must lie on this line, we have that RA is tangent to ω at A, as desired.

16 2015-01-20

16.1 Own

Suppose we have a polynomial $P(x)$ defined as follows:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$$

where $a_i \geq 0$ for $0 \leq i \leq a_n$. Further, suppose that:

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} \frac{1}{a_i}$$

and:

$$P(2) = \frac{124}{5}$$

What is the minimum possible value of

$$P\left(\frac{1}{3}\right)$$

?

Use $P(x)P(1/x) \geq P(1)$. Minimize $P(1)$ by applying Cauchy on the second condition. Answer is $\boxed{\frac{n+1}{7}}$. NOTE THAT THIS DOES NOT WORK; NEED TO FIX

16.2 Own

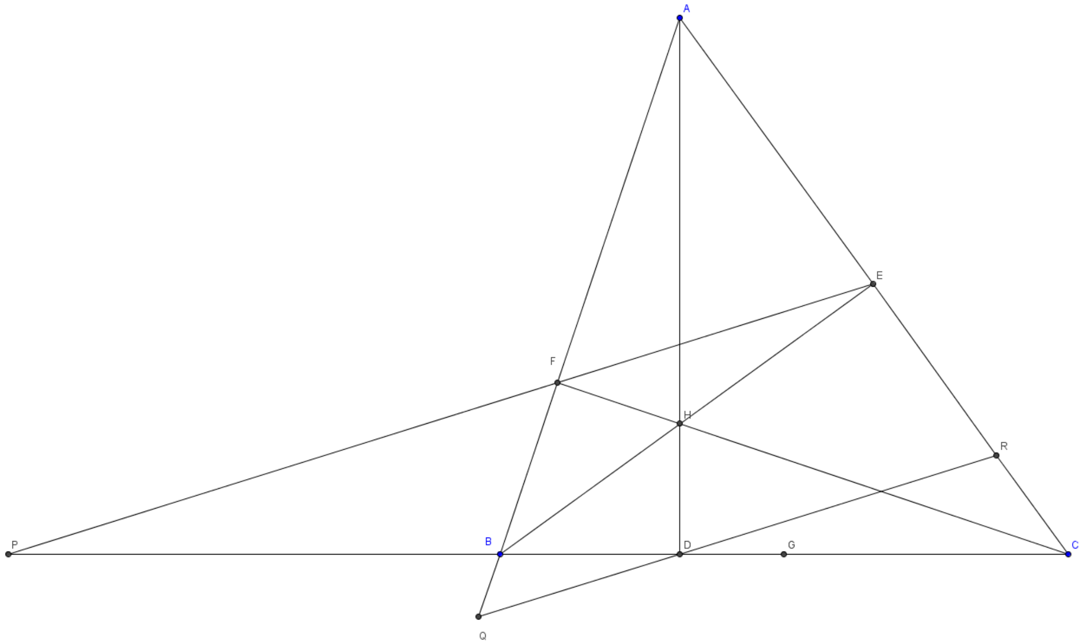
Let $\triangle ABC$ be a triangle such that $AB = 13$, $BC = 14$, and $AC = 15$. Let the altitudes to sides AB, AC, BC be CH_C, BH_B, AH_A, respectively, and let the orthocenter of $\triangle ABC$ be H. Extend CH to the circumcircle of $\triangle ABC$ such that CH intersects it at D. Extend H_BH_C such that it intersects the circumcircle of $\triangle AH_BB$ at E. Determine the length of H_CE.

Cyclic quads ftw, just evaluate $HH_C \cdot CH_C/H_CH_B$

17 2015-01-24

17.1 1997 ISL 18

The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of the triangle PQR passes through the midpoint of BC.



Define the midpoint of BC to be G.
Notice that it suffices to prove that $PD \cdot DG = QD \cdot DR$. Since $QR || EF$, we have $\angle AQR = \angle AFE$. Furthermore, note that since $\angle BFC = \angle BEC = \frac{\pi}{2}$, we have $BFEC$ cyclic. Then, $\angle C = \pi - \angle EFB = \angle AFE$. Hence, $\angle AQR = \angle AFE = \angle C$. It then follows that $QBRC$ is cyclic. Hence, by power of a point, $BD \cdot CD = QD \cdot QR$.
Now, by Menelaus (and ignoring the signed lengths), we have $\frac{CP}{PB} \cdot \frac{BF}{AF} \cdot \frac{AE}{EC} = 1$. Notice that if we multiply both sides of this by $\frac{CD}{BD}$, we obtain

$$\frac{CP}{PB} \cdot \frac{BF}{AF} \cdot \frac{AE}{EC} \cdot \frac{CD}{BD} = \frac{CP}{PB} = \frac{CD}{BD}$$

where the above follows from Ceva's Theorem. Without loss of generality, let $BD = l, BC = 1$. Let $PB = y$. We then have:

$$\frac{y+1}{y} = \frac{1-l}{l}$$

$$1 + \frac{1}{y} = \frac{1}{l} - 1$$

$$\frac{1}{y} = \frac{1}{l} - 2 = \frac{1-2l}{l}$$

$$y = \frac{l}{1-2l}$$

Then, we have:

$$BD \cdot CD = l \cdot (1-l)$$

and:

$$PD \cdot DG = (\frac{l}{1-2l} + l)(\frac{1}{2} - l) = \frac{2l - 2l^2}{1-2l} \cdot \frac{1-2l}{2} = l(1-l)$$

Therefore, we have $BD \cdot CD = PD \cdot DG$, and since we established that $BD \cdot CD = QD \cdot QR$, we have $PD \cdot DG = QD \cdot DR$, as desired.

18 2015-01-25

18.1 USA TST 2010/2

Let a, b, c be positive reals such that $abc = 1$. Prove that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}$$

Use the following substitution: $\frac{1}{a} = x, \frac{1}{b} = y, \frac{1}{c} = z$. We now wish to prove

$$\frac{x^3}{(z+2y)^2} + \frac{y^3}{(2z+x)^2} + \frac{z^3}{(2x+y)^2} \geq \frac{1}{3}$$

By Holder's Inequality, we can write

$$9(x+y+z)^2\left(\frac{x^3}{(z+2y)^2} + \frac{y^3}{(2z+x)^2} + \frac{z^3}{(2x+y)^2}\right) \geq (x+y+z)^3$$

$$\frac{x^3}{(z+2y)^2} + \frac{y^3}{(2z+x)^2} + \frac{z^3}{(2x+y)^2} \geq \frac{x+y+z}{9}$$

By AM-GM, we have $x + y + z \geq 3(xyz)^{\frac{1}{3}} = 3$. With this substitution, we obtain the desired.

18.2 1986/1 IMO

Let d be any positive integer not equal to $2, 5,$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

We proceed by contradiction. Suppose that we cannot find any pair of distinct integers in this set such that one less than their product is not a square. Then, we must have

$$2d - 1 \equiv 1 (mod 4)$$

$$d \equiv 1 (mod 4)$$

or

$$d \equiv 3 (mod 4)$$

However, if the second statement is true, then we have $13d - 1 \equiv 38 \equiv 2 (mod 4)$ which is not a perfect square, contradiction.

If the first statement is true, then we have to consider the set $\{2, 5, 13, 4x + 1\}$ for some positive integer x. We then must have

$$2(4x + 1) - 1 = 8x + 1$$

$$5(4x + 1) - 1 = 20x + 4 = 4(5x + 1)$$

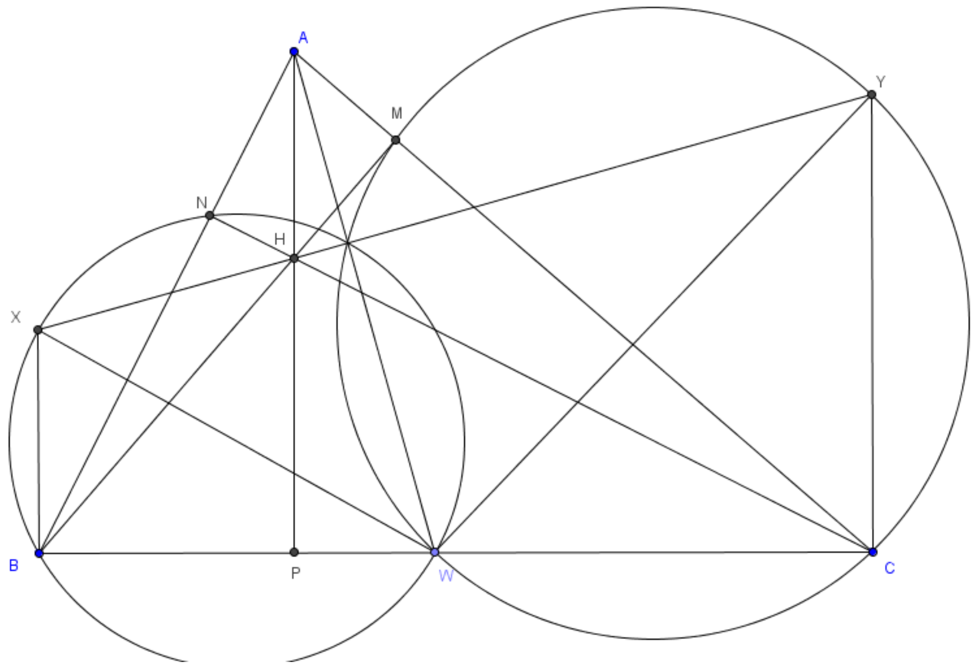
$$13(4x + 1) - 1 = 52x + 12 = 4(13x + 3)$$

to be perfect squares. In other words, $8x + 1, 5x + 1, 13x + 3$ must be perfect squares for some x. We can now have $5x + 1 \equiv 1(mod 4)$, or $x \equiv 0(mod 4)$: however, if this is the case, $13x + 3 \equiv 3(mod 4)$, contradiction. The other possibility is that $5x + 1 \equiv 0(mod 4)$, or $5x \equiv 3(mod 4)$ Solving this linear congruence gives us $x \equiv 3(mod 4)$. However, $13x + 3 \equiv 42 \equiv 2(mod 4)$, contradiction. Since we have arrived at contradictions at all possibilities, we are finished.

19 2015-02-06

19.1 2013/4 IMO

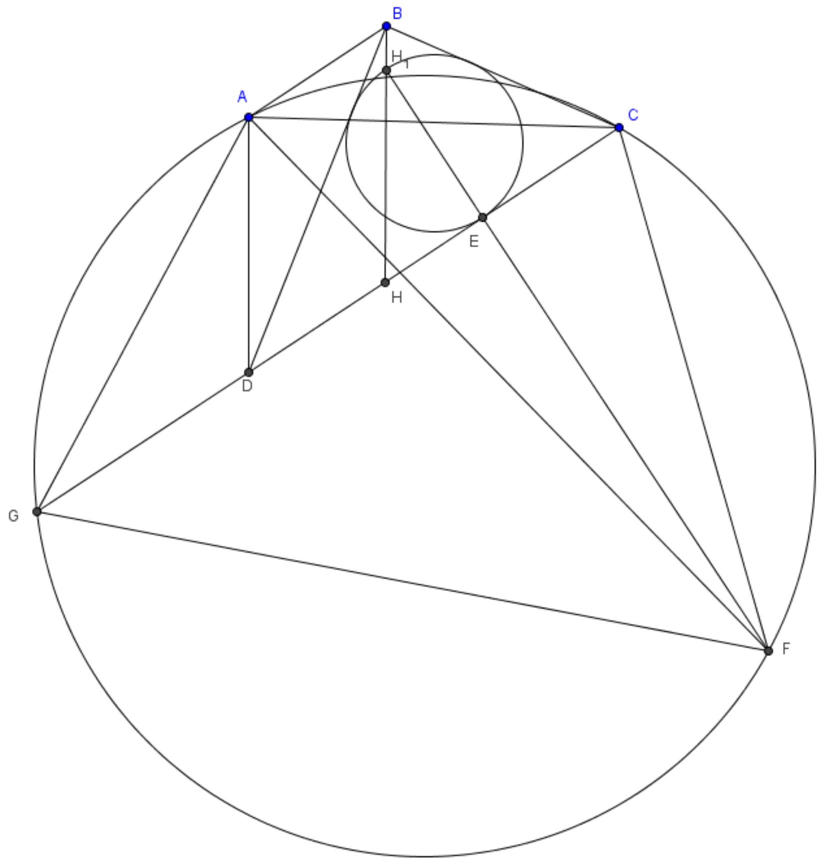
Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 as the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogoously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X,Y and H are collinear.



Define K such that $K = \omega_1 \cap \omega_2, K \neq W$. Since $\angle WBX + \angle XKW = \pi$ and since $\angle WBX = \frac{\pi}{2}$, we have $\angle XKW = \frac{\pi}{2}$. With similar logic, we have $\angle YKW = \frac{\pi}{2}$. It then follows that X, K, Y are collinear. Also, note that since $BNMC$ is cyclic, we have $AN \cdot AB = AM \cdot AC$, which means A must lie on the radical axis (KW) of ω_1, ω_2. Additionally, since $BNHP$ is cyclic, we have $AN \cdot AB = AH \cdot AP = AK \cdot AW$. It then follows that $HKWP$ is cyclic, which indicates that $\angle HKW = \pi - \angle APW = \frac{\pi}{2}$. Since $\angle HKW = \angle XKW = \frac{\pi}{2}$, we know that X, H, K and therefore X, H, Y are collinear, as desired.

19.2 1999/6 USAMO

Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

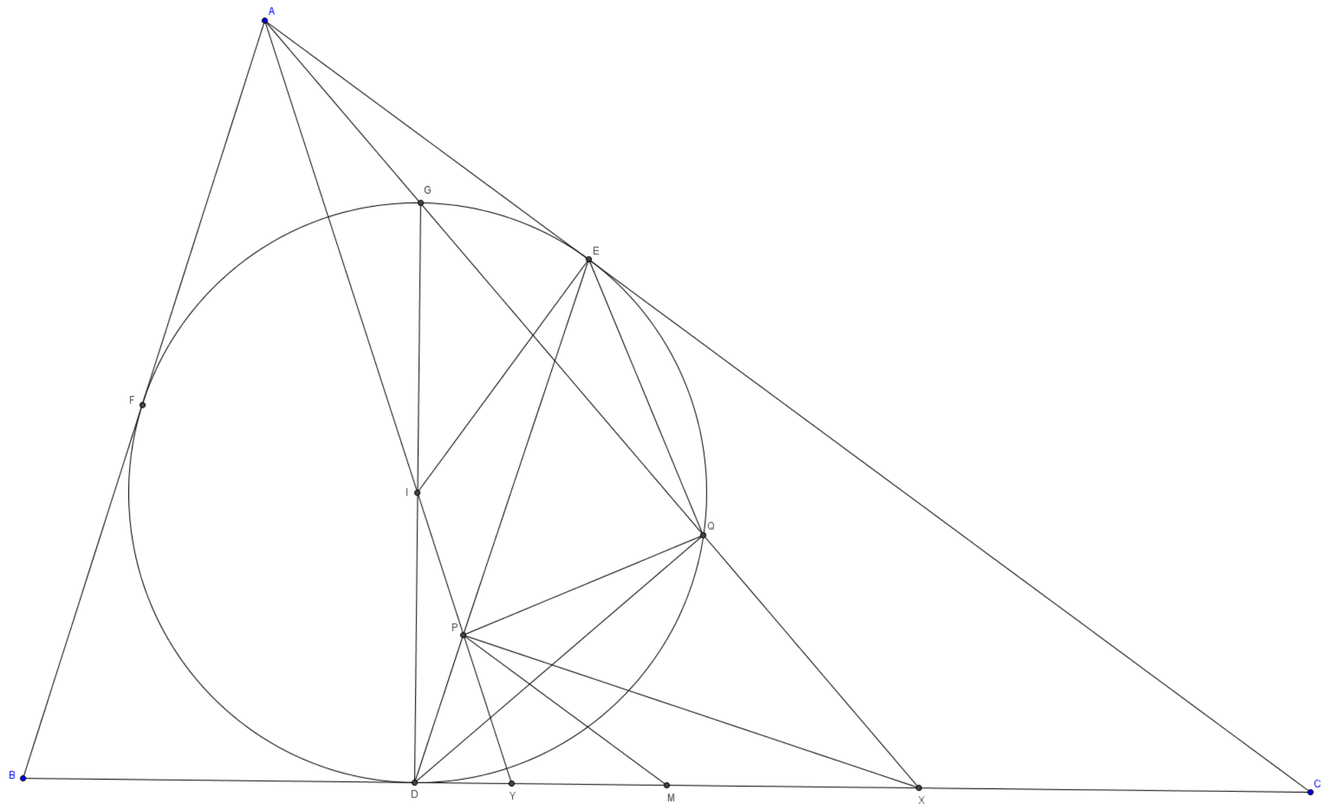


By some fancy lemma in geometry, we have $DH = EC$. Since $FE \perp CG$ and F lies on the angle bisector of $\angle DAC$, we know that F must be the A-excenter of $\triangle DAC$. Then, we angle chase:

20 2015-02-08

20.1 2015/1 USA December TST

Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.



We first prove D, P, E collinear (TO PROVE STILL).
Next, we prove $\angle DPX = \frac{\pi}{2}$. This is true because X is the tangency point of the A-excircle; hence, $BD = CX$ and it follows $DM = MX$. Since $DM = MP$, we have that M is the circumcenter of the circumcircle of $\triangle DPX$ and from this it follows that $\angle DPX = \frac{\pi}{2}$.
Now, let $\angle CEQ = \alpha, \angle DEQ = \beta$. It follows that $\angle PDQ = \alpha, \angle QDX = \beta$. Then, $\angle DPX = \frac{\pi}{2} - \alpha - \beta$. Also, note that $\angle GDE = \angle GQE = \frac{\pi}{2} - \alpha - \beta$. By a well-known theorem, $\angle GAE = \angle EGQ - \angle GQE$. Also, since $\angle EGQ = \angle QEC = \alpha$, we can determine $\angle GAE$ to be:

$$\angle GAE = \angle EGQ - \angle GQE = \alpha - (\frac{\pi}{2} - \alpha - \beta) = 2\alpha - \beta - \frac{\pi}{2}$$

Also, since $MP \parallel CE$, we have $\angle ECM = \angle PMD = \pi - 2(\alpha + \beta)$. Hence,

$$\angle AXC = \pi - (\angle GAE + \angle ACX) = \frac{\pi}{2} + \beta$$

.
Since $\angle PXD = \frac{\pi}{2} - (\alpha + \beta)$, we have $\angle QXP = \alpha$. Hence, $DPQX$ is cyclic, and we therefore obtain $\angle QPX = \angle QDX = \beta$. Since $\angle XPE = \frac{\pi}{2}$, we know that $\angle EPQ = \frac{\pi}{2} - \beta$, and solving for $\angle PQE$ gives us $\angle PQE = \frac{\pi}{2}$ as desired.

21 2015-02-13

21.1 1974/1 USAMO

Let a, b, and c denote three distinct integers, and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$.
Suppose the assumption is true. Note that for a polynomial with integral coefficients, for integers x, y, we have $x - y | P(x) - P(y)$. Now, we have that

$$\frac{P(a) - P(b)}{a - b} = \frac{b - c}{a - b}$$
$$\frac{P(b) - P(c)}{b - c} = \frac{c - a}{b - c}$$

$$\frac{P(a) - P(c)}{a - c} = \frac{b - a}{a - c}$$

must all be integers. Then, multiplying the first two fractions must yield an integer. However, we then have

$$\frac{c - a}{a - b}$$

and

$$\frac{b - a}{a - c}$$

are integers. Hence, both of these are equal to 1 or -1. In the second case, we have $c - a = b - a$ which implies $b = c$, contradicting the distinct restriction. In the first case, we have $b + c = 2a$, or equivalently, $b = 2a - c$. We then have

$$\frac{P(b) - P(c)}{b - c} = \frac{c - a}{b - c} = \frac{c - a}{2a - 2c} = -\frac{1}{2}$$

which is a contradiction since we established that this must be an integer. Since we have arrived at contradictions in all possible cases, we are done.