eCHT course on Stable Homotopy Theory

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1 Homotpy limits and colimits

1.1 Motivation

Notation: $\mathcal{C}$ homotopical category. $\mathcal{M}$ model category.

Usual limits colimts use 1-category structure. usual notion of colimits cannot be correct for topological structure.

colim $: Top \to Top$ is not homotopical functor.

We want to give a modern correct formulation for this. A cofibrant replacement in the correct sense.

1.2 Preliminaries

Definition 1.1. A left deformation of $\mathcal{C}$ is an endofunctor

$$Q : \mathcal{C} \to \mathcal{C}$$

together with a natural weak equivalence

$$\eta : Q \xrightarrow{\simeq} \mathrm{id}_{\mathcal{C}}$$

$\mathcal{C}_Q$ full subcat of $\mathcal{X}$ that contains image of Q.

in model cat $\mathcal{C}_Q$ is the category of cofibrant objects.

Definition 1.2. A left deformation of a functor $F : \mathcal{C} \to \mathcal{D}$. is a left derived functor

$$q : Q \to \mathrm{id}_{\mathcal{C}}$$

of $\mathcal{C}$ and a choice of $\mathcal{C}_Q$ such that F is homotopical on C_Q.

Example 1.3. If M is a model cat. F is left Quillen, then the cofibrant replacement is a left def of F by Ken Brown lemma

Example 1.4. $M = CGWH$. Fact: CW complexes are cofibrant.

Theorem 1.5. If $q : Q \implies id$ is a left def of $F : C \to D$, tehn

$$LF = FQ$$

is a left derived functor of F.

Proof. 2.2.8 Riehl □

1.3 Homotopy colimts

Let J be small cat

Definition 1.6. the homotopy colimit functor, if it exists, is a left derived functor

$$\text{hocolim} := \mathbb{L}\,\text{colim} : \mathcal{C}^J \to \mathcal{C}$$

of colim : $\mathcal{C}^J \to \mathcal{C}$.

To producte it find a left deformation functor Q of $\mathcal{C}$ and consider FQ.

David's recap L4 25/1

We can compute the left derived functor of $F : C \to D$ between homotpical categories using a left def $Q : C \to C, q : Q \implies 1$ such that $F|_{\text{im}Q}$ is a homotpical functor by $\mathbb{L}F = FQ$.

In Top, Q is CW approximation in $CH(R)$ Q is projective resolution

colim : $C^I \to C$ wher C^I homotopical cat with pointwise weak equivalence . Then we copmute hocolim $= \mathbb{L}\,$colim using a bar complex.

Homotopy pushouts: hocolim $Z \leftarrow X \to Y$ replace one of the morphisms with a cofibration. so the above is colim($QZ \leftarrow QX \to QY$)

you replace one of the objects with $Q(-)$ and then use the cofibration factorization on the other. But in top you only need one of f and g to be made into cofibration.

L5 30/1

2 Stable phenomenon of Spaces

Spaces of interest -Top,Top$_*$ These spaces arenot closed under products,even for CW complexes. CGWH is a cateogry that satisfies these properties and we take these to mean spaces.

Definition 2.1. $(X, \cdot), (Y, \cdot) \in$ Top$_*$ Define Y^X to be maps and $F(X, Y)$to be pointed maps equipped with compactopen topology.

Smash Product defined.

(Top$_*, \wedge, S^0$) make a closedsymmetric monoidal category. Internal hom $= F(X, Y)$.

$\text{hom}_{\text{Top}}(X \wedge Y, Z) \cong \text{hom}(X, F(Y, Z))$. $F(X \wedge Y, Z) \cong F(X, (Y, Z))$.

$\pi_0(F(X, Y), *) \cong [S^0, F(X, Y)] \cong [X, Y]$

If we choose $Y = S^1$, then we get $F(\Sigma X, Z) \cong F(X, \Omega Z)$. If you apply yne π_0 functor we get homootpy classes of maps bijection. in general $- \wedge Z \dashv F(-, Z)$

ΣX is the pushout of $* \leftarrow X \to *$.

ΩX is the pullback of $* \to X \leftarrow *$.

ΩX is an H group. ΣX gives a co H group structure.

If Y is a H group then $[-, Y]$ is a group. If X is a co gropu then $[X, Y]$ is a group. So the loopspace suspension adjunction is an isomorphism.

If $[X, y]$ for X co H group and Y H group, then $[X, Y]$ is an abelian group.

Freudenthal Suspension Theorem

$\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X)$ is an isomorphism for $0 \leq i \leq 2n$ and epimorphism for $i = 2n + 1$.

$\text{colim}(\pi_{i+n+1(\Sigma^n X)}) = \pi_i^s(X)$ is the i^{th} stablehomotpy group. If we take $X = Sp$, it is called ith stable stem.

$\pi^*(S)$ is a graded ring, with graded commutativestructure...

$X \to \Omega\Sigma X$, is $(2n + 1)$ connected if X is n-connected.

Define $QX = \text{colim}(X \to \Omega\Sigma X \to (\Omega\Sigma)^2 X \to \cdots)$ $\pi_*(QX) = \pi_*^s(X)$.

Excision does nothold for $X = S^2, A = X\mathbb{N}, B = X\S$ and $A \cap B = S^1$. then $\pi_3(A, A \cap B) \cong \pi_2(S^1) = 0$ and $\pi_3(S^2, B) \cong \pi_3(S^2) \neq 0$.

Excision holds for higher homtopy groups upto connectivity.

Theorem 2.2 (Blaker's Massey Excision). If $X = A \cup B$, $A \cap B \neq \varphi$ and A, C is m connected and (B, C) is n connected

$$\pi_i(A, C) \xrightarrow{i_*} \pi_i(X, B)$$

isomorphism for $i < m + n$ and surjective for $i = m + n$.

Theorem 2.3. If

$$\begin{tikzcd}
Q \arrow[rr, "n-connected"] \arrow[dr, dashed, "h\, m+n-1connected"] \arrow[dd, "m-connected"'] & & Y \arrow[dd] \\
& F \arrow[ur] \arrow[dl] & \\
X \arrow[rr] & & P
\end{tikzcd}$$

here F is the pullback of the inner square. his gives a version of van kampen.

Corollary 2.4. $X =*$. $Q \xrightarrow{g} Y \xrightarrow{c} P$ Q is m connected and g, n connected. Then hofib $= F$ $Q \to F$ is $m + n$ connected

If X, Y are p, q, connected then $X \vee Y \to X \times Y$ is $p + q + 1$ connected.

Theorem 2.5. Hurewicz Theorem fsdafasjkdfhljkh