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Problem 1

Part a

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[dr, "ba"'] \arrow[drr, dashed, "cba"] & \bullet \arrow[d, "b"'] \arrow[dr, "cb"] & \\ & \bullet \arrow[r, "c"'] & \bullet \end{tikzcd}$$

We have the category as above. If $g, fg \in \mathcal{W}$. Then set $a = f, b = g, h = 1$ and from the diagram above we have $f \in \mathcal{W}$. Similarly for $f, fg \in \mathcal{W}$, set $a = 1, b = f, c = g$ to get $g \in \mathcal{W}$. If $f, g \in \mathcal{W}$, by settin $a = f, b = 1, c = h$ we get $fg \in \mathcal{W}$.

Part b

The 2-out-of-6 is stronger than the 2-out-of-3 property.

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dashed, "hgf"] & \bullet \arrow[d, "g"'] \arrow[dr, "hg"] & \\ & \bullet \arrow[r, "h"'] & \bullet \end{tikzcd}$$

If we take $\mathcal{C}$ to be the catgory above and $\mathcal{W}$ to be morhpisms gf, hg and identity morhpisms. Then 2-out-of-3 does not imply that f, g, h or $hgf \in \mathcal{W}$.

Problem 2

Part a

$\mathcal{W}$ is saturated. We have

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr, "L(gf)"'] \arrow[drr, dashed] & \bullet \arrow[d] \arrow[dr, "L(hg)"] & \\ & \bullet \arrow[r] & \bullet \end{tikzcd}$$

with $fg, gf \in \mathcal{W}$. In the category $ho\mathcal{C}$, we have $L(fg), L(hg)$ are isomorphisms. Isomorphisms satisfy the 2-out-of-6 property. So $Lf, Lh, Lg, Lhgf$ are all isomorphisms. Saturated now implies that $f, g, h, hgf \in \mathcal{W}$.

Problem 3

$C \underset{F}{\overset{G}{\leftrightarrows}} D$. Let $\boldsymbol{L}F$ be the total left derived functor and $\boldsymbol{R}G$ the total right derived functor. So we have

$$hoC \underset{\boldsymbol{L}F}{\overset{\boldsymbol{R}G}{\leftrightarrows}} hoD$$

F, G are adjoint functors, so we have $\eta : 1_C \implies GF, \varepsilon : FG \implies 1_D$. Let γ, δ be the localization maps for C and D respectively.

$$\begin{tikzcd}
C \arrow[rr, "\delta F"] \arrow[dr, "\gamma"'] & & hoD \arrow[r, "RG"'] & hoC \\
& hoC \arrow[u, phantom, "\Uparrow\alpha"] \arrow[ur, "LF"'] \arrow[urr, dashed, bend right] & &
\end{tikzcd}$$

$\boldsymbol{R}G$ is an absolute Kan extension. So $\boldsymbol{R}G\boldsymbol{L}F$ is a kan extension. Similarly $\boldsymbol{L}F\boldsymbol{R}G$ is a Kan extension and we have the following diagrams.

$$\begin{tikzcd}
C \arrow[rr, "RG\delta F"] \arrow[dr, "\gamma"'] & & hoC \\
& hoC \arrow[u, phantom, "RG\alpha\Uparrow"] \arrow[ur, "RGLF"] \arrow[ur, bend right=60, "1_{hoC}"'] \arrow[ur, phantom, shift right=3, "\Uparrow\sigma"] &
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[rr, "RG\delta F"] \arrow[dr, "\delta"'] & & hoD \\
& hoD \arrow[u, phantom, "\Downarrow LF\beta"] \arrow[ur, "LFRG"] \arrow[ur, bend right=60, "1_{hoD}"'] \arrow[ur, phantom, shift right=3, "\tau\Downarrow"] &
\end{tikzcd}$$

We also have the following commutative diagrams on the natural transformations

$$\begin{tikzcd}
\gamma GF \arrow[r, Rightarrow, "\beta F"] & RG\delta F \\
\gamma \arrow[u, Rightarrow, "\gamma\eta"] \arrow[r, Rightarrow, "\sigma\gamma"'] & RGLF\gamma \arrow[u, Rightarrow, "RG\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
LF\gamma G \arrow[r, Rightarrow, "\alpha G"] \arrow[d, Rightarrow, "LF\beta"'] & \delta FG \arrow[d, Rightarrow, "\delta\varepsilon"] \\
LFRG\delta \arrow[r, Rightarrow, "\tau\delta"'] & \delta
\end{tikzcd}$$

We have $\sigma : 1 \implies \boldsymbol{R}G\boldsymbol{L}F$ and $\tau : \boldsymbol{L}F\boldsymbol{R}G \implies 1$. We need to verify the commutativity of the triangle, i.e.,

$$(\tau\boldsymbol{L}F)(\boldsymbol{L}F\sigma) = 1 \quad (\boldsymbol{R}G\tau)(\sigma\boldsymbol{R}G) = 1$$

Consider the following Kan extension

$$\begin{tikzcd}
D \arrow[rr, "G\gamma"] \arrow[dr, "\delta"'] & & hoC \\
& hoD \arrow[u, phantom, "\Downarrow\beta"] \arrow[ur, "RG"'] &
\end{tikzcd}$$

We will show that $\overbrace{(\boldsymbol{R}G\tau)(\sigma\boldsymbol{R}G)}^{\text{nat. tran. from } \boldsymbol{R}G \text{ to } \boldsymbol{R}G}(\delta)\beta = \beta$. Then from the universal property of kan extension any other $(\boldsymbol{R}G, \varphi)$ functor, natural transformation pair, φ factors through β. So this will show that $(\boldsymbol{R}G\tau)(\sigma\boldsymbol{R}G) = 1$.

$$\begin{tikzcd} D \arrow[r, bend left, "\gamma G"] \arrow[r, bend right, "RG\delta"'] \arrow[r, phantom, "\Downarrow\beta"] & hoC \arrow[r, bend left, "1"] \arrow[r, bend right, "RGLF"'] \arrow[r, phantom, "\Downarrow\sigma"] & hoC \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, bend left, "FG"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & D \arrow[r, bend left, "\gamma G"] \arrow[r, bend right, "RG\delta"'] \arrow[r, phantom, "\Downarrow\beta"] & hoC \end{tikzcd}$$

From whiskering arguments, we have,

$$(\sigma\boldsymbol{R}G\delta)\beta = (\boldsymbol{R}G\boldsymbol{L}F\beta)(\sigma\gamma G) \quad \beta(\sigma G\varepsilon) = (\boldsymbol{R}G\delta\varepsilon)(\beta FG)$$

$$\gamma G \xrightarrow{\beta} \boldsymbol{R}G\delta \xrightarrow{\sigma\boldsymbol{R}G\delta} \boldsymbol{R}G\boldsymbol{L}F\boldsymbol{R}G\delta \xrightarrow{\boldsymbol{R}G\tau\delta} \boldsymbol{R}G\delta \tag{1}$$
$$\gamma G \xrightarrow{\sigma\gamma G} \boldsymbol{R}G\boldsymbol{L}F\gamma G \xrightarrow{\boldsymbol{R}G\boldsymbol{L}F\beta} \boldsymbol{R}G\boldsymbol{L}F\boldsymbol{R}G\delta \xrightarrow{\boldsymbol{R}G\tau\delta} \boldsymbol{R}G\delta \tag{2}$$
$$\gamma G \xrightarrow{\sigma\gamma G} \boldsymbol{R}G\boldsymbol{L}F\gamma G \xrightarrow{\boldsymbol{R}G\alpha G} \boldsymbol{R}G\delta FG \xrightarrow{\boldsymbol{R}G\delta\varepsilon} \boldsymbol{R}G\delta \tag{3}$$
$$\gamma G \xrightarrow{\gamma\eta G} \gamma GFG \xrightarrow{\beta FG} \boldsymbol{R}G\delta FG \xrightarrow{\boldsymbol{R}G\delta\varepsilon} \boldsymbol{R}G\delta \tag{4}$$
$$\gamma G \xrightarrow{\gamma\eta G} \gamma GFG \xrightarrow{\gamma G\varepsilon} \gamma G \xrightarrow{\beta} \boldsymbol{R}G\delta \tag{5}$$
$$\gamma G \xrightarrow{\beta} \boldsymbol{R}G\delta \tag{6}$$

eq 1 to eq 2 is whiskering argument. eq3 is obtained by noticing that

$$\boldsymbol{R}G\tau\delta \cdot \boldsymbol{R}G\boldsymbol{L}F\beta = \boldsymbol{R}G(\tau\delta \cdot \boldsymbol{L}F\beta) = \boldsymbol{R}G(\delta\varepsilon \cdot \alpha G)$$

eq4 is obtained by using

$$(\boldsymbol{R}G\alpha)(\sigma\gamma) = (\beta F)(\gamma\eta)$$

Another whiskering gives us eq5 and finally

$$\gamma(G\varepsilon \cdot \eta G) = \gamma$$

This shows that $\boldsymbol{L}F \dashv \boldsymbol{R}G$

Problem 4

Part a

The commutative diagram at the left below gives unique $\tilde{F}$ such that, $\tilde{F}L = F$. We know from Yoneda lemma, that

$$\hom(\mathcal{C}(c, -), F) \cong Fc = \tilde{F}Lc = \hom(ho\mathcal{C}(Lc, -), \tilde{F}c)$$

This gives the diagram at the right.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \textbf{Set} \\ ho\mathcal{C} \arrow[ur, "!\tilde{F}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(c,-) \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "L"'] & F \\ ho\mathcal{C}(Lc,-) \arrow[ur, Rightarrow, "\tilde{\alpha}"'] & \end{tikzcd}$$

So given α a natural transformation as above, we have a unique map $\tilde{\alpha}$ such that $\tilde{\alpha}L = \alpha$. But $Lc = c$ on $c \in \text{obj}\mathcal{C}$. This gives us the factorization through $ho\mathcal{C}(c,-)$.

Part b

Consider id : $\mathcal{C}(c,-) \implies \mathcal{C}(c,-)$. This set of natural transformations is isomorphic to $\mathcal{C}(c,c)$. We have

$$\begin{tikzcd} \mathcal{C}(c,d) \arrow[r, "\text{id}"] \arrow[d, "L"'] & \mathcal{C}(c,d) \\ ho\mathcal{C}(c,d) \arrow[ur, "\tilde{\text{id}}"'] & \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r] \arrow[d, "L"'] & f \\ \tilde{f} \arrow[ur, "\tilde{\text{id}}"'] & \end{tikzcd}$$

This is clearly surjective, and it is injective because if $\tilde{f}, \tilde{g}$ go to some h, then $\tilde{\text{id}}L(-) = \text{id}(-)$ implies $f = g$ and $\tilde{f} = \tilde{g}$.