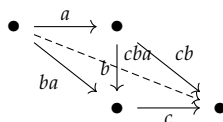


## PROBLEM 1

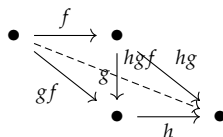
### PART A



We have the category as above. If  $g, fg \in \mathcal{W}$ . Then set  $a = f, b = g, h = 1$  and from the diagram above we have  $f \in \mathcal{W}$ . Similarly for  $f, fg \in \mathcal{W}$ , set  $a = 1, b = f, c = g$  to get  $g \in \mathcal{W}$ . If  $f, g \in \mathcal{W}$ , by setting  $a = f, b = 1, c = h$  we get  $fg \in \mathcal{W}$ .

### PART B

The 2-out-of-6 is stronger than the 2-out-of-3 property.

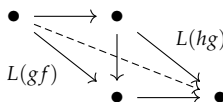


If we take  $\mathcal{C}$  to be the category above and  $\mathcal{W}$  to be morphisms  $gf, hg$  and identity morphisms. Then 2-out-of-3 does not imply that  $f, g, h$  or  $hgf \in \mathcal{W}$ .

## PROBLEM 2

### PART A

$\mathcal{W}$  is saturated. We have



with  $fg, gf \in \mathcal{W}$ . In the category  $ho\mathcal{C}$ , we have  $L(fg), L(hg)$  are isomorphisms. Isomorphisms satisfy the 2-out-of-6 property. So  $Lf, Lh, Lg, Lhgf$  are all isomorphisms. Saturated now implies that  $f, g, h, hgf \in \mathcal{W}$ .

### PROBLEM 3

$C \xrightleftharpoons[F]{G} D$ . Let  $LF$  be the total left derived functor and  $RG$  the total right derived functor. So we have

$$hoC \xrightleftharpoons[LF]{RG} hoD$$

$F, G$  are adjoint functors, so we have  $\eta : 1_C \Rightarrow GF, \varepsilon : FG \Rightarrow 1_D$ . Let  $\gamma, \delta$  be the localization maps for  $C$  and  $D$  respectively.

$$\begin{array}{ccccc} C & \xrightarrow{\delta F} & hoD & \xrightarrow{RG} & hoC \\ & \searrow \gamma & \uparrow \alpha & \nearrow LF & \\ & & hoC & & \end{array}$$

(A dashed curved arrow also points from  $hoC$  to  $hoC$ .)

$RG$  is an absolute Kan extension. So  $RG\delta F$  is a Kan extension. Similarly  $LFRG$  is a Kan extension and we have the following diagrams.

$$\begin{array}{ccc} C & \xrightarrow{RG\delta F} & hoC \\ \gamma \searrow & \uparrow RG\alpha & \nearrow RGLF \\ & hoC & \end{array}$$

(A curved arrow labeled  $1_{hoC}$  goes from  $hoC$  to  $hoC$ .)

$$\begin{array}{ccc} D & \xrightarrow{RG\delta F} & hoD \\ \delta \searrow & \downarrow LF\beta & \nearrow LFRG \\ & hoD & \end{array}$$

(A curved arrow labeled  $1_{hoD}$  goes from  $hoD$  to  $hoD$ .)

We also have the following commutative diagrams on the natural transformations

$$\begin{array}{ccc} \gamma GF & \xrightarrow{\beta F} & RG\delta F \\ \gamma \eta \uparrow & & \uparrow RG\alpha \\ \gamma & \xrightarrow{\sigma \gamma} & RGLF\gamma \end{array} \quad \begin{array}{ccc} LF\gamma G & \xrightarrow{\alpha G} & \delta FG \\ LF\beta \downarrow & & \downarrow \delta \varepsilon \\ LFRG\delta & \xrightarrow{\tau \delta} & \delta \end{array}$$

We have  $\sigma : 1 \Rightarrow RGLF$  and  $\tau : LFRG \Rightarrow 1$ . We need to verify the commutativity of the triangle, i.e.,

$$(\tau LF)(LF\sigma) = 1 \quad (RG\tau)(\sigma RG) = 1$$

Consider the following Kan extension

$$\begin{array}{ccc} D & \xrightarrow{G\gamma} & hoC \\ \delta \searrow & \downarrow \beta & \nearrow RG \\ & hoD & \end{array}$$

We will show that  $\overbrace{(RG\tau)(\sigma RG)}^{\text{nat. tran. from } RG \text{ to } RG} (\delta)\beta = \beta$ . Then from the universal property of kan extension any other  $(RG, \varphi)$  functor, natural transformation pair,  $\varphi$  factors through  $\beta$ . So this will show that  $(RG\tau)(\sigma RG) = 1$ .

$$\begin{array}{ccc}
 D & \xrightarrow{\gamma G} & hoC \\
 \Downarrow \beta & & \Downarrow \sigma \\
 D & \xrightarrow{RG\delta} & hoC
 \end{array}
 \quad
 \begin{array}{ccc}
 D & \xrightarrow{FG} & D \\
 \Downarrow \varepsilon & & \Downarrow \beta \\
 D & \xrightarrow{RG\delta} & hoC
 \end{array}$$

From whiskering arguments, we have,

$$(\sigma RG\delta)\beta = (RGLF\beta)(\sigma\gamma G) \quad \beta(\sigma G\varepsilon) = (RG\delta\varepsilon)(\beta FG)$$

$$\gamma G \xrightarrow{\beta} RG\delta \xrightarrow{\sigma RG\delta} RGLFRG\delta \xrightarrow{RG\tau\delta} RG\delta \quad (1)$$

$$\gamma G \xrightarrow{\sigma\gamma G} RGLF\gamma G \xrightarrow{RGLF\beta} RGLFRG\delta \xrightarrow{RG\tau\delta} RG\delta \quad (2)$$

$$\gamma G \xrightarrow{\sigma\gamma G} RGLF\gamma G \xrightarrow{RG\alpha G} RG\delta FG \xrightarrow{RG\delta\varepsilon} RG\delta \quad (3)$$

$$\gamma G \xrightarrow{\gamma\eta G} \gamma GFG \xrightarrow{\beta FG} RG\delta FG \xrightarrow{RG\delta\varepsilon} RG\delta \quad (4)$$

$$\gamma G \xrightarrow{\gamma\eta G} \gamma GFG \xrightarrow{\gamma G\varepsilon} \gamma G \xrightarrow{\beta} RG\delta \quad (5)$$

$$\gamma G \xrightarrow{\beta} RG\delta \quad (6)$$

eq 1 to eq 2 is whiskering argument. eq3 is obtained by noticing that

$$RG\tau\delta \cdot RGLF\beta = RG(\tau\delta \cdot LF\beta) = RG(\delta\varepsilon \cdot \alpha G)$$

eq4 is obtained by using

$$(RG\alpha)(\sigma\gamma) = (\beta F)(\gamma\eta)$$

Another whiskering gives us eq5 and finally

$$\gamma(G\varepsilon \cdot \eta G) = \gamma$$

This shows that  $LF \dashv RG$

## PROBLEM 4

### PART A

The commutative diagram at the left below gives unique  $\tilde{F}$  such that,  $\tilde{F}L = F$ . We know from Yoneda lemma, that

$$\text{hom}(\mathcal{C}(c, -), F) \cong Fc = \tilde{F}Lc = \text{hom}(ho\mathcal{C}(Lc, -), \tilde{F}c)$$

This gives the diagram at the right.

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathbf{Set} \\
 L \downarrow & \nearrow !\tilde{F} & \\
 ho\mathcal{C} & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{C}(c, -) & \xrightarrow{\alpha} & F \\
 L \downarrow & \nearrow \tilde{\alpha} & \\
 ho\mathcal{C}(Lc, -) & & 
 \end{array}$$

So given  $\alpha$  a natural transformation as above, we have a unique map  $\tilde{\alpha}$  such that  $\tilde{\alpha}L = \alpha$ . But  $Lc = c$  on  $c \in \text{obj}\mathcal{C}$ . This gives us the factorization through  $ho\mathcal{C}(c, -)$ .

PART B

Consider  $\text{id} : \mathcal{C}(c, -) \Rightarrow \mathcal{C}(c, -)$ . This set of natural transformations is isomorphic to  $\mathcal{C}(c, c)$ . We have

$$\begin{array}{ccc}
 \mathcal{C}(c, d) & \xrightarrow{\text{id}} & \mathcal{C}(c, d) \\
 L \downarrow & \nearrow \tilde{\text{id}} & \\
 ho\mathcal{C}(c, d) & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 f & \longrightarrow & f \\
 L \downarrow & \nearrow \tilde{\text{id}} & \\
 \tilde{f} & & 
 \end{array}$$

This is clearly surjective, and it is injective because if  $\tilde{f}, \tilde{g}$  go to some  $h$ , then  $\tilde{\text{id}}L(-) = \text{id}(-)$  implies  $f = g$  and  $\tilde{f} = \tilde{g}$ .