Sequential Spectra
Lecture 5

February 9, 2024

We work in the category $\mathcal{T}$ of based(compactly generated weak Hausdorff) spaces and basepoint preserving maps.

Definition 0.1. A (sequential) spectrum X is a sequence of spaces X_n for $n \geq 0$ and structure maps $\sigma_n : \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$.

A map of spectra $f : X \to Y$ is a sequence of maps $f_n : X_n \to Y_n$ such that eacg square of the following form commutes

S^1 commute with $\wedge$

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "\Sigma f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

The category of spectra and the maps described above is denoted by $\mathcal{S}p$.

Note that the bonding maps $\sigma_i : \Sigma X_i \to \Sigma_{i+1}$ have adjoints $\tilde{\sigma}_i : \Omega X_i \to \Omega X_{i+1}$ and we could equivalently define a spectrum using the adjoint bonding maps.

Definition 0.2. The degree $k \in \mathbb{Z}$ stable homotopy group of a spectrum X is the abelian group

$$\pi_k(X) = \mathrm{colim}_n\, \pi_{k+n}(X_n)$$

Here $\pi_{k+n}(X_n) \to \pi_{k+n+1(X_{n+1})}$ maps the homotopy class of $\phi : S^{k+n} \to X_n$ to the class of $\sigma(S^1 \wedge \phi) : S^1 \wedge S^{k+n} \to X_{n+1}$.

Definition 0.3. A map of spectra $f : X \to Y$ is a stable equivalence (π_* isomorphism) if the induced map $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Consider $\mathcal{W} \subset \mathcal{S}p$ to be the wide subcategory of stable equivalences. The stable homotopy category ho$\mathcal{S}p$ is the localization of $\mathcal{S}p$ at $\mathcal{W}$. The functor

$$\mathcal{S}p \to \mathcal{S}p[\mathcal{W}^{-1}] = \mathrm{ho}\mathcal{S}p$$

sends stable equivalences to isomorphisms.

Initial functor?

Example 0.4.
1. The suspension spectrum $\Sigma^\infty X$ of a space X is the spectrum, with $(\Sigma^\infty X)_n = \Sigma^n X$, with the bonding maps id $: S^1 \wedge \Sigma^i X \to \Sigma^{i+1} X$. The suspension spectrum for an unpointed space X is obtained by adding a disjoint basepoint: $\Sigma^\infty_+ X := \Sigma^\infty(X_+)$.
   The homotopy groups of $\Sigma^\infty X$ are the stable homotopy groups of the space X.
2. An important example of a suspension spectrum is the *sphere spectrum*

   $$\mathbb{S} := \Sigma^\infty S^0$$

   This is the sequence of spheres $S^0, S^1, S^2, \ldots$
   The homotopy groups of the sphere spectrum are called the *stable stems* . They are the stable homotopy groups of S^0, often written $\pi_i^s := \pi_i \mathbb{S}$.
3. The zero spectrum $\Sigma^\infty *$ is the suspension spectrum of a point, with the bonding maps

   $$\Sigma(*) \cong * \to *$$

Every spectrum X admits a unique maps of spectra $* \to X \to *$. X is weakly contractible if one (equivalently both) of these maps are stable equivalences. The suspension spectrum has all of its homotopy groups zero.

Given two spectra X and Y, the zero map $X \to Y$ is the unique map of spectra that factors through $*$

$$\begin{tikzcd} X_n \arrow[rr] \arrow[dr] & & Y_n \\ & * \arrow[ur] & \end{tikzcd}$$

4. Eilenberg-Maclane spaces are infinite loopspaces. We can define teh Eilenberg-Maclane spectrumm HA by

$$(HA)_n = K(A, n)$$

with bonding maps adjoing to the canonical equivalences $K(A, n) \simeq \Omega K(A, n+1)$. We use the letter H because the infinite loopspace represents cohomology with coefficients in A. todo

The homotopy grops of HA are given by

$$\pi_i HA = \begin{cases} A & i = 0 \\ 0 & i \neq 0 \end{cases}$$

5. The complex K theory spectrum KU

$$KU_n = \begin{cases} \mathbb{Z} \times BU & n \text{ even} \\ U & n \text{ odd} \end{cases}$$

The structure maps adjoint to the equivalences

$$\Omega U \simeq \mathbb{Z} \times BU \quad \Omega(\mathbb{Z} \times BU) \simeq U$$

The homotopy groups of KU are given by Bott elemen generator

$$\pi_i KU = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Definition 0.5. An $\Omega-$ spectrum (or fibrant spectrum) is a spectrum X in which the adjunct bonding maps

$$X_n \xrightarrow{\simeq} \Omega X_{n+1}$$

are weak homotopy equivalences.

The zeroeth space X_0 of an Ω -spectrum is an infinite loop space. Examples 3,4,5 above are Ω-spectra.