Sequential Spectra
Lecture 5

February 7, 2024

We work in the category $\mathcal{T}$ of based(compactly generated weak Hausdorff spacess) and basepoint preserving maps.

Definition 0.1. A (sequential) spectrum X is a sequence of spaces X_n for $n \geq 0$ and structure maps $\sigma_n : \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$.

A map of spectra $f : X \to Y$ is a sequence of maps $f_n : X_n \to Y_n$ such that eacg square of the following form commutes

$$
\begin{array}{ccc}
\Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\
{\scriptstyle \Sigma f_n} \downarrow & & \downarrow {\scriptstyle \Sigma f_{n+1}} \\
\Sigma Y_n & \xrightarrow[\sigma_n]{} & Y_{n+1}
\end{array}
$$

The category of spectra and the maps described above is denoted by $\mathcal{S}p$.

Note that the bonding maps $\sigma_i : \Sigma X_i \to \Sigma_{i+1}$ have adjoints $\tilde{\sigma}_i : \Omega X_i \to \Omega X_{i+1}$ and we could equivalently define a spectrum using the adjoint bonding maps.

Definition 0.2. The degree $k \in \mathbb{Z}$ stable homotopy group of a spectrum X is the abelian group

$$\pi_k(X) = \text{colim}_n \, \pi_{k+n}(X_n)$$

Here $\pi_{k+n}(X_n) \to \pi_{k+n+1(X_{n+1})}$ maps the homotopy class of $\phi : S^{k+n} \to X_n$ to the class of $\sigma(S^1 \wedge \phi) : S^1 \wedge S^{k+n} \to X_{n+1}$.

Definition 0.3. A map of spectra $f : X \to Y$ is a stable equivalence (π_* isomorphism) if the induced map $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Consider $\mathcal{W} \subset \mathcal{S}p$ to be the wide subcategory of stable equivalences. The stable homotopy category $\text{ho}\mathcal{S}p$ is the localization of $\mathcal{S}p$ at $\mathcal{W}$. The functor

$$\mathcal{S}p \to \mathcal{S}p[\mathcal{W}^{-1}] = \text{ho}\mathcal{S}p$$

sends stable equivalences to isomorphisms.

Initial functor?

Example 0.4. 1. The suspension spectrum $\Sigma^\infty X$ of a space X is the spectrum, with $(\Sigma^\infty X)_n = \Sigma^n X$, with the bonding maps id $: S^1 \wedge \Sigma^i X \to \Sigma^{i+1} X$. The suspension spectrum for an unpointed space X is obtained by adding a disjoint basepoint: $\Sigma_+^\infty X := \Sigma^\infty(X_+)$.
The homotopy groups of $\Sigma^\infty X$ are the stable homotopy groups of the space X.

2. An important example of a suspension spectrum is the *sphere spectrum*

$$\mathbb{S} := \Sigma^\infty S^0$$

This is the sequence of spheres $S^0, S^1, S^2, \ldots$
The homotopy groups of the sphere spectrum are called the *stable stems* . They are the stable homotopy groups of S^0, often written $\pi_i^s := \pi_i \mathbb{S}$.