Sequential Spectra Lecture 5

February 9, 2024

We work in the category \mathcal{T} of based(compactly generated weak Hausdorff) spaces and basepoint preserving maps.

Definition 0.1. A (sequential) spectrum X is a sequence of spaces X_n for $n \ge 0$ and structure maps $\sigma_n : \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$.

A map of spectra $f: X \to Y$ is a sequence of maps $f_n: X_n \to Y_n$ such that eacg square of the following form commutes

 S^1 commute with \land

$$\begin{array}{ccc}
\Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\
\Sigma f_n \downarrow & & \downarrow \Sigma f_{n+1} \\
\Sigma Y_n & \xrightarrow{\sigma_n} & Y_{n+1}
\end{array}$$

The category of spectra and the maps described above is denoted by Sp.

Note that the bonding maps $\sigma_i : \Sigma X_i \to \Sigma_{i+1}$ have adjoints $\tilde{\sigma}_i : \Omega X_i \to \Omega X_{i+1}$ and we could equivalently define a spectrum using the adjoint bonding maps.

Definition o.2. The degree $k \in \mathbb{Z}$ stable homotopy group of a spectrum X is the abelian group

$$\pi_k(X) = \operatorname{colim}_n \pi_{k+n}(X_n)$$

Here $\pi_{k+n}(X_n) \to \pi_{k+n+1(X_{n+1})}$ maps the homotopy class of $\phi: S^{k+n} \to X_n$ to the class of $\sigma(S^1 \land \phi): S^1 \land S^{k+n} \to X_{n+1}$.

Definition 0.3. A map of spectra $f: X \to Y$ is a stable equivalence (π_* isomorphism) if the induced map $f_*: \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Consider $W \subset Sp$ to be the wide subcategory of stable equivalences. The stable homotopy category hoSp is the localization of Sp at W. The functor

$$Sp \to Sp[W^{-1}] = \text{ho}Sp$$

sends stable equivalences to isomorphisms.

Initial functor?

Example 0.4. 1. The suspension spectrum $\Sigma^{\infty}X$ of a space X is the spectrum, with $(\Sigma^{\infty}X)_n = \Sigma^n X$, with the bonding maps $\mathrm{id}: S^1 \wedge \Sigma^i X \to \Sigma^{i+1} X$. The suspension spectrum for an unpointed space X is obtained by adding a disjoint basepoint: $\Sigma^{\infty}_+ X := \Sigma^{\infty}(X_+)$.

The homotopy groups of $\Sigma^{\infty}X$ are the stable homotopy groups of the space X.

2. An important example of a suspension spectrum is the sphere spectrum

$$\mathbb{S} \coloneqq \Sigma^{\infty} S^0$$

This is the sequence of spheres S^0, S^1, S^2, \dots

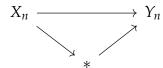
The homotopy groups of the sphere spectrum are called the *stable stems* . They are the stable homotopy groups of S^0 , often written $\pi_i^s := \pi_i S$.

3. The zero spectrum $\Sigma^{\infty}*$ is the suspension spectrum of a point, with the bonding maps

$$\Sigma(*) \cong * \rightarrow *$$

Every spectrum X admits a unique maps of spectra $* \to X \to *$. X is weakly contractible if one (equivalently both) of these maps are stable equivalences. The suspension spectrum has all of its homotopy groups zero.

Given two spectra X and Y, the zero map $X \to Y$ is the unique map of spectra that factors through *



4. Eilenberg-Maclane spaces are infinite loopspaces. We can define teh Eilenberg-Maclane spectrumm *HA* by

$$(HA)_n = K(A, n)$$

with bonding maps adjoing to the canonical equivalences $K(A, n) \simeq \Omega K(A, n + 1)$. We use the letter H because the infinite loopspace represents cohomology with coefficients in A.

todo

The homotopy grops of HA are given by

$$\pi_i H A = \begin{cases} A & i = 0 \\ 0 & i \neq 0 \end{cases}$$

5. The complex *K* theory spectrum *KU*

$$KU_n = \begin{cases} \mathbb{Z} \times BU & n \text{ even} \\ U & n \text{ odd} \end{cases}$$

The structure maps adjoint to the equivalences

$$\Omega U \simeq \mathbb{Z} \times BU \quad \Omega(\mathbb{Z} \times BU) \simeq U$$

The homotopy groups of *KU* are given by

Bott elemen generator

$$\pi_i K U = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Definition 0.5. An Ω - spectrum (or fibrant spectrum) is a spectrum X in which the adjunct bonding maps

$$X_n \xrightarrow{\simeq} \Omega X_{n+1}$$

are weak homotopy equivalences.

The zeroeth space X_0 of an Ω -spectrum is an infinite loop space. Examples 3,4,5 above are Ω -spectra.