## Sequential Spectra Lecture 5

February 8, 2024

We work in the category  $\mathcal{T}$  of based( compactly generated weak Hausdorff spacess) and basepoint preserving maps.

**Definition 0.1.** A (sequential) spectrum X is a sequence of spaces  $X_n$  for  $n \ge 0$  and structure maps  $\sigma_n : \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$ .

A map of spectra  $f: X \to Y$  is a sequence of maps  $f_n: X_n \to Y_n$  such that eacg square of the following form commutes

 $S^1$  commute with  $\land$ 

$$\begin{array}{ccc}
\Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\
\Sigma f_n \downarrow & & \downarrow \Sigma f_{n+1} \\
\Sigma Y_n & \xrightarrow{\sigma_n} & Y_{n+1}
\end{array}$$

The category of spectra and the maps described above is denoted by Sp.

Note that the bonding maps  $\sigma_i : \Sigma X_i \to \Sigma_{i+1}$  have adjoints  $\tilde{\sigma}_i : \Omega X_i \to \Omega X_{i+1}$  and we could equivalently define a spectrum using the adjoint bonding maps.

**Definition o.2.** The degree  $k \in \mathbb{Z}$  stable homotopy group of a spectrum X is the abelian group

$$\pi_k(X) = \operatorname{colim}_n \pi_{k+n}(X_n)$$

Here  $\pi_{k+n}(X_n) \to \pi_{k+n+1(X_{n+1})}$  maps the homotopy class of  $\phi: S^{k+n} \to X_n$  to the class of  $\sigma(S^1 \land \phi): S^1 \land S^{k+n} \to X_{n+1}$ .

**Definition 0.3.** A map of spectra  $f: X \to Y$  is a stable equivalence ( $\pi_*$  isomorphism) if the induced map  $f_*: \pi_*(X) \to \pi_*(Y)$  is an isomorphism.

Consider  $W \subset Sp$  to be the wide subcategory of stable equivalences. The stable homotopy category hoSp is the localization of Sp at W. The functor

$$Sp \to Sp[W^{-1}] = \text{ho}Sp$$

sends stable equivalences to isomorphisms.

Initial functor?

**Example 0.4.** 1. The suspension spectrum  $\Sigma^{\infty}X$  of a space X is the spectrum, with  $(\Sigma^{\infty}X)_n = \Sigma^n X$ , with the bonding maps  $\mathrm{id}: S^1 \wedge \Sigma^i X \to \Sigma^{i+1} X$ . The suspension spectrum for an unpointed space X is obtained by adding a disjoint basepoint:  $\Sigma^{\infty}_+ X := \Sigma^{\infty}(X_+)$ .

The homotopy groups of  $\Sigma^{\infty}X$  are the stable homotopy groups of the space X.

2. An important example of a suspension spectrum is the sphere spectrum

$$\mathbb{S} := \Sigma^{\infty} S^0$$

This is the sequence of spheres  $S^0, S^1, S^2, \dots$ 

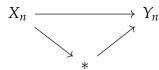
The homotopy groups of the sphere spectrum are called the *stable stems* . They are the stable homotopy groups of  $S^0$ , often written  $\pi_i^s := \pi_i S$ .

3. The zero spectrum  $\Sigma^{\infty}*$  is the suspension spectrum of a point, with the bonding maps

$$\Sigma(*) \cong * \to *$$

Every spectrum X admits a unique maps of spectra  $* \to X \to *$ . X is weakly contractible if one (equivalently both) of these maps are stable equivalences. The suspension spectrum has all of its homotopy groups zero.

Given two spectra X and Y, the zero map  $X \to Y$  is the unique map of spectra that factors through \*



Eilenberg-Maclane spaces are infinite loopspaces. We can define teh Eilenberg-Maclane spectrumm HA by

$$(HA)_n = K(A, n)$$

with bonding maps adjoing to the canonical equivalences  $K(A, n) \simeq \Omega K(A, n+1)$ . We use the letter H because the infinite loopspace represents cohomology with coefficients in A.

todo

The homotopy grops of HA are given by

$$\pi_i H A = \begin{cases} A & i = 0 \\ 0 & i \neq 0 \end{cases}$$

The complex *K* theory spectrum *KU* 

$$KU_n = \begin{cases} \mathbb{Z} \times BU & n \text{ even} \\ U & n \text{ odd} \end{cases}$$

The structure maps adjoint to the equivalences

$$\Omega U \simeq \mathbb{Z} \times BU \quad \Omega(\mathbb{Z} \times BU) \simeq U$$

The homotopy groups of KU are given by

$$\pi_i K U = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Bott elemen generator