Lecture 5

Sequential Spectra

We work in the category $\mathscr{T}$ of based(compactly generated weak Hausdorff) spaces and basepoint preserving maps.

Definition 1. A (sequential) spectrum X is a sequence of spaces X_n for $n \geq 0$ and structure maps $\sigma_n : \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$.

A map of spectra $f : X \to Y$ is a sequence of maps $f_n : X_n \to Y_n$ such that each square of the following form commutes

S^1 commutes with $\wedge$

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "\Sigma f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

The category of spectra and the maps described above is denoted by $\mathscr{S}p$.

Note that the bonding maps $\sigma_i : \Sigma X_i \to \Sigma_{i+1}$ have adjoints $\tilde{\sigma}_i : \Omega X_i \to \Omega X_{i+1}$ and we could equivalently define a spectrum using the adjoint bonding maps.

Definition 2. The degree $k \in \mathbb{Z}$ stable homotopy group of a spectrum X is the abelian group

$$\pi_k(X) = \mathrm{colim}_n \, \pi_{k+n}(X_n)$$

Here $\pi_{k+n}(X_n) \to \pi_{k+n+1(X_{n+1})}$ maps the homotopy class of $\phi : S^{k+n} \to X_n$ to the class of $\sigma(S^1 \wedge \phi) : S^1 \wedge S^{k+n} \to X_{n+1}$.

Definition 3. A map of spectra $f : X \to Y$ is a stable equivalence (π_* isomorphism) if the induced map $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Consider $\mathscr{W} \subset \mathscr{S}p$ to be the wide subcategory of stable equivalences. The stable homotopy category ho$\mathscr{S}p$ is the localization of $\mathscr{S}p$ at $\mathscr{W}$. The functor

$$\mathscr{S}p \to \mathscr{S}p[\mathscr{W}^{-1}] = \mathrm{ho}\mathscr{S}p$$

sends stable equivalences to isomorphisms.

Initial functor?

Example 4.
1. The suspension spectrum $\Sigma^\infty X$ of a space X is the spectrum, with $(\Sigma^\infty X)_n = \Sigma^n X$, with the bonding maps $\mathrm{id} : S^1 \wedge \Sigma^i X \to \Sigma^{i+1} X$. The suspension spectrum for an unpointed space X is obtained by adding a disjoint basepoint: $\Sigma^\infty_+ X := \Sigma^\infty(X_+)$. The homotopy groups of $\Sigma^\infty X$ are the stable homotopy groups of the space X.
2. An important example of a suspension spectrum is the *sphere spectrum*

   $$\mathbb{S} := \Sigma^\infty S^0$$

   This is the sequence of spheres $S^0, S^1, S^2, \ldots$
   The homotopy groups of the sphere spectrum are called the *stable stems* . They are the stable homotopy groups of S^0, often written $\pi_i^s := \pi_i \mathbb{S}$.
3. The zero spectrum $\Sigma^\infty *$ is the suspension spectrum of a point, with the bonding maps

   $$\Sigma(*) \cong * \to *$$

Every spectrum X admits a unique maps of spectra $* \to X \to *$. X is weakly contractible if one (equivalently both) of these maps are stable equivalences.
The suspension spectrum has all of its homotopy groups zero.

Given two spectra X and Y, the zero map $X \to Y$ is the unique map of spectra that factors through $*$

$$\begin{tikzcd} X_n \arrow[rr] \arrow[dr] & & Y_n \\ & * \arrow[ur] & \end{tikzcd}$$

4. Eilenberg-Maclane spaces are infinite loopspaces. We can define teh Eilenberg-Maclane spectrumm HA by

$$(HA)_n = K(A,n)$$

with bonding maps adjoing to the canonical equivalences $K(A,n) \simeq \Omega K(A,n+1)$. We use the letter H because the infinite loopspace represents cohomology with coefficients in A. todo

The homotopy grops of HA are given by

$$\pi_i HA = \begin{cases} A & i = 0 \\ 0 & i \neq 0 \end{cases}$$

5. The complex K theory spectrum KU

$$KU_n = \begin{cases} \mathbb{Z} \times BU & n \text{ even} \\ U & n \text{ odd} \end{cases}$$

The structure maps adjoint to the equivalences

$$\Omega U \simeq \mathbb{Z} \times BU \quad \Omega(\mathbb{Z} \times BU) \simeq U$$

The homotopy groups of KU are given by Bott element generator

$$\pi_i KU = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Definition 5. An $\Omega-$ spectrum (or fibrant spectrum) is a spectrum X in which the adjunct bonding maps

$$X_n \xrightarrow{\simeq} \Omega X_{n+1}$$

are weak homotopy equivalences.

The zeroeth space X_0 of an Ω -spectrum is an infinite loop space. Examples 3,4,5 above are Ω-spectra.

Proposition 6. If X is a spectrum, there exists an $\Omega-$spectrum RX with a stable equivalence

$$X \to RX$$

which is natural.

R here is a right deformation of the category $\mathscr{S}p$. homotopy category?

For a spectrum X, with spaces X_n and bonding maps σ_n, RX is defined as

$$(RX)_n = \mathrm{hocolim}(X_n \xrightarrow{\sigma_n} \Omega X_{n+1} \xrightarrow{\Omega\sigma_{n+1}} \Omega^2 X_{n+2} \to \cdots)$$

The commuting maps

$$\begin{tikzcd} X \arrow[r, "\sigma_n"] \arrow[d, "\sigma_n"'] & \Omega X_{n+1} \arrow[r, "\Omega\sigma_{n+1}"] \arrow[d, "\Omega\sigma_{n+1}"] \arrow[dl, dashed, "\mathrm{id}"] & \Omega^2 X_{n+2} \arrow[r] \arrow[d] \arrow[dl, dashed, "\mathrm{id}"] & {} \\ \Omega X_{n+1} \arrow[r, "\Omega\sigma_{n+1}"'] & \Omega^2 X_{n+2} \arrow[r] & \Omega^3 X_{n+3} \arrow[r] & \cdots \end{tikzcd}$$

induce a map of homotopy colimits

$$\underset{m\to\infty}{\mathrm{hocolim}}\, \Omega^m X_{n+m} \xrightarrow{\simeq} \underset{m\to\infty}{\mathrm{hocolim}}\, \Omega^{1+m} X_{n+1+m} \xrightarrow{\simeq} \Omega(\underset{m\to\infty}{\mathrm{hocolim}}\, \Omega^m X_{n+1+m})$$

We define the bonding maps $(RX)_n \xrightarrow{\simeq} \Omega(RX)_{n+1}$ to be the composite map from above.

Definition 7. For any spectrum X, we define its infinite loop space to be

$$\begin{aligned} \Omega^\infty : \mathscr{S}p &\to \mathscr{T} \\ X &\mapsto (RX)_0 \end{aligned}$$

For example, $\Omega^\infty \mathbb{S} = \mathrm{colim}\, \Omega^n \mathbb{S}^n$.

Proposition 8. The functors Ω^∞ and Σ^∞ are adjoint

$$\begin{tikzcd} \mathscr{S}p \arrow[r, bend left, "\Omega^\infty"] \arrow[r, phantom, "\dashv"] & \mathscr{T} \arrow[l, bend left, "\Sigma^\infty"] \end{tikzcd}$$

remark abou QX

Definition 9. If $X \in \mathscr{S}p$ and $K \in \mathscr{T}$, then we can define the tensor or smash product $K \wedge X$ by lettign $(K \wedge X)_n = K \wedge X_n$ and the bonding maps $K \wedge X_n \wedge S^1 \to K \wedge X_{n+1}$.

We define

$$\begin{aligned} \Sigma : \mathscr{S}p &\to \mathscr{S}p \\ X &\mapsto S^1 \wedge X \end{aligned}$$

Definition 10. If X is a spectrum and K is a based space, we form the cotensor or function spectrum $F(K,X)$ by applying $\mathrm{Map}_*(K,-)$ to every spectrum level of X. So $F(K,X)$ is the spectrum whose nth level is the space of based maps $\mathrm{Map}_*(K,X_n)$. The bonding maps are

$$\mathrm{Map}_*(K,X_n) \xrightarrow{\mathrm{Map}_*(K,\Omega X_{n+1})} \mathrm{Map}_*(K,\Omega X_{n+1}) \xleftrightarrow{\cong} \Omega \mathrm{Map}_*(K,X_{n+1})$$

As a special case, we define

$$\begin{aligned} \Omega : \mathscr{S}p &\to \mathscr{S}p \\ X &\mapsto F(S^1,X) \end{aligned}$$

These two operations are adjoint functors on spectra.

Proposition 11. For any spectrum X , there are isomorphisms PROOF

$$\pi_{k+1}(\Sigma X) \cong \pi_k(X) \cong \pi_{k-1}(\Omega X)$$

Corollary 12. There are natural stable equivalences $X \to \Sigma\Omega X$ and $\Sigma\Omega X \to X$.

Thus Σ and Ω are inverse functors upto stable equivalence.

Definition 13. A cofibers sequence is a sequence of the form

$$X \xrightarrow{f} Y \to C_f$$

where Cf is the homotopy cofiber, which is defined as

$$(Cf)_n = \text{homotopy cofiber of } (X \xrightarrow{f_n} Y_n)$$

Similarly a fiber sequence is anything of the form

$$Fg \to Y \xrightarrow{g} Z$$

where Fg is the homotopy fiber.

Theorem 14. For each fiber sequence $X \to Y \to Z$, we get a long exact sequence

$$\cdots \to [W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{g_*} [W,Z] \to \dots$$

Theorem 15. For each cofiber sequence $X \to Y \to Z$, we get a long exact sequence check proof

$$\cdots \leftarrow [X,W] \leftarrow [Y,W] \leftarrow [Z,W] \leftarrow \cdots$$

and also a long exact sequence

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} [W,Z]$$

Proof. Suppose we have a map $g : W \to Y$ so that $ig \simeq 0$. Let h be a null homotopy of this composite. This is precisely a map $b : CW \to Cf$. We thus have the following diagram, where the horizontal parts are cofibre sequences,

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & Cf \arrow[r] & \Sigma X \arrow[r] & \Sigma Y \arrow[r] & \cdots \\
W \arrow[u, "-\Sigma^{-1}j"] \arrow[r, "1"'] & W \arrow[u, "g"] \arrow[r, "i"'] & CW \arrow[u, "h"] \arrow[r] & \Sigma W \arrow[u, dashed, "k"] \arrow[r, "-1"'] & \Sigma W \arrow[u, "\Sigma g"] \arrow[r] & \cdots
\end{tikzcd}$$

Because these are cofibre sequences, we get the dashed arrow k making the diagram commute for free, and the rest is automatic. But Σ has an inverse in SHC. Applying Σ^{-1} gives us a map $\Sigma^{-1}k : W \to X$, making the necessary diagram commute (functoriality of Σ^{-1}). This proves exactness.

$\square$

Lemma 16. For each map of spectra $f : X \to Y$ there is a natrual stable equivalence

$$\varepsilon : \Sigma Ff \to Cf$$

or equivalently its adjoint.

Proof. We define ε by

$$\begin{aligned} \varepsilon : \Sigma(X \times_Y PY) &\to Y \cup_X CX \\ (t, x, \gamma) &= \begin{cases} \gamma(2t) & t \le 1/2 \\ (x, 2-2t) & t \ge 1/2 \end{cases} \end{aligned}$$

We check the commutativity upto homotopy of the following squares by looking at the suspension of the fiber sequence of f and cofiber sequence of f.

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma(X \times_Y PY) \arrow[r] \arrow[d, "\varepsilon"] & \Sigma X \arrow[r, "\Sigma f"] \arrow[d, "\text{flip}"'] & \Sigma Y \arrow[d, "\text{flip}"] \\
X \arrow[r, "f"'] & Y \arrow[r] & Y \cup_X CX \arrow[r] & \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y
\end{tikzcd}$$

This gives that ε is a π_* isomorphism. ☐

Proposition 17. $X \to Y \to Z$ is a cofiber sequence iff it is a fiber sequence.

Proof. We look at the squares in the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] \arrow[d, equal] & Cf \arrow[r] \arrow[d, "h\cup g"] & \Sigma X \arrow[r] \arrow[d, "h\cup Cf"] \arrow[dd, bend right, "\Sigma(f\times h)"'] & \Sigma Y \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r, "g"'] & Z \arrow[r] & Cg \arrow[r] & \Sigma Y \\
 & & & \Sigma(Y \times_Z F(I,Z)) \arrow[u, "\simeq"', "\varepsilon"] &
\end{tikzcd}$$

We have

$$\begin{aligned} X \to Y \to Z \text{ a cofiber sequence} &\iff h \cup g \text{ is a stable equivalence} \\ &\iff h \cup Cf \text{ is a stable equivalence} \\ &\iff \Sigma(f \times h) \text{ is a stable equivalence} \\ &\iff f \times h \text{ is a stable equivalence} \\ &\iff X \to Y \to Z \text{ is a fiber sequence} \end{aligned}$$

☐

Lemma 18. A commuting square of spectra is a homotopy pushout iff it is a homotopy pullback.

Proposition 19. Finite coproducts and finite products coincide in $\mathscr{S}p$ just this?

$$X \vee Y \cong X \times Y$$