Revision Notes

Seb

Complex Analysis

1. If u and v have continuous parital derivatives which satisfy the Cauchy-Riemann equations then the function $f(z) = u(z) + iv(z)$ is analytic with derivative conitnuous $f'(z)$, and conversely. hello there

2. (Lucas' theorem)If all zeroes of a polyonomial $p(z)$ lie in a half plane then all the zeroes of the derivative $p'(z)$ lie in the same half plane. Also, the smallest convex polygon that contains the zeroes of $p(z)$ also contains the zeroes of $p'(z)$.

3. (Abel's Limit theorem) If $\sum a_n$ is convergent then $f(z) = \sum a_n z^n$ tends to $f(1)$ as $z \to 1$ in such a way that $|1-z|/(1-|z|)$ stays bounded. (approach takes place in the Stolz angle)([**Ahl**] Ch2 Thm 3 page 41)

4. If $f(z)$ is analytic in an open disc containing a closed curve γ or homotopic to a constant curve, then for a point a not on the curve, we have

$$\eta(\gamma;a)\cdot f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}$$

where $\eta(\gamma;a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$. When $\eta(\gamma;a) = 1$, we have the standard formula

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

and also

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

5. Moreva's theorem, Liouville theorem

6. Let z_j be the zeros of a function $f(z)$ which is analytic in a disk Δ and does not vanish identically, each zero being counted as many times as its order indicates. For every closed curve γ in Δ which does not pass through a zero

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

where the sum has only a finite number of terms $\neq 0$.

7. Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. If $\varepsilon \geq 0$ is sufficiently small, there exists a corresponding $\delta \geq 0$ such that for all a with $|a - w_0| \leq 0$ the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| \leq \varepsilon$.

8. A nonconstant analytic function is an open map

9. IF $f(z)$ is analytic with $f'(z_0)$ nonzero at z_0 then f maps a neighborhood of z_0 conformally and topologically onto a region.

10. If $f(z)$ is defined and continuous on a closed bounded set E, analytic on the intereior of E, then the maximum of $|f(z)|$ on E is attained on the boundary of E.

11. If $f(z)$ is analytic on $|z| \leq 1$ and is $|f(z)| \leq 1$, $f(0) = 1$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for any $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ with $|| = 1$.

12. If $f(z)$ is analytic in Ω, then $\int_\gamma f(z)dz = 0$ for all γ which are homotopic to the constant curve.(Cauchy's theorem)

13. Let z_0 be an isolated singularity, then $\text{Res}(f, z_0) = a_{-1}$ where a_n are coefficients of the Laurent series expansion around z_0.

14. Let $f(z)$ be analytic except for isolated singularites at z_j in a region Ω. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j \eta(\gamma; z_j)Res(z_j, f)$$

for any cycle γ which is homootopic to a constant loop in Ω and does notpass through any points z_j.(Residue theorem)

15. If $f(z)$ is meromorphic in Ω with the zeros z_j and the poles p_k, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k)$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles.

16. Let C be circle and $f(z)$ and $g(z)$ be analytic functions such that $|f(z) - g(z)| \leq |g(z)|$ on C. Then $f(z)$ and $g(z)$ have the same number of zeroes bounded by C.

Differential Geometry

1. The linear map

$$\begin{aligned}\phi : & T_p(\mathbb{R}^n) \to \mathfrak{D}_p\mathbb{R}^n (\text{Set of derivations}) \\ & v \mapsto D_v = \sum v_i \frac{\partial}{\partial x_i}\bigg|_p\end{aligned}$$

   is an isomorphism of vector spaces.

2. Wedge Product $f \wedge g = \frac{1}{k!l!} Alt(f \otimes g)$,

$$\begin{aligned}&(f \wedge g)(v_1, \ldots, v_{k+\ell}) \\ &\quad = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right).\end{aligned}$$

   $f \wedge g = (-1)^{kl} g \wedge f$

3. Let $U \subset \mathbb{R}^n$ be open and $S \subset \mathbb{R}^n$ be an arbitrary subset. If $f : U \to S$ is a diffeomorphism, then S is open in $\mathbb{R}^n$.

4. $\int_{\partial M} \omega = \int_M d\omega$ (Stokes theorem).
   If we take ω to be the 1-form $Pdx + Qdy$ then we have Greens theorem for surfaces $\int_{\partial D} Pdx + Qdy = \int_D (Q_x - P_y) dxdy$

5. $U \in \mathbb{R}^n$ be an open set and $S \subset \mathbb{R}^n$ be an arbitrary subset. $f : U \to S$ diffeomorphism then S is open in $\mathbb{R}^n$ ([**Tu**] Theorem 22.3)
   $\dim U = \dim S$, $]0,1[\subset \mathbb{R}^1$ and $]0,1[\times \{0\} \subset \mathbb{R}^2$ diffeomorphic but the latter is not open in $\mathbb{R}^2$

6. The multiplication map $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ $A, B \mapsto AB$ is C^∞. Same with $SL_n(\mathbb{R})$.

Topology

Quotient Topology

1. A surjective continuous map that is open or closed is a quotent map.
2. Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}y$, for $y \in Y$.Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map iff g is a quotient map

Connectedness

1. The union of collection of connected subspaces that have a point in common is connected.
2. Arbitrary product of connected spaces is connected in product topology but not in box topology
3. Topologist's sine curve is connected but not path connected, not locally connected.
   $\mathbb{Q}$ is neither connected nor locally connected.
4. A space X is locally (path) connected if and only if for every open set U of X, each (path) component of U is open in X.
5. Examples[1]

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+--------------------------+
|Connected                 |
|             1      +-----+----------------------------+
|                    |  3  |           Locally connected|
|   +----------------+-----+     6                      |
|   |Path-connected  |  4  |                            |
|   |                +-----+------------------------+   |
|   |    2           |  5  |  Locally path-connected|   |
+---+----------------+-----+                        |   |
                  8  |           7                  |   |
                     +------------------------------+---|
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   1 The topologist's sine curve

   2 The comb space

   3 The ordered square

[1] https://math.stackexchange.com/questions/843173/path-connected-and-locally-connected-space-that-is-not-843261#843261

4 See here

5 The real line

6 The disjoint union of two spaces of the 3rd type

7 $[0,1] \cup [2,3]$

8 The rationals

Compact spaces

1. Every compact subspace of a Hausdorff space is closed.

2. Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

3. X is compact if an only if for every collection $\mathscr{C}$ of closed sets in X having the finite intersection property, the intersection $\cap_{C \in \mathscr{C}} C$ of all elements of $\mathscr{C}$.

4. X is limit point compact if every infinite subset of X has a limit point. Compactness implies limit point compactness.

5. For a metrizable space, the following are equivalent

   (a) Compact

   (b) Limit point compact

   (c) Sequentially compact

6. X is locally compact T_2 if and only if there exists a space Y satisfying:

   (a) X is a subspace of Y

   (b) The set $Y - X$ consists of a single point.

   (c) Y is a compact T_2 space.

   If Y and Y' are two spaces satsifying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

7. A metric spaces (X, d) is said to be totally bounded if for every $\varepsilon > 0$, there is a finite covering of X by ε balls.
   A metric space is compact if and only if it is compact and totally bounded.

8. Let X be a space and (Y, d) be a metric space. If the subset $\mathscr{F}$ of $\mathscr{C}(X, Y)$ is totally bouded under the uniform metric corresponding to d, then $\mathscr{F}$ is equicontinuous under d.

Baire spaces

1. X is a Baire space if : Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.

2. X is a Baire space iff given any countable $\{U_n\}$ of open sets in X, each of which is dense in X, their intersection $\cap U_n$ is also dense in X.

3. (Baire category theorem) If X is a compact T_2 space or a complete metric space, then X is a Baire space.

Misc.

1. For a finite group G that acts on a set X, let X/G be the set of orbits of X. Then Burnside's lemma states that
$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$
where X^g is the subset of X that is fixed by g.
Orbit stablizer theorem says that $|G| = |G_x| \times |G \cdot x|$.

References

[Ahl] Ahlfors, Complex analysis

[Tu] Tu, An introduction to smooth manifolds

[Munk] Munkres, Topology