# Revision Notes

Seb

# Complex Analysis

1. If $u$ and $v$ have continuous parital derivatives which satisfy the Cauchy-Riemann equations then the function $f(z) = u(z) + iv(z)$ is analytic with derivative conitnuous $f'(z)$, and conversely. hello there

2. (Lucas' theorem)If all zeroes of a polyonomial $p(z)$ lie in a half plane then all the zeroes of the derivative $p'(z)$ lie in the same half plane. Also, the smallest convex polygon that contains the zeroes of $p(z)$ also contains the zeroes of $p'(z)$.

3. (Abel's Limit theorem) If $\sum a_n$ is convergent then $f(z) = \sum a_n z^n$ tends to $f(1)$ as $z \to 1$ in such a way that $|1-z|/(1-|z|)$ stays bounded. (approach takes place in the Stolz angle)([**Ahl**] Ch2 Thm 3 page 41 )

4. If $f(z)$ is analytic in an open disc containing a closed curve $\gamma$ or homotopic to a constant curve, then for a point $a$ not on the curve, we have

   $$\eta(\gamma; a) \cdot f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-a}$$

   where $\eta(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$. When $\eta(\gamma; a) = 1$, we have the standard formula

   $$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

   and also

   $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

5. Moreva's theorem, Liouville theorem

6. Let $z_j$ be the zeros of a function $f(z)$ which is analytic in a disk $\Delta$ and does not vanish identically, each zero being counted as many times as its order indicates. For every closed curve $\gamma$ in $\Delta$ which does not pass through a zero

   $$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

   where the sum has only a finite number of terms $\neq 0$.

7. Suppose that $f(z)$ is analytic at $z_0$, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order $n$ at $z_0$. If $\varepsilon \geq 0$ is sufficiently small, there exists a corresponding $\delta \geq 0$ such that for all $a$ with $|a - w_0| \leq 0$ the equation $f(z) = a$ has exactly $n$ roots in the disk $|z - z_0| \leq \varepsilon$.

8. A nonconstant analytic function is an open map
9. IF $f(z)$ is analytic with $f'(z_0)$ nonzero at $z_0$ then $f$ maps a neighborhood of $z_0$ conformally and topologically onto a region.
10. If $f(z)$ is defined and continuous on a closed bounded set $E$, analytic on the intereior of $E$, then the maximum of $|f(z)|$ on $E$ is attained on the boundary of $E$.
11. If $f(z)$ is analytic on $|z| \leq 1$ and is $|f(z)| \leq 1$, $f(0) = 1$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for any $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ with $|| = 1$.
12. If $f(z)$ is analytic in $\Omega$, then $\int_\gamma f(z)dz = 0$ for all $\gamma$ which are homotopic to the constant curve.(Cauchy's theorem)
13. Let $z_0$ be an isolated singularity, then $\operatorname{Res}(f, z_0) = a_{-1}$ where $a_n$ are coefficients of the Laurent series expansion around $z_0$.
14. Let $f(z)$ be analytic except for isolated singularites at $z_j$ in a region $\Omega$. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j \eta(\gamma; z_j) Res(z_j, f)$$

for any cycle $\gamma$ which is homootopic to a constant loop in $\Omega$ and does notpass through any points $z_j$.(Residue theorem)

15. If $f(z)$ is meromorphic in $\Omega$ with the zeros $z_j$ and the poles $p_k$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k)$$

for every cycle $\gamma$ which is homologous to zero in $\Omega$ and does not pass through any of the zeros or poles.

16. Let $C$ be circle and $f(z)$ and $g(z)$ be analytic functions such that $|f(z) - g(z)| \leq |g(z)|$ on $C$. Then $f(z)$ and $g(z)$ have the same number of zeroes bounded by $C$.
17. $log(z) = ln|z| + i\arg(z)$

# Differential Geometry

1. $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$
2. The linear map

$$\begin{aligned}\phi :& T_p(\mathbb{R}^n) \to \mathfrak{D}_p\mathbb{R}^n \text{(Set of derivations)}\\ &v \mapsto D_v = \sum v_i \frac{\partial}{\partial x_i}\bigg|_p\end{aligned}$$

   is an isomorphism of vector spaces.
3. Wedge Product $f \wedge g = \frac{1}{k!l!} Alt(f \otimes g)$,

$$\begin{aligned}&(f \wedge g)(v_1, \ldots, v_{k+\ell})\\ &\quad = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right).\end{aligned}$$

   $f \wedge g = (-1)^{kl} g \wedge f$
4. Let $U \subset \mathbb{R}^n$ be open and $S \subset \mathbb{R}^n$ be an arbitrary subset. If $f : U \to S$ is a diffeomorphism, then $S$ is open in $\mathbb{R}^n$.
5. $\int_{\partial M} \omega = \int_M d\omega$ (Stokes theorem).
   If we take $\omega$ to be the 1-form $Pdx + Qdy$ then we have Greens theorem for surfaces $\int_{\partial D} Pdx + Qdy = \int_D (Q_x - P_y) dxdy$
6. $U \in \mathbb{R}^n$ be an open set and $S \subset \mathbb{R}^n$ be an arbitrary subset. $f : U \to S$ diffeomorphism then $S$ is open in $\mathbb{R}^n$ ([**Tu**] Theorem 22.3)
   $\dim U = \dim S$, $]0,1[ \subset \mathbb{R}^1$ and $]0,1[ \times \{0\} \subset \mathbb{R}^2$ diffeomorphic but the latter is not open in $\mathbb{R}^2$
7. The multiplication map $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ $A, B \mapsto AB$ is $C^\infty$. Same with $SL_n(\mathbb{R})$.

# Topology

### Quotient Topology

1. A surjective continuous map that is open or closed is a quotent map.

2. Let $p : X \to Y$ be a quotient map. Let $Z$ be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}y$, for $y \in Y$.Then $g$ induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map $f$ is continuous if and only if $g$ is continuous; $f$ is a quotient map iff $g$ is a quotient map

### Connectedness

1. The union of collection of connected subspaces that have a point in common is connected.

2. Arbitrary product of connected spaces is connected in product topology but not in box topology

3. Topologist's sine curve is connected but not path connected, not locally connected.
   $\mathbb{Q}$ is neither connected nor locally connected.

4. A space $X$ is locally (path) connected if and only if for every open set $U$ of $X$, each (path) component of $U$ is open in $X$.

5. Examples[1]

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+--------------------------+
|Connected                 |
|             1      +-----+----------------------------+
|                    |  3  |           Locally connected|
|   +----------------+-----+     6                      |
|   |Path-connected  |  4  |                            |
|   |                +-----+------------------------+   |
|   |    2           |  5  |  Locally path-connected|   |
+---+----------------+-----+                        |   |
                  8  |           7                  |   |
                     +------------------------------+---|
```

   1 The topologist's sine curve

   2 The comb space

   3 The ordered square

[1] `https://math.stackexchange.com/questions/843173/path-connected-and-locally-connected-space-that-is-not-843261#843261`

4 See here

5 The real line

6 The disjoint union of two spaces of the 3rd type

7 $[0,1] \cup [2,3]$

8 The rationals

**Compact spaces**

1. Every compact subspace of a Hausdorff space is closed.

2. Let $f : X \to Y$ be a bijective continuous function. If $X$ is compact and $Y$ is Hausdorff, then $f$ is a homeomorphism.

3. $X$ is compact if an only if for every collection $\mathscr{C}$ of closed sets in $X$ having the finite intersection property, the intersection $\cap_{C \in \mathscr{C}} C$ of all elements of $\mathscr{C}$ is non-empty.

4. $X$ is limit point compact if every infinite subset of $X$ has a limit point. Compactness implies limit point compactness.

5. For a metrizable space, the following are equivalent

   (a) Compact

   (b) Limit point compact

   (c) Sequentially compact

6. $X$ is locally compact $T_2$ if and only if there exists a space $Y$ satisfying:

   (a) $X$ is a subspace of $Y$

   (b) The set $Y - X$ consists of a single point.

   (c) $Y$ is a compact $T_2$ space.

   If $Y$ and $Y'$ are two spaces satsifying these conditions, then there is a homeomorphism of $Y$ with $Y'$ that equals the identity map on $X$.

7. A metric spaces $(X,d)$ is said to be totally bounded if for every $\varepsilon > 0$, there is a finite covering of $X$ by $\varepsilon$ balls.
   A metric space is compact if and only if it is complete and totally bounded.

8. Let $X$ be a space and $(Y,d)$ be a metric space. If the subset $\mathscr{F}$ of $\mathscr{C}(X,Y)$ is totally bouded under the uniform metric corresponding to $d$, then $\mathscr{F}$ is equicontinuous under $d$.

9. $(X,d)$ is sequentially compact if for every sequence in $X$ there exists a convergent subsequence.

10. sequentially compact metric space is complete.

11. $Y \subset X$ is closed iff it is sequentially compact.

12. Seuqntially compact implies the space is toally bounded.

13. Metric space is complate and totally bounded then it is a compact metric space and converse also holds true.

14. If a sequence $\mathscr{F} = \{f_n\}$ in $C(X)$ is bounded and equicontinuous, then it has a uniformly convergent subsequence.
    A subset $\mathscr{F}$ of $C(X)$ is compact (or relatively compact?) if and only if it is closed, bounded and equicontinuous.

**Baire spaces**

1. $X$ is a Baire space if : Given any countable collection $\{A_n\}$ of closed sets of $X$ each of which has empty interior in $X$, their union $\cup A_n$ also has empty interior in $X$.

2. $X$ is a Baire space iff given any countable $\{U_n\}$ of open sets in $X$, each of which is dense in $X$, their intersection $\cap U_n$ is also dense in $X$.

3. (Baire category theorem) If $X$ is a compact $T_2$ space or a complete metric space, then $X$ is a Baire space.

4. $l^2(\mathbb{R})$ is a metric space where closed does not imply compact.

5. Subgroup of $(\mathbb{R}, +)$ are either dense or cyclic.

# Algebra

1. The irreducible polynomial $x^4 + 1 \in \mathbb{Z}[x]$ is reducible modulo $p$ for every prime $p$.
2. If $P$ is a group of order $p^n$ , then $Z(P) \neq 1$.
3. $G/Z(G) = Inn(G)$ yclic implies $G$ is abelian.
4. If $G$ is a simple group of order 60 then $G \cong A_5$.
5. Every irreducible polynomial over a finite field is separable.
6. $\mathbb{F}_{p^n} = \text{split}(x^{p^n} - x, \mathbb{F}_p)$.
7. Every matrix $M$, can be foactored as $UDV^*$ where $U$ has orthonormal columsn, $D$ is a diagonal matrix of rank $M$, $V$ rows orthonormal.
8. Every group is a quotient of a free group.
9. $GL_n(\mathbb{Z})$ has finitely many finite subgroups, each of which is isomorphic to $GL_n(\mathbb{F}_q)$
10. If $G$ is a finite group then it is isomorphic to a subgroup of $GL_n(\mathbb{Z})$.
11. Finite subgroups of $F^*$ are cyclic.
12. For every $f \in V^*$ there exits a unique $w \in V$ such that $f(v) = \langle v, w \rangle$.
13. A finitely generated free module over a PID is free. If $M$ is a fg module over $R$, then $M = R^n \oplus Tor(M)$.
14. $T$ be a normal operator on $\mathbb{C}$ vector space (self adjoint operator on finite dimensional $\mathbb{R}$ vsp). Let $c_i$ be eigenvalues of $T$ adn $W_i$ be their corresponding eigenspaces. Let $P_i$ be orthogonal projection onto $W_i$. Then
    (a) $W_i \perp W_j$ if $i \neq j$.
    (b) $V = \oplus W_i$
    (c) $T = \sum_{i=1}^{k} c_i P_i$

# Analysis

1. If $f$ is continuous and open, inverse image of a dense set is dense.
2. $l^p$ is complete.$l^\infty$ is a Banach space(Complete normed linear space)
3. 

# Misc.

1. For a finite group $G$ that acts on a set $X$, let $X/G$ be the set of orbits of $X$. Then Burnside's lemma states that
$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$
where $X^g$ is the subset of $X$ that is fixed by $g$.
Orbit stablizer theorem says that $|G| = |G_x| \times |G \cdot x|$.

# Functional Analysis

1. (a) A complete normed linear space is called a Banach space.

   (b) A Hilbert space is an inner product space $H$ such that the norm $\|x\| = |\langle x, x\rangle|^{1/2}$. makes $H$ complete.

2. A closed subspace of a banach space is a banach space.

3. (a) $l^p = \{(x_n)_n | \sum |x_i|^p < \infty\}$ is a complete space.

   (b) $\|x\|_\infty = \sup |x_i|$ makes the space $l^\infty$ a Banach space.

   (c) $l^2$ is a Hilbert space.

   (d) $C_0 = \{x \in l^\infty | \lim x_n = 0\}$ and $C = \{x \in l^\infty | \lim x_n \text{exists}\}$ are Banach spaces with $l^\infty$ norm.

   (e) $(c_{00}, \|\cdot\|_p)$ is a nls. where $c_{00} = \{x_n \in l_\infty | x_n = 0 \text{ for infinitely many } n\}$. The closure of $c_{00}$ under $l_p$ and $l_\infty$ norm give $l_p$ and $c_0$ spaces respectively.

4. $X$ locally compact Hausdorff space and $\mu$ regular $\sigma$ finite Borel measure . Then $C_c(X)$ is dense in $L^p(X, \mu)$, for $q \leq p < \infty$.

5. $X$ be a measure space and $\mu$ a positive measure, then $L^p(X, \mu)$ is a Banach space.

6. $X$ locally compact $T_2$ space. Then for $C_0(X)$ with sup norm, we have

   (a) $C_c(X)$ is dense in $C_0(X)$.

   (b) $C_0(X)$ is a Banach space.

7. The norm function is uniformly continuous.

8. Quotient space.

9. $X$ normed linear space and $Y$ closed subspace of $X$. If $Z \subset X$ is finite dimensional then $Y + Z$ is closed.

10. Two equivalent norms induce the same topology on a normed linear space.

11. Any finite dimensional subspace of a normed linear space is closed.

12. For $X$ a nls, the following are equivalent

    (a) Any closed and bounded subset of $X$ is compact

    (b) $S = \{x : \|x\| \leq 1\}$ is compact

(c) $X$ is finite dimensional

13. (Riesz) $Y$ closed proper subspace of $X$. If $\alpha \in (0,1)$ then there exists $x_\alpha \in X$ such that $\|x_\alpha\| = 1$ and $\alpha \leq d(x_\alpha, Y)$.

14. If $X$ is a locally compact normed linear space then $X$ is finite dimensional.

15. If $X$ is a Banach space, any Hamel basis of $X$ is uncountable unless $X$ is finite dimensional

16. A proper subspace of a normed linear space has empty interior

17. The set of all polynomials with sup norm over $[0,1]$ is not a complete space

18. Polarization identity: $4\langle x,y\rangle = \sum_{k=0}^{3} i^k \left\|x+i^k y\right\|^2$.

19. Let $C$ be a closed convex subset of a Hilvert space $H$. tThen there exists a unique element in $C$ of smalles norm.

20. $M$ closed subspace of $H$. Then there exists a unique $y_0 \in M$ such that $\|x-y_0\| = d(x,M)$

21. Let $M \subset H$ closed subspace, then $H = M \oplus M^{\perp}$.

Suppose $X,Y$ are nls and $B(X,Y)$ is equipped with operator norm ($\|A\|_{op} = \inf\{c \geq 0 : \|Av\| \leq c\|v\|$ for all $v \in V\}$).

22. If $Y$ is complete, then $B(X,Y)$ is a banach space. $X^* = B(X,\mathbb{C})$ is a banach space.

23. Let $y \in H$

$$\begin{aligned} \psi :& H \to H^* \\ & y \mapsto f_y(x) = \langle x,y\rangle \end{aligned}$$

then $\psi$ is an antilinear isometric isomorphism onto $H^*$.

24. (Riesz Representation theorem) $f \in H^*$ then there exists unique $y \in H$ such that $f(x) = \langle x,y\rangle$ for all $x \in H$.

25. A complete orthonormal set (maximal) in a hilbert space is called an orthonromal basis. In any hilbert space there exists a complete orthonormal set.

26. H hilbert space, tfae

    (a) $\{e_i\}$ is complete

    (b) $x \perp e_i \forall i \implies x = 0$.

    (c) $x = \sum_{i\in I}\langle x,e_i\rangle \forall x \in H$.

    (d) $\|x\|^2 = \sum i \in I|\langle x,e_i\rangle|^2$

27. If $\{e_i\}$ is an orthonormal set in $H$, then $\sum_{i\in I}|\langle x,e_i\rangle|^2 \leq \|x\|^2$.

28. (a) an orthonormal basis is not a hamel basis.

    (b) $H$ is separable iff any onb is countable.

    (c) If $H$ is not separable then any onb is uncountable.

(d) any separable hilbert space is isometrically isomorphic to $l^2(\mathbb{N})$.

(e) The set of all trignometric polyomials $=\{e^{inx} : n \in \mathbb{Z}\}$ is dense in $C(S^1)$.

29. Suppose $N_1$ is a nls and $N_2$ is a banach space. Suppose $W$ is a dense subspace of $N_1$ and $T : W \to N_2$ is such that $\|T(w)\| \leq k\|w\|$ for all $w \in W$. Then there exists a unique extension $\tilde{T}$ of $T$. $\tilde{T} : N_1 \to N_2$ such that $\tilde{T} \in B(N_1, N_2)$. Moreover, $\|\tilde{T}\| = \|T\|$.

30. (Hahn-Banach Theorem) Let $Y$ be a subspace of nls $X$ and $f \in Y^*$, then there exists an $F \in X^*$ such that $F|_Y = f$ and $\|F\| = \|f\|$.

31. Let $X$ be a nls and $x_0 \neq 0 \in X$. Then there exists $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.
If $x \neq y$, there exists $f \in X^*$ such that $f(x) \neq f(y)$.

32. Suppose $X, Y$ are nls such that $B(X,Y)$ is banach, then $Y$ is banach.

33. If $X^*$ is separable, then $X$ is separable.

34. $l^q$ is separable but $(l^1)^* = l^\infty$ is not separable.

35. There is an isometric isomorphism $(L^p)^* \simeq (L^{p'})$.

36. There is an isometry from $X \simeq X^{**}$. If $j(X) = X^{**}$ then the nls $X$ is called reflexive. Hilbert spaces are reflexive. $L^p$ spaces are reflexive for $1 < p < \infty$, for $\sigma$ finite measure space. $l^1$ is not reflexive.

37. Suppose $X, Y$ are Banach spaces and $T : X \to Y$ is bounded and surjective then $T$ is open.

38. If $T \in B(X,Y)$ is bijective, then $T^{-1}$ is continuous, i.e., $T$ is a homeomorphism.

# References

[Ahl] Ahlfors, Complex analysis

[Tu] Tu, An introduction to smooth manifolds

[Munk] Munkres, Topology