# Stable Homotopy theory and Spectral sequences

Naren

## Contents

# ※ Underlying Theorems

We begin with noting down some theorems from algebraic topology and later about Brown Representability theorem. For the following theorems we refer to [H] for proofs.

**Theorem 1.1.** *Let $X$ be a $CW$ complex decomposed as the union of subcomplexes $A$ and $B$ with nonempty connected intersection $C = A \cap B$. If $(A,C)$ is m-connected and $(B,C)$ is $n$-connected, $m,n \geq 0$, then the map $\pi_i(A,C) \to \pi_i(X,B)$ induced by inclusion is an isomorphism for $i < m+n$ and a surjection for $i = m+n$.*

This yields the Freudenthal Suspension theorem

**Theorem 1.2.** *The suspension map $\pi_i(S^n) \to \pi_{i+1}\left(S^{n+1}\right)$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$. More generally this holds for the suspension $\pi_i(X) \to \pi_{i+1}(SX)$ whenever $X$ is an $(n-1)$-connected CW complex.*

Let $X$ and $Y$ be CW complexes with basepoints. The suspension $\Sigma X$ , or equivalently reduced suspension, be either $S^1 \wedge X$ or $X \wedge S^1$. Suspension induces a function

$$S : [X,Y] \to [\Sigma X, \Sigma Y]$$

**Theorem 1.3.** *Suppose that $Y$ is $(n-1)$-connected. Then $S$ is onto if $\dim X \leq 2n-1$ and is a $1-1$ correspondence if $\dim X < 2n-1$.*

Under these circumstances we call an element of $[X,Y]$ a stable homotopy class of maps.

We define the notion of a cohomology operation. It is a natural transformation

$$\phi : H^n(X,Y;\pi) \to H^m(X,Y;G)$$

where $n$ runs over $\mathbb{Z}$. The map is subject to the axiom: if $f : X,Y \to X',Y'$ and $h \in H^n(X',Y';\pi)$ then $\phi(f^*h) = f^*(\phi h)$ (See [H] ).

A stable cohomology operation is said to be a collection of cohomology operations, say

$$\phi_n : H^n(X,Y;\pi) \to H^{n+d}(X,Y;G)$$

Here $n$ runs over $\mathbb{Z}$. Each $\phi_n$ is required to be natural, as above and the following diagram be commutative for each $n$.

$$\begin{tikzcd}
H^n(Y,Z;\pi) \arrow[r, "\delta"] \arrow[d, "\phi_n"'] & H^{n+1}(Y,Z;\pi) \arrow[d, "\phi_{n+1}"] \\
H^{n+d}(Y,Z;G) \arrow[r, "\delta"] & H^{n+d+1}(Y,Z;G)
\end{tikzcd}$$

## 1.1 Brown Representability Theorem

Let $\mathcal{C}$ be a locally small category, i.e., a category such that for any object $C$ and $C'$ in $\mathcal{C}$, the class of morphisms $\mathcal{C}(C,C')$ is a set. Let $C_0$ be a fixed object of $\mathcal{C}$. We define the contravariant functor :

$$\begin{aligned} \mathcal{C}(-,C_0) : \mathcal{C} &\longrightarrow \text{ Set} \\ C &\longmapsto \mathcal{C}(C,C_0) \\ C \xrightarrow{f} C' &\longmapsto f^* : \mathcal{C}\left(C',C_0\right) \to \mathcal{C}(C,C_0) \end{aligned}$$

where $f^*(\varphi) = \varphi \circ f$, for any $\varphi$ in $\mathcal{C}(C',C_0)$

**Definition 1.4** (Representable Contravariant Functor)**.** Let $\mathcal{C}$ be a locally small category. A contravariant functor $F : \mathcal{C} \to \text{Set}$ is said to be representable if there is an object $C_0$ in $\mathcal{C}$ and a natural isomorphism :

$$e : \mathcal{C}(-,C_0) \Rightarrow F$$

We say that $C_0$ represents $F$, and $C_0$ is a classifying object for $F$.

**Lemma 1.5** (Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category. Let $F : \mathcal{C} \to \text{Set}$ be a contravariant functor. For any object $C_0$ in $\mathcal{C}$, there is a one-to-one correspondance between natural transformation e: $\mathcal{C}(-,C_0) \Rightarrow F$ and elements $u$ in $F(C_0)$, which is given, for any object $C$ in $\mathcal{C}$, by:

$$\begin{aligned} e_C : \mathcal{C}(C,C_0) &\longrightarrow F(C) \\ \varphi &\longmapsto F(\varphi)(u). \end{aligned}$$

We now introduce Brown functors and discuss about their representabiliy.

**Definition 1.6** (Brown Functors)**.** Let $\mathcal{T}$ be a full subcategory of Top $_*$. A Brown functor $h : \mathcal{T} \to \text{Set}$ is a contravariant homotopy functor, which respects the following axioms.

Additivity/Wedge axiom For any collection $\left\{X_j \mid j \in \mathcal{J}\right\}$ of based spaces in $\mathcal{T}$, the inclusion maps $i_j : X_j \hookrightarrow \bigvee_{j\in\}} X_j$ induce an isomorphism on Set:

$$\left(h\left(i_j\right)\right)_{j\in\mathcal{Z}} : h\left(\bigvee_{j\in\mathcal{Z}} X_j\right) \xrightarrow{\cong} \prod_{j\in\mathcal{Z}} h\left(X_j\right).$$

Mayer-Vietoris For any excisive triad $(X;A,B)$ in $\mathcal{T}$, if $a$ is in $h(A)$, and $b$ is in $h(B)$, such that $a|_{A\cap B} = b|_{A\cap B}$, then there exists $x$ in $h(X)$, such that $x|_A = a$ and $x|_B = b$.

Any generalised cohomology theory on $CW_*$ defines a Brown functor in each dimension.

**Proposition 1.7.** Let $h$ be a Brown functor. If $X$ is a co-H-group then $h(X)$ is a group.

**Theorem 1.8** (Brown Representability Theorem)**.** $h : CW_* \to Set_*$ be a brown fucntor. Then $h$ is representable.

So when the functor $h$ on $CW_*$ is representable, then thre exists a vased CW complex $E$ such that there exists a natural isomorphism,

$$\begin{aligned} e : [-E]_* &\Longrightarrow h \\ e_X([f]_*) &= h(f)(u) \end{aligned}$$

where $f : X \to E$ and $u \in h(E)$ is the universal element of $h$.

The representability theorem is also valid in the category of CW spectra and morphisms of degree 0.

## ※ Spectra

A *spectrum* $E$ is a sequence of spaces $E_n$ with basepoint, provided with structure maps, $\varepsilon_n : \Sigma E_n \to E_{n+1}$ or equivalently $\varepsilon'_n : E_n \to \Omega E_{n+1}$.

**Example.** To illustrate this we give an example where cohomology theory gives rise to a CW spectrum. Let $K^*$ be a generalized cohomology theory, defined on CW pairs. We have $K^n(X) = K^n(X, pt.) + K^n(pt.)$ aand define $\tilde{K^n}(X) = K^n(X, pt)$. We assume $K^*$ satifies the wedge axiom.
We can apply Brown representability theorem and say that there exist connected CW-complexes $E_n$ with base point and natural equivalences such that

$$\tilde{K^n}(X) \cong [X, E_n]$$

Consider the suspension isomorphism $\Sigma : \tilde{K^n}(X) \xrightarrow{\cong} K^{\tilde{n+1}}(\Sigma X)$. The suspension isomorphism is defined with the follwing commutative diagram:

$$\begin{tikzcd}
K^n(X,pt) \arrow[r, "\delta\cong"'] \arrow[ddr, "\Sigma"] & K^{n+1}(CX,X) \\
& K^{n+1}(\Sigma X, C'X) \arrow[u, "\cong\text{excision}"'] \arrow[d, "\cong C'X\text{contractible}"] \\
& K^{n+1}(\Sigma X, pt)
\end{tikzcd}$$

The map $\delta$ is the coboudnary for the exact sequence fo the triple $(CX, X, pt.)$.( Here $CX$ and $C'X$ are the two cones that make up $\Sigma X$)

We have now natural equivalences

$$\begin{aligned}
[X, E_n] &\cong \widetilde{K^n}(X) \cong \widetilde{K^{n+1}}(\Sigma X) \\
&\cong [\Sigma X, E_{n+1}] \cong [X, \Omega_0 E_{n+1}].
\end{aligned}$$

This natural equivalence must be induced by a weak equivalence:

$$\varepsilon'_n : E_n \to \Omega E_{n+1}$$

So this sequence of spaces becomes a spectrum. This spectrum is also called $\Omega-$*spectrum* (assuming all the spaces are connected for convenience )

We define a spectrum $F$ to be a *suspension spectrum* or *S-spectrum* if

$$\varphi_n : \Sigma F_n \longrightarrow F_{n+1}$$

is a weak homotopy equivalence for $n$ sufficiently large.

**Example.** Given a CW-complex $X$, let $E_n = \begin{cases} \Sigma^n X & (n \geq 0) \\ pt & (n < 0) \end{cases}$ with the obvious maps. Then this spectrum $E$ would be an S-spectrum, but need not be an $\Omega$-spectrum. $E$ is called the suspension spectrum of $X$.
In particular, the sphere spectrum $S$ is the suspension spectrum of $S^0$; it has $n^{\text{th}}$ term $S^n$ for $n \geq 0$.

We now define homotopy groups of a spectrum. consider the following homomorphisms

$$\pi_{n+r}(E_n) \longrightarrow \pi_{n+r+1}(\Sigma E_{n+1}) \xrightarrow{(\varepsilon_n)_*} \pi_{n+r+1}(E_{n+1})$$

We define the stable homotopy groups:

$$\pi_r(E) = \varinjlim_n \pi_{n+r}(E_n)$$ [1]

If $E$ is an $\Omega$-spectrum then the homomorphism

$$\pi_{n+r}(E_n) \longrightarrow \pi_{n+r+1}(E_{n+1})$$

is an isomorphism for $n+r \geq 1$; the limit is attained, and we have

$$\pi_r(E) = \pi_{n+r}(E_n) \quad \text{for } n+r \geq 1.$$

In the case of Suspension spectrum, we have $\pi_r(E) = \varinjlim_n \pi_{n+r}(\Sigma^n X)$. The limit is attained for $n > r+1$. In this case we have the homotopy groups of $E$ are the stable homotopy grops of $X$.

Similarly we define relative homotopy groups. Let $X$ be a spectrum, then a subspectrum $A$ of $X$ consist of subspaces $A_n \subset X_n$ such that the spectrum maps $\xi_n : \Sigma X_n \to X_{n+1}$ maps $\Sigma A_n$ into $A_{n+1}$. We define the relative homotopy groups as

$$\pi_r(X,A) = \varinjlim_n \pi_{n+r}(X_n, A_n)$$

and we get a exact sequence

$$\cdots \to \pi_*(A) \to \pi_*(X) \to \pi_*(X,A) \to \pi_*(A) \to \cdots$$

## 2.1 Stable Homotopy Category

$E$ is called a CW spectrum if

[1] in this case colimit=$\lim_n$

1. the terms $E_n$ are CW-complexes with base point and
2. each map $\varepsilon_n : \Sigma E_n \to E_{n+1}$ is an isomorphism from $\Sigma E_n$ to a sub-complex of $E_{n+1}$.

A subspectrum $A$ of a CW spectrum $E$ is as defined before, with the added condition that $A_n \subset X_n$ for each $n$. Let $E$ be a CW-spectrum, $E'$ a subspectrum of $E$. We say $E'$ is cofinal in $E$ if for each $n$ and each finite subcomplex $K \subset E_n$ there is an $m$(depnding on $n$ and $K$) such that $\Sigma^m K$ maps into $E'_{m+n}$ under the map

$$\Sigma^m E_n \stackrel{\Sigma^{m-1}\varepsilon_n}{\longrightarrow} \Sigma^{m-1} E_{n+1} \longrightarrow \ldots \longrightarrow E_{m+n-1} \stackrel{\varepsilon_{m+n-1}}{\longrightarrow} E_{m+n}.$$

A *function* $f$ from one spectrum $E$ to another spectrum $F$ of degree $r$ is a sequence of maps $f_n : E_n \to F_{n-r}$ such that the following diagram is structly commutative for each $n$

$$\begin{tikzcd}
\Sigma E_n \arrow[r, "\varepsilon_n"] \arrow[d, "\Sigma f_n"] & E_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma F_{n-r} \arrow[r, "\phi_{n-r}"] & F_{n-r+1}
\end{tikzcd}$$

or equivalently maps in the $\Omega$ spectrum. Note here that the diagrams are to be strictly commutative and not commutative up to homotpy.

Let $E$ be a CW spectrum and $F$ be a CW spectrum. take all cofinal subspectra $E' \subset E$ and all functions $f' : E' \to F$. Say that two functions $f' : E' \to F$ and $f" : E'' \to F$ are equivalent if there is a cofinal subspectrum $E'''$ contained in $E'$ and $E''$ such that the restrictions of $f'$ and $f''$ to $E'''$ coincide.

A *map* from $E$ to $F$ is an equivalence class of such functions. This is saying that if we have a cell $c$ in $E_n$, a map need not be defined on it at once; we can wait till $E_{m+n}$ before defining the map on $\Sigma^m c$. This is equivalent to saying that two functios $f' : E' \to F$ and $f'' : E'' \to F$ are equivalent if their restrictions to $E' \cap E''$ coincide.

**Lemma 2.1.** Let $f : E \to F$ be a function and $F'$ a cofinal subspectrum of $F$.. Then there is a cofinal subspectrum $E'$ of $E$ such that $f$ maps $E'$ into $F'$.

*Proof.* Consider the collection of all subspectra $G$ such that $f(G) \subseteq F'$. This collection is nonempty because it contains the basepoint. Let $E'$ be the union of these subspectra. Then $f(E') \subseteq F'$. It remains to show that $E'$ is a cofinal subspectrum. Let $K$ be a finite complex in $E_n$. Consider $f_n(K)$, this is contained in a finite subcomplex $H \subseteq F_n$. This is because $f_n$ is cellular. As $F'$ is cofinal, there is a $d$ such that $\Sigma^d H \subseteq F'_{n+d}$. Thus $f_{n+d}\left(\Sigma^d K\right) \subseteq F'_{n+d}$. So $\Sigma^d K \subseteq E'_{n+d}$.

□

Let $I^+$ be the union of the unit interval and a disjoint base-point. For $E$ a spectrum, we define $Cyl(E)$ is the cylinder spectrum and has terms

$$Cyl(E)_n = I^+ \wedge E_n$$

and maps

$$(I^+ \wedge E_n) \wedge S^1 \xrightarrow{1 \wedge \varepsilon_n} I^+ \wedge E_{n+1}$$

The cylinder spectrum is a functor: a map $f : E \to F$ indues a map $Cyl(f) : Cyl(E) \to Cyl(F)$.

Two maps $f, g : E \to F$ are homotopic if there is a map $h : Cyl(E) \to F$ such that the following diagram commutes

$$\begin{tikzcd} E \wedge 0^+ \arrow[r] \arrow[dr, "f"'] & Cyl(E) \arrow[d, "h"] & E \wedge 1^+ \arrow[l] \arrow[dl, "g"] \\ & F & \end{tikzcd}$$

A *morphism* in the category $CWsp$ will be a homotopy class of maps. We write $[E,F]_r$ for the set of homotopy classes of maps with degree $r$.

The *stable homotopy category*, denoted SHC is the category whose objects are the same as those of $CWSp$ and whose morphisms are the homotopy classes of maps. That is $\mathrm{SHC}(E,F) := [E,F]$ for CW spectra $E$ and $F$.

As long as we deal entirely with CW spectra we can restrict attention to functions whose components $f_n : E_n \to F_{n-r}$ are cellular maps.

**Proposition 2.2.** Let $K$ be a finite CW-complex and let $R$ be its suspension spectrum, so that $E_n = \Sigma^n K$ for $n \geq 0$. Let $F$ be any spectrum.
We have

$$[E,F]_r = \varinjlim_n [\Sigma^{n+r} K, F_n]$$

In particular,

$$[S,F]_r = \pi_r(F)$$

*Proof.* [A, Pg 164] □

Let $C_n$ be the set of cells in $E_n$ other than the base-point. Then we get a function $C_n \to C_{n+1}$ by $C_\alpha \mapsto \varepsilon_n(\Sigma C_\alpha)$. This function is an injection. Let $C = \lim_{n\to\infty} C_n$. An element of $C$ is called *stable cell*. A stable cell is essentially an equivalence class that contains atmost one cell in $E_n$. If there are only finite number of stable cells in a spectra we call that a finite spectra.

We state the homotopy extension theorem in $CWsp$ and refer to [A] for proof.

**Lemma 2.3.** Let $X,A$ be a pair of $CW$-spectra, and $Y,B$ a pair of spectra such that $\pi_*(Y,B) = 0$. Suppose given a map $f : X \longrightarrow Y$ and a homotopy $h : \mathrm{Cyl}(A) \longrightarrow Y$ from $f|_A$ to a map $g : A \longrightarrow B$. Then the homotopy can be extended over $\mathrm{Cyl}(X)$ so as to deform $f$ to a map $X \longrightarrow B$.

$$\begin{tikzcd}
A \arrow[rr, "in_0"] \arrow[dd, "\iota"] & & A \wedge I_+ \arrow[dl, "h"] \arrow[dd] & & A \arrow[ll, "in_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Y & & B \arrow[ll] & \\
X \arrow[ur, "f"] \arrow[rr] & & X \wedge I_+ \arrow[ul, dotted] & & X \arrow[ll, "in_1"] \arrow[ul, dotted]
\end{tikzcd}$$

The homotopy extension theorem is a special case when $B = Y$.

**Lemma 2.4.** Suppose $\pi_*(Y) = 0$ and $X,A$ is a pair of CW-spectra. Then any map $f : A \to Y$ can be extended over $X$.

*Proof.* Applying the previous lemma to the pair $(A,*)$ and $(Y,*)$ we get that $f$ is null-homotopic. We have $h : Cyl(A) \to Y$ a homotopy from $f$ to a map $g : A \to *$. Then there exists an extension of $h$, $\tilde{h} : Cyl(X) \to Y$.

□

**Theorem 2.5.** Let $f : E \to F$ be function between spectra(need not be CW) such that $f_* : \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then for any CW-spectrum $X$,

$$f_* : [X,E]_* \to [X,F]_*$$

is an isomorphism.

*Proof.* We can replace $F$ by the spcetrum $M$ in which $M_n$ is the mapping cylinder of $f_n$ and assume that $f$ is an inclusion. Then $\pi_*(F,E) = 0$ by the exact sequence. Now consider $(X,*)$ and apply 2.4. This gives us $f_*$ is an epimorphism. For proving monomorphism consider 2.4 for the pair $(X \wedge I_+, X \wedge (\partial I)_+)$ (i.e. $Cyl(X)$ mod its ends).

□

**Corollary 2.6.** Let $f: E \to F$ be a morphism between CW-spectra such that $f_*: \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then $f$ is an equivalence in our category.

**Lemma 2.7.** Any CW spectrum $Y$ is equivalent in the SHC to an $\Omega$ spectrum.

*Proof.* Let us consider a functor $T^{(n)}$ from CW complexes to spectra given by

$$(T^{(n)}X)_r = \begin{cases} \Sigma^{r-n}X & r \geq n \\ \text{pt.} & r < n \end{cases}$$

Form the set of morphisms $[T^{(n)}X, Y]_0$, which is a Brown functor and is representable. Let it be represented by $Z_n$.

$$[X, Z_n] \approx [T^{(n)}X, Y] \approx [T^{(n+1)}(\Sigma X), Y] \approx [\Sigma X, Z_{n+1}] \approx [X, \Omega Z_{n+1}]$$

Thus $Z$ is an $\Omega$ spectrum. Take $X = Y_n$,

$$[T^{(n)}Y_n, Y] \approx [Y_n, Z_n].$$

Take the map $f_n : Y_n \to Z_n$ that corresponds to the equivalence class of functions $\phi_n : (T^{(n)}Y_n)_n = Y_n \to Y_n$. Since $[Y_n, Z_n]$ is a group $f_n$ has an inverse. Consider function $f$ with the sequence of maps $f_n$, this induces isomorphism

$$f_* : \pi_*(Y) \to \pi_*(Z).$$

Applying 2.6 gives the desired conclusion. □

If $X$ is a spectrum, let $Cone(X)$ be the spectrum whose n$^{th}$ term is $I \wedge X_n$ with maps

$$(I \wedge X_n) \wedge S^1 \xrightarrow{1 \wedge \varepsilon_n} I \wedge X_{n+1}$$

**Theorem 2.8.** Let $f : E, A \longrightarrow F, B$ be a function between pairs of spectra such that

$$f_* : \pi_*(E, A) \longrightarrow \pi_*(F, B)$$

is an isomorphism. Then for any CW-spectrum X,

$$f_* : [\text{Cone}(X), X; E, A]_* \longrightarrow [\text{Cone}(X), X; F, B]_*$$

is an isomorphism.

For any spectrum $X$, define $Susp(X)$ to be the spectrum whose n$^{th}$ terms is $S^1 \wedge X_n$ and its structure maps are

$$(S^1 \wedge X_n) \wedge s^1 \xrightarrow{1 \wedge \xi_n} S^1 \wedge X_{n+1}.$$

*Susp* is a functor.

**Theorem 2.9.** $Susp : [X,Y]_* \to [Susp(X), Susp(Y)]_*$ is an isomorphism.

*Proof.* [A, Theorem 3.7]

□

This shows the the sets of morphism $[X,Y]$ are abelian groups and also that the compositions are bilinear.

We would like to show that we have an additive category. We take the spectrum $E_n$=pt. for all $n$ to be the trivial object. This category has arbitrary sums(=coproducts). Given spectra $X_\alpha$ for $\alpha \in A$, we form $X = \bigvee_\alpha X_\alpha$ by $X_n = \bigvee (X_\alpha)_n$, with structure maps

$$X_n \wedge S^1 = \left( \bigvee_\alpha (X_\alpha)_n \right) \wedge S^1 = \bigvee_\alpha (X_\alpha) \wedge S^1 \xrightarrow{\bigvee_\alpha \xi_{\alpha n}} \bigvee_\alpha (X_\alpha)_{n+1}$$

This has the required property:

$$\left[ \bigvee_\alpha X_\alpha, Y \right] \xrightarrow{\cong} [X_\alpha, Y]$$

We now look at cofiber sequences for CW spectra. Suppose given a map $f : X \longrightarrow Y$ between CW-spectra. Let it be represented by a function $f' : X' \longrightarrow Y$, where $X'$ is a cofinal subspectrum. Without loss of generality we can suppose $f'$ is cellular. We form the mapping cone $Y \cup_{f'} CX$ as follows: its $n^{\text{th}}$ terms is $Y_n \cup_{f'_n} (I \wedge X'_n)$ and the structure maps are the obvious ones. If we replace $X'$ by a smaller cofinal subspectrum $X''$, we get $Y \cup_{f''} CX''$ which is smaller than $Y \cup_{f'} CX'$, but cofinal in it, and so equivalent. So the construct depends essentially only on the map $f$, and we can write it $Y \cup_f CX$. If we vary $f$ by a homotopy, $Y \cup_{f_0} CX$ and $Y \cup_{f_1} CX$ are equivalent, but the equivalence depends on the choice of homotopy.

Let $X$ be a CW-spectrum, $A$ a subspectrum. $A$ is closed if for ever finite subcomplex $K \subset X_n, \Sigma^m K \subset A_{m+n}$ implies $K \subset A_n$. It is equivalent to saying that $A \subset B \subset X, A$ cofinal in $B$ implies that $A = B$.

**Proposition 2.10.** Suppose we have a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$$

Then for each $Z$ the sequence

$$[Y \cup_f CX, Z] \xrightarrow{i^*} [Y, Z] \xrightarrow{f^*} [X, Z]$$

is exact.

**Proposition 2.11.** Suppose we have a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$$

The sequence

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} \left[W, Y \cup_f CX\right]$$

is exact.

We can extend the cofibre sequence further and similar to CW complexes we get

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{j} Susp(X) \xrightarrow{Susp(f)} Susp(Y)$$

In other words, in SHC cofiberings are the same as fibering.

**Proposition 2.12.** Finite sums are products.

*Proof.* We have

$$X \to X \vee Y \to Y$$

which is clearly a cofiber. So we have the exact sequence

$$[W,X] \to [W, X \vee Y] \to [W,Y].$$

The map $Y \xrightarrow{i} X \vee Y$ is a section so the exact sequence splits.

$$[W, X \vee Y] \cong [W,X] \oplus [W,Y]$$

and $X \vee Y$ is also the product of $X$ and $Y$ □

This proves that SHC is an additive category.

As noted down earlier, the Brown representability theorem is valid in the category of CW-spectra and morphisms of degree 0.

**Proposition 2.13.** Any spectrum $Y$ is weakly equivalent to a CW-spectrum.

*Proof.* Consider the representible functor $[X,Y]_0$. $[X,K] \approx [X,Y]_0$ for some CW spectrum $K$. We consider $X = K$ and take the image of $id$. □

**Proposition 2.14.** The SHC has arbitrary product.

*Proof.* The functor of $X$ given by $\prod_\alpha [X, Y_\alpha]_0$ is a Brown Functor and is representable. (This works out for maps of degree r as well but how?) □

For any collection of $X_\alpha$ we have a morphism $\bigvee_\alpha X_\alpha \to \prod_\alpha X_\alpha$.

**Proposition 2.15.** Suppose that for each $n$ $\pi_n(X_\alpha) = 0$ for all but a finite number of $\alpha$ then the map

$$\bigvee_\alpha X_\alpha \to \prod_\alpha X_\alpha$$

is an equivalence.

*Proof.* We have

$$\pi_n(X_1 \vee \cdots \vee X_m) \cong \sum_{i=1}^{m} \pi_n(X_i)$$

for finite wedges. By passing to direct limits(how?) we have

$$\pi_n(\bigvee_\alpha X_\alpha) = \sum_\alpha \pi_n(X_\alpha)$$

Also

$$\pi_n\left(\prod_\alpha X_\alpha\right) = \prod_\alpha \pi_n\left(X_\alpha\right)$$

Now the data was chosen precisely so that $\sum_\alpha \pi_n\left(X_\alpha\right) \longrightarrow \prod_\alpha \pi_n\left(X_\alpha\right)$ is an isomorphism. Therefore $\bigvee_\alpha X_\alpha \longrightarrow \prod_\alpha X_\alpha$ is an equivalence. $\square$

## ※ Smash Products

In this section we will construct smash product. Given two CW spectra $X$ and $Y$, we construct a CW spectrum $X \wedge Y$ so as to have the properties stated in the following theorem, among other properties.

**Theorem 3.1.** 1. $X \wedge Y$ is a functor of two variables, with arguments and values in the (graded) SHC.

2. The smash-product is associative, commutative and has the sphere spectrum $S$ as a unit, up to coherent natural equivalences.

Statement 1 is to be taken in the graded sense. That is for $f \in [X,X']_r$ and $g \in [Y,Y']_s, f \wedge g \in [X \wedge Y, X' \wedge Y']_{r+s}$ and also $(f \wedge g)(h \wedge k) = (-1)^{bc}(fh) \wedge (gk)$ if $f \in [X',X'']_a, h \in [X,X']_b, g \in [Y',Y'']_c, k \in [Y,Y']_d$.

The following equivalences hold true in our category.

$$
\begin{array}{cc}
a & a(X,Y,Z) : (X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z) \\
c = & C(X,Y) : X \wedge Y \longrightarrow Y \wedge X \\
l = & l(Y) : S \wedge Y \longrightarrow Y \\
r = & r(X) : X \wedge S \overset{\longrightarrow}{\longrightarrow} Y
\end{array}
$$

These are all maps of degree 0 and are natural. The naturality for commutativity is up to a sign $(-1)^{rs}$, if $f \in [X,X']_r$ and $g \in [Y,Y']_s$.

$$
\begin{array}{ccc}
X \wedge Y & \xrightarrow{c} & W \wedge Y \\
\Big\downarrow{\scriptstyle f\wedge g} & & \Big\downarrow{\scriptstyle g\wedge f} \\
X' \wedge Y' & \xrightarrow{c} & Y' \wedge X'
\end{array}
$$

Let $A$ be an ordered set isomorphic to $\{0,1,2,3,\ldots\}$. Suppose we have a partition of $A$ into two subsets $B$ and $C$, so that $A = B \cup C$ and $B \cap C = \phi$. Then we define a smash product functor which assigns to any two CW spectra $X$ and $Y$ a CW spectrum $X \wedge_{BC} Y$. The terms of this product spectrum $P = X \wedge_{BC} Y$ are given by by $P_{\alpha(a)} = X_{\beta(a)} \wedge Y_{\gamma(a)}$.Here $\alpha$ is an isomorphism from $A = B \cup C$ to the set $\{0,1,2,3,\ldots\}$ and $\beta, \gamma$ are monotonic functions. such that $\beta(a) + \gamma(a) = \alpha(a)$. This is called handicrafted or naive smash products.

The maps of the prodcut spectrum are defined as follows. We have

$$P_{\alpha(a)} \wedge S^1 = X_{\beta(a)} \wedge Y_{\gamma(a)} \wedge S^1.$$

We regard $S^1$ as one point compactification of $\mathbb{R}$, where infinity becomes the base point. This allows us to define a map of degree $-1$ from $S^1$ to $S^1$. by $t \mapsto -t$.

If $a \in B$, then

$$P_{\alpha(a)+1} = X_{\beta(a)+1} \wedge Y_{\gamma(a)}$$

and we define the map

$$\pi_{\alpha(a)} : SP_{\alpha(a)} \longrightarrow P_{\alpha(a)+1}$$

by

$$\pi_{\alpha(a)}(x \wedge y \wedge t) = \xi_{\beta(a)}\left(x \wedge (-1)^{\gamma(a)} t\right) \wedge y$$

If $a \in C$, then

$$P_{\alpha(a)+1} = X_{\beta(a)} \wedge Y_{\gamma(a)+1}$$

and we define the map

$$\pi_{\alpha(a)}(x \wedge y \wedge t) = x \wedge \eta_{\gamma(a)}(y \wedge t).$$

Here

$$x \in X_{\beta(a)}, \quad y \in Y_{\gamma(a)}, \quad t \in S^1,$$

and

$$\xi_{\beta(a)} : X_{\beta(a)} \wedge S^1 \longrightarrow X_{\beta(a)}, \quad \eta_{\gamma(a)} : Y_{\gamma(a)} \wedge S^1 \longrightarrow Y_{\alpha(a)+1}$$

are the appropriate maps from the spectra $X, Y$. The sign$(-1)^{\gamma(a)}$ is introduced, of course, because we have moved $S^1$ across $Y_{\gamma(a)}$.
The product $P$ is functorial for functios of $X$ and $Y$. If $B$ is infinite and $X'$ is cofinal in $X$, then $X' \wedge_{BC} Y$ is cofinal in $X \wedge_{BC} Y.Cyl(X) \wedge_{BC} Y$ and $X \wedge_{BC} Cyl(Y)$ can be identified with $Cyl(X \wedge_{BC} Y)$.

$X \wedge Y$ is constructed so that it has the following properties.

**Theorem 3.2.** For each choice of $B, C$ there is a morphism

$$\mathrm{eq}_{BC} : X \wedge_{BC} Y \longrightarrow X \wedge Y \quad (\text{ of degree } 0)$$

with the following properties.

1. If $B$ is infinite and $f : X \longrightarrow X'$ is a morphism of degree 0 , then the following diagram is commutative.

$$\begin{tikzcd}
X \wedge_{BC} Y \arrow[r, "eq_{BC}"] \arrow[d, "f \wedge_{BC} 1"'] & X \wedge Y \arrow[d, "f \wedge 1"] \\
X' \wedge_{BC} Y \arrow[r, "eq_{BC}"] & X' \wedge Y
\end{tikzcd}$$

2. If $C$ is infinite and $g : Y \longrightarrow Y'$ is a morphism of degree $0$ , then the following diagram is commutative.

$$\begin{tikzcd} X \wedge_{BC} Y \arrow[r, "eq_{BC}"] \arrow[d, "1\wedge_{BC} g"'] & X \wedge Y \arrow[d, "1\wedge g"] \\ X \wedge_{BC} Y' \arrow[r, "eq_{BC}"] & X' \wedge Y \end{tikzcd}$$

3. The morphism $eq_{BC} : X \wedge_{BC} Y \to X \wedge Y$ is an equivalence if any one of the following conditios is satisfied.

   (a) $B$ and $C$ are infinite.

   (b) $B$ is finite, say with $d$ elements and $\xi_r : \Sigma X_r \to X_{r+1}$ is an isomorphism for $r \geq d$.

   (c) $C$ is finite, say with $d$ elmemetns and $\eta_r : \Sigma Y_r \to Y_{r+1}$ is an isomorphism for $r \geq d$.

The handicrafted smash products are commutative for the right choice of $B, C$ at each point. We partition the sets accordingly with the following condition.
*Condition* Elements number $0,1,2,3$ in $A$ are either four elemnts in $B$ or four elements in $C$. similarly for elments number $4,5,6,7$ in $A$ and similarly for elementss numver $4r, 4r+1, 4r+2, 4r+3$ for each $r$. The smash product has the following property regarding commutativity

**Theorem 3.3.** The equivalence $c : X \wedge Y \to Y \wedge X$ makes the following diagram commutative for each choice of $B, C$ satsifying the condition stated above

$$\begin{tikzcd} X \wedge Y \arrow[r, "c"] & Y \wedge X \\ X \wedge_{BC} Y \arrow[u, "eq_{BC}"] \arrow[r, "c_{BC}"] & Y \wedge_{CB} X \arrow[u, "eq_{CB}"'] \end{tikzcd}$$

The handicrafted smash products have $S$ as a unit if we pick the right product at each point. Say, we partition $A = \phi \cup A$ satisfying the condition we have $S$ as a unit.

Define

$$l : S \wedge Y \to Y$$

to be the composite

$$S \wedge Y \xleftarrow[\text{equivalence}]{eq_{\phi,A}} A \wedge_{\phi A} Y \cong Y (eq_{\phi,A} \text{ is an equivalence})$$

We also have the isomorphisms $S^0 \wedge Y \cong Y$ and $X \wedge S^0 \cong X$ with the obvious componentwise isomorphism.This is also natural for morphisms of degree 0. we noe define

$$r : X \wedge S \to X$$

to be the composite

$$X \wedge S \xleftarrow{eq_{\phi,A}} X \wedge_{A\phi} S \cong X (eq_{\phi,A} \text{ is an equivalence})$$

Since $S \wedge S$ is equivalent to $S$, we have $[S \wedge S, S \wedge S]_0 \cong [S,S]_0 \cong \mathbb{Z}$. Also we construct the smash product so that the map $c : S \wedge S \to S \wedge S$ has degree 1.

We refer to [A] for details regarding the construction. We will note down some more properties of the smash product especially regarding sphere-spectra.

Let us define sphere-spectra of different stable dimensions.

$$(\underline{S}^i)_n = \begin{cases} S^{n+1} & n+i \geq 0 \\ \text{pt.} & n+i < 0 \end{cases}$$

**Proposition 3.4.** We have an equivalence $\underline{S}^i \wedge \underline{S}^j \xrightarrow{e} \underline{S}^{i+j}$ such that the following diagrams are commutative.

$$\begin{tikzcd}
(\underline{S}^i \wedge \underline{S}^j) \wedge \underline{S}^k \arrow[rr, "a"] \arrow[d, "e\wedge 1"] & & \underline{S}^i \wedge (\underline{S}^j \wedge \underline{S}^k) \arrow[d, "1\wedge e"] \\
\underline{S}^{i+j} \wedge \underline{S}^k \arrow[dr, "e"] & & \underline{S}^i \wedge \underline{S}^{j+k} \arrow[dl, "e"] \\
& \underline{S}^{i+j+k} &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{S}^i \wedge \underline{S}^j \arrow[r, "c"] \arrow[d, "e"] & \underline{S}^j \wedge \underline{S}^i \arrow[d, "e"] \\
\underline{S}^{i+j} \arrow[r, "(-1)^{ij}"] & \underline{S}^{j+i}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{S}^0 \wedge \underline{S}^j \arrow[r, bend left, "e"] \arrow[r, bend right, "l"'] & \underline{S}^j
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{S}^i \wedge \underline{S}^0 \arrow[r, bend left, "e"] \arrow[r, bend right, "r"'] & \underline{S}^i
\end{tikzcd}$$

**Proposition 3.5.** We have the equialences

$$\gamma_r : X \to \underline{(S)}^r \wedge X \text{of degree r}$$

with the following properties

1. (i) $\gamma_r$ is natural for maps of $X$ of degree 0 . (This is all we can ask, because we have not yet made $\underline{S}^r \wedge X$ functorial for maps of non-zero degree.).

2. $\gamma_0 = \ell^{-1}$.

3. The following diagram is commutative for each $r$ and $s$.

$$\begin{tikzcd} \underline{(S)}^{r+s} \wedge X & & (\underline{S}^r \wedge \underline{S}^s) \wedge X \arrow[ll, "e\wedge 1"'] \arrow[d, "a"] \\ & & \underline{S}^r \wedge (\underline{S}^s \wedge X) \\ X \arrow[uu, "\gamma_{r+s}"] \arrow[rr, "\gamma_s"] & & \underline{S}^s \wedge X \arrow[u, "\gamma_r"'] \end{tikzcd}$$

With the above proposition at hand we can the original SHC with a new category where the objects are still CW spectra, but the morphisms of degree r are given by $[\underline{S}^r \wedge X, Y]_0$ in the old category.
Composition is as follows. If we have $\underline{S}^r \wedge X \xrightarrow{f} Y$ and $\underline{S}^r \wedge Y \xrightarrow{g} Z$ of degree 0, take their composite to be

$$\underline{(S)}^{s+r} \wedge X \xleftarrow{e\wedge 1} (\underline{S}^s \wedge \underline{S}^r) \wedge X \xrightarrow{a} \underline{S}^s \wedge (\underline{S}^r \wedge X) \xrightarrow{1\wedge f} \underline{S}^s \wedge Y \xrightarrow{g} Z.$$

The composition is associative and $\ell : \underline{S}^0 \wedge X \longrightarrow X$ is an identity map.

**Proposition 3.6.** The new graded category is isomorphic to the old SHC, under the isomorphism sending

$$\left\{ \begin{array}{c} \underline{S}^r \wedge X \xrightarrow{f} Y \\ \text{(in the new category)} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} X \xrightarrow{\gamma_r} \underline{S}^r \wedge X \xrightarrow{f} Y \\ \text{(in the old category)} \end{array} \right\}$$

It is an easy to show the naturality of $\gamma_r$ with respect to maps of degree $s$: the diagram is commutative up toa sighn of $(-1)^{rs}$ if $f \in [X,Y]_s$.

$$\begin{tikzcd} X \arrow[r, "\gamma_r"'] \arrow[d, "f"] \arrow[dr, phantom, "(-1)^{rs}"] & \underline{S}^r \wedge X \arrow[d, "1\wedge f"] \\ Y \arrow[r, "\gamma_r"] & \underline{S}^r \wedge Y \end{tikzcd}$$

The smash product is distributive over the wedge-sum.Let $X = \bigvee_\alpha X_\alpha$; let $i_\alpha : X_\alpha \longrightarrow X$ be a typical inclusion. Then the morphism

$$\bigvee_\alpha (X_\alpha \wedge Y) \xrightarrow{i_\alpha \wedge 1} \left( \bigvee_\alpha X_\alpha \right) \wedge Y$$

is an equivalence.

We end this section with a remark on cofiber sequence of smash product.

**Proposition 3.7.** Let $X \xrightarrow{f} Y \xrightarrow{i} Z$ be a cofibering (it is sufficient to consider morphisms of degree zero). Then

$$W \wedge X \xrightarrow{1 \wedge f} W \times Y \xrightarrow{1 \wedge i} W \wedge Z$$

is also a cofibering.

*Proof.* [A] It suffices to check for the case in which $f : X \longrightarrow Y$ is the inclusion of a closed subspectrum, $i : Y \longrightarrow Z$ is the projection $Y \longrightarrow Y/X$ and $\bigwedge = \bigwedge_{BC}$. $\square$

# ※ Duality

If $X$ is a compact subset embedded in $S^n$, then by Alexander duality theorem, we have

$$\tilde{H}^k(X) \cong \tilde{H}_{n-k-1}(S^n - X).$$

Let $K$ be homotopy equivalent to $S^n - K$ for $K$ a compact set embedded in $S^n$.We would like to prove that $K$ determines the stable homotopy type of $L$. The homotopy type of $L$ in general is not determined by $K$ as it dependes on the embedding of $K$.

Embed $S^n$ as the equatorial sphere in $S^{n+1}$ and embed the suspension $\Sigma K$ of $K$ in $S^{n+1}$ by joining to the two poles. Then $S^{n+1} - \Sigma K \simeq S^n - K$. So if we have $K \subset S^n$ and $M \subset S^m$ and a homtopy equivalence $f : \Sigma^p K \to \Sigma^q M$, we can embed $\Sigma^p K$ in $S^{n+p}$ and $\Sigma^q M$ in $S^{m+q}$, since the complements remain homotopy equivalent. So WLOG, we can say we have $K' \subset S^{n'}$ and $M' \subset S^{m'}$ and a homotopy equivalence $f : K' \to M'$. We can even assume $f$ is piecewise linear.

Now suppose $K \subset S^n$ and embed $S^n$ as an equiatorial sphere in $S^{n+1}$ without changin $K$. Then $S^{n+1} - K = \Sigma(S^n - K)$. Consider the join of to spheres in which $S^n * S^m \simeq S^{m+n+1}$, $K$ and $M$ are embedded, $S^n$ and $S^m$ respectively. We can embed the mapping cylinder $M_c$ of $f$ in the join. In this sphere we have

$$S^{m+n+1} - K = \Sigma^{m+1}(S^n - K)$$

$$S^{m+n+1} - M = \Sigma^{n+1}(S^m - M)$$

and two maps

$$S^{m+n+1} - K \xleftarrow{f} S^{m+n+1} - M_c \xrightarrow{g} S^{m+n+1} - M$$

But the injective maps

$$K \to M_c \leftarrow M$$

indcue isomorphismc of cohomology. The alexander duality isomorphism is natural for inclusion maps and therefore $f$ and $g$ induce isoorphism of homology. We can suspend further to make everything simply connected. So by Whitehead's theorem, $f$ and $g$ are stable homotopy equivalences. Thus we've proved the assignment $K \mapsto L$ is well-defined, up to stable equivalence, for the suspension spectrum of $K$. The desuspension is made so that degrees are as expected.

Let $X$ be CW spectrum. Consider the set $[W \wedge X, S]_0$. With $X$ fixed this is a contravariant functor of $W$ and this is now a Brown functor. So it is representable, say by $X^*$ and there is a natural isomorphism

$$[W, X^*]_0 \xrightarrow{T} [W \wedge X, S]_0$$

Taking $W = X^*$ and the *id* map we see that there is a map $e : X \wedge X^* \to S$. Since $T$ is natural it carries, $f \to X^*$ into $W \wedge X \xrightarrow{f \wedge 1} X^* \wedge X \xrightarrow{e} S$.We can then extend this isomorphism to maps of degree $r$

$$[W, X^*]_r \xrightarrow{T} [W \wedge X, S]_r$$

We can think of $X^*$ as the dual. The dual $X^*$ is a contravariant functor of $X$. If $g : X \to Y$ is a map, then it induces

$$[W, Y^*] \xrightarrow{(1 \wedge g)^*} [W, X^*]$$

adn this natural transformation must be induced by a unique map $g^* : Y^* \to X^*$. We have the following commutative map

$$\begin{array}{ccc} Y^* \wedge X & \xrightarrow{1 \wedge g} & Y^* \wedge Y \\ {\scriptstyle g^* \wedge 1}\big\downarrow & & \big\downarrow{\scriptstyle e_Y} \\ X^* \wedge X & \xrightarrow[e_X]{} & S \end{array}$$

Let $Z$ be a spectrum , we can make a natural transformation

$$[W, Z \wedge X^*]_r \xrightarrow{T} [W \wedge X, Z]_r$$

as follows: Given $W \xrightarrow{f \wedge 1} Z \wedge X^*$ we take $W \wedge X \xrightarrow{f \wedge 1} Z \wedge X^* \wedge X \xrightarrow{1 \wedge e} Z$.
Note that $T$ is an isomorphism if $Z = S^n$.

**Proposition 4.1.** Suppose we have cofiber sequence $Z_1 \to Z_2 \to Z_3 \to Z_4 \to Z_5$ adn $T$ is an isomorphism for $Z_1, z_2, Z_4, Z_5$ the it is an isomorphism for $Z_3$

*Proof.* The proof is a simple application of five lemma.

□

**Proposition 4.2.** $T$ is an isomorphism if $Z$ is any finite spectrum.

*Proof.* We have a cofiber sequence,

$$S \to X \to (X \cup_f D) \to \Sigma S \to \Sigma X$$

We then proceed by induction and the previous remark. □

**Proposition 4.3.** If $W$ and $X$ are finite spectra , then

$$T : [W, Z \wedge X^*]_r \to [W \wedge X, Z]_r$$

is an isomorphism for any spectrum $Z$.

*Proof.* I have to use direct limits. Writing an infinite spectra as direct limit of finte spectra. Not sure how to do it.

□

**Lemma 4.4.** If $X$ is a finite spectrum then $X^*$ is equivalent to a finite spectrum.

*Proof.* The proof involves homology theories of a spectra and is postponed till next chapter.

□

**Proposition 4.5.** Let $X$ be a finite spectrum, $Y$ any spectrum. Then we have an equivalence $(X \wedge Y)^* \xrightarrow{h} X^* \wedge Y^*$ which makes the follwoing diagram commute

$$\begin{tikzcd}
(X \wedge Y)^* \wedge X \wedge Y \arrow[rr, "e_{X \wedge Y}"] \arrow[d, "h \wedge 1"] & & S \\
X^* \wedge Y^* \wedge X \wedge Y \arrow[r, "1 \wedge c \wedge 1"] & X^* \wedge X \wedge Y^* \wedge Y \arrow[ur, "e_X \wedge e_Y"'] &
\end{tikzcd}$$

*Proof.* By 4.4 we can assume that $X^*$ is a finite spectrum. Then,

$$[W, X^* \wedge Y^*]_r \xrightarrow{T_Y} [W \wedge Y, X^*]_r$$

is an isomorphism for any spectrum $W$, and so is

$$[W \wedge Y, X^*]_r \xrightarrow{T_X} [W \wedge Y \wedge X, S]_r$$

by the original property of $X^*$ applied to the spectrum $W \wedge Y$. This state of affairs reveals $X^* \wedge Y^*$ as the dual of $Y \wedge X$ with $T_{Y \wedge X} = T_X T_Y$. Writing this equation in terms of maps $e$, we obtain the desired.

□

# ※ Homology and Cohomology

We define $E$-homology and $E$-cohomology for a given spectrum $E$ and study their properties.

The $E$-homology is defined as

$$E_n(X) = [S, E \wedge X]_n$$

and $E$-cohomology is defined as

$$E^n(X) = [X, E]_{-n}$$

These functors satisfy the properties that generalised homology and cohomology functors satisfy. They give an analog for a theoy defined on spectra of the Eilenberg-Steenrod axioms. We record the properties in the proposition below. These are easy to check.

Consider the Eilenberg-Maclane spectrum $H\mathbb{Z}$. Define

$$H_n(X) = [S, H\mathbb{Z} \wedge X]$$

and

$$H^n(X) = [X, H\mathbb{Z}]$$

**Proposition 5.1.**

1. $E_*(X)$ is a covariant functor of two variables $E, X$ in SHC with values in the category of graded abelian groups. $E^*(X)$ is a covariant fucntor in $E$ and contravariant in $X$.

2. If we vary $E$ or $X$ along a cofibering, we obtain an exact sequence, That is, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a cofiber sequence, then

$$E_n(X) \xrightarrow{f_*} E_n(Y) \xrightarrow{g_*} E_n(Z)$$

and

$$E^n(X) \xleftarrow{f^*} E^n(Y) \xleftarrow{g^*} E^n(Z)$$

are exact; if $E \xrightarrow{i} F \xrightarrow{j} G$ is a cofiber sequence, then

$$E_n(X) \xrightarrow{i_*} F_n(X) \xrightarrow{j_*} G_n(X)$$

and

$$E^n(X) \xrightarrow{i_*} F^n(X) \xrightarrow{j_*} G^n(X)$$

are exact.

3. There area natural isomorphisms

$$E_n(X) \cong E_{n+1}\left(S^1 \wedge X\right)$$
$$E_n(X) \cong E^{n+1}\left(S^1 \wedge X\right)$$

4.
$$E_n(S) = E^{-n}(S) = \pi_n(E)$$

For a CW complex $L$ we define homology and cohomology to be $E_n$ or $E^n$ applied to the suspension spectrum of the complex.

$$\tilde{E}_n(L) = E_n(\Sigma^\infty L)$$

$$\tilde{E}^n(L) = E^n(\Sigma^\infty L)$$

The following fact holds
$$E_n(X) \cong X_n(E).$$

**Proposition 5.2.** If $X$ is a finite spectrum $E_n(X^*) \cong E^{-n}(X)$.

*Proof.* The proof is a simple application of 4.3.

□

*Proof of 4.4.* Let $X$ be a finite spectrum. Then $[S, X^*] \cong [X, S]$ and the right hadn side is zero if $n$ is negative for large absolute values. But $H_n(X^*) = H^{-n}(X)$ is finitely generated in each dimension and zero for all except for finite number of dimensions. This proves that $X^*$ has only finite stable cells and hence is a finite spectrum □

We now discuss homology and cohomology groups with coefficients.

**Moore spectrum** Let $G$ be an abelian group. conisder a free resolution $0 \to R \xrightarrow{i} F \to G \to 0$. Take $\vee_\alpha S, \vee_\beta S$ such that $\pi_0$ of the two spectra are $R$ and $F$ respectively. take a map $f : \vee_\alpha S \to \vee_\beta S$ inducing $i$[2]. Form another spectrum $M = \vee_\alpha S \bigcup_f C(\vee_\beta S)$. This is a *Moore spectrum of type G*.
So we have

$$\begin{aligned} \pi_r(M) &= 0 && \text{for } r < 0 \\ \pi_0(M) &= H_0(M) = G \\ H_r(M) &= 0 && \text{for } r > 0 \end{aligned}$$

For any spectrum $E$, we define the corresponding spectrum with coefficients in $G$ by

$$EG = E \wedge M$$

[2]See here for more details on the induced map $f$

**Proposition 5.3.** 1. There exists an exact sequence

$$0 \longrightarrow \pi_n(E) \otimes G \longrightarrow \pi_n(EG) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{n-1}(E)) \longrightarrow 0$$

(This need not split, e.g., take $E = \mathrm{KO}, G = \mathbb{Z}_2$. )

2. More generally, there exists exact sequences

$$0 \longrightarrow E_n(X) \otimes G \longrightarrow (EG)_n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(E_{n-1}(X), G) \longrightarrow 0$$

and (if $X$ is a finite spectrum or $G$ is finitely generated)

$$0 \longrightarrow E^n(E) \otimes G \longrightarrow (EG)^n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}\left(E^{n+1}(X), G\right) \longrightarrow 0$$

*Proof.* [A, Page 221] for proof □

The Moore spectrum for $\mathbb{Q}$ is same as the Eilenberg-Maclane spectrum for $\mathbb{Q}$. With this fact one can show that the rational stble homotopy is same as rational homology,i.e.

$$\pi_*(X) \otimes \mathbb{Q} \to H_*(X) \otimes \mathbb{Q}.$$

The isomorhism is induced by the map $i : S \to H$ representing a generator of $\pi_0(H) = \mathbb{Z}$.

## References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology*