1 Introduction

We saye $\mathscr{C}$ is a model category if $\mathscr{C}$ is a cat with three classes of maps

- fibrations
- cofibrations
- weak equivalences

satisfying following axioms

M0 There exists finite limits and colimits.

M1 Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where
i is a cofibration and a weak equivalence(trivial cofibration) and p is a fibration or
i is a cofibration and p is a fibration(trivial fibration) and weak equivalence,
then $\exists$ a lift $B \to X$.

M2 Any map f may be factored as
$f = pi$ where i=trivial cofibration and p=fibration and
$f = pi$ where i=cofibration and p =trivial fibration.

M3 Fibrations are stable under composition, base change and any isomorphism is a fibration.
Cofibrations are stable under composition, cobase change and an isomorphism is a cofibration.

M4 The base extension of a map which is a trivial fibration is a weak equivalence.
The cobase extension of a map which is trivial fibration is a weak equivalence.

M5 if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is in $\mathscr{C}$. Then if two of f, g, gf are weak equivalences, so is the third. Any isomorphism is a weak equivalence.

An intial object in a category $\mathscr{C}$ is an object ϕ such that for all objects C in $\mathscr{C}$ there is a unique morphism $\phi \to C$. The dual notion of this is the terminal object $*$. These objects exist in $\mathscr{C}$ because of M0 and they are unique.

X is cofibrant if $\phi \to X$ is a cofibration. X is fibrant if $X \to e$ is a fibration.

Let $f, g : A \to B$ be maps. We say that f is left-homtopic to g if there is a diagram of the form where σ is a weak equivalence.

$$\begin{tikzcd} A \vee A \arrow[r, "f+g"] \arrow[d, "\nabla"] \arrow[dr, "\partial_0+\partial_1"] & B \\ A & \tilde{A} \arrow[l, "\sigma"] \arrow[u, "h"'] \end{tikzcd} \tag{1}$$

Dually we say that f is right homotopic to g if there is a diagram of the form where s is a weak equivalence.

$$\begin{tikzcd} \tilde{B} \arrow[dr] & B \arrow[l, "s"] \\ A \arrow[u, "k"] \arrow[u, "(d_0,d_1)"'] \arrow[r, "(f,g)"'] & B \times B \arrow[u, "\triangle"'] \end{tikzcd} \tag{2}$$

By cylinder object for an object A we mean an object $A \times I$ together with maps

$$A \vee A \xrightarrow{\partial_0+\partial_1} A \times I \xrightarrow{\sigma} A$$

with $\sigma(\partial_0 + \partial_1) = \nabla_A$ such that $\partial_0 + \partial_1$ is a cofibration and σ is a weak equivalence.
Dually, a path object for B shall be an onject B^I together with a factoriazation

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B$$

of $\triangle_B$ where s is a weak equivalence and (d_0, d_1) is a fibration.

By a left homotopy from f to g, we mean a diagram 1 where $\partial_0 + \partial_1$ is a cofibration and hence $\tilde{A}$ is a cylinder object for A. This is also saying that there exists a cylinder object such that the map $A \vee B \xrightarrow{f+g} B$ extends to a map $h : A \times I \to B$ with obvious commutative relations
Similarly a right homotopy from f to g is a diagram 2 where $\tilde{B}$ is a path object for B. Equivalently the map $A \xrightarrow{(f,g)} B \times B$ extends to a map $B^I \to B \times B$ with relevant commutative relations.

Lemma 1. If $f, g \in \mathbf{hom}(A, B)$ and $f \overset{l}{\sim} g$, then thre is a left homotopy $h : A \times I \to B$ from f to g.

Lemma 2. Le A be a cofibrant object and let $A \times I$ be a cylinder object for A. Then $\partial_0 : A \to A \times I$ and $\partial_1 : A \to A \times I$ are trivial cofibrations.

Lemma 3. Let A be cofibrant and let $A \times I$ and $A \times I'$ be two cylinder objects for A.Then the result of gluing $A \times I$ and $A \times I'$ by identification $\partial_1 A = \partial'_0 A$ defined precisely to be the object $\tilde{A}$ is also a cylinder object.

Lemma 4. If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on $\mathbf{hom}(A, B)$.

Lemma 5. Let A be cofibrant and let $f, g \in \mathbf{hom}(A, B)$ Then

1. $f \stackrel{l}{\sim} g \implies f \stackrel{r}{\sim} g$
   (dual)If B is fibrant then $f \stackrel{r}{\sim} g \implies f \stackrel{l}{\sim} g$

2. $f \stackrel{r}{\sim} g \implies$ there exists a right homotopy $k : A \to B^I$ from f to g with $s : B \to B^I$ a trivial cofibration.

3. If $u : B \to C$, then $f \stackrel{r}{\sim} g \implies uf \stackrel{r}{\sim} ug$

Let A and B be objects of $\mathscr{C}$ let $\pi^r(A,B)$(similar for $\pi^l(A,B)$) be the set of equivalence classes of $\hom(A,B)$ with repsect to the equivalence relation generated by $\stackrel{r}{\sim}$. When A cofibrant and B is fibrant, in which case left and right homotopies coincide and are already equivalence relations, we shall denote the relation by $\sim$, call it homotopy and $\pi_0(A,B)$.

Lemma 6. If A is cofibrant, then composition in $\mathscr{C}$ induces a map $\pi^r(A,B) \times \pi^r(B,C) \to \pi^r(A,C)$.

Lemma 7. Let A be cofibrant and let $p : X \to Y$ be a trivial fibration. Then p induces a bijection $p_* : \pi^l(A,X) \to \pi^l(A,Y)$.
(dual) Let B be fibrant and $i : X \to Y$ be a tivial cofibration, then i induces a bijection $i_* : \pi^r(Y,B) \simeq \pi^r(X,B)$

Let $\mathscr{C}_c, \mathscr{C}_f, \mathscr{C}_{cf}$ be full subcategories[1] consisting of the cofibrant, fibrant and both cofibrant and fibrant objects of $\mathscr{C}$ respectively. Define

$$\pi\mathscr{C}_c \text{ with objects} = Obj(\mathscr{C}_c) \text{ and morphisms} = \pi^r(A,B)$$

If we denote the right homotopy class of a map $f : A \to B$ by $\bar{f}$ we obtain a functor $\mathscr{C}_c \to \pi\mathscr{C}_c$ given by $X \to X, f \to \bar{f}$. Similarly we define $\pi_{\mathscr{C}_f}$ and $\pi\mathscr{C}_{cf}$.

Let $\mathscr{C}$ be an arbitrary category and let S be a subclass of the class of maps of $\mathscr{C}$. By localization of $\mathscr{C}$ with respect to S we mean a category $S^{-1}\mathscr{C}$ together with a functor $\gamma : \mathscr{C} \to S^{-1}\mathscr{C}$ having the following universal porperty: For every $s \in S$, $\gamma(s)$ is an isomorphism; given any functor $F : \mathscr{C} \to \mathscr{B}$ with $F(s)$ an isomorphism for all $s \in S$ there is a unique functor $\theta : S^{-1}\mathscr{C} \to \mathscr{B}$ such that $\theta \circ \gamma = F$.

Let $\mathscr{C}$ be a model category. Then the homotopy category of $\mathscr{C}$ is the localization of ¸with respect to the class of weak equivalences and is denoted by $\gamma : \mathscr{C} \to Ho\mathscr{C}$. $\gamma : \mathscr{C}_c \to Ho\mathscr{C}_c$ and $\gamma : \mathscr{C}_f \to Ho\mathscr{C}_f$ will denote the localization of $\mathscr{C}_c$ and $\mathscr{C}_f$ with repect to the class of maps in the respective categories which are weak equivalences in $\mathscr{C}$. $[X,Y] := \hom_{Ho\mathscr{C}}(X,Y)$.

Lemma 8.
1. Let $F : \mathscr{C} \to \mathscr{B}$ carry weak equivalences in $\mathscr{C}$ 1nto isomorphisms in $\mathscr{B}$. If $f \stackrel{l}{\sim} g$ or $f \stackrel{r}{\sim} g$, then $F(f) = F(g)$ in $\mathscr{B}$.

2. Let $F : \mathscr{C}_c \to \mathscr{B}$ carry weak equivalences in $\mathscr{C}_c$ into isomorphisms in $\mathscr{B}$. If $f \stackrel{r}{\sim} g$, then $F(f) = F(g)$ in $\mathscr{B}$.

[1]some objects but all morphisms

The above lemma implies the functors $\gamma_c, \gamma_f, \gamma$ induce functors $\overline{\gamma_c} : \pi\mathscr{C}_c \to Ho\mathscr{C}_c, \overline{\gamma_f} : \pi\mathscr{C}_f \to Ho\mathscr{C}_f$, $\overline{\gamma} : \pi\mathscr{C}_{cf} \to Ho\mathscr{C}$.

The homotopy category is the category

$$\color{red}{Ho\mathscr{C}} \text{ with objects} = Obj(\mathscr{C}) \text{and} \, \mathbf{hom}_{Ho\mathscr{C}}(X,Y) = \mathbf{hom}_{\pi\mathscr{C}_{cf}}(RQX, RQY) = \pi(RQX, RQY)$$

For each object X choose a trivial fibration $p_X : Q(X) \to X$ with $Q(X)$ cofibrant and a trivial cofibration $i_X : X \to R(X)$ with $R(X)$ fibrant. For each map $f : X \to Y$, we may choose a map $\underline{Q(f)} : \underline{Q(X)} \to Q(Y)$ and $\underline{R(f)} : R(X) \to R(Y)$. By mapping $X \to Q(X)$ or $R(X)$ and $f \to \overline{Q(F)}$ or $\overline{R(f)}$ we get functors $\overline{Q} : \mathscr{C} \to \pi\mathscr{C}_c$ and $\overline{R} : \mathscr{C} \to \pi\mathscr{C}_f$. Some more math and we get a well-defined functor

$$\overline{RQ} : \mathscr{C} \to \pi\mathscr{C}_{cf}$$

$$X \to RQX$$

$$f \to \overline{RQ(f)}$$

Theorem 1. $Ho\mathscr{C}, Ho\mathscr{C}_c, Ho\mathscr{C}_f$ exist and there is a diagram of functors

$$\begin{tikzcd}
\pi\mathscr{C}_c \arrow[r, "\overline{\gamma_c}"] & Ho\mathscr{C}_c \arrow[d, "\sim"'] \\
\pi\mathscr{C}_{cf} \arrow[u, hook] \arrow[d, hook] \arrow[r, "\overline{\gamma}", "\sim"'] & Ho\mathscr{C} \\
\pi\mathscr{C}_f \arrow[r, "\overline{\gamma_f}"] & Ho\mathscr{C}_f \arrow[u, "\sim"]
\end{tikzcd}$$

where $\hookrightarrow$ denotes a full embedding and $\xrightarrow{\sim}$ denotes an equivalence of categories. Furthermore if $(\overline{\gamma})^{-1}$ is a quasi-inverse[2] for $\overline{\gamma}$, then the fully faithful functor

$$Ho\mathscr{C}_c \xrightarrow{\sim} Ho\mathscr{C} \xrightarrow[\sim]{(\overline{\gamma})^{-1}} \pi\mathscr{C}_{cf} \hookrightarrow \pi\mathscr{C}_c$$

is right adjoint to $\overline{\gamma_c}$ and the fully faithful functor

$$Ho\mathscr{C}_f \xrightarrow{\sim} Ho\mathscr{C} \xrightarrow[\sim]{(\overline{\gamma})^{-1}} \pi\mathscr{C}_{cf} \hookrightarrow \pi\mathscr{C}_{cf} \hookrightarrow \pi\mathscr{C}_{cf}$$

is left adjoint to $\overline{\gamma_f}$.

Corollary 1. If A is cofibrant and B is fibrant, then

$$\mathbf{hom}_{Ho\mathscr{C}}(A,B) = \pi(A,B)$$

The category $\mathscr{C}$ can have different model structures on it, but same $Ho\mathscr{C}$, i.e. the weak equivalences are same but fibrations and cofibrations can be different.

[2] Definition

2 Loop and suspension functors

Let $\mathscr{C}$ be a fixed model category and $f, g : A \to B$ be two maps with A cofibrant and B fibrant.
Define left homotopy between left homotopies and right homotopy between right homotopies in the analogous way.

Let $h : A \times I \to B$ be a left homotopy from f to g and $k : A \to B^I$ be a right homotopy from f to g. By a correspondence between h and k we mean a map $H : A \times I \to B^I$ such that $H\partial_0 = k, H\partial_1 = sg, d_0 H = h, d_1 H = g\sigma$. (here $\sigma : A \times I \to A$ and $s : B \to B^I$ are weak equivalences.) We use the following diagrams to indicate the situation:

$$\begin{tikzcd} f \arrow[r, dash, "h"] & g \end{tikzcd} \qquad \begin{tikzcd} g \arrow[d, dash, "k"] \\ f \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, dash, "g\sigma"] \arrow[d, dash, "k"'] & g \arrow[d, dash, "sg"] \\ f \arrow[r, "h"] \arrow[ur, phantom, "H"] & g \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "sg"] \arrow[d, "\partial_1"'] & B^I \arrow[d, "{(d_0,d_1)}"] \\ A \times I \arrow[r, "{(h,g\sigma)}"'] \arrow[ur, dashed, "H"] & B \times B \end{tikzcd}$$

Lemma 1. Given $A \times I$ and a right homotopy $k; A \to B^I$ there is a left homotopy $h : A \times I \to B$ corresponding to k. Dually given B^I and h, there is a k corresponding to h.

Lemma 2. Suppose that $h : A \times I \to B$ and $h' : A \times I' \to B$ are two left homotopies from f to g and that $k : A \to B^I$ is a right homotopy from f to g. Suppose that h and k correspond, then h' and k correspond iff h' is left homotopic to h.

These two lemmas make left homotopy between left homotopies an equivalence relation, denoted by $\pi_1^l(A, B; f, g)$. Correspondence yields a bijection $\pi_1^l(A, B; f, g) \simeq \pi_1^r(A, B; f, g)$. So denoting this as $\pi_1(A, B; f, g)$, an element of this is a homotopy class of homtopies from f to g.

Theorem 1. Composition of left homotopies induces maps $\pi_1^l(A, B; f_1, f_2) \times \pi_1^l(A, B; f_2, f_3) \to \pi_1^l(A, B; f_1, f_3)$ and similarly for right homotopies. Composition of left and right homotopies is compatible with the correspondence bijection of the corollary to above lemma. Finally the category with objects $\mathbf{hom}(A, B)$, with a morphism from f to g defined to be an element of $\pi_1(A, B; f, g)$, and with composition of morphisms defined to be induced by composition of homotopies, is a groupoid. The inverse of an element of $\pi_1^l(A, B; f, g)$ represented by h being represented by h^{-1}.

Lemma 3. The diagram commutes

$$\begin{tikzcd}
\pi_1(A,B;f,g) \arrow[r, "i^*"] \arrow[d, "j_*"] & \pi_1(A',B;fi,gi) \arrow[d, "j_*"] \\
\pi_1(A,B';jf,jg) \arrow[r, "i^*"] & \pi_1(A',B';jfi,jgi)
\end{tikzcd}$$

Definition 1. A pointed category is a category $\mathscr{C}$, in which the initial object and final object exist and are isomorphic, denoted by $\star$ and call it the null object of $\mathscr{C}$. If X and Y are arbitrary objects of $\mathscr{C}$, we denote by $0 \in \mathrm{hom}(X,Y)$ teh composition $X \to \star \to Y$. If $f : X \to Y$ is a map in $\mathscr{C}$, we dfine the fibre of f to be the fibre product $\star \times_Y X$ and the cofibre of f to be the cofiber product of $\star \vee_Y X$.

By a pointed model category we mean a model category $\mathscr{C}$, which is also a pointed category. If A is in $\mathscr{C}_c$ and $B \in \mathscr{C}_f$, the we write $\pi_1(A,B;0,0)$ as $\pi_1(A,B)$ which is a group.

Theorem 2. Let $\mathscr{C}$ be a pointed model category. Then there is a functor $A,B \to [A,B]_1$ from $(HoC)^{op} \times HoC \to \{groups\}$ which is determined up to canonical isomorphism by $[\mathrm{A},\mathrm{B}]_1 = \pi_1(\mathrm{A},\mathrm{B})$ if A is cofibrant and B is fibrant. Furthermore, there are two functors from $Ho\mathscr{C}$ to $Ho\mathscr{C}$, the suspension functor Σ and the loop functor Ω and canonical isomorphisms

$$[\Sigma \mathrm{A}, \mathrm{B}] \simeq [\mathrm{A},\mathrm{B}]_1 \simeq [\mathrm{A}, \Omega \mathrm{B}]$$

of functors $(\mathrm{HoC})^{op} \times (\mathrm{HoC}) \to$ (sets) where $[\mathrm{X},\mathrm{Y}] = \mathrm{Hom}(\mathrm{X},\mathrm{Y})$.

We also use ΣE to denote the cofiber of map $\partial_0 + \partial_1 : A \vee A \to A \times I$. $\underline{\underline{L}}\Sigma$ is used when needed to denote the functor. Σ and Ω are adjoint functors on $Ho\mathscr{C}$ and are unique up to canonical isomoprhism. Also for any X, $\Sigma^n X$ is a cogroup object for $n \geq 1$ and $\Omega^n X$ is a group object in $Ho\mathscr{C}$, which is commutative for $n \geq 2$.

3 Fibration and cofibration sequences

We have a cartesian map,

$$\begin{tikzcd}
F \times_E E^I \times_E F \arrow[r, hook, "pr_2"] \arrow[d, "\pi"] & E^I \arrow[d, "(d_0{,}p^I) \text{ trivial fibration}"] \\
F \times \Omega B \arrow[r, "i \times j"'] & E \times_B B^I
\end{tikzcd}$$

where $\pi = (pr_1, j^{-1} p^I pr_2)$ where $j : \Omega B \to B^I$ is the fiber. In $Ho\mathscr{C}$, we have a map

$$m : F \times \Omega B \to F$$

given by $F \times \Omega B \xrightarrow{\pi^{-1}} F \times_E E^I \times_E F \xrightarrow{pr_3} F$.

Proposition 1. Let A be cofibrant and let the map

$$\begin{aligned} m_* : [A, F] \times [A, \Omega B] &\to [A, F] \\ \alpha, \gamma &\mapsto \alpha \cdot \gamma \end{aligned}$$

If $\alpha \in [A, F]$ is represented bu $u : A \to F$, if $\gamma \in [A, \Omega B] = [A, B]_1$ is represented by $h : A \times I \to B$ with $h(\partial_0 + \partial_1) = 0$ and if h' is a dotted arrow in

$$\begin{tikzcd} A \arrow[r, "iu"] \arrow[d, "\partial_0"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "h"] \arrow[ur, dashed, "h'"] & B \end{tikzcd}$$

then $\alpha \cdot \gamma$ is represented by $i^{-1} h' \partial_1 : A \to F$.

The group action followed by the identity map of $F \times \Omega B$(taking $A = F \times \Omega B$) gives a map in $Ho\mathscr{C}$

$$m : F \times \Omega B \to F$$

Proposition 2. The map m is a right action of the group object ΩB on F in $Ho\mathscr{C}$.

Definition 1. A fibration sequence in $Ho\mathscr{C}$ where $\mathscr{C}$ is a pointed model category is a diagram in $Ho\mathscr{C}$ of the form

$$X \to Y \to Z$$

that is isomorphic in $Ho\mathscr{C}$ to a diagram $F \xrightarrow{i} E \xrightarrow{p} B$. Further more the diagram is equipped with a right action in $Ho\mathscr{C}$,

$$X \times \Omega Z \to X$$

that is isomorphic to the action $F \times \Omega B \xrightarrow{m} F$.

By dualizing we can consruct

$$A \xrightarrow{u} X \xrightarrow{v} C$$

with co-action isomorphic to the action

$$C \to C \vee \Sigma A$$

where we have a cofibration u in $\mathscr{C}_C$. This defines the notion of a cofibration sequence in $Ho\mathscr{C}$.

Proposition 3. If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence so is

$$\Omega B \xrightarrow{\partial} F \xrightarrow{i} E \quad \Omega B \times \Omega E \xrightarrow{n} \Omega B$$

where ∂ is the composition $\Omega B \xrightarrow{(0,id)} F \times \Omega B \xrightarrow{m} F$ and where $n_* : [A, \Omega B] \times [A, \Omega E] \to [A, \Omega B]$ is given by $(\lambda, \mu) \to ((\Omega p)_* \mu)^{-1} \circ \lambda$. (Here $\circ$ is the group operation in $[A, \Omega B]$).

Proposition 4. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration sequence in $Ho\mathscr{C}$, let A be any object of $H \circ C$. Then the sequence

$$\begin{aligned} \ldots \to \left[A, \Omega^{q+1}B\right] \xrightarrow{(\Omega^q\partial)_*} [A, \Omega^q F] \xrightarrow{\left(\Omega^q i\right)_*} [A, \Omega^q E] \xrightarrow{(\Omega^q p)_*} \ldots \\ \to [A, \Omega E] \xrightarrow{(\Omega p)_*} [A, \Omega B] \xrightarrow{\partial_*} [A, F] \xrightarrow{i_*} [A, E] \xrightarrow{p_*} [A, B] \end{aligned}$$

is exact in the following sense:

1. $(p_*)^{-1}(0) = \mathrm{Im}\,(i_*)$
2. $i_*\partial_* = 0$ and $i_*\alpha_1 = i_*\alpha_2 \Longleftrightarrow \alpha_2 = \alpha_1 \cdot \lambda$ for some $\lambda \in [A, \Omega B]$
3. $\partial_*(\Omega i)_* = 0$ and $\partial_*\lambda_1 = \partial_*\lambda_2 \Longleftrightarrow \lambda_2 = (\Omega p)_*\mu \cdot \lambda_1$ for some $\mu \in [A, \Omega E]$
4. The sequence of group homomorphisms from $[A, \Omega E]$ to the left is exact in the usual sense.

The dual proposition holds for cofibration seqeunces.

Proposition 5. The class of fibration sequences in $Ho\mathscr{C}$ has the following properties

1. Any map $f : X \to Y$ may be embedded in a fibration sequences $F \to X \xrightarrow{f} Y, F{\times}\Omega Y \to F$.
2. Given a diagram of solid arrows where the rows are fibration sequences, the dotted arrow γ exists

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, dotted, "\gamma"] & E \arrow[r, "p"] \arrow[d, "\beta"] & B \arrow[d, "\alpha"] \\ F' \arrow[r, "i'"] & E' \arrow[r, "p'"] & B' \end{tikzcd} \qquad \begin{tikzcd} F \times \Omega B \arrow[r, "m"] \arrow[d, dotted, "\gamma\times\Omega\alpha"] & F \arrow[d, dotted, "\gamma"] \\ F' \times \Omega B' \arrow[r, "m'"] & F' \end{tikzcd}$$

3. In the above diagram where the rows are fibration sequences, if α and β are isomrophism so is γ.

We will look at Toda bracket next.