Poincare Duality

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1 Orientations

We know orientation on a topological $n-$manifold is defined as the choice of a maximal oriented atlas. Here an atlas $(U_i, \varphi_i : U_i \to V_i \subset \mathbb{R}^n)$ is called oriented if all coordinate changes $\varphi_i\varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$are orientation preserving. An oriented atlas is called maximal if it cannot be enlarged to an oriented atlas by adding another chart.

The relative homology groups $H_i(M, M - \{x\})$ have the following property by excision:

$$\begin{aligned} H_i(M, M - \{x\}; \mathbb{Z}) &\approx H_i\left(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}\right) \quad \text{by excision} \\ &\approx \widetilde{H}_{i-1}\left(\mathbb{R}^n - \{0\}; \mathbb{Z}\right) \\ &\approx \widetilde{H}_{i-1}\left(S^{n-1}; \mathbb{Z}\right) \end{aligned}$$

This motivates us to define orientation of $\mathbb{R}^n$ at a point x as a choice of generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, where the coefficients are in $\mathbb{Z}$ unless otherwise mentioned.This is called the local orientation $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \approx H_{n-1}(\mathbb{R}^n - x) \approx H_{n-1}(S^{n-1})$. rotations of S^{n-1} have degree 1 and is homotopc to identity while reflections have degree -1. so the generator satisfies the property of an orientation.

If y is any other point $\neq x$ and B is a ball conatining x and y, then $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \approx H_n(\mathbb{R}^n, \mathbb{R}^n - B) \approx H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$

An orientation of an n- dimensional manifold M is a function $x \to \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M|x)$ satisifying the local consistency condition mentioned earlier. If an orientation exists for M then M is orientable.

Every manifold M has an orientable 2-sheeted covering space $\tilde{M}$. We start by defining $\tilde{M} = \{\mu_x | x \in M \text{and } \mu_x \text{ is a local orientation of M at x}\}$. The map $\mu_x \to x$ is a 2-1 surjection. Given an open ball $B \in \mathbb{R}^n \in M$ and a generator $\mu_B \in H_n(M|B)$. $U(\mu_B) = \{\mu_x \in$

$\tilde{M}|\mu_x = im(H_n(M|B) \to H_n(M|x))\}$
$\tilde{M}$ is orientable because of $H_n(\tilde{M}|\mu_x) \approx H_n(U(\mu_b)|\mu_x) \approx H_n(B|x)$

Proposition. *If M is connected them M is orientable iff $\tilde{m}$ has two components. In particular, M is orientable if its simply connected or more generally if $\pi_1(M)$ has no subgroup of index two.*

A more general definition of orientation is obtained by replacing the coefficient group $\mathbb{Z}$ by any commutative ring R with identity.Then an R-orientation of M assigns to each $x \in M$ a generator of $H_n(M|x;R) \approx R$. The isomorphism here is an R-module isomorphism.

The canonical isomorphism $H_n(M|x;R) \approx H_n(M|x) \otimes R$ implies that an orientable manifold is R-manifold for all R. On the other hand a non orientable manifold is R- orientable iff R contains a unit of order 2 (since $\mu_x \otimes r = -\mu_x \otimes r$). So every manifild is $\mathbb{Z}_2$ orientable.

The orientability of a closed manifold is reflected in the structure of its homology.

Theorem. *Let M be a closed connected n-manifold. Then:*
(a) If M is R-orientable, the map $H_n(M;R) \to H_n(M \mid x;R) \approx R$ is an isomorphism for all $x \in M$
(b) If M is not R-orientable, the map $H_n(M;R) \to H_n(M \mid x;R) \approx R$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$
(c) $H_i(M;R) = 0$ for $i > n$.
In particular, $H_n(M;\mathbb{Z})$ is $\mathbb{Z}$ or 0 depending on whether M is orientable or not, and in either case $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$

To prove this theorem we prove another lemma:

Lemma. *Let M be a manifold of dimention n and let $A \subset M$ be a compact subset. Then:*
(a) if $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|A;R)$ is α_x for all $x \in A$
(b) $H_i(M|A;R) = 0$ for $i > n$

The covering space map $\tilde{M} \to M$ can be embedded in a larger covering space $M_R \to M$ where M_R consists of all elements $\alpha_x \in H_n(M|x;R)$ as x rangs over M.This is an infinite sheeted covering map. We topologize M_R via the basis of sets $U(\alpha_B)$ consisting of α_x 's with $x \in B$ and α_x the image of an element $\alpha_B \in H_n(M \mid B)$ under the map $H_n(M \mid B) \to H_n(M \mid x)$.
A continuous map $M \to M_R$ of the form $x \mapsto \alpha_x \in H_n(M \mid x)$ is called a section of the covering space. An orientation of M is the same thing as a section $x \mapsto \mu_x$ such that μ_x is a generator of $H_n(M \mid x)$ for each x.
The lemma implies the theorem:

M_R is a disjoint union of the covering spaces M_r, which are isomorphic to the trivial cover M if $r = -r$ and to $\widetilde{M}$ otherwise.

Let $\Gamma_R(M)$ be the set of sections $M_R \to M$. this is an $R-$module.there is a homomorphism $H_n(M;R) \to \Gamma_R(M)$ sending a class α to the section $x \mapsto \alpha_x$ where α_x is the image of α under the map $H_n(M;R) \to H_n(M \mid x;R)$.

For (a) when M is orientable, M_R is a union of copies of M. Hence any section α_x is determined by its value at a single point and is just a homeomorphism from M to the relevant component of M_R. The R-module of sections is thus isomorphic to R.

For (b) when M is not orientable, M_R is a union of copies of M corresponding to the elements of R with $r = -r$ along with copies of $\widetilde{M}$ corresponding to the other elements of R. By the lifting criterion,(each $\tilde{M}$ is a 2 sheeted covering and since it is not R- orientable there are no subgroups of index 2) there are no sections $M \to \widetilde{M}$, so the R-module of sections is isomorphic to $\{r \in R : r = -r\}$.

An element of $H_n(M; R)$ whose image in $H_n(M \mid x; R)$ is a generator for all x is called a fundamental class for M with coefficients in R. By the theorem, a fundamental class exists if M is closed and R-orientable. To show that the converse is also true, let $\mu \in H_n(M; R)$ be a fundamental class and let μ_x denote its image in $H_n(M \mid x; R)$. The function $x \mapsto \mu_x$ is then an R-orientation since the map $H_n(M; R) \to H_n(M \mid x; R)$ factors through $H_n(M \mid B; R)$ for B any open ball in M containing x. Furthermore, M must be compact since μ_x can only be nonzero for x in the image of a cycle representing μ, and this image is compact. So if the manifold M is not compact then mu_x is nonzer for some points of then maifold, which is not right.

In view of these remarks a fundamental class could also be called an orientation class for M.

2 The Duality Theorem

For an arbitrary space X and coefficient ring R, define an R-bilinear cap product $\frown$: $C_k(X; R) \times C^\ell(X; R) \to C_{k-\ell}(X; R)$ for $k \geq \ell$ by setting

$$\sigma \frown \varphi = \varphi(\sigma \mid [v_0, \cdots, v_\ell])\, \sigma \mid [v_\ell, \cdots, v_k]$$

for $\sigma : \Delta^k \to X$ and $\varphi \in C^\ell(X; R)$. This induces a cap product in homology and cohomology:

$$\frown: H_k(X; R) \times H^\ell(X; R) \to H_{k-\ell}(X; R)$$

We also have

$$\begin{tikzcd} H_k(X) \times H^l(X) \arrow[r, "\frown"] \arrow[d, "f_*"] & H_{k-l}(X) \arrow[d, "f_*"] \\ H_k(Y) \times H^l(Y) \arrow[u, "f^*"'] \arrow[r, "\frown"'] & H_{k-l}(Y) \end{tikzcd} \qquad f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$$

2.1 Cohomology with compact supports

$C^i_c(X; G) = \{\phi : C_i(X) \to G \mid \exists \text{ compact set } K = K_\phi \subset X \text{ such that } \phi \equiv 0 \text{ on } X - K\} \subset C^i(X; G)$

If ϕ is zero on chains in $X - K$, then $\partial^*\phi \in C^{I+1}(X; G)$ is also zero on chains in $X - K$. So the $C^i_c(X; G)$ form a subcomplex of the sisngular cochain complex of C. the cohomology groups $H^i_c(X; G)$ of this subcomplex are the chomology groups with compact supports.

The stronger condition that cochains be nonzero only on singular simplices contained in some compact set, in general, does not form a subcomplex of the cochain complex. ∂^* of a cochain need not even have compact support.

There is an alternative and an easier to work with definition in terms of direct limits.

$C^i_c(X) = \bigcup\limits_{K \subset X, cpt} C^i(X, X-K)$ Here $C^i(X, X-K) \subseteq C^i_c(X)$ with basis the cochains whose value is zero on $X - K$

Suppose now $X = \bigcup X_\alpha$ with X_α forming a directed system with respect to the inclusion relation. Then the groups $H_i(X_\alpha)$ for fixed i and G form a directed system, using the homomorphism induced by inclusions. The natural maps $H_i(X_\alpha) \to H_i(X)$ induce homomorphisms $\varinjlim H_i(X_\alpha) \to H_i(X)$

$X = \bigcup\limits_{\alpha \in I} X_\alpha$ with the property that aeach compact set in X is contained in some X_α the the map $\varinjlim H_i(X_\alpha) \to H_i(X)$ is an isomorphism for all i.

With this we can give an alternative definition of chomoology with compact supports in terms of direct limits.

The compact subsets $K \subset X$ form a directed st under inclusion. To each compact $K \subset X$ we associate the group $H^i(X|K)$ and inclusion induces natural homomorphism between the corresponding groups. Then

Theorem. $\varinjlim_K H_i(X|K) = H^i_c(X)$

Proof. The map $H^i_c(X) \to \varinjlim_K H_i(X|K)$ that takes some g to 0 for some representative g which is a cocycle in $C^i(X|K)$ for some compact K. This is zero under the map iff it is the coboudnary of a cochain in $C^{i-1}(X|L)$ for some compact $K \subset L$. So $g = 0$. $\square$

Cohomology with compact supports is ant an invariant of homotopy type. For example to compute $H^i_c(\mathbb{R}^n)$ with the direct limit definition, we can let compact sets range over valls B_k of radius k centered at origin. only the $i = n$ cohomology group is non zero and is equal to G. so we have $H^i_c(\mathbb{R}^n) = 0$ for $i \neq n$

Proper maps are those maps between topological spaces which induce maps on H^*_c , for which the inverse image of each compact set is compact.

2.2 Duality for Noncompact manifolds

For compact manifolds, we have existence of fundamental clans with which we defined cap product and the duality map.

$$D_M : H^k_c(M) \longmapsto H_{n-k}(M)$$

For compact sets $K \subset L \subset M$

$$\begin{tikzcd}
H_n(M|L) \arrow[dd, "i_*"] & \times & H^k(M|L) \arrow[dr, "\frown"] & \\
 & & & H_{n-k}(M) \\
H_n(M|K) & \times & H^k(M|K) \arrow[uu, "i^*"] \arrow[ur, "\frown"'] &
\end{tikzcd}$$

By an earlier theorem that guarentees the existence of fundamental class there exists unique μ_K, μ_L that restricts to orientation at a given point. Uniqueness implies $i_*(\mu_L) = \mu_K$ and by naturality $\mu_L \frown i^*(x) = i_*(\mu_L) \frown x$ where $x \in H^k(M|K)$ letting K vary over compact sets in M, the homomorphisms

$$H^k(M \mid K; R) \to H_{n-k}(M; R)$$

$$x \mapsto \mu_K \frown x$$

induce in the limit a duality homomorphism

$$D_M : H^k_c(M; R) \to H_{n-k}(M; R)$$

We state the Poincare duality theorem:

Theorem. *The duality map*

$$D_M : H^k_c(M; R) \to H_{n-k}(M; R)$$

is a n isomorphism for all k whenever M is an $R-$ oriented $n-$ manifold.

We first prove a lemma that would help us prove the theorem in-turn:

Lemma. *If M is the union of two open sets U and V, then there is a diagram of Mayer-Vietoris sequences, commutative up to sign:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^k_c(U)\oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^k_c(M) \arrow[r] \arrow[d, "D_M"] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

We can prove poincare duality with this. The only place where I felt more information was needed was in step 1:

$$H^k_c(\mathbb{R}^n) = \varinjlim H^k(\mathbb{R}^n|K)$$

From the long exact cohomology sequence for the pair $(\mathbb{R}^n|K)$, we get isomorphisms induced by the boundary map δ between $H_n(\mathbb{R}^n - |K)$ and $H^{n-1}(\mathbb{R}^n - K) \approx H^{n-1}(S^{n-1})$. The quotient map h betweeh cohomology groups and $\hom(*, G)$ is also an isomorphism .

$$\begin{tikzcd}
H^{n-1}(\mathbb{R}^n - K) \arrow[r, "\delta"] \arrow[d, "h"] & H^n(\mathbb{R}^n - |K) \arrow[d, "h"] \\
\hom(H_{n-1}(\mathbb{R}^n - K), G) \arrow[r, "\partial^*"'] & \hom(H_n(\mathbb{R}^n|K), G)
\end{tikzcd}$$

We know $H_n(S^n)$ is generated by simplices $\triangle^n_1 - \triangle^n_2$ and is cyclic. So under the inverse of the boundary map we see that its preimage is the identity map $i_n : \triangle^n \to \triangle^n$ as a singular simplex. The corresponding group hom group is generated by the function that takes value 1 on the generator. So from this commutative diagram we can see that $H^n(\mathbb{R}^n|K)$ is generated by cocycle that takes 1 on $\triangle^n$.

Cup and cap product are realted by the formula

$$\psi(\alpha \frown \phi) = (\phi \smile \psi)(\alpha)$$

for $\alpha \in C_{k+l}, \phi \in C^k, \psi \in C^l$. It is easy to see this commutative diagram arises from the above expression:

$$\begin{tikzcd} H^l(X) \arrow[r, "h"] \arrow[d, "\phi\smile"'] & \hom(H_l(X), R) \arrow[d, "(\frown\phi)^*"] \\ H^{k+l}(X) \arrow[r, "h"'] & \hom(H_{k+l}(X, R)) \end{tikzcd}$$

We can construct a bilinear map :

$$H^k(M) \times H^{n-k}(M) \to R \ \ (\phi, \psi) \mapsto (\phi \smile \psi)[M]$$

Now suppose the coefficient group R is a field or $=\mathbb{Z}$ and torsion in $H^*(M)$ is factored out, the cup product pairing we defined above is a non singular map.

2.3 Other forms of duality

For a manifold M with boundary, the points x at boundary have $H_n(M|x) \approx H_n(\mathbb{R}^n_+|0) = 0$. These points form an $n-1$ dimensional manifold with empty boundary.

If M is a manifold with boudnary, then a collar neighbourhood of ∂M in M is an open neighborhood homeomorphic to $\partial M \times [0, 1)$ by homeomorphism taking ∂M to $\partial M \times \{0\}$

Every compact manifold with boundary , has a collar neighborhood for ∂M.

A compact manifold M with boundary is defined to be R- orientable if $M - \partial M$ is R-orientable as a manifold without boundary.

If $\partial M \times [0, 1)$ is a collar neighbourhood, then $H_i(M, \partial M)$ is naturally isomorphic to $H_i(M - \partial M, \partial M \times (0, \epsilon))$ We have the inclusion map $i : M - \partial M \to M$. Consider the map $g : M \to M - \partial M$. We know $M \cong M \sqcup \partial M \times [0, 1)$. We rename M to be the manifold $M - \partial M$. So $M - \partial M \cong M \cup \partial M \times (0, 1)$. Now outside this collar we can define g to be identity and on the collar we define g to be the homotopy map taking $\partial M \times [0, 1) \mapsto \partial M \times (0, 1)$

Thus there exists a relative fundamental class $[M]$ in $H_n(M, \partial M)$ restricting to a given orientation at each point of $M - \partial M$. This follows from the fact that $H_i(M - \partial M, M - \partial M - A) \approx H_i(M - \partial M, \partial M \times (0, \epsilon))$ The existence of fundamental class follow from a previous lemma.

With this we can state Poincare duality for manifolds wih boundary.

Theorem. *Suppose M is a compact R-orientable n-manifold whose boundary ∂M is decomposed as the union of two compact $(n-1)$ - dimensional manifolds A and B with a common boundary $\partial A = \partial B = A \cap B$. Then cap product with a fundamental class $[M] \in H_n(M, \partial M; R)$ gives isomorphisms $D_M : H^k(M, A; R) \to H_{n-k}(M, B; R)$ for all k.*