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1 Introduction

We say \mathcal{C} is a model category if \mathcal{C} is a cat with three classes of maps

- fibrations
- cofibrations
- weak equivalences

satisfying following axioms

M0 There exists finite limits and colimits.

M1 Given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where

i is a cofibration and a weak equivalence(trivial cofibration) and p is a fibration or

i is a cofibration and p is a fibration(trivial fibration) and weak equivalence,

then \exists a lift $B \rightarrow X$.

M2 Any map f may be factored as

$f = pi$ where i =trivial cofibration and p =fibration and

$f = pi$ where i =cofibration and p =trivial fibration.

M3 Fibrations are stable under composition, base change and any isomorphism is a fibration.

Cofibrations are stable under composition, cobase change and an isomorphism is a cofibration.

M4 The base extension of a map which is a trivial fibration is a weak equivalence.

The cobase extension of a map which is trivial fibration is a weak equivalence.

M5 if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is in \mathcal{C} . Then if two of f, g, gf are weak equivalences, so is the third. Any isomorphism is a weak equivalence.

An initial object in a category \mathcal{C} is an object ϕ such that for all objects C in \mathcal{C} there is a unique morphism $\phi \rightarrow C$. The dual notion of this is the terminal object $*$. These objects exist in \mathcal{C} because of M0 and they are unique.

X is **cofibrant** if $\phi \rightarrow X$ is a cofibration. X is **fibrant** if $X \rightarrow e$ is a fibration.

Let $f, g : A \rightarrow B$ be maps. We say that f is **left-homotopic** to g if there is a diagram of the form where σ is a weak equivalence.

$$\begin{array}{ccc} A \vee A & \xrightarrow{f+g} & B \\ \downarrow \nabla & \searrow \partial_0 + \partial_1 & \uparrow h \\ A & \xleftarrow{\sigma} & \tilde{A} \end{array} \quad (1)$$

Dually we say that f is **right homotopic** to g if there is a diagram of the form where s is a weak equivalence.

$$\begin{array}{ccc} \tilde{B} & \xleftarrow{s} & B \\ \uparrow k_{(d_0, d_1)} & \searrow & \uparrow \Delta \\ A & \xrightarrow{(f, g)} & B \times B \end{array} \quad (2)$$

By **cylinder object** for an object A we mean an object $A \times I$ together with maps

$$A \vee A \xrightarrow{\partial_0 + \partial_1} A \times I \xrightarrow{\sigma} A$$

with $\sigma(\partial_0 + \partial_1) = \nabla_A$ such that $\partial_0 + \partial_1$ is a cofibration and σ is a weak equivalence. Dually, a **path object** for B shall be an object B^I together with a factorization

$$B \xrightarrow{s} B^I \xrightarrow{(d_0, d_1)} B \times B$$

of Δ_B where s is a weak equivalence and (d_0, d_1) is a fibration.

By a **left homotopy** from f to g , we mean a diagram 1 where $\partial_0 + \partial_1$ is a cofibration and hence \tilde{A} is a cylinder object for A . This is also saying that there exists a cylinder object such that the map $A \vee B \xrightarrow{f+g} B$ extends to a map $h : A \times I \rightarrow B$ with obvious commutative relations

Similarly a **right homotopy** from f to g is a diagram 2 where \tilde{B} is a path object for B . Equivalently the map $A \xrightarrow{(f, g)} B \times B$ extends to a map $B^I \rightarrow B \times B$ with relevant commutative relations.

Lemma 1. If $f, g \in \text{hom}(A, B)$ and $f \stackrel{l}{\sim} g$, then there is a left homotopy $h : A \times I \rightarrow B$ from f to g .

Lemma 2. Let A be a cofibrant object and let $A \times I$ be a cylinder object for A . Then $\partial_0 : A \rightarrow A \times I$ and $\partial_1 : A \rightarrow A \times I$ are trivial cofibrations.

Lemma 3. Let A be cofibrant and let $A \times I$ and $A \times I'$ be two cylinder objects for A . Then the result of gluing $A \times I$ and $A \times I'$ by identification $\partial_1 A = \partial'_0 A$ defined precisely to be the object \tilde{A} is also a cylinder object.

Lemma 4. If A is cofibrant, then $\stackrel{l}{\sim}$ is an equivalence relation on $\text{hom}(A, B)$.

Lemma 5. Let A be cofibrant and let $f, g \in \text{hom}(A, B)$. Then

1. $f \stackrel{l}{\sim} g \implies f \stackrel{r}{\sim} g$
(dual) If B is fibrant then $f \stackrel{r}{\sim} g \implies f \stackrel{l}{\sim} g$
2. $f \stackrel{r}{\sim} g \implies$ there exists a right homotopy $k : A \rightarrow B^I$ from f to g with $s : B \rightarrow B^I$ a trivial cofibration.
3. If $u : B \rightarrow C$, then $f \stackrel{r}{\sim} g \implies uf \stackrel{r}{\sim} ug$

Let A and B be objects of \mathcal{C} let $\pi^r(A, B)$ (similar for $\pi^l(A, B)$) be the set of equivalence classes of $\text{hom}(A, B)$ with respect to the equivalence relation generated by $\stackrel{r}{\sim}$. When A cofibrant and B is fibrant, in which case left and right homotopies coincide and are already equivalence relations, we shall denote the relation by \sim , call it homotopy and $\pi_0(A, B)$.

Lemma 6. If A is cofibrant, then composition in \mathcal{C} induces a map $\pi^r(A, B) \times \pi^r(B, C) \rightarrow \pi^r(A, C)$.

Lemma 7. Let A be cofibrant and let $p : X \rightarrow Y$ be a trivial fibration. Then p induces a bijection $p_* : \pi^l(A, X) \rightarrow \pi^l(A, Y)$.

(dual) Let B be fibrant and $i : X \rightarrow Y$ be a trivial cofibration, then i induces a bijection $i_* : \pi^r(Y, B) \simeq \pi^r(X, B)$

Let $\mathcal{C}_c, \mathcal{C}_f, \mathcal{C}_{cf}$ be full subcategories¹ consisting of the cofibrant, fibrant and both cofibrant and fibrant objects of \mathcal{C} respectively. Define

$$\pi\mathcal{C}_c \text{ with objects } = \text{Obj}(\mathcal{C}_c) \text{ and morphisms } = \pi^r(A, B)$$

If we denote the right homotopy class of a map $f : A \rightarrow B$ by \bar{f} we obtain a functor $\mathcal{C} \rightarrow \pi\mathcal{C}_c$ given by $X \rightarrow X, f \rightarrow \bar{f}$. Similarly we define $\pi\mathcal{C}_f$ and $\pi\mathcal{C}_{cf}$.

Let \mathcal{C} be an arbitrary category and let S be a subclass of the class of maps of \mathcal{C} . By localization of \mathcal{C} with respect to S we mean a category $S^{-1}\mathcal{C}$ together with a functor $\gamma : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ having the following universal property: For every $s \in S$, $\gamma(s)$ is an isomorphism; given any functor $F : \mathcal{C} \rightarrow \mathcal{B}$ with $F(s)$ an isomorphism for all $s \in S$ there is a unique functor $\theta : S^{-1}\mathcal{C} \rightarrow \mathcal{B}$ such that $\theta \circ \gamma = F$.

Let \mathcal{C} be a model category. Then the **homotopy category** of \mathcal{C} is the localization of \mathcal{C} with respect to the class of weak equivalences and is denoted by $\gamma : \mathcal{C} \rightarrow Ho\mathcal{C}$. $\gamma : \mathcal{C}_c \rightarrow Ho\mathcal{C}_c$ and $\gamma : \mathcal{C}_f \rightarrow Ho\mathcal{C}_f$ will denote the localization of \mathcal{C}_c and \mathcal{C}_f with respect to the class of maps in the respective categories which are weak equivalences in \mathcal{C} . $[X, Y] := \text{hom}_{Ho\mathcal{C}}(X, Y)$.

- Lemma 8.**
1. Let $F : \mathcal{C} \rightarrow \mathcal{B}$ carry weak equivalences in \mathcal{C} into isomorphisms in \mathcal{B} . If $f \stackrel{l}{\sim} g$ or $f \stackrel{r}{\sim} g$, then $F(f) = F(g)$ in \mathcal{B} .
 2. Let $F : \mathcal{C}_c \rightarrow \mathcal{B}$ carry weak equivalences in \mathcal{C}_c into isomorphisms in \mathcal{B} . If $f \stackrel{r}{\sim} g$, then $F(f) = F(g)$ in \mathcal{B} .

¹some objects but all morphisms

The above lemma implies the functors $\gamma_c, \gamma_f, \gamma$ induce functors $\bar{\gamma}_c : \pi\mathcal{C}_c \rightarrow Ho\mathcal{C}_c, \bar{\gamma}_f : \pi\mathcal{C}_f \rightarrow Ho\mathcal{C}_f, \bar{\gamma} : \pi\mathcal{C}_{cf} \rightarrow Ho\mathcal{C}$.

The homotopy category is the category

$$Ho\mathcal{C} \text{ with objects } = Obj(\mathcal{C}) \text{ and } \text{hom}_{Ho\mathcal{C}}(X, Y) = \text{hom}_{\pi\mathcal{C}_{cf}}(RQX, RQY) = \pi(RQX, RQY)$$

For each object X choose a trivial fibration $p_X : Q(X) \rightarrow X$ with $Q(X)$ cofibrant and a trivial cofibration $i_X : X \rightarrow R(X)$ with $R(X)$ fibrant. For each map $f : X \rightarrow Y$, we may choose a map $\underline{Q}(f) : \underline{Q}(X) \rightarrow \underline{Q}(Y)$ and $\underline{R}(f) : \underline{R}(X) \rightarrow \underline{R}(Y)$. By mapping $X \rightarrow Q(X)$ or $R(X)$ and $f \rightarrow \underline{Q}(f)$ or $\underline{R}(f)$ we get functors $\bar{Q} : \mathcal{C} \rightarrow \pi\mathcal{C}_c$ and $\bar{R} : \mathcal{C} \rightarrow \pi\mathcal{C}_f$. Some more math and we get a well-defined functor

$$\begin{aligned} \bar{RQ} : \mathcal{C} &\rightarrow \pi\mathcal{C}_{cf} \\ X &\rightarrow RQX \\ f &\rightarrow \bar{RQ}(f) \end{aligned}$$

Theorem 1. $Ho\mathcal{C}, Ho\mathcal{C}_c, Ho\mathcal{C}_f$ exist and there is a diagram of functors

$$\begin{array}{ccc} \pi\mathcal{C}_c & \xrightarrow{\bar{\gamma}_c} & Ho\mathcal{C}_c \\ \uparrow & & \sim \downarrow \\ \pi\mathcal{C}_{cf} & \xrightarrow[\sim]{\bar{\gamma}} & Ho\mathcal{C} \\ \downarrow & & \sim \uparrow \\ \pi\mathcal{C}_f & \xrightarrow{\bar{\gamma}_f} & Ho\mathcal{C}_f \end{array}$$

where \hookrightarrow denotes a full embedding and $\tilde{\rightarrow}$ denotes an equivalence of categories. Furthermore if $(\bar{\gamma})^{-1}$ is a quasi-inverse² for $\bar{\gamma}$, then the fully faithful functor

$$Ho\mathcal{C}_c \xrightarrow[\sim]{\tilde{\rightarrow}} Ho\mathcal{C} \xrightarrow[\sim]{(\bar{\gamma})^{-1}} \pi\mathcal{C}_{cf} \hookrightarrow \pi\mathcal{C}_c$$

is right adjoint to $\bar{\gamma}_c$ and the fully faithful functor

$$Ho\mathcal{C}_f \xrightarrow[\sim]{\tilde{\rightarrow}} Ho\mathcal{C} \xrightarrow[\sim]{(\bar{\gamma})^{-1}} \pi\mathcal{C}_{cf} \hookrightarrow \pi\mathcal{C}_f \hookrightarrow \pi\mathcal{C}_{cf}$$

is left adjoint to $\bar{\gamma}_f$.

Corollary 1. If A is cofibrant and B is fibrant, then

$$\text{hom}_{Ho\mathcal{C}}(A, B) = \pi(A, B)$$

The category \mathcal{C} can have different model structures on it, but same $Ho\mathcal{C}$, i.e. the weak equivalences are same but fibrations and cofibrations can be different.

²Definition

2 Loop and suspension functors

Let \mathcal{C} be a fixed model category and $f, g : A \rightarrow B$ be two maps with A cofibrant and B fibrant.

Define left homotopy between left homotopies and right homotopy between right homotopies in the analogous way.

Let $h : A \times I \rightarrow B$ be a left homotopy from f to g and $k : A \rightarrow B^I$ be a right homotopy from f to g . By a **correspondence** between h and k we mean a map $H : A \times I \rightarrow B^I$ such that $H\partial_0 = k, H\partial_1 = sg, d_0H = h, d_1H = g\sigma$. (here $\sigma : A \times I \rightarrow A$ and $s : B \rightarrow B^I$ are weak equivalences.) We use the following diagrams to indicate the situation:

$$\begin{array}{ccc}
 & g & \\
 & \downarrow k & \\
 f & \xrightarrow{h} & g \\
 & f & \\
 & \downarrow k & \\
 & f & \\
 & \xrightarrow{h} & g \\
 & \downarrow sg & \\
 & g &
 \end{array}
 \quad
 \begin{array}{ccc}
 & g & \xrightarrow{g\sigma} & g \\
 & \downarrow k & & \downarrow sg \\
 & f & \xrightarrow{h} & g
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{sg} & B^I \\
 \partial_1 \downarrow & \nearrow H & \downarrow (d_0, d_1) \\
 A \times I & \xrightarrow{(h, g\sigma)} & B \times B
 \end{array}$$

Lemma 1. Given $A \times I$ and a right homotopy $k; A \rightarrow B^I$ there is a left homotopy $h : A \times I \rightarrow B$ corresponding to k . Dually given B^I and h , there is a k corresponding to h .

Lemma 2. Suppose that $h : A \times I \rightarrow B$ and $h' : A \times I' \rightarrow B$ are two left homotopies from f to g and that $k : A \rightarrow B^I$ is a right homotopy from f to g . Suppose that h and k correspond, then h' and k correspond iff h' is left homotopic to h .

These two lemmas make left homotopy between left homotopies an equivalence relation, denoted by $\pi_1^l(A, B; f, g)$. Correspondence yields a bijection $\pi_1^l(A, B; f, g) \simeq \pi_1^r(A, B; f, g)$. So denoting this as $\pi_1(A, B; f, g)$, an element of this is a homotopy class of homotopies from f to g .

Theorem 1. Composition of left homotopies induces maps $\pi_1^l(A, B; f_1, f_2) \times \pi_1^l(A, B; f_2, f_3) \rightarrow \pi_1^l(A, B; f_1, f_3)$ and similarly for right homotopies. Composition of left and right homotopies is compatible with the correspondence bijection of the corollary to above lemma. Finally the category with objects $\text{hom}(A, B)$, with a morphism from f to g defined to be an element of $\pi_1(A, B; f, g)$, and with composition of morphisms defined to be induced by composition of homotopies, is a groupoid. The inverse of an element of $\pi_1^l(A, B; f, g)$ represented by h being represented by h^{-1} .

Lemma 3. The diagram commutes

$$\begin{array}{ccc} \pi_1(A, B; f, g) & \xrightarrow{i^*} & \pi_1(A', B; fi, gi) \\ \downarrow j_* & & \downarrow j_* \\ \pi_1(A, B'; jf, jg) & \xrightarrow{i^*} & \pi_1(A', B'; jfi, jgi) \end{array}$$

Definition 1. A **pointed category** is a category \mathcal{C} , in which the initial object and final object exist and are isomorphic, denoted by \star and call it the null object of \mathcal{C} . If X and Y are arbitrary objects of \mathcal{C} , we denote by $0 \in \text{hom}(X, Y)$ the composition $X \rightarrow \star \rightarrow Y$. If $f : X \rightarrow Y$ is a map in \mathcal{C} , we define the **fibre** of f to be the fibre product $\star \times_Y X$ and the **cofibre** of f to be the cofiber product of $\star \vee_Y X$.

By a **pointed model category** we mean a model category \mathcal{C} , which is also a pointed category. If A is in \mathcal{C}_c and $B \in \mathcal{C}_f$, then we write $\pi_1(A, B; 0, 0)$ as $\pi_1(A, B)$ which is a group.

Theorem 2. Let \mathcal{C} be a pointed model category. Then there is a functor $A, B \rightarrow [A, B]_1$ from $(\text{HoC})^{op} \times \text{HoC} \rightarrow \{\text{groups}\}$ which is determined up to canonical isomorphism by $[A, B]_1 = \pi_1(A, B)$ if A is cofibrant and B is fibrant. Furthermore, there are two functors from HoC to HoC , the suspension functor Σ and the loop functor Ω and canonical isomorphisms

$$[\Sigma A, B] \simeq [A, B]_1 \simeq [A, \Omega B]$$

of functors $(\text{HoC})^{op} \times (\text{HoC}) \rightarrow (\text{sets})$ where $[X, Y] = \text{Hom}(X, Y)$.

We also use ΣE to denote the cofiber of map $\partial_0 + \partial_1 : A \vee A \rightarrow A \times I$. $\underline{\Sigma}$ is used when needed to denote the functor. Σ and Ω are adjoint functors on HoC and are unique up to canonical isomorphism. Also for any X , $\Sigma^n X$ is a cogroup object for $n \geq 1$ and $\Omega^n X$ is a group object in HoC , which is commutative for $n \geq 2$.

3 Fibration and cofibration sequences

We have a cartesian map,

$$\begin{array}{ccc} F \times_E E^I \times_E F & \xhookrightarrow{pr_2} & E^I \\ \downarrow \pi & & \downarrow (d_0, p^I) \text{ trivial fibration} \\ F \times \Omega B & \xrightarrow{i \times j} & E \times_B B^I \end{array}$$

where $\pi = (pr_1, j^{-1} p^I pr_2)$ where $j : \Omega B \rightarrow B^I$ is the fiber. In HoC , we have a map

$$m : F \times \Omega B \rightarrow F$$

given by $F \times \Omega B \xrightarrow{\pi^{-1}} F \times_E E^I \times_E F \xrightarrow{pr_3} F$.

Proposition 1. Let A be cofibrant and let the map

$$m_* : [A, F] \times [A, \Omega B] \rightarrow [A, F]$$

$$\alpha, \gamma \mapsto \alpha \cdot \gamma$$

If $\alpha \in [A, F]$ is represented by $u : A \rightarrow F$, if $\gamma \in [A, \Omega B] = [A, B]_1$ is represented by $h : A \times I \rightarrow B$ with $h(\partial_0 + \partial_1) = 0$ and if h' is a dotted arrow in

$$\begin{array}{ccc} A & \xrightarrow{iu} & E \\ \downarrow \partial_0 & \nearrow h' & \downarrow p \\ A \times I & \xrightarrow{h} & B \end{array}$$

then $\alpha \cdot \gamma$ is represented by $i^{-1}h'\partial_1 : A \rightarrow F$.

The group action followed by the identity map of $F \times \Omega B$ (taking $A = F \times \Omega B$) gives a map in $Ho\mathcal{C}$

$$m : F \times \Omega B \rightarrow F$$

Proposition 2. The map m is a right action of the group object ΩB on F in $Ho\mathcal{C}$.

Definition 1. A **fibration sequence** in $Ho\mathcal{C}$ where \mathcal{C} is a pointed model category is a diagram in $Ho\mathcal{C}$ of the form

$$X \rightarrow Y \rightarrow Z$$

that is isomorphic in $Ho\mathcal{C}$ to a diagram $F \xrightarrow{i} E \xrightarrow{p} B$. Further more the diagram is equipped with a right action in $Ho\mathcal{C}$,

$$X \times \Omega Z \rightarrow X$$

that is isomorphic to the action $F \times \Omega B \xrightarrow{m} F$.

By dualizing we can construct

$$A \xrightarrow{u} X \xrightarrow{v} C$$

with co-action isomorphic to the action

$$C \rightarrow C \vee \Sigma A$$

where we have a cofibration u in \mathcal{C} . This defines the notion of a **cofibration sequence** in $Ho\mathcal{C}$.

Proposition 3. If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence so is

$$\Omega B \xrightarrow{\partial} F \xrightarrow{i} E \quad \Omega B \times \Omega E \xrightarrow{n} \Omega B$$

where ∂ is the composition $\Omega B \xrightarrow{(0, id)} F \times \Omega B \xrightarrow{m} F$ and where $n_* : [A, \Omega B] \times [A, \Omega E] \rightarrow [A, \Omega B]$ is given by $(\lambda, \mu) \rightarrow ((\Omega p)_* \mu)^{-1} \circ \lambda$. (Here \circ is the group operation in $[A, \Omega B]$).

Proposition 4. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration sequence in $Ho\mathcal{C}$, let A be any object of $H \circ C$. Then the sequence

$$\begin{aligned} \dots \rightarrow [A, \Omega^{q+1}B] &\xrightarrow{(\Omega^q \partial)_*} [A, \Omega^q F] \xrightarrow{(\Omega^{q-1} i)_*} [A, \Omega^q E] \xrightarrow{(\Omega^{q-1} p)_*} \dots \\ &\rightarrow [A, \Omega E] \xrightarrow{(\Omega p)_*} [A, \Omega B] \xrightarrow{\partial_*} [A, F] \xrightarrow{i_*} [A, E] \xrightarrow{p_*} [A, B] \end{aligned}$$

is exact in the following sense:

1. $(p_*)^{-1}(0) = \text{Im}(i_*)$
2. $i_* \partial_* = 0$ and $i_* \alpha_1 = i_* \alpha_2 \iff \alpha_2 = \alpha_1 \cdot \lambda$ for some $\lambda \in [A, \Omega B]$
3. $\partial_*(\Omega i)_* = 0$ and $\partial_* \lambda_1 = \partial_* \lambda_2 \iff \lambda_2 = (\Omega p)_* \mu \cdot \lambda_1$ for some $\mu \in [A, \Omega E]$
4. The sequence of group homomorphisms from $[A, \Omega E]$ to the left is exact in the usual sense.

The dual proposition holds for cofibration sequences.

Proposition 5. The class of fibration sequences in $Ho\mathcal{C}$ has the following properties

1. Any map $f : X \rightarrow Y$ may be embedded in a fibration sequences $F \rightarrow X \xrightarrow{f} Y, F \times \Omega Y \rightarrow F$.
2. Given a diagram of solid arrows where the rows are fibration sequences, the dotted arrow γ exists

$$\begin{array}{ccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha \\ F' & \xrightarrow{i'} & E' & \xrightarrow{p'} & B' \end{array} \qquad \begin{array}{ccc} F \times \Omega B & \xrightarrow{m} & F \\ \downarrow \gamma \times \Omega \alpha & & \downarrow \gamma \\ F' \times \Omega B' & \xrightarrow{m'} & F' \end{array}$$

3. In the above diagram where the rows are fibration sequences, if α and β are isomorphism so is γ .

Proposition 6. Let

$$\begin{array}{ccccccc} A & \xrightarrow{u} & X & \xrightarrow{v} & C & \xrightarrow{\partial'} & \Sigma A \\ \downarrow \alpha & & \downarrow \beta & \searrow f & \downarrow \gamma & & \downarrow \delta \\ \Omega B & \xrightarrow{\partial} & F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{n} & C \vee \Sigma A \\ F \times \Omega B & \xrightarrow{m} & F \end{array}$$

be a solid arrow diagram in $Ho\mathcal{C}$, where the first row except for ∂' is a cofibration sequence, and where the second row except for ∂ is a fibration sequence. We suppose that $\partial' = (id_C + 0) \cdot n$ and $\partial = m \cdot (0, id_{\Omega B})$. We suppose that $fu = pf = 0$. The dotted arrows $\alpha, \beta, \gamma, \delta$ exist and the set of possibilities of α form a double coset:

$$\Omega p_*[A, \Omega E] \cdots u^*[X, \omega B]$$

and the set of possibilities for ∂ also forms a double coset:

$$\Sigma u^*[\Sigma X, B] \cdots p_*[\Sigma A, E]$$

Furthermore under the identification $[A, \Omega B] = [\Sigma A, B]$ the first double coset is the inverse of the second.

Definition 2. Let $A \xrightarrow{u} X \xrightarrow{f} E \xrightarrow{p} B$ be a sequence in $Ho\mathcal{C}$ such that $fu = pf = 0$. Form a diagram

$$\begin{array}{ccccc} A & \xrightarrow{u} & X & \xrightarrow{v} & C & \xrightarrow{\partial'} & \Sigma A & & C & \xrightarrow{n} & C \vee \Sigma A \\ & & & \searrow f & \downarrow \gamma & & \downarrow \delta & & & & \\ & & & & E & \xrightarrow{p} & B & & & & \end{array}$$

by choosing a cofibration sequence containing u . Then the set of possibilities for ∂ is a double coset in $[\Sigma A, B]$ which is called the **Toda Bracket** of u, f, p and is denoted $\langle u, f, p \rangle$.

The Toda bracket is independent of the choice of the top row. It can also be computed by choosing the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & X & & & & \\ \downarrow \alpha & & \downarrow \beta & \searrow f & & & \\ \Omega B & \xrightarrow{\partial} & F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array} \quad F \times \Omega B \xrightarrow{m} F$$