Stable Homotopy Theory and Adams Spectral Sequences

Narendran E

This document contains notes on stable homotopy category and Adams specseq. We first introduce stable homotopy category and study it's structure. Later we discuss construction and convergence of Adams specseq.
This is part of my project under Prof Ramesh Kasilingam (IIT-M).

Underlying Theorems

We begin with noting down some theorems from algebraic topology and later about Brown Representability theorem. For the following theorems we refer to [**H**] for proofs.

Theorem 1.0.1. *Let X be a CW complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cap B$. If (A,C) is m-connected and (B,C) is n-connected, $m,n \geq 0$, then the map $\pi_i(A,C) \to \pi_i(X,B)$ induced by inclusion is an isomorphism for $i < m+n$ and a surjection for $i = m+n$.*

This yields the Freudenthal Suspension theorem

Theorem 1.0.2. *The suspension map $\pi_i(S^n) \to \pi_{i+1}\left(S^{n+1}\right)$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$. More generally this holds for the suspension $\pi_i(X) \to \pi_{i+1}(SX)$ whenever X is an $(n-1)$-connected CW complex.*

Let X and Y be CW complexes with basepoints. The suspension ΣX , or equivalently reduced suspension, be either $S^1 \wedge X$ or $X \wedge S^1$. Suspension induces a function

$$S : [X,Y] \to [\Sigma X, \Sigma Y]$$

Theorem 1.0.3. *Suppose that Y is $(n-1)$-connected. Then S is onto if $\dim X \leq 2n-1$ and is a $1-1$ correspondence if $\dim X < 2n-1$.*

Under these circumstances we call an element of $[X,Y]$ a stable homotopy class of maps.

We define the notion of a cohomology operation. It is a natural transformation

$$\phi : H^n(X,Y;\pi) \to H^m(X,Y;G)$$

where n runs over $\mathbb{Z}$. The map is subject to the axiom: if $f : X,Y \to X',Y'$ and $h \in H^n(X',Y';\pi)$ then $\phi(f^*h) = f^*(\phi h)$ (See [**H**]).

A stable cohomology operation is said to be a collection of cohomology operations, say

$$\phi_n : H^n(X,Y;\pi) \to H^{n+d}(X,Y;G)$$

Here n runs over $\mathbb{Z}$. Each ϕ_n is required to be natural, as above and the following diagram be commutative for each n.

$$\begin{tikzcd}
H^n(Y,Z;\pi) \arrow[r, "\delta"] \arrow[d, "\phi_n"'] & H^{n+1}(Y,Z;\pi) \arrow[d, "\phi_{n+1}"] \\
H^{n+d}(Y,Z;G) \arrow[r, "\delta"] & H^{n+d+1}(Y,Z;G)
\end{tikzcd}$$

1.1 Brown Representability Theorem

Let $\mathscr{C}$ be a locally small category, i.e., a category such that for any object C and C' in $\mathscr{C}$, the class of morphisms $\mathscr{C}(C,C')$ is a set. Let C_0 be a fixed object of $\mathscr{C}$. We define the contravariant functor:

$$\begin{aligned} \mathscr{C}(-,C_0) : \mathscr{C} &\longrightarrow \text{Set} \\ C &\longmapsto \mathscr{C}(C,C_0) \\ C \xrightarrow{f} C' &\longmapsto f^* : \mathscr{C}\left(C',C_0\right) \to \mathscr{C}(C,C_0) \end{aligned}$$

where $f^*(\varphi) = \varphi \circ f$, for any φ in $\mathscr{C}(C',C_0)$

Definition 1.1.1 (Representable Contravariant Functor)**.** *Let $\mathscr{C}$ be a locally small category. A contravariant functor $F : \mathscr{C} \to Set$ is said to be representable if there is an object C_0 in $\mathscr{C}$ and a natural isomorphism :*

$$e : \mathscr{C}(-,C_0) \Rightarrow F$$

We say that C_0 represents F, and C_0 is a classifying object for F.

Lemma 1.1.2 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a locally small category. Let $F : \mathscr{C} \to Set$ be a contravariant functor. For any object C_0 in $\mathscr{C}$, there is a one-to-one correspondance between natural transformation e: $\mathscr{C}(-,C_0) \Rightarrow F$ and elements u in $F(C_0)$, which is given, for any object C in $\mathscr{C}$, by:*

$$\begin{aligned} e_C : \mathscr{C}(C,C_0) &\longrightarrow F(C) \\ \varphi &\longmapsto F(\varphi)(u). \end{aligned}$$

We now introduce Brown functors and discuss about their representabiliy.

Definition 1.1.3 (Brown Functors)**.** *Let $\mathscr{T}$ be a full subcategory of Top $_*$. A Brown functor $h : \mathscr{T} \to$ Set is a contravariant homotopy functor, which respects the following axioms.*

Additivity/Wedge axiom For any collection $\left\{X_j \mid j \in \mathscr{J}\right\}$ of based spaces in $\mathscr{T}$, the inclusion maps $i_j : X_j \hookrightarrow \bigvee_{j\in} X_j$ induce an isomorphism on Set:

$$\left(h\left(i_j\right)\right)_{j\in\mathscr{Z}} : h\left(\bigvee_{j\in\mathscr{Z}} X_j\right) \xrightarrow{\cong} \prod_{j\in\mathscr{Z}} h\left(X_j\right).$$

Mayer-Vietoris For any excisive triad $(X;A,B)$ in $\mathscr{T}$, if a is in $h(A)$, and b is in $h(B)$, such that $a|_{A\cap B} = b|_{A\cap B}$, then there exists x in $h(X)$, such that $x|_A = a$ and $x|_B = b$.

Any generalised cohomology theory on CW_* defines a Brown functor in each dimension.

Proposition. *Let h be a Brown functor. If X is a co-H-group then $h(X)$ is a group.*

Theorem 1.1.4 (Brown Representability Theorem)**.** *$h : CW_* \to Set_*$ be a brown fucntor. Then h is representable.*

So when the functor h on CW_* is representable, then thre exists a vased CW complex E such that there exists a natural isomorphism,

$$
\begin{aligned}
e :& [-E]_* \Longrightarrow h \\
& e_X([f]_*) = h(f)(u)
\end{aligned}
$$

where $f : X \to E$ and $u \in h(E)$ is the universal element of h.

The representability theorem is also valid in the category of CW spectra and morphisms of degree 0.

1.2 Spectra

A *spectrum* E is a sequence of spaces E_n with basepoint, provided with structure maps, $\varepsilon_n : \Sigma E_n \to E_{n+1}$ or equivalently $\varepsilon'_n : E_n \to \Omega E_{n+1}$.

Example. *To illustrate this we give an example where cohomology theory gives rise to a CW spectrum. Let K^* be a generalized cohomology theory, defined on CW pairs. We have $K^n(X) = K^n(X, pt.) + K^n(pt.)$ aand define $\tilde{K^n}(X) = K^n(X, pt)$. We assume K^* satifies the wedge axiom.*
We can apply Brown representability theorem and say that there exist connected CW-complexes E_n with base point and natural equivalences such that

$$\tilde{K^n}(X) \cong [X, E_n]$$

Consider the suspension isomorphism $\Sigma : \tilde{K^n}(X) \xrightarrow{\cong} \tilde{K^{n+1}}(\Sigma X)$. We have now natural equivalences

$$
\begin{aligned}
[X, E_n] &\cong \widetilde{K^n}(X) \cong \widetilde{K^{n+1}}(\Sigma X) \\
&\cong [\Sigma X, E_{n+1}] \cong [X, \Omega_0 E_{n+1}] .
\end{aligned}
$$

This natural equivalence must be induced by a weak equivalence:

$$\varepsilon'_n : E_n \to \Omega E_{n+1}$$

So this sequence of spaces becomes a spectrum. This spectrum is also called $\Omega-$spectrum (assuming all the spaces are connected for convenience)

We define a spectrum F to be a *suspension spectrum* or *S-spectrum* if

$$\varphi_n : \Sigma F_n \longrightarrow F_{n+1}$$

is a weak homotopy equivalence for n sufficiently large.

Example. *Given a CW-complex X, let $E_n = \begin{cases} \Sigma^n X & (n \geq 0) \\ pt & (n < 0) \end{cases}$ with the obvious maps. Then this spectrum E would be an S-spectrum, but need not be an Ω-spectrum. E is called the suspension spectrum of X.*
In particular, the sphere spectrum S is the suspension spectrum of S^0; it has n^{th} term S^n for $n \geq 0$.

We now define homotopy groups of a spectrum. consider the following homomorphisms

$$\pi_{n+r}(E_n) \longrightarrow \pi_{n+r+1}(\Sigma E_{n+1}) \xrightarrow{(\varepsilon_n)_*} \pi_{n+r+1}(E_{n+1})$$

We define the stable homotopy groups:

$$\pi_r(E) = \underrightarrow{\operatorname{colim}}_n \pi_{n+r}(E_n)^{1}$$

If E is an Ω-spectrum then the homomorphism

$$\pi_{n+r}(E_n) \longrightarrow \pi_{n+r+1}(E_{n+1})$$

is an isomorphism for $n+r \geq 1$; the limit is attained, and we have

$$\pi_r(E) = \pi_{n+r}(E_n) \quad \text{for } n+r \geq 1.$$

In the case of Suspension spectrum, we have $\pi_r(E) = \underrightarrow{\operatorname{colim}}_n \pi_{n+r}(\Sigma^n X)$. The limit is attained for $n > r+1$. In this case we have the homotopy groups of E are the stable homotopy grops of X.

Similarly we define relative homotopy groups. Let X be a spectrum, then a subspectrum A of X consist of subspaces $A_n \subset X_n$ such that the spectrum maps $\xi_n : \Sigma X_n \to X_{n+1}$ maps ΣA_n into A_{n+1}. We define the relative homotopy groups as

$$\pi_r(X,A) = \underrightarrow{\operatorname{colim}}_n \pi_{n+r}(X_n, A_n)$$

and we get a exact sequence

$$\cdots \to \pi_*(A) \to \pi_*(X) \to \pi_*(X,A) \to \pi_*(A) \to \cdots$$

1.3 Stable Homotopy Category

E is called a CW spectrum if

1. the terms E_n are CW-complexes with base point and
2. each map $\varepsilon_n : \Sigma E_n \to E_{n+1}$ is an isomorphism from ΣE_n to a sub-complex of E_{n+1}.

A subspectrum A of a CW spectrum E is as defined before, with the added condition that $A_n \subset X_n$ for each n. Let E be a CW-spectrum, E' a subspectrum of E. We say E' is cofinal in E if for each n and each finite subcomplex $K \subset E_n$ there is an m(depnding on n and K) such that $\Sigma^m K$ maps into E'_{m+n} under the map

$$\Sigma^m E_n \xrightarrow{\Sigma^{m-1}\varepsilon_n} \Sigma^{m-1} E_{n+1} \longrightarrow \ldots \longrightarrow E_{m+n-1} \xrightarrow{\varepsilon_{m+n-1}} E_{m+n}.$$

[1] in this case colimit=$\lim_n$

A *function* f from one spectrum E to another spectrum F of degree r is a sequence of maps $f_n : E_n \to F_{n-r}$ such that the following diagram is structly commutative for each n

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\varepsilon_n} & E_{n+1} \\ \Big\downarrow{\scriptstyle \Sigma f_n} & & \Big\downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_{n-r} & \xrightarrow{\phi_{n-r}} & F_{n-r+1} \end{array}$$

or equivalently maps in the Ω spectrum. Note here that the diagrams are to be strictly commutative and not commutative up to homotpy.

Let E be a CW spectrum and F be a CW spectrum. take Say that two functions $f' : E' \to F$ and $f" : E'' \to F$ are equivalent if there is a cofinal subspectrum E''' contained in E' and E'' such that the restrictions of f' and f'' to E''' coincide.

A *map* from E to F is an equivalence class of such functions.

Lemma 1.3.1. *Let $f : E \to F$ be a function and F' a cofinal subspectrum of F.. Then there is a cofinal subspectrum E' of E such that f maps E' into F'.*

Let I^+ be the union of the unit interval and a disjoint base-point. For E a spectrum, we define $Cyl(E)$ is the cylinder spectrum and has terms

$$Cyl(E)_n = I^+ \wedge E_n$$

and maps

$$(I^+ \wedge E_n) \wedge S^1 \xrightarrow{1 \wedge \varepsilon_n} I^+ \wedge E_{n+1}$$

The cylinder spectrum is a functor: a map $f : E \to F$ indues a map $Cyl(f) : Cyl(E) \to Cyl(F)$.

Two maps $f, g : E \to F$ are homotopic if there is a map $h : Cyl(E) \to F$ such that the following diagram commutes

$$\begin{tikzcd} E \wedge 0^+ \arrow[r] \arrow[dr, "f"'] & Cyl(E) \arrow[d, "h"] & E \wedge 1^+ \arrow[l] \arrow[dl, "g"] \\ & F & \end{tikzcd}$$

A *morphism* in the category $CWsp$ will be a homotopy class of maps. We write $[E, F]_r$ for the set of homotopy classes of maps with degree r.

The *stable homotopy category*, denoted SHC is the category whose objects are the same as those of $CWSp$ and whose morphisms are the homotopy classes of maps. That is $\mathrm{SHC}(E, F) := [E, F]$ for CW spectra E and F.

As long as we deal entirely with CW spectra we can restrict attention to functions whose components $f_n : E_n \to F_{n-r}$ are cellular maps.

Proposition. *Let K be a finite CW-complex and let R be its suspension spectrum, so that $E_n = \Sigma^n K$ for $n \geq 0$. Let F be any spectrum.*
We have

$$[E,F]_r = \underrightarrow{\operatorname{colim}}_n [\Sigma^{n+r}K, F_n]$$

In particular,

$$[S,F]_r = \pi_r(F)$$

Let C_n be the set of cells in E_n other than the base-point. Then we get a function $C_n \to C_{n+1}$ by $C_\alpha \mapsto \varepsilon_n(\Sigma C_\alpha)$. This function is an injection. Let $C = \lim_{n\to\infty} C_n$. An element of C is called *stable cell*. A stable cell is essentially an equivalence class that contains atmost one cell in E_n. If there are only finite number of stable cells in a spectra we call that a finite spectra.

We state the homotopy extension theorem for $CWsp$ and refer to [**A**] for proof.

Lemma 1.3.2. *Let X,A be a pair of CW-spectra, and Y,B a pair of spectra such that $\pi_*(Y,B) = 0$. Suppose given a map $f : X \longrightarrow Y$ and a homotopy $h : \mathrm{Cyl}(A) \longrightarrow Y$ from $f|_A$ to a map $g : A \longrightarrow B$. Then the homotopy can be extended over $\mathrm{Cyl}(X)$ so as to deform f to a map $X \longrightarrow B$.*

$$\begin{tikzcd}
A \arrow[rr, "in_0"] \arrow[dd, "\iota"] & & A \wedge I_+ \arrow[dl, "h"] \arrow[dd] & & A \arrow[ll, "in_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Y & & B \arrow[ll] & \\
X \arrow[ur, "f"] \arrow[rr] & & X \wedge I_+ \arrow[ul, dotted] & & X \arrow[ll, "in_1"] \arrow[ul, dotted]
\end{tikzcd}$$

The homotopy extension theorem is a special case when $B = Y$.

Lemma 1.3.3. *Suppose $\pi_*(Y) = 0$ and X,A is a pair of CW-spectra. Then any map $f : A \to Y$ can be extended over X.*

Proof. Applying the previous lemma to the pair $(A,*)$ and $(Y,*)$ we get that f is nullhomotopic. We have $h : Cyl(A) \to Y$ a homotopy from f to a map $g : A \to *$. Then there exists an extension of h, $\tilde{h} : Cyl(X) \to Y$.

□

Theorem 1.3.4. *Let $f : E \to F$ be function between spectra(need not be CW) such that $f_* : \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then for any CW-spectrum X,*

$$f_* : [X,E]_* \to [X,F]_*$$

is an isomorphism.

Proof. We can replace F by the spectrum M in which M_n is the mapping cylinder of f_n and assume that f is an inclusion. Then $\pi_*(F,E) = 0$ by the exact sequence. Now consider $(X,*)$ and apply 1.3.3. This gives us f_* is an epimorphism. For proving monomorphism consider 1.3.3 for the pair $(X \wedge I_+, X \wedge (\partial I)_+)$ (i.e. $Cyl(X)$ mod its ends).

□

Corollary 1.3.4.1. *Let $f : E \to F$ be a morphism between CW-spectra such that $f_* : \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then f is an equivalence in our category.*

Lemma 1.3.5. *Any CW spectrum Y is equivalent in the SHC to an Ω spectrum.*

Proof. Let us consider a functor $T^{(n)}$ from CW complexes to spectra given by

$$(T^{(n)}X)_r = \begin{cases} \Sigma^{r-n}X & r \geq n \\ \text{pt.} & r < n \end{cases}$$

Form the set of morphisms $[T^{(n)}X, Y]_0$, which is a Brown functor and is representable. Let it be represented by Z_n.

$$[X, Z_n] \approx [T^{(n)}X, Y] \approx [T^{(n+1)}(\Sigma X), Y] \approx [\Sigma X, Z_{n+1}] \approx [X, \Omega Z_{n+1}]$$

Thus Z is an Ω spectrum. Take $X = Y_n$,

$$[T^{(n)}Y_n, Y] \approx [Y_n, Z_n].$$

Take the map $f_n : Y_n \to Z_n$ that corresponds to the equivalence class of functions $\phi_n : (T^{(n)}Y_n)_n = Y_n \to Y_n$. Since $[Y_n, Z_n]$ is a group f_n has an inverse. Consider function f with the sequence of maps f_n, this induces isomorphism

$$f_* : \pi_*(Y) \to \pi_*(Z).$$

Applying 1.3.4.1 gives the desired conclusion. □

If X is a spectrum, let $Cone(X)$ be the spectrum whose nth term is $I \wedge X_n$ with maps

$$(I \wedge X_n) \wedge S^1 \xrightarrow{1 \wedge \varepsilon_n} I \wedge X_{n+1}$$

Theorem 1.3.6. *Let $f : E, A \longrightarrow F, B$ be a function between pairs of spectra such that*

$$f_* : \pi_*(E, A) \longrightarrow \pi_*(F, B)$$

is an isomorphism. Then for any CW-spectrum X,

$$f_* : [\text{Cone}(X), X; E, A]_* \longrightarrow [\text{Cone}(X), X; F, B]_*$$

is an isomorphism.

For any spectrum X, define $Susp(X)$ to be the spectrum whose nth terms is $S^1 \wedge X_n$ and its structure maps are

$$(S^1 \wedge X_n) \wedge s^1 \xrightarrow{1 \wedge \xi_n} S^1 \wedge X_{n+1}.$$

Susp is a functor.

Theorem 1.3.7. *$Susp : [X, Y]_* \to [Susp(X), Susp(Y)]_*$ is an isomorphism.*

The above theorem shows the the sets of morphism $[X,Y]$ are abelian groups and also that the compositions are bilinear.

We would like to show that we have an additive category. We take the spectrum E_n=pt. for all n to be the trivial object. This category has arbitrary sums(=coproducts). Given spectra X_α for $\alpha \in A$, we form $X = \bigvee_\alpha X_\alpha$ by $X_n = \bigvee (X_\alpha)_n$, with structure maps

$$X_n \wedge S^1 = \left(\bigvee_\alpha (X_\alpha)_n\right) \wedge S^1 = \bigvee_\alpha (X_\alpha) \wedge S^1 \overset{\bigvee_\alpha \xi_{\alpha n}}{\longrightarrow} \bigvee_\alpha (X_\alpha)_{n+1}$$

This has the required property:

$$\left[\bigvee_\alpha X_\alpha, Y\right] \xrightarrow{\cong} [X_\alpha, Y]$$

This proves that SHC is an additive category.

We now look at cofiber sequences for CW spectra. Suppose given a map $f : X \longrightarrow Y$ between CW-spectra. Let it be represented by a function $f' : X' \longrightarrow Y$, where X' is a cofinal subspectrum. Without loss of generality we can suppose f' is cellular. We form the mapping cone $Y \cup_{f'} CX$ as follows: its n^{th} terms is $Y_n \cup_{f'_n} (I \wedge X'_n)$ and the structure maps are the obvious ones. If we replace X' by a smaller cofinal subspectrum X'', we get $Y \cup_{f''} CX''$ which is smaller than $Y \cup_{f'} CX'$, but cofinal in it, and so equivalent. So the construct depends essentially only on the map f, and we can write it $Y \cup_f CX$. If we vary f by a homotopy, $Y \cup_{f_0} CX$ and $Y \cup_{f_1} CX$ are equivalent, but the equivalence depends on the choice of homotopy.

Let X be a CW-spectrum, A a subspectrum. A is closed if for ever finite subcomplex $K \subset X_n, \Sigma^m K \subset A_{m+n}$ implies $K \subset A_n$. It is equivalent to saying that $A \subset B \subset X, A$ cofinal in B implies that $A = B$.

Proposition. *Suppose we have a cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$$

Then for each Z the sequence

$$[Y \cup_f CX, Z] \xrightarrow{i^*} [Y,Z] \xrightarrow{f^*} [X,Z]$$

is exact.

Proposition. *Suppose we have a cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$$

The sequence

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} \left[W, Y \cup_f CX\right]$$

is exact.

We can extend the cofibre sequence further and similar to CW complexes we get

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{j} Susp(X) \xrightarrow{Susp(f)} Susp(Y)$$

This shows that in SHC cofiberings are the same as fibering.

As noted down earlier, the Brown representability theorem is valid in the category of CW-spectra and morphisms of degree 0.

Proposition. *Any spectrum Y is weakly equivalent to a CW-spectrum.*

Proof. Consider the representible functor $[X,Y]_0$. $[X,K] \approx [X,Y]_0$ for some CW spectrum K. We consider $X = K$ and take the image of id. $\square$

Smash Products

In this section we will construct smash product of two CW spectra X and Y - a CW spectrum $X \wedge Y$ - such that it has the properties we describe in this section.

Theorem 2.0.1. *1. $X \wedge Y$ is a functor of two variables, with arguments and values in the (graded) SHC.*

2. The smash-product is associative, commutative and has the sphere spectrum S as a unit, up to coherent natural equivalences.

Statement 1 is to be taken in the graded sense. That is for $f \in [X,X']_r$ and $g \in [Y,Y']_s, f \wedge g \in [X \wedge Y, X' \wedge Y']_{r+s}$ and also $(f \wedge g)(h \wedge k) = (-1)^{bc}(fh) \wedge (gk)$ if $f \in [X',X'']_a, h \in [X,X']_b, g \in [Y',Y'']_c, k \in [Y,Y']_d$.

The following equivalences hold true in our category.

$$\begin{array}{cc} a & a(X,Y,Z) : (X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z) \\ c = & C(X,Y) : X \wedge Y \longrightarrow Y \wedge X \\ l = & l(Y) : S \wedge Y \longrightarrow Y \\ r = & r(X) : X \wedge S \overset{\longrightarrow}{\longrightarrow} Y \end{array}$$

These are all maps of degree 0 and are natural. The naturality for commutativity is up to a sign $(-1)^{rs}$, if $f \in [X,X']_r$ and $g \in [Y,Y']_s$.

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{c} & W \wedge Y \\ \Big\downarrow{\scriptstyle f\wedge g} & & \Big\downarrow{\scriptstyle g\wedge f} \\ X' \wedge Y' & \xrightarrow{c} & Y' \wedge X' \end{array}$$

Let A be an ordered set isomorphic to $\{0,1,2,3,\dots\}$. Suppose we have a partition of A into two subsets B and C, so that $A = B \cup C$ and $B \cap C = \phi$. Then we define a smash product functor which assigns to any two CW spectra X and Y a CW spectrum $X \wedge_{BC} Y$,called the naive smash product. The terms of this product spectrum $P = X \wedge_{BC} Y$ are given by by $P_{\alpha(a)} = X_{\beta(a)} \wedge Y_{\gamma(a)}$.Here α is an isomorphism from $A = B \cup C$ to the set $\{0,1,2,3,\dots\}$ and β, γ are monotonic functions on A such that $\beta(a) + \gamma(a) = \alpha(a)$.

The maps of the product spectrum are defined as follows. We have

$$P_{\alpha(a)} \wedge S^1 = X_{\beta(a)} \wedge Y_{\gamma(a)} \wedge S^1.$$

We regard S^1 as one point compactification of $\mathbb{R}$, where infinity becomes the base point. This allows us to define a map of degree -1 from S^1 to S^1. by $t \mapsto -t$.

If $a \in B$, then

$$P_{\alpha(a)+1} = X_{\beta(a)+1} \wedge Y_{\gamma(a)}$$

and we define the map

$$\pi_{\alpha(a)} : SP_{\alpha(a)} \longrightarrow P_{\alpha(a)+1}$$

by

$$\pi_{\alpha(a)}(x \wedge y \wedge t) = \xi_{\beta(a)}\left(x \wedge (-1)^{\gamma(a)} t\right) \wedge y$$

If $a \in C$, then

$$P_{\alpha(a)+1} = X_{\beta(a)} \wedge Y_{\gamma(a)+1}$$

and we define the map

$$\pi_{\alpha(a)}(x \wedge y \wedge t) = x \wedge \eta_{\gamma(a)}(y \wedge t).$$

Here

$$x \in X_{\beta(a)}, \quad y \in Y_{\gamma(a)}, \quad t \in S^1,$$

and

$$\xi_{\beta(a)} : X_{\beta(a)} \wedge S^1 \longrightarrow X_{\beta(a)}, \quad \eta_{\gamma(a)} : Y_{\gamma(a)} \wedge S^1 \longrightarrow Y_{\alpha(a)+1}$$

are the appropriate maps from the spectra X, Y.

$X \wedge Y$ also has the following equivalences with the naive smash products. We only look at the ones that are of interest to us.

Theorem 2.0.2. *For each choice of B, C there is a morphism*

$$\mathrm{eq}_{BC} : X \wedge_{BC} Y \longrightarrow X \wedge Y \quad (\textit{ of degree } 0)$$

with the following properties.

1. *The morphism $eq_{BC} : X \wedge_{BC} Y \to X \wedge Y$ is an equivalence if any one of the following conditios is satisfied.*
   - *(a) B and C are infinite.*
   - *(b) B is finite, say with d elements and $\xi_r : \Sigma X_r \to X_{r+1}$ is an isomorphism for $r \geq d$.*
   - *(c) C is finite, say with d elmemetns and $\eta_r : \Sigma Y_r \to Y_{r+1}$ is an isomorphism for $r \geq d$.*

The handicrafted smash products are commutative for the right choice of B, C at each point. We partition the sets accordingly with the following condition.

Condition Elements number $0,1,2,3$ in A are either four elemnts in B or four elements in C. similarly for elments number $4,5,6,7$ in A and similarly for elementss numver $4r, 4r+1, 4r+2, 4r+3$ for each r. The smash product has the following property regarding commutativity

Theorem 2.0.3. *The equivalence $c : X \wedge Y \to Y \wedge X$ makes the following diagram commutative for each choice of B, C satsifying the condition stated above*

$$\begin{tikzcd} X \wedge Y \arrow[r, "c"] & Y \wedge X \\ X \wedge_{BC} Y \arrow[u, "eq_{BC}"] \arrow[r, "c_{BC}"] & Y \wedge_{CB} X \arrow[u, "eq_{CB}"'] \end{tikzcd}$$

The handicrafted smash products have S as a unit if we pick the right product at each point. Say, we partition $A = \phi \cup A$ satisfying the condition we have S as a unit.

Define

$$l : S \wedge Y \to Y$$

to be the composite

$$S \wedge Y \xleftarrow[\text{equivalence}]{eq_{\phi,A}} A \wedge_{\phi A} Y \cong Y (eq_{\phi,A} \text{ is an equivalence})$$

We also have the isomorphisms $S^0 \wedge Y \cong Y$ and $X \wedge S^0 \cong X$ with the obvious component-wise isomorphism.This is also natural for morphisms of degree 0. we noe define

$$r : X \wedge S \to X$$

to be the composite

$$X \wedge S \xleftarrow{eq_{\phi,A}} X \wedge_{A\phi} S \cong X (eq_{\phi,A} \text{ is an equivalence})$$

Since $S \wedge S$ is equivalent to S, we have $[S \wedge S, S \wedge S]_0 \cong [S, S]_0 \cong \mathbb{Z}$. Also we construct the smash product so that the map $c : S \wedge S \to S \wedge S$ has degree 1.

We refer to [**A**] for details regarding the construction.

Duality

We begin by describing *Spanier-Whitehead duality* that talks about the homotopy type of a compact subset in S^n and it's complement. If X is a compact subset embedded in S^n, then by Alexander duality theorem, we have

$$\tilde{H}^k(X) \cong \tilde{H}_{n-k-1}(S^n - X).$$

Let K be homotopy equivalent to $S^n - K$ for K a compact set embedded in S^n.We would like to prove that K determines the stable homotopy type of L. The homotopy type of L in general is not determined by K as it dependes on the embedding of K.

Embed S^n as the equatorial sphere in S^{n+1} and embed the suspension ΣK of K in S^{n+1} by joining to the two poles. Then $S^{n+1} - \Sigma K \simeq S^n - K$. So if we have $K \subset S^n$ and $M \subset S^m$ and a homtopy equivalence $f : \Sigma^p K \to \Sigma^q M$, we can embed $\Sigma^p K$ in S^{n+p} and $\Sigma^q M$ in S^{m+q}, since the complements remain homotopy equivalent. So WLOG, we can say we have $K' \subset S^{n'}$ and $M' \subset S^{m'}$ and a homotopy equivalence $f : K' \to M'$. We can even assume f is piecewise linear.

Now suppose $K \subset S^n$ and embed S^n as an equiatorial sphere in S^{n+1} without changin K. Then $S^{n+1} - K = \Sigma(S^n - K)$. Consider the join of to spheres in which $S^n * S^m \simeq S^{m+n+1}$, K and M are embedded, S^n and S^m respectively. We can embed the mapping cylinder M_c of f in the join. In this sphere we have

$$S^{m+n+1} - K = \Sigma^{m+1}(S^n - K)$$
$$S^{m+n+1} - M = \Sigma^{n+1}(S^m - M)$$

and two maps

$$S^{m+n+1} - K \xleftarrow{f} S^{m+n+1} - M_c \xrightarrow{g} S^{m+n+1} - M$$

But the injective maps

$$K \to M_c \leftarrow M$$

indcue isomorphismc of cohomology. The alexander duality isomorphism is natural for inclusion maps and therefore f and g induce isoorphism of homology. We can suspend further to make everything simply connected. So by Whitehead's theorem, f and g are stable homotopy equivalences. Thus we've proved the assignment $K \mapsto L$ is well-defined, up to stable equivalence, for the suspension spectrum of K. The desuspension is made so that degrees are as expected.

Definition 3.0.1. *A dual of X is an object X^* equipped with maps*

$$e : X \wedge X^* \to S^0$$
$$\eta : S^0 \to X^* \wedge X$$

such that the compositions

$$X \wedge S^0 \overset{1\wedge\eta}{\to} X \wedge X^* \wedge X \overset{e\wedge 1}{\to} X \wedge S^0$$

and

$$S^0 \wedge X^* \overset{\eta\wedge 1}{\to} X^* \wedge X \wedge X^* \overset{1\eta e}{\to} X^* \wedge S^0$$

are the canonical isomorphisms.

We will look at the construction of a dual in SHC. Let X be CW spectrum. Consider the set $[W \wedge X, S]_0$. With X fixed this is a contravariant functor of W and this is now a Brown functor. So it is representable, say by X^* and there is a natural isomorphism

$$[W, X^*]_0 \xrightarrow{T} [W \wedge X, S]_0$$

Taking $W = X^*$ and the *id* map we see that there is a map $e : X \wedge X^* \to S$. Since T is natural it carries, $f \to X^*$ into $W \wedge X \xrightarrow{f\wedge 1} X^* \wedge X \xrightarrow{e} S$.We can then extend this isomorphism to maps of degree r

$$[W, X^*]_r \xrightarrow{T} [W \wedge X, S]_r$$

We can think of X^* as the dual. The dual X^* is a contravariant functor of X. If $g : X \to Y$ is a map, then it induces

$$[W, Y^*] \xrightarrow{(1\wedge g)^*} [W, X^*]$$

and this natural transformation must be induced by a unique map $g^* : Y^* \to X^*$.

Let Z be a spectrum , we can make a natural transformation

$$[W, Z \wedge X^*]_r \overset{T}{\to} [W \wedge X, Z]_r$$

as follows: Given $W \xrightarrow{f\wedge 1} Z \wedge X^*$ we take $W \wedge X \xrightarrow{f\wedge 1} Z \wedge X^* \wedge X \xrightarrow{1\wedge e} Z$.

Proposition. *If W and X are finite spectra , then*

$$T : [W, Z \wedge X^*]_r \to [W \wedge X, Z]_r$$

is an isomorphism for any spectrum Z.

Lemma 3.0.2. *If X is a finite spectrum then X^* is equivalent to a finite spectrum.*

Proof. The proof involves homology theories of a spectra and is postponed till next chapter. □

Homology and Cohomology

We define E-homology and E-cohomology for a given spectrum E and study their properties. The E-homology is defined as

$$E_n(X) = [S, E \wedge X]_n$$

and E-cohomology is defined as

$$E^n(X) = [X, E]_{-n}$$

These functors satisfy the properties that generalised homology and cohomology functors satisfy. They give an analog for a theoy defined on spectra of the Eilenberg-Steenrod axioms. We record the properties in the proposition below. These are easy to check.

Consider the Eilenberg-Maclane spectrum $H\mathbb{Z}$. Define

$$H_n(X) = [S, H\mathbb{Z} \wedge X]$$

and

$$H^n(X) = [X, H\mathbb{Z}]$$

Proposition. *1. $E_*(X)$ is a covariant functor of two variables E, X in SHC with values in the category of graded abelian groups. $E^*(X)$ is a covariant fucntor in E and contravariant in X.*

2. If we vary E or X along a cofibering, we obtain an exact sequence, That is, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a cofiber sequence, then

$$E_n(X) \xrightarrow{f_*} E_n(Y) \xrightarrow{g_*} E_n(Z)$$

and

$$E^n(X) \xleftarrow{f^*} E^n(Y) \xleftarrow{g^*} E^n(Z)$$

are exact; if $E \xrightarrow{i} F \xrightarrow{j} G$ is a cofiber sequence, then

$$E_n(X) \xrightarrow{i_*} F_n(X) \xrightarrow{j_*} G_n(X)$$

and

$$E^n(X) \xrightarrow{i_*} F^n(X) \xrightarrow{j_*} G^n(X)$$

are exact.

3. *There area natural isomorphisms*

$$E_n(X) \cong E_{n+1}\left(S^1 \wedge X\right)$$
$$E_n(X) \cong E^{n+1}\left(S^1 \wedge X\right)$$

4. $$E_n(S) = E^{-n}(S) = \pi_n(E)$$

For a CW complex L we define homology and cohomology to be E_n or E^n applied to the suspension spectrum of the complex.

$$\tilde{E}_n(L) = E_n(\Sigma^\infty L)$$

$$\tilde{E}^n(L) = E^n(\Sigma^\infty L)$$

The following fact holds

$$E_n(X) \cong X_n(E).$$

Proposition. *If X is a finite spectrum $E_n(X^*) \cong E^{-n}(X)$.*

Proof. The proof is a simple application of 3.

□

Proof of 3.0.2. Let X be a finite spectrum. Then $[S, X^*] \cong [X, S]$ and the right hadn side is zero if n is negative for large absolute values. But $H_n(X^*) = H^{-n}(X)$ is finitely generated in each dimension and zero for all except for finite number of dimensions. This proves that X^* has only finite stable cells and hence is a finite spectrum □

We now discuss homology and cohomology groups with coefficients.

Moore spectrum Let G be an abelian group. conisder a free resolution $0 \to R \xrightarrow{i} F \to G \to 0$. Take $\vee_\alpha S, \vee_\beta S$ such that π_0 of the two spectra are R and F respectively. take a map $f : \vee_\alpha S \to \vee_\beta S$ inducing i[1]. Form another spectrum $M = \vee_\alpha S \bigcup_f C(\vee_\beta S)$. This is a *Moore spectrum of type G*. So we have

$$\begin{aligned} &\pi_r(M) = 0 && \text{for } r < 0 \\ &\pi_0(M) = H_0(M) = G && \\ &H_r(M) = 0 && \text{for } r > 0 \end{aligned}$$

For any spectrum E, we define the corresponding spectrum with coefficients in G by

$$EG = E \wedge M$$

Proposition. 1. *There exists an exact sequence*

$$0 \longrightarrow \pi_n(E) \otimes G \longrightarrow \pi_n(EG) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}\left(\pi_{n-1}(E)\right) \longrightarrow 0$$

(This need not split, e.g., take $E = \mathrm{KO}, G = \mathbb{Z}_2$.)

[1] See here for more details on the induced map f

2. *More generally, there exists exact sequences*

$$0 \longrightarrow E_n(X) \otimes G \longrightarrow (EG)_n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}\left(E_{n-1}(X), G\right) \longrightarrow 0$$

and (if X is a finite spectrum or G is finitely generated)

$$0 \longrightarrow E^n(E) \otimes G \longrightarrow (EG)^n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}\left(E^{n+1}(X), G\right) \longrightarrow 0$$

Proof. [**A**] for proof □

The Moore spectrum for $\mathbb{Q}$ is same as the Eilenberg-Maclane spectrum for $\mathbb{Q}$. With this fact one can show that the rational stble homotopy is same as rational homology,i.e.

$$\pi_*(X) \otimes \mathbb{Q} \to H_*(X) \otimes \mathbb{Q}.$$

The isomorphism is induced by the map $i : S \to H$ representing a generator of $\pi_0(H) = \mathbb{Z}$.

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology*