Model Categories

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Learning Model categories from Quillen's book. Recording down stuff to refer.

We saye $\mathscr{C}$ is a model category if $\mathscr{C}$ is a cat with three classes of maps

- fibrations
- cofibrations
- weak equivalences

satisfying following axioms

M0 There exists finite limits and colimits.

M1 Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where
i is a cofibration and a weak equivalence(trivial cofibration) and p is a fibration or
i is a cofibration and p is a fibration(trivial fibration) and weak equivalence,
then $\exists$ a lift $B \to X$.

M2 Any map f may be factored as
$f = pi$ where i=trivial cofibration and p=fibration and
$f = pi$ where i=cofibration and p =trivial fibration.

M3 Fibrations are stable under composition, base change and any isomorphism is a fibration.
Cofibrations are stable under composition, cobase change and an isomorphism is a cofibration.

M4 The base extension of a map which is a trivial fibration is a weak equivalence.
The cobase extension of a map which is trivial fibration is a weak equivalence.

M5 if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is in $\mathscr{C}$. Then if two of f, g, gf are weak equivalences, so is the third. Any isomorphism is a weak equivalence.

An intial object in a category $\mathscr{C}$ is an object ϕ such that for all objects C in $\mathscr{C}$ there is a unique morphism $\phi \to C$. The dual notion of this is the terminal object $*$. These objects exist in $\mathscr{C}$ because of M0 and they are unique.

X is cofibrant if $\phi \to X$ is a cofibration. X is fibrant if $X \to e$ is a fibration.

Let $f, g : A \to B$ be maps. We say that f is left-homtopic to g if there is a diagram of the form where σ is a weak equivalence.

$$\begin{tikzcd} A \vee A \arrow[r, "f+g"] \arrow[d, "\nabla"] \arrow[dr, "\partial_0+\partial_1"] & B \\ A & \tilde{A} \arrow[l, "\sigma"] \arrow[u, "h"'] \end{tikzcd} \tag{1}$$

Dually we say that f is right homotopic to g if there is a diagram of the form where s is a weak equivalence.

$$\begin{tikzcd} \tilde{B} \arrow[dr, "(d_0,d_1)"] & B \arrow[l, "s"'] \\ A \arrow[u, "k"] \arrow[r, "(f,g)"'] & B \times B \arrow[u, "\triangle"'] \end{tikzcd} \tag{2}$$

By cylinder object for an object A we mean an object $A \times I$ together with maps

$$A \vee A \xrightarrow{\partial_0+\partial_1} A \times I \xrightarrow{\sigma} A$$

with $\sigma(\partial_0 + \partial_1) = \nabla_A$ such that $\partial_0 + \partial_1$ is a cofibration and σ is a weak equivalence.
Dually, a path object for B shall be an onject B^I together with a factoriazation

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B$$

of $\triangle_B$ where s is a weak equivalence and (d_0, d_1) is a fibration.

By a left homotopy from f to g, we mean a diagram 1 where $\partial_0 + \partial_1$ is a cofibration and hence $\tilde{A}$ is a cylinder object for A. This is also saying that there exists a cylinder object such that the map $A \vee B \xrightarrow{f+g} B$ extends to a map $h : A \times I \to B$ with obvious commutative relations
Similarly a right homotopy from f to g is a diagram 2 where $\tilde{B}$ is a path object for B. Equivalently the map $A \xrightarrow{(f,g)} B \times B$ extends to a map $B^I \to B \times B$ with relevant commutative relations.

lemma 1. If $f, g \in \hom(A, B)$ and $f \overset{l}{\sim} g$, then thre is a left homotopy $h : A \times I \to B$ from f to g.

lemma 2. Le A be a cofibrant object and let $A \times I$ be a cylinder object for A. Then $\partial_0 : A \to A \times I$ and $\partial_1 : A \to A \times I$ are trivial cofibrations.

lemma 3. Let A be cofibrant and let $A \times I$ and $A \times I'$ be two cylinder objects for A.Then the result of gluing $A \times I$ and $A \times I'$ by identification $\partial_1 A = \partial_0' A$ defined precisely to be the object $\tilde{A}$ is also a cylinder object.

lemma 4. If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on $\mathbf{hom}(A, B)$.

lemma 5. Let A be cofibrant and let $f, g \in \mathbf{hom}(A, B)$ Then

1. $f \overset{l}{\sim} g \implies f \overset{r}{\sim} g$
   (dual)If B is fibrant then $f \overset{r}{\sim} g \implies f \overset{l}{\sim} g$
2. $f \overset{r}{\sim} g \implies$ there exists a right homotopy $k : A \to B^I$ from f to g with $s : B \to B^I$ a trivial cofibration.
3. If $u : B \to C$, then $f \overset{r}{\sim} g \implies uf \overset{r}{\sim} ug$

Let A and B be objects of $\mathscr{C}$ let $\pi^r(A, B)$(similar for $\pi^l(A, B)$) be the set of equivalence classes of $\mathbf{hom}(A, B)$ with repsect to the equivalence relation generated by $\overset{r}{\sim}$. When A cofibrant and B is fibrant, in which case left and right homotopies coincide and are already equivalence relations, we shall denote the relation by $\sim$, call it homotopy and $\pi_0(A, B)$.

lemma 6. If A is cofibrant, then composition in $\mathscr{C}$ induces a map $\pi^r(A, B) \times \pi^r(B, C) \to \pi^r(A, C)$.

lemma 7. Let A be cofibrant and let $p : X \to Y$ be a trivial fibration. Then p induces a bijection $p_* : \pi^l(A, X) \to \pi^l(A, Y)$.
(dual) Let B be fibrant and $i : X \to Y$ be a tivial cofibration, then i induces a bijection $i_* : \pi^r(Y, B) \simeq \pi^r(X, B)$

Let $\mathscr{C}_c, \mathscr{C}_f, \mathscr{C}_{cf}$ be full subcategories[1] consisting of the cofibrant, fibrant and both cofibrant and fibrant objects of $\mathscr{C}$ respectively. Define

$$\pi\mathscr{C}_c \text{ with objects} = Obj(\mathscr{C}_c) \text{ and morphisms} = \pi^r(A, B)$$

If we denote the right homotopy class of a map $f : A \to B$ by $\bar{f}$ we obtain a functor $\mathscr{C}_c \to \pi\mathscr{C}_c$ given by $X \to X, f \to \bar{f}$. Similarly we define $\pi_{\mathscr{C}_f}$ and $\pi\mathscr{C}_{cf}$.

Let $\mathscr{C}$ be an arbitrary category and let S be a subclass of the class of maps of $\mathscr{C}$. By localization of $\mathscr{C}$ with respect to S we mean a category $S^{-1}\mathscr{C}$ together with a functor $\gamma : \mathscr{C} \to S^{-1}\mathscr{C}$ having the following universal porperty: For every $s \in S$, $\gamma(s)$ is an isomorphism; given any functor $F : \mathscr{C} \to \mathscr{B}$ with $F(s)$ an isomorphism for all $s \in S$ there is a unique functor $\theta : S^{-1}\mathscr{C} \to \mathscr{B}$ such that $\theta \circ \gamma = F$.

Let $\mathscr{C}$ be a model category. Then the homotopy category of $\mathscr{C}$ is the localization of ¸with respect to the class of weak equivalences and is denoted by $\gamma : \mathscr{C} \to Ho\mathscr{C}$. $\gamma : \mathscr{C}_c \to Ho\mathscr{C}_c$ and $\gamma : \mathscr{C}_f \to Ho\mathscr{C}_f$ will denote the localization of $\mathscr{C}_c$ and $\mathscr{C}_f$ with repect to the class of maps in the respective categories which are weak equivalences in $\mathscr{C}$. $[X, Y] := \mathbf{hom}_{Ho\mathscr{C}}(X, Y)$.

lemma 8. 1. Let $F : \mathscr{C} \to \mathscr{B}$ carry weak equivalences in $\mathscr{C}$ 1nto isomorphisms in $\mathscr{B}$. If $f \overset{l}{\sim} g$ or $f \overset{r}{\sim} g$, then $F(f) = F(g)$ in $\mathscr{B}$.

[1]some objects but all morphisms

2. Let $F : \mathscr{C}_c \to \mathscr{B}$ carry weak equivalences in $\mathscr{C}_c$ into isomorphisms in $\mathscr{B}$. If $f \overset{r}{\sim} g$, then $F(f) = F(g)$ in $\mathscr{B}$.

The above lemma implies the functors $\gamma_c, \gamma_f, \gamma$ induce functors $\overline{\gamma_c} : \pi\mathscr{C}_c \to Ho\mathscr{C}_c, \overline{\gamma_f} : \pi\mathscr{C}_f \to Ho\mathscr{C}_f$, $\overline{\gamma} : \pi\mathscr{C}_{cf} \to Ho\mathscr{C}$.

The homotopy category is the category

$$Ho\mathscr{C} \text{ with objects} = Obj(\mathscr{C})\text{and}\,\hom_{Ho\mathscr{C}}(X,Y) = \hom_{\pi\mathscr{C}_{cf}}(RQX, RQY) = \pi(RQX, RQY)$$

For each object X choose a trivial fibration $p_X : Q(X) \to X$ with $Q(X)$ cofibrant and a trivial cofibration $i_X : X \to R(X)$ with $R(X)$ fibrant. For each map $f : X \to Y$, we may choose a map $\underline{Q(f)} : \underline{Q(X)} \to Q(Y)$ and $R(f) : R(X) \to R(Y)$. By mapping $X \to Q(X)$ or $R(X)$ and $f \to \overline{Q(F)}$ or $\overline{R(f)}$ we get functors $\overline{Q} : \mathscr{C} \to \pi\mathscr{C}_c$ and $\overline{R} : \mathscr{C} \to \pi\mathscr{C}_f$. Some more math and we get a well-defined functor

$$\overline{RQ} : \mathscr{C} \to \pi\mathscr{C}_{cf}$$
$$X \to RQX$$
$$f \to \overline{RQ(f)}$$

Theorem 1. $Ho\mathscr{C}, Ho\mathscr{C}_c, Ho\mathscr{C}_f$ exist and there is a diagram of functors

$$\begin{tikzcd}
\pi\mathscr{C}_c \arrow[r, "\overline{\gamma_c}"] & Ho\mathscr{C}_c \arrow[d, "\sim"'] \\
\pi\mathscr{C}_{cf} \arrow[u, hook] \arrow[d, hook] \arrow[r, "\overline{\gamma}", "\sim"'] & Ho\mathscr{C} \\
\pi\mathscr{C}_f \arrow[r, "\overline{\gamma_f}"] & Ho\mathscr{C}_f \arrow[u, "\sim"]
\end{tikzcd}$$

where $\hookrightarrow$ denotes a full embedding and $\xrightarrow{\sim}$ denotes an equivalence of categories. Furthermore if $(\overline{\gamma})^{-1}$ is a quasi-inverse[2] for $\overline{\gamma}$, then the fully faithful functor

$$Ho\mathscr{C}_c \xrightarrow{\sim} Ho\mathscr{C} \xrightarrow[\sim]{(\overline{\gamma})^{-1}} \pi\mathscr{C}_{cf} \hookrightarrow \pi\mathscr{C}_c$$

is right adjoint to $\overline{\gamma_c}$ and the fully faithful functor

$$Ho\mathscr{C}_f \xrightarrow{\sim} Ho\mathscr{C} \xrightarrow[\sim]{(\overline{\gamma})^{-1}} \pi\mathscr{C}_{cf} \hookrightarrow \pi\mathscr{C}_{cf} \hookrightarrow \pi\mathscr{C}_{cf}$$

is left adjoint to $\overline{\gamma_f}$.

Corollary 1. If A is cofibrant and B is fibrant, then

$$\hom_{Ho\mathscr{C}}(A,B) = \pi(A,B)$$

The category $\mathscr{C}$ can have different model structures on it, but same $Ho\mathscr{C}$, i.e. the weak equivalences are same but fibrations and cofibrations can be different.

[2] Definition