

| | | |
|---|------------------------------|---|
| 1 | Introduction | 1 |
| 2 | Loop and suspension functors | 5 |

1 Introduction

We say \mathcal{C} is a model category if \mathcal{C} is a cat with three classes of maps

- fibrations
- cofibrations
- weak equivalences

satisfying following axioms

M0 There exists finite limits and colimits.

M1 Given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where

i is a cofibration and a weak equivalence(trivial cofibration) and p is a fibration or

i is a cofibration and p is a fibration(trivial fibration) and weak equivalence,

then \exists a lift $B \rightarrow X$.

M2 Any map f may be factored as

$f = pi$ where i =trivial cofibration and p =fibration and

$f = pi$ where i =cofibration and p =trivial fibration.

M3 Fibrations are stable under composition, base change and any isomorphism is a fibration.

Cofibrations are stable under composition, cobase change and an isomorphism is a cofibration.

M4 The base extension of a map which is a trivial fibration is a weak equivalence.

The cobase extension of a map which is trivial fibration is a weak equivalence.

M5 if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is in \mathcal{C} . Then if two of f, g, gf are weak equivalences, so is the third. Any isomorphism is a weak equivalence.

An initial object in a category \mathcal{C} is an object ϕ such that for all objects C in \mathcal{C} there is a unique morphism $\phi \rightarrow C$. The dual notion of this is the terminal object $*$. These objects exist in \mathcal{C} because of M0 and they are unique.

X is **cofibrant** if $\phi \rightarrow X$ is a cofibration. X is **fibrant** if $X \rightarrow e$ is a fibration.

Let $f, g : A \rightarrow B$ be maps. We say that f is **left-homotopic** to g if there is a diagram of the form where σ is a weak equivalence.

$$\begin{array}{ccc} A \vee A & \xrightarrow{f+g} & B \\ \downarrow \nabla & \searrow \partial_0 + \partial_1 & \uparrow h \\ A & \xleftarrow{\sigma} & \tilde{A} \end{array} \quad (1)$$

Dually we say that f is **right homotopic** to g if there is a diagram of the form where s is a weak equivalence.

$$\begin{array}{ccc} \tilde{B} & \xleftarrow{s} & B \\ \uparrow k_{(d_0, d_1)} & \searrow & \uparrow \Delta \\ A & \xrightarrow{(f, g)} & B \times B \end{array} \quad (2)$$

By **cylinder object** for an object A we mean an object $A \times I$ together with maps

$$A \vee A \xrightarrow{\partial_0 + \partial_1} A \times I \xrightarrow{\sigma} A$$

with $\sigma(\partial_0 + \partial_1) = \nabla_A$ such that $\partial_0 + \partial_1$ is a cofibration and σ is a weak equivalence. Dually, a **path object** for B shall be an object B^I together with a factorization

$$B \xrightarrow{s} B^I \xrightarrow{(d_0, d_1)} B \times B$$

of Δ_B where s is a weak equivalence and (d_0, d_1) is a fibration.

By a **left homotopy** from f to g , we mean a diagram 1 where $\partial_0 + \partial_1$ is a cofibration and hence \tilde{A} is a cylinder object for A . This is also saying that there exists a cylinder object such that the map $A \vee B \xrightarrow{f+g} B$ extends to a map $h : A \times I \rightarrow B$ with obvious commutative relations

Similarly a **right homotopy** from f to g is a diagram 2 where \tilde{B} is a path object for B . Equivalently the map $A \xrightarrow{(f, g)} B \times B$ extends to a map $B^I \rightarrow B \times B$ with relevant commutative relations.

lemma 1. If $f, g \in \text{hom}(A, B)$ and $f \stackrel{l}{\sim} g$, then there is a left homotopy $h : A \times I \rightarrow B$ from f to g .

lemma 2. Let A be a cofibrant object and let $A \times I$ be a cylinder object for A . Then $\partial_0 : A \rightarrow A \times I$ and $\partial_1 : A \rightarrow A \times I$ are trivial cofibrations.

lemma 3. Let A be cofibrant and let $A \times I$ and $A \times I'$ be two cylinder objects for A . Then the result of gluing $A \times I$ and $A \times I'$ by identification $\partial_1 A = \partial'_0 A$ defined precisely to be the object \tilde{A} is also a cylinder object.

lemma 4. If A is cofibrant, then $\stackrel{l}{\sim}$ is an equivalence relation on $\text{hom}(A, B)$.

lemma 5. Let A be cofibrant and let $f, g \in \text{hom}(A, B)$ Then

1. $f \stackrel{l}{\sim} g \implies f \stackrel{r}{\sim} g$
(dual) If B is fibrant then $f \stackrel{r}{\sim} g \implies f \stackrel{l}{\sim} g$
2. $f \stackrel{r}{\sim} g \implies$ there exists a right homotopy $k : A \rightarrow B^I$ from f to g with $s : B \rightarrow B^I$ a trivial cofibration.
3. If $u : B \rightarrow C$, then $f \stackrel{r}{\sim} g \implies uf \stackrel{r}{\sim} ug$

Let A and B be objects of \mathcal{C} let $\pi^r(A, B)$ (similar for $\pi^l(A, B)$) be the set of equivalence classes of $\text{hom}(A, B)$ with respect to the equivalence relation generated by $\stackrel{r}{\sim}$. When A cofibrant and B is fibrant, in which case left and right homotopies coincide and are already equivalence relations, we shall denote the relation by \sim , call it homotopy and $\pi_0(A, B)$.

lemma 6. If A is cofibrant, then composition in \mathcal{C} induces a map $\pi^r(A, B) \times \pi^r(B, C) \rightarrow \pi^r(A, C)$.

lemma 7. Let A be cofibrant and let $p : X \rightarrow Y$ be a trivial fibration. Then p induces a bijection $p_* : \pi^l(A, X) \rightarrow \pi^l(A, Y)$.

(dual) Let B be fibrant and $i : X \rightarrow Y$ be a trivial cofibration, then i induces a bijection $i_* : \pi^r(Y, B) \simeq \pi^r(X, B)$

Let $\mathcal{C}_c, \mathcal{C}_f, \mathcal{C}_{cf}$ be full subcategories¹ consisting of the cofibrant, fibrant and both cofibrant and fibrant objects of \mathcal{C} respectively. Define

$$\pi\mathcal{C}_c \text{ with objects } = \text{Obj}(\mathcal{C}_c) \text{ and morphisms } = \pi^r(A, B)$$

If we denote the right homotopy class of a map $f : A \rightarrow B$ by \bar{f} we obtain a functor $\mathcal{C}_c \rightarrow \pi\mathcal{C}_c$ given by $X \rightarrow X, f \rightarrow \bar{f}$. Similarly we define $\pi\mathcal{C}_f$ and $\pi\mathcal{C}_{cf}$.

Let \mathcal{C} be an arbitrary category and let S be a subclass of the class of maps of \mathcal{C} . By localization of \mathcal{C} with respect to S we mean a category $S^{-1}\mathcal{C}$ together with a functor $\gamma : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ having the following universal property: For every $s \in S$, $\gamma(s)$ is an isomorphism; given any functor $F : \mathcal{C} \rightarrow \mathcal{B}$ with $F(s)$ an isomorphism for all $s \in S$ there is a unique functor $\theta : S^{-1}\mathcal{C} \rightarrow \mathcal{B}$ such that $\theta \circ \gamma = F$.

Let \mathcal{C} be a model category. Then the **homotopy category** of \mathcal{C} is the localization of \mathcal{C} with respect to the class of weak equivalences and is denoted by $Ho\mathcal{C}$. $\gamma : \mathcal{C}_c \rightarrow Ho\mathcal{C}_c$ and $\gamma : \mathcal{C}_f \rightarrow Ho\mathcal{C}_f$ will denote the localization of \mathcal{C}_c and \mathcal{C}_f with respect to the class of maps in the respective categories which are weak equivalences in \mathcal{C} . $[X, Y] := \text{hom}_{Ho\mathcal{C}}(X, Y)$.

- lemma 8.**
1. Let $F : \mathcal{C} \rightarrow \mathcal{B}$ carry weak equivalences in \mathcal{C} into isomorphisms in \mathcal{B} . If $f \stackrel{l}{\sim} g$ or $f \stackrel{r}{\sim} g$, then $F(f) = F(g)$ in \mathcal{B} .
 2. Let $F : \mathcal{C}_c \rightarrow \mathcal{B}$ carry weak equivalences in \mathcal{C}_c into isomorphisms in \mathcal{B} . If $f \stackrel{r}{\sim} g$, then $F(f) = F(g)$ in \mathcal{B} .

¹some objects but all morphisms

The above lemma implies the functors $\gamma_c, \gamma_f, \gamma$ induce functors $\bar{\gamma}_c : \pi\mathcal{C}_c \rightarrow Ho\mathcal{C}_c, \bar{\gamma}_f : \pi\mathcal{C}_f \rightarrow Ho\mathcal{C}_f, \bar{\gamma} : \pi\mathcal{C}_{cf} \rightarrow Ho\mathcal{C}$.

The homotopy category is the category

$$Ho\mathcal{C} \text{ with objects } = Obj(\mathcal{C}) \text{ and } \text{hom}_{Ho\mathcal{C}}(X, Y) = \text{hom}_{\pi\mathcal{C}_{cf}}(RQX, RQY) = \pi(RQX, RQY)$$

For each object X choose a trivial fibration $p_X : Q(X) \rightarrow X$ with $Q(X)$ cofibrant and a trivial cofibration $i_X : X \rightarrow R(X)$ with $R(X)$ fibrant. For each map $f : X \rightarrow Y$, we may choose a map $\underline{Q}(f) : Q(X) \rightarrow Q(Y)$ and $\underline{R}(f) : R(X) \rightarrow R(Y)$. By mapping $X \rightarrow Q(X)$ or $R(X)$ and $f \rightarrow \underline{Q}(f)$ or $\underline{R}(f)$ we get functors $\bar{Q} : \mathcal{C} \rightarrow \pi\mathcal{C}_c$ and $\bar{R} : \mathcal{C} \rightarrow \pi\mathcal{C}_f$. Some more math and we get a well-defined functor

$$\begin{aligned} \bar{RQ} : \mathcal{C} &\rightarrow \pi\mathcal{C}_{cf} \\ X &\rightarrow RQX \\ f &\rightarrow \bar{RQ}(f) \end{aligned}$$

Theorem 1. $Ho\mathcal{C}, Ho\mathcal{C}_c, Ho\mathcal{C}_f$ exist and there is a diagram of functors

$$\begin{array}{ccc} \pi\mathcal{C}_c & \xrightarrow{\bar{\gamma}_c} & Ho\mathcal{C}_c \\ \uparrow & & \sim \downarrow \\ \pi\mathcal{C}_{cf} & \xrightarrow[\sim]{\bar{\gamma}} & Ho\mathcal{C} \\ \downarrow & & \sim \uparrow \\ \pi\mathcal{C}_f & \xrightarrow{\bar{\gamma}_f} & Ho\mathcal{C}_f \end{array}$$

where \hookrightarrow denotes a full embedding and $\tilde{\rightarrow}$ denotes an equivalence of categories. Furthermore if $(\bar{\gamma})^{-1}$ is a quasi-inverse² for $\bar{\gamma}$, then the fully faithful functor

$$Ho\mathcal{C}_c \xrightarrow[\sim]{\tilde{\rightarrow}} Ho\mathcal{C} \xrightarrow[\sim]{(\bar{\gamma})^{-1}} \pi\mathcal{C}_{cf} \hookrightarrow \pi\mathcal{C}_c$$

is right adjoint to $\bar{\gamma}_c$ and the fully faithful functor

$$Ho\mathcal{C}_f \xrightarrow[\sim]{\tilde{\rightarrow}} Ho\mathcal{C} \xrightarrow[\sim]{(\bar{\gamma})^{-1}} \pi\mathcal{C}_{cf} \hookrightarrow \pi\mathcal{C}_f \hookrightarrow \pi\mathcal{C}_{cf}$$

is left adjoint to $\bar{\gamma}_f$.

Corollary 1. If A is cofibrant and B is fibrant, then

$$\text{hom}_{Ho\mathcal{C}}(A, B) = \pi(A, B)$$

The category \mathcal{C} can have different model structures on it, but same $Ho\mathcal{C}$, i.e. the weak equivalences are same but fibrations and cofibrations can be different.

²Definition

2 Loop and suspension functors

Let \mathcal{C} be a fixed model category and $f, g : A \rightarrow B$ be two maps with A cofibrant and B fibrant.

Define left homotopy between left homotopies and right homotopy between right homotopies in the analogous way.

Let $h : A \times I \rightarrow B$ be a left homotopy from f to g and $k : A \rightarrow B^I$ be a right homotopy from f to g . By a **correspondence** between h and k we mean a map $H : A \times I \rightarrow B^I$ such that $H\partial_0 = k, H\partial_1 = sg, d_0H = h, d_1H = g\sigma$. (here $\sigma : A \times I \rightarrow A$ and $s : B \rightarrow B^I$ are weak equivalences.) We use the following diagrams to indicate the situation:

$$\begin{array}{ccc}
 & & g \\
 & & \downarrow k \\
 f & \xrightarrow{h} & g \\
 & & f
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & g & \xrightarrow{g\sigma} & g \\
 & & \downarrow k & & \downarrow sg \\
 & & f & \xrightarrow{h} & g
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{sg} & B^I \\
 \partial_1 \downarrow & \nearrow H & \downarrow (d_0, d_1) \\
 A \times I & \xrightarrow{(h, g\sigma)} & B \times B
 \end{array}$$

lemma 1. Given $A \times I$ and a right homotopy $k; A \rightarrow B^I$ there is a left homotopy $h : A \times I \rightarrow B$ corresponding to k . Dually given B^I and h , there is a k corresponding to h .

lemma 2. Suppose that $h : A \times I \rightarrow B$ and $h' : A \times I' \rightarrow B$ are two left homotopies from f to g and that $k : A \rightarrow B^I$ is a right homotopy from f to g . Suppose that h and k correspond, then h' and k correspond iff h' is left homotopic to h .

These two lemmas make left homotopy between left homotopies an equivalence relation, denoted by $\pi_1^l(A, B; f, g)$. Correspondence yields a bijection $\pi_1^l(A, B; f, g) \simeq \pi_1^r(A, B; f, g)$. So denoting this as $\pi_1(A, B; f, g)$, an element of this is a homotopy class of homotopies from f to g .

Definition 1. Let $f_1, f_2, f_3 \in \text{hom}(A, B)$. Let $h : A \times I \rightarrow B$ be a left homotopy from f_1 to f_2 and $h' : A \times I' \rightarrow B$ be a left homotopy from f_2 to f_3 . By the **composition** of h and h' , denoted $h \cdot h'$, we mean the homotopy $h'' : A \times I'' \rightarrow B$ given by $h''in_1 = h, h''in_2 = h'$ where $A \times I''$ is the path object constructed from amalgamating two path objects in 3. If $f, g \in \text{hom}(A, B)$ and $h : A \times I \rightarrow B$ is a left homotopy from f to g , then by the **inverse** of h , denoted h^{-1} , we mean the left homotopy h' from g to f , where $A \times I'$ is the path object for A given by $A \times I' = A \times I, \partial'_0 = \partial_1, \partial'_1 = \partial_0, \sigma' = \sigma, h' = h$.

Lemma 2.1. Given $A \times I$ and a right homotopy $k : A \rightarrow B^I$ there is a left homotopy $h : A \times I \rightarrow B$ corresponding to k . Dually given B^I and h , there is a k corresponding to h .

lemma 3. Suppose that $h : A \times I \rightarrow B$ and $h' : A \times I' \rightarrow B$ are two left homotopies from f to g and that $k : A \rightarrow B'$ is a right homotopy from f to g . Suppose that h and k correspond. Then h' and k correspond iff h' is left homotopic to h .

Corollary 1. "is left homotopic to" is an equivalence relation on the class of left homotopies from f to g and the equivalence classes form a set $\pi_1^l(A, B; f, g)$. Dually right homotopy classes of right homotopies form a set $\pi_1^r(A, B; f, g)$. Correspondence yields a bijection $\pi_1^l(A, B; f, g) \simeq \pi_1^r(A, B; f, g) = \pi_1(A, B; f, g)$.

Definition 2. Let $f_1, f_2, f_3 \in \text{hom}(A, B)$, let $h : A \times I \rightarrow B$ be a left homotopy from f_1 to f_2 and let $h' : A \times I' \rightarrow B$ be a left homotopy from f_2 to f_3 . By the composition of h and h' , denoted $h \cdot h'$, we mean the homotopy $h'' : A \times I'' \rightarrow B$ given by $h''in_1 = h, h''in_2 = h'$ where $A \times I''$ is the path object constructed in 3. If $f, g \in \text{Hom}(A, B)$ and $h : A \times I \rightarrow B$ is a left homotopy from f to g , then by the inverse of h , denoted h^{-1} , we mean the left homotopy $h' : A \times I' \rightarrow B$ from g to f , where $A \times I'$ is the path object for A given by $A \times I' = A \times I$, $\partial'_0 = \partial_1, \partial'_1 = \partial_0, \sigma' = \sigma$ and where $h' = h$.

Proposition. Composition of left homotopies induces maps $\pi_1^l(A, B; f_1, f_2) \times \pi_1^l(A, B; f_2, f_3) \rightarrow \pi_1^l(A, B; f_1, f_3)$ and similarly for right homotopies. Composition of left and right homotopies is compatible with the correspondence bijection of the corollary to above lemma. Finally the category with objects $\text{hom}(A, B)$, with a morphism from f to g defined to be an element of $\pi_1(A, B; f, g)$, and with composition of morphisms defined to be induced by composition of homotopies, is a groupoid. The inverse of an element of $\pi_1^l(A, B; f, g)$ represented by h being represented by h^{-1} .

lemma 4. The diagram commutes

$$\begin{array}{ccc} \pi_1(A, B; f, g) & \xrightarrow{i^*} & \pi_1(A', B; fi, gi) \\ \downarrow j_* & & \downarrow j_* \\ \pi_1(A, B'; jf, jg) & \xrightarrow{i^*} & \pi_1(A', B'; jfi, jgi) \end{array}$$

Definition 3. A **pointed category** is a category \mathcal{C} , in which the initial object and final object exist and are isomorphic, denoted by \star and call it the null object of \mathcal{C} . If X and Y are arbitrary objects of \mathcal{C} , we denote by $0 \in \text{hom}(X, Y)$ the composition $X \rightarrow \star \rightarrow Y$. If $f : X \rightarrow Y$ is a map in \mathcal{C} , we define the **fibre** of f to be the fibre product $\star \times_Y X$ and the **cofibre** of f to be the cofiber product of $\star \vee_Y X$.

By a **pointed model category** we mean a model category \mathcal{C} , which is also a pointed category. If A is in \mathcal{C}_c and $B \in \mathcal{C}_f$, then we write $\pi_1(A, B; 0, 0)$ as $\pi_a(A, B)$ which is a group.

Theorem 2.2. Let \mathcal{C} be a pointed model category. Then there is a functor $A, B \rightarrow [A, B]_1$ from $(\text{HoC})^\circ \times \text{HoC} \rightarrow \{\text{groups}\}$ which is determined up to canonical isomorphism by $[A, B]_1 = \pi_1(A, B)$ if A is cofibrant and B is fibrant. Furthermore, there are two functors from HoC to HoC , the suspension functor Σ and the loop functor Ω and canonical isomorphisms

$$[\Sigma A, B] \simeq [A, B]_1 \simeq [A, \Omega B]$$

of functors $(\text{HoC})^\circ \times (\text{HoC}) \rightarrow (\text{sets})$ where $[X, Y] = \text{Hom}(X, Y)$.

We also use ΣE to denote the cofiber of map $\partial_0 + \partial_1 : A \vee A \rightarrow A \times I$. $\underline{\Sigma}$ is used when needed to denote the functor. Σ and Ω are adjoint functors on $Ho\mathcal{C}$ and are unique up to canonical isomorphism. Also for any X , $\Sigma^n X$ is a cogroup object for $n \geq 1$ and $\Omega^n X$ is a group object in $Ho\mathcal{C}$, which is commutative for $n \geq 2$.