Stable Homotopy theory and Spectral sequences

Naren

Contents

※ Underlying Theorems

We begin with noting down some theorems from algebraic topology and later about Brown Representability theorem. For the following theorems we refer to [H] for proofs.

Theorem 1.1. *Let X be a CW complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cap B$. If (A,C) is m-connected and (B,C) is n-connected, $m,n \geq 0$, then the map $\pi_i(A,C) \to \pi_i(X,B)$ induced by inclusion is an isomorphism for $i < m+n$ and a surjection for $i = m+n$.*

This yields the Freudenthal Suspension theorem

Theorem 1.2. *The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$. More generally this holds for the suspension $\pi_i(X) \to \pi_{i+1}(SX)$ whenever X is an $(n-1)$-connected CW complex.*

Let X and Y be CW complexes with basepoints. The suspension ΣX , or equivalently reduced suspension, be either $S^1 \wedge X$ or $X \wedge S^1$. Suspension induces a function

$$S : [X,Y] \to [\Sigma X, \Sigma Y]$$

Theorem 1.3. *Suppose that Y is $(n-1)$-connected. Then S is onto if $\dim X \leq 2n-1$ and is a $1-1$ correspondence if $\dim X < 2n-1$.*

Under these circumstances we call an element of $[X,Y]$ a stable homotopy class of maps.

We define the notion of a cohomology operation. It is a natural transformation

$$\phi : H^n(X,Y;\pi) \to H^m(X,Y;G)$$

where n runs over $\mathbb{Z}$. The map is subject to the axiom: if $f : X,Y \to X',Y'$ and $h \in H^n(X',Y';\pi)$ then $\phi(f^*h) = f^*(\phi h)$ (See [H]).

A stable cohomology operation is said to be a collection of cohomology operations, say

$$\phi_n : H^n(X,Y;\pi) \to H^{n+d}(X,Y;G)$$

Here n runs over $\mathbb{Z}$. Each ϕ_n is required to be natural, as above and the following diagram be commutative for each n.

$$\begin{tikzcd}
H^n(Y,Z;\pi) \arrow[r, "\delta"] \arrow[d, "\phi_n"'] & H^{n+1}(Y,Z;\pi) \arrow[d, "\phi_{n+1}"] \\
H^{n+d}(Y,Z;G) \arrow[r, "\delta"] & H^{n+d+1}(Y,Z;G)
\end{tikzcd}$$

1.1 Brown Representability Theorem

Let $\mathcal{C}$ be a locally small category, i.e., a category such that for any object C and C' in $\mathcal{C}$, the class of morphisms $\mathcal{C}(C,C')$ is a set. Let C_0 be a fixed object of $\mathcal{C}$. We define the contravariant functor :

$$\begin{aligned}\mathcal{C}(-,C_0) : \mathcal{C} &\longrightarrow \text{ Set}\\ C &\longmapsto \mathcal{C}(C,C_0)\\ C \xrightarrow{f} C' &\longmapsto f^* : \mathcal{C}\left(C',C_0\right) \to \mathcal{C}(C,C_0)\end{aligned}$$

where $f^*(\varphi) = \varphi \circ f$, for any φ in $\mathcal{C}(C',C_0)$

Definition 1.4 (Representable Contravariant Functor)**.** Let $\mathcal{C}$ be a locally small category. A contravariant functor $F : \mathcal{C} \to$ Set is said to be representable if there is an object C_0 in $\mathcal{C}$ and a natural isomorphism :

$$e : \mathcal{C}(-,C_0) \Rightarrow F$$

We say that C_0 represents F, and C_0 is a classifying object for F.

Lemma 1.5 (Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category. Let $F : \mathcal{C} \to$ Set be a contravariant functor. For any object C_0 in $\mathcal{C}$, there is a one-to-one correspondance between natural transformation e: $\mathcal{C}(-,C_0) \Rightarrow F$ and elements u in $F(C_0)$, which is given, for any object C in $\mathcal{C}$, by:

$$\begin{aligned}e_C : \mathcal{C}(C,C_0) &\longrightarrow F(C)\\ \varphi &\longmapsto F(\varphi)(u).\end{aligned}$$

We now introduce Brown functors and discuss about their representabiliy.

Definition 1.6 (Brown Functors)**.** Let $\mathcal{T}$ be a full subcategory of Top $_*$. A Brown functor $h : \mathcal{T} \to$ Set is a contravariant homotopy functor, which respects the following axioms.

Additivity/Wedge axiom For any collection $\left\{X_j \mid j \in \mathcal{J}\right\}$ of based spaces in $\mathcal{T}$, the inclusion maps $i_j : X_j \hookrightarrow \bigvee_{j\in\}} X_j$ induce an isomorphism on Set:

$$\left(h\left(i_j\right)\right)_{j\in\mathcal{Z}} : h\left(\bigvee_{j\in\mathcal{Z}} X_j\right) \xrightarrow{\cong} \prod_{j\in\mathcal{Z}} h\left(X_j\right).$$

Mayer-Vietoris For any excisive triad $(X;A,B)$ in $\mathcal{T}$, if a is in $h(A)$, and b is in $h(B)$, such that $a|_{A\cap B} = b|_{A\cap B}$, then there exists x in $h(X)$, such that $x|_A = a$ and $x|_B = b$.

Any generalised cohomology theory on CW_* defines a Brown functor in each dimension.

Proposition 1.7. Let h be a Brown functor. If X is a co-H-group then $h(X)$ is a group.

Theorem 1.8 (Brown Representability Theorem)**.** $h : CW_* \to Set_*$ be a brown fucntor. Then h is representable.

So when the functor h on CW_* is representable, then thre exists a vased CW complex E such that there exists a natural isomorphism,

$$e : [-E]_* \Longrightarrow h$$
$$e_X([f]_*) = h(f)(u)$$

where $f : X \to E$ and $u \in h(E)$ is the universal element of h.

The representability theorem is also valid in the category of CW spectra and morphisms of degree 0.

※ Spectra

A *spectrum E* is a sequence of spaces E_n with basepoint, provided with structure maps, $\varepsilon_n : \Sigma E_n \to E_{n+1}$ or equivalently $\varepsilon_n' : E_n \to \Omega E_{n+1}$.

Example. To illustrate this we give an example where cohomology theory gives rise to a CW spectrum. Let K^* be a generalized cohomology theory, defined on CW pairs. We have $K^n(X) = K^n(X, pt.) + K^n(pt.)$ aand define $\tilde{K^n}(X) = K^n(X, pt)$. We assume K^* satifies the wedge axiom.
We can apply Brown representability theorem and say that there exist connected CW-complexes E_n with base point and natural equivalences such that

$$\tilde{K^n}(X) \cong [X, E_n]$$

Consider the suspension isomorphism $\Sigma : \tilde{K^n}(X) \xrightarrow{\cong} K^{\tilde{n+1}}(\Sigma X)$. The suspension isomorphism is defined with the follwing commutative diagram:

$$\begin{tikzcd}
K^n(X,pt) \arrow[r, "\delta\cong"'] \arrow[ddr, "\Sigma"] & K^{n+1}(CX,X) \\
 & K^{n+1}(\Sigma X, C'X) \arrow[u, "\cong\text{excision}"] \arrow[d, "\cong C'X\text{contractible}"'] \\
 & K^{n+1}(\Sigma X, pt)
\end{tikzcd}$$

The map δ is the coboudnary for the exact sequence fo the triple $(CX, X, pt.)$.(Here CX and $C'X$ are the two cones that make up ΣX)

We have now natural equivalences

$$\begin{aligned}
[X, E_n] &\cong \widetilde{K^n}(X) \cong \widetilde{K^{n+1}}(\Sigma X) \\
&\cong [\Sigma X, E_{n+1}] \cong [X, \Omega_0 E_{n+1}] .
\end{aligned}$$

This natural equivalence must be induced by a weak equivalence:

$$\varepsilon_n' : E_n \to \Omega E_{n+1}$$

So this sequence of spaces becomes a spectrum. This spectrum is also called $\Omega-$*spectrum* (assuming all the spaces are connected for convenience)

We define a spectrum F to be a *suspension spectrum* or *S-spectrum* if

$$\varphi_n : \Sigma F_n \longrightarrow F_{n+1}$$

is a weak homotopy equivalence for n sufficiently large.

Example. Given a CW-complex X, let $E_n = \begin{cases} \Sigma^n X & (n \geq 0) \\ pt & (n < 0) \end{cases}$ with the obvious maps. Then this spectrum E would be an S-spectrum, but need not be an Ω-spectrum. E is called the suspension spectrum of X.
In particular, the sphere spectrum S is the suspension spectrum of S^0; it has n^{th} term S^n for $n \geq 0$.

We now define homotopy groups of a spectrum. consider the following homomorphisms

$$\pi_{n+r}(E_n) \longrightarrow \pi_{n+r+1}(\Sigma E_{n+1}) \xrightarrow{(\varepsilon_n)_*} \pi_{n+r+1}(E_{n+1})$$

We define the stable homotopy groups:

$$\pi_r(E) = \varinjlim_n \pi_{n+r}(E_n)\ ^{1}$$

If E is an Ω-spectrum then the homomorphism

$$\pi_{n+r}(E_n) \longrightarrow \pi_{n+r+1}(E_{n+1})$$

is an isomorphism for $n + r \geq 1$; the limit is attained, and we have

$$\pi_r(E) = \pi_{n+r}(E_n) \qquad \text{for } n + r \geq 1.$$

In the case of Suspension spectrum, we have $\pi_r(E) = \varinjlim_n \pi_{n+r}(\Sigma^n X)$. The limit is attained for $n > r + 1$. In this case we have the homotopy groups of E are the stable homotopy grops of X.

Similarly we define relative homotopy groups. Let X be a spectrum, then a subspectrum A of X consist of subspaces $A_n \subset X_n$ such that the spectrum maps $\xi_n : \Sigma X_n \to X_{n+1}$ maps ΣA_n into A_{n+1}. We define the relative homotopy groups as

$$\pi_r(X, A) = \varinjlim_n \pi_{n+r}(X_n, A_n)$$

and we get a exact sequence

$$\cdots \to \pi_*(A) \to \pi_*(X) \to \pi_*(X, A) \to \pi_*(A) \to \cdots$$

2.1 Stable Homotopy Category

E is called a CW spectrum if

[1] in this case colimit=$\lim_n$

1. the terms E_n are CW-complexes with base point and
2. each map $\varepsilon_n : \Sigma E_n \to E_{n+1}$ is an isomorphism from ΣE_n to a sub-complex of E_{n+1}.

A subspectrum A of a CW spectrum E is as defined before, with the added condition that $A_n \subset X_n$ for each n. Let E be a CW-spectrum, E' a subspectrum of E. We say E' is cofinal in E if for each n and each finite subcomplex $K \subset E_n$ there is an m(depnding on n and K) such that $\Sigma^m K$ maps into E'_{m+n} under the map

$$\Sigma^m E_n \xrightarrow{\Sigma^{m-1}\varepsilon_n} \Sigma^{m-1} E_{n+1} \longrightarrow \ldots \longrightarrow E_{m+n-1} \xrightarrow{\varepsilon_{m+n-1}} E_{m+n}.$$

A *function* f from one spectrum E to another spectrum F of degree r is a sequence of maps $f_n : E_n \to F_{n-r}$ such that the following diagram is structly commutative for each n

$$\begin{tikzcd}
\Sigma E_n \arrow[r, "\varepsilon_n"] \arrow[d, "\Sigma f_n"] & E_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma F_{n-r} \arrow[r, "\phi_{n-r}"] & F_{n-r+1}
\end{tikzcd}$$

or equivalently maps in the Ω spectrum. Note here that the diagrams are to be strictly commutative and not commutative up to homotpy.

Let E be a CW spectrum and F be a CW spectrum. take all cofinal subspectra $E' \subset E$ and all functions $f' : E' \to F$. Say that two functions $f' : E' \to F$ and $f" : E'' \to F$ are equivalent if there is a cofinal subspectrum E''' contained in E' and E'' such that the restrictions of f' and f'' to E''' coincide.

A *map* from E to F is an equivalence class of such functions. This is saying that if we have a cell c in E_n, a map need not be defined on it at once; we can wait till E_{m+n} before defining the map on $\Sigma^m c$. This is equivalent to saying that two functios $f' : E' \to F$ and $f'' : E'' \to F$ are equivalent if their restrictions to $E' \cap E''$ coincide.

Lemma 2.1. Let $f : E \to F$ be a function and F' a cofinal subspectrum of F.. Then there is a cofinal subspectrum E' of E such that f maps E' into F'.

Proof. Consider the collection of all subspectra G such that $f(G) \subseteq F'$. This collection is nonempty because it contains the basepoint. Let E' be the union of these subspectra. Then $f(E') \subseteq F'$. It remains to show that E' is a cofinal subspectrum. Let K be a finite complex in E_n. Consider $f_n(K)$, this is contained in a finite subcomplex $H \subseteq F_n$. This is because f_n is cellular. As F' is cofinal, there is a d such that $\Sigma^d H \subseteq F'_{n+d}$. Thus $f_{n+d}\left(\Sigma^d K\right) \subseteq F'_{n+d}$. So $\Sigma^d K \subseteq E'_{n+d}$.

□

Let I^+ be the union of the unit interval and a disjoint base-point. For E a spectrum, we define $Cyl(E)$ is the cylinder spectrum and has terms

$$Cyl(E)_n = I^+ \wedge E_n$$

and maps

$$(I^+ \wedge E_n) \wedge S^1 \xrightarrow{1\wedge\varepsilon_n} I^+ \wedge E_{n+1}$$

The cylinder spectrum is a functor: a map $f : E \to F$ indues a map $Cyl(f) : Cyl(E) \to Cyl(F)$.

Two maps $f, g : E \to F$ are homotopic if there is a map $h : Cyl(E) \to F$ such that the following diagram commutes

$$\begin{tikzcd} E \wedge 0^+ \arrow[r] \arrow[dr, "f"'] & Cyl(E) \arrow[d, "h"] & E \wedge 1^+ \arrow[l] \arrow[dl, "g"] \\ & F & \end{tikzcd}$$

A *morphism* in the category $CWsp$ will be a homotopy class of maps. We write $[E,F]_r$ for the set of homotopy classes of maps with degree r.

The *stable homotopy category*, denoted SHC is the category whose objects are the same as those of $CWSp$ and whose morphisms are the homotopy classes of maps. That is $\mathrm{SHC}(E,F) := [E,F]$ for CW spectra E and F.

As long as we deal entirely with CW spectra we can restrict attention to functions whose components $f_n : E_n \to F_{n-r}$ are cellular maps.

Proposition 2.2. Let K be a finite CW-complex and let R be its suspension spectrum, so that $E_n = \Sigma^n K$ for $n \geq 0$. Let F be any spectrum.
We have

$$[E,F]_r = \underrightarrow{\mathrm{colim}}_n [\Sigma^{n+r} K, F_n]$$

In particular,

$$[S,F]_r = \pi_r(F)$$

Proof. [A, Pg 164] □

Let C_n be the set of cells in E_n other than the base-point. Then we get a function $C_n \to C_{n+1}$ by $C_\alpha \mapsto \varepsilon_n(\Sigma C_\alpha)$. This function is an injection. Let $C = \lim_{n\to\infty} C_n$. An element of C is called *stable cell*. A stable cell is essentially an equivalence class that contains atmost one cell in E_n. If there are only finite number of stable cells in a spectra we call that a finite spectra.

We state the homotopy extension theorem in $CWsp$ and refer to [A] for proof.

Lemma 2.3. Let X, A be a pair of CW-spectra, and Y, B a pair of spectra such that $\pi_*(Y, B) = 0$. Suppose given a map $f : X \longrightarrow Y$ and a homotopy $h : \mathrm{Cyl}(A) \longrightarrow Y$ from $f|_A$ to a map $g : A \longrightarrow B$. Then the homotopy can be extended over $\mathrm{Cyl}(X)$ so as to deform f to a map $X \longrightarrow B$.

$$\begin{tikzcd}
A \arrow[rr, "in_0"] \arrow[dd, "\iota"'] & & A \wedge I_+ \arrow[dl, "h"] \arrow[dd] & & A \arrow[ll, "in_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Y & & B \arrow[ll] & \\
X \arrow[rr] \arrow[ur, "f"] & & X \wedge I_+ \arrow[ul, dotted] & & X \arrow[ll, "in_1"] \arrow[ul, dotted]
\end{tikzcd}$$

The homotopy extension theorem is a special case when $B = Y$.

Lemma 2.4. Suppose $\pi_*(Y) = 0$ and X, A is a pair of CW-spectra. Then any map $f : A \to Y$ can be extended over X.

Proof. Applying the previous lemma to the pair $(A, *)$ and $(Y, *)$ we get that f is null-homotopic. We have $h : Cyl(A) \to Y$ a homotopy from f to a map $g : A \to *$. Then there exists an extension of h, $\tilde{h} : Cyl(X) \to Y$.

$\square$

Theorem 2.5. Let $f : E \to F$ be function between spectra(need not be CW) such that $f_* : \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then for any CW-spectrum X,

$$f_* : [X, E]_* \to [X, F]_*$$

is an isomorphism.

Proof. We can replace F by the spcetrum M in which M_n is the mapping cylinder of f_n and assume that f is an inclusion. Then $\pi_*(F, E) = 0$ by the exact sequence. Now consider $(X, *)$ and apply 2.4. This gives us f_* is an epimorphism. For proving monomorphism consider 2.4 for the pair $(X \wedge I_+, X \wedge (\partial I)_+)$ (i.e. $Cyl(X)$ mod its ends).

$\square$

Corollary 2.6. Let $f : E \to F$ be a morphism between CW-spectra such that $f_* : \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then f is an equivalence in our category.

Lemma 2.7. Any CW spectrum Y is equivalent in the SHC to an Ω spectrum.

Proof. Let us consider a functor $T^{(n)}$ from CW complexes to spectra given by

$$(T^{(n)}X)_r = \begin{cases} \Sigma^{r-n}X & r \geq n \\ \text{pt.} & r < n \end{cases}$$

Form the set of morphisms $[T^{(n)}X, Y]_0$, which is a Brown functor and is representable. Let it be represented by Z_n.

$$[X, Z_n] \approx [T^{(n)}X, Y] \approx [T^{(n+1)}(\Sigma X), Y] \approx [\Sigma X, Z_{n+1}] \approx [X, \Omega Z_{n+1}]$$

Thus Z is an Ω spectrum. Take $X = Y_n$,

$$[T^{(n)}Y_n, Y] \approx [Y_n, Z_n].$$

Take the map $f_n : Y_n \to Z_n$ that corresponds to the equivalence class of functions $\phi_n : (T^{(n)}Y_n)_n = Y_n \to Y_n$. Since $[Y_n, Z_n]$ is a group f_n has an inverse. Consider function f with the sequence of maps f_n, this induces isomorphism

$$f_* : \pi_*(Y) \to \pi_*(Z).$$

Applying 2.6 gives the desired conclusion. $\square$

If X is a spectrum, let $Cone(X)$ be the spectrum whose nth term is $I \wedge X_n$ with maps

$$(I \wedge X_n) \wedge S^1 \xrightarrow{1\wedge\varepsilon_n} I \wedge X_{n+1}$$

Theorem 2.8. Let $f : E, A \longrightarrow F, B$ be a function between pairs of spectra such that

$$f_* : \pi_*(E, A) \longrightarrow \pi_*(F, B)$$

is an isomorphism. Then for any CW-spectrum X,

$$f_* : [\text{Cone}(X), X; E, A]_* \longrightarrow [\text{Cone}(X), X; F, B]_*$$

is an isomorphism.

For any spectrum X, define $Susp(X)$ to be the spectrum whose nth terms is $S^1 \wedge X_n$ and its structure maps are

$$(S^1 \wedge X_n) \wedge s^1 \xrightarrow{1\wedge\xi_n} S^1 \wedge X_{n+1}.$$

Susp is a functor.

Theorem 2.9. *Susp* : $[X,Y]_* \to [Susp(X), Susp(Y)]_*$ is an isomorphism.

Proof. [A, Theorem 3.7]

□

This shows the the sets of morphism $[X,Y]$ are abelian groups and also that the compositions are bilinear.

We would like to show that we have an additive category. We take the spectrum E_n=pt. for all n to be the trivial object. This category has arbitrary sums(=coproducts). Given spectra X_α for $\alpha \in A$, we form $X = \bigvee_\alpha X_\alpha$ by $X_n = \bigvee (X_\alpha)_n$, with structure maps

$$X_n \wedge S^1 = \left(\bigvee_\alpha (X_\alpha)_n\right) \wedge S^1 = \bigvee_\alpha (X_\alpha) \wedge S^1 \xrightarrow{\bigvee_\alpha \xi_{\alpha n}} \bigvee_\alpha (X_\alpha)_{n+1}$$

This has the required property:

$$\left[\bigvee_\alpha X_\alpha, Y\right] \xrightarrow{\cong} [X_\alpha, Y]$$

We now look at cofiber sequences for CW spectra. Suppose given a map $f : X \longrightarrow Y$ between CW-spectra. Let it be represented by a function $f' : X' \longrightarrow Y$, where X' is a cofinal subspectrum. Without loss of generality we can suppose f' is cellular. We form the mapping cone $Y \cup_{f'} CX$ as follows: its n^{th} terms is $Y_n \cup_{f'_n} (I \wedge X'_n)$ and the structure maps are the obvious ones. If we replace X' by a smaller cofinal subspectrum X'', we get $Y \cup_{f''} CX''$ which is smaller than $Y \cup_{f'} CX'$, but cofinal in it, and so equivalent. So the construct depends essentially only on the map f, and we can write it $Y \cup_f CX$. If we vary f by a homotopy, $Y \cup_{f_0} CX$ and $Y \cup_{f_1} CX$ are equivalent, but the equivalence depends on the choice of homotopy.

Let X be a CW-spectrum, A a subspectrum. A is closed if for ever finite subcomplex $K \subset X_n, \Sigma^m K \subset A_{m+n}$ implies $K \subset A_n$. It is equivalent to saying that $A \subset B \subset X, A$ cofinal in B implies that $A = B$.

Proposition 2.10. Suppose we have a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$$

Then for each Z the sequence

$$[Y \cup_f CX, Z] \xrightarrow{i^*} [Y,Z] \xrightarrow{f^*} [X,Z]$$

is exact.

Proposition 2.11. Suppose we have a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$$

The sequence

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} \left[W, Y \cup_f CX\right]$$

is exact.

We can extend the cofibre sequence further and similar to CW complexes we get

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{j} Susp(X) \xrightarrow{Susp(f)} Susp(Y)$$

In other words, in SHC cofiberings are the same as fibering.

Proposition 2.12. Finite sums are products.

Proof. We have

$$X \to X \vee Y \to Y$$

which is clearly a cofiber. So we have the exact sequence

$$[W,X] \to [W, X \vee Y] \to [W,Y].$$

The map $Y \xrightarrow{i} X \vee Y$ is a section so the exact sequence splits.

$$[W, X \vee Y] \cong [W,X] \oplus [W,Y]$$

and $X \vee Y$ is also the product of X and Y □

This proves that SHC is an additive category.

As noted down earlier, the Brown representability theorem is valid in the category of CW-spectra and morphisms of degree 0.

Proposition 2.13. Any spectrum Y is weakly equivalent to a CW-spectrum.

Proof. Consider the representible functor $[X,Y]_0$. $[X,K] \approx [X,Y]_0$ for some CW spectrum K. We consider $X = K$ and take the image of id. □

Proposition 2.14. The SHC has arbitrary product.

Proof. The functor of X given by $\prod_\alpha [X, Y_\alpha]_0$ is a Brown Functor and is representable. (This works out for maps of degree r as well but how?) □

For any collection of X_α we have a morphism $\bigvee_\alpha X_\alpha \to \prod_\alpha X_\alpha$.

Proposition 2.15. Suppose that for each n $\pi_n(X_\alpha) = 0$ for all but a finite number of α then the map

$$\bigvee_\alpha X_\alpha \to \prod_\alpha X_\alpha$$

is an equivalence.

Proof. We have

$$\pi_n(X_1 \vee \cdots \vee X_m) \cong \sum_{i=1}^{m} \pi_n(X_i)$$

for finite wedges. By passing to direct limits(how?) we have

$$\pi_n(\bigvee_\alpha X_\alpha) = \sum_\alpha \pi_n(X_\alpha)$$

Also

$$\pi_n\left(\prod_\alpha X_\alpha\right) = \prod_\alpha \pi_n\left(X_\alpha\right)$$

Now the data was chosen precisely so that $\sum_\alpha \pi_n\left(X_\alpha\right) \longrightarrow \prod_\alpha \pi_n\left(X_\alpha\right)$ is an isomorphism. Therefore $\bigvee_\alpha X_\alpha \longrightarrow \prod_\alpha X_\alpha$ is an equivalence. $\square$

※ Smash Products

In this section we will construct smash product. Given two CW spectra X and Y, we construct a CW spectrum $X \wedge Y$ so as to have the properties stated in the following theorem, among other properties.

Theorem 3.1. 1. $X \wedge Y$ is a functor of two variables, with arguments and values in the (graded) SHC.

2. The smash-product is associative, commutative and has the sphere spectrum S as a unit, up to coherent natural equivalences.

Statement 1 is to be taken in the graded sense. That is for $f \in [X,X']_r$ and $g \in [Y,Y']_s, f \wedge g \in [X \wedge Y, X' \wedge Y']_{r+s}$ and also $(f \wedge g)(h \wedge k) = (-1)^{bc}(fh) \wedge (gk)$ if $f \in [X',X'']_a, h \in [X,X']_b, g \in [Y',Y'']_c, k \in [Y,Y']_d$.

The following equivalences hold true in our category.

$$\begin{array}{cc} a & a(X,Y,Z) : (X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z) \\ c = & C(X,Y) : X \wedge Y \longrightarrow Y \wedge X \\ l = & l(Y) : S \wedge Y \longrightarrow Y \\ r = & r(X) : X \wedge S \overset{\longrightarrow}{\longrightarrow} Y \end{array}$$

These are all maps of degree 0 and are natural. The naturality for commutativity is up to a sign $(-1)^{rs}$, if $f \in [X,X']_r$ and $g \in [Y,Y']_s$.

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{c} & W \wedge Y \\ \Big\downarrow{\scriptstyle f\wedge g} & & \Big\downarrow{\scriptstyle g\wedge f} \\ X' \wedge Y' & \xrightarrow{c} & Y' \wedge X' \end{array}$$

Let A be an ordered set isomorphic to $\{0,1,2,3,\dots\}$. Suppose we have a partition of A into two subsets B and C, so that $A = B \cup C$ and $B \cap C = \phi$. Then we define a smash product functor which assigns to any two CW spectra X and Y a CW spectrum $X \wedge_{BC} Y$. The terms of this product spectrum $P = X \wedge_{BC} Y$ are given by by $P_{\alpha(a)} = X_{\beta(a)} \wedge Y_{\gamma(a)}$.Here α is an isomorphism from $A = B \cup C$ to the set $\{0,1,2,3,\dots\}$ and β, γ are monotonic functions. such that $\beta(a) + \gamma(a) = \alpha(a)$. This is called handicrafted or naive smash products.

The maps of the prodcut spectrum are defined as follows. We have

$$P_{\alpha(a)} \wedge S^1 = X_{\beta(a)} \wedge Y_{\gamma(a)} \wedge S^1.$$

We regard S^1 as one point compactification of $\mathbb{R}$, where infinity becomes the base point. This allows us to define a map of degree -1 from S^1 to S^1. by $t \mapsto -t$.

If $a \in B$, then

$$P_{\alpha(a)+1} = X_{\beta(a)+1} \wedge Y_{\gamma(a)}$$

and we define the map

$$\pi_{\alpha(a)} : SP_{\alpha(a)} \longrightarrow P_{\alpha(a)+1}$$

by

$$\pi_{\alpha(a)}(x \wedge y \wedge t) = \xi_{\beta(a)}\left(x \wedge (-1)^{\gamma(a)} t\right) \wedge y$$

If $a \in C$, then

$$P_{\alpha(a)+1} = X_{\beta(a)} \wedge Y_{\gamma(a)+1}$$

and we define the map

$$\pi_{\alpha(a)}(x \wedge y \wedge t) = x \wedge \eta_{\gamma(a)}(y \wedge t).$$

Here

$$x \in X_{\beta(a)}, \quad y \in Y_{\gamma(a)}, \quad t \in S^1,$$

and

$$\xi_{\beta(a)} : X_{\beta(a)} \wedge S^1 \longrightarrow X_{\beta(a)}, \quad \eta_{\gamma(a)} : Y_{\gamma(a)} \wedge S^1 \longrightarrow Y_{\alpha(a)+1}$$

are the appropriate maps from the spectra X, Y. The sign$(-1)^{\gamma(a)}$ is introduced, of course, because we have moved S^1 across $Y_{\gamma(a)}$.
The product P is functorial for functios of X and Y. If B is infinite and X' is cofinal in X, then $X' \wedge_{BC} Y$ is cofinal in $X \wedge_{BC} Y.Cyl(X) \wedge_{BC} Y$ and $X \wedge_{BC} Cyl(Y)$ can be identified with $Cyl(X \wedge_{BC} Y)$.

$X \wedge Y$ is constructed so that it has the following properties.

Theorem 3.2. For each choice of B, C there is a morphism

$$\mathrm{eq}_{BC} : X \wedge_{BC} Y \longrightarrow X \wedge Y \quad (\text{ of degree } 0)$$

with the following properties.

1. If B is infinite and $f : X \longrightarrow X'$ is a morphism of degree 0 , then the following diagram is commutative.

$$\begin{tikzcd}
X \wedge_{BC} Y \arrow[r, "eq_{BC}"] \arrow[d, "f \wedge_{BC} 1"'] & X \wedge Y \arrow[d, "f \wedge 1"] \\
X' \wedge_{BC} Y \arrow[r, "eq_{BC}"] & X' \wedge Y
\end{tikzcd}$$

2. If C is infinite and $g : Y \longrightarrow Y'$ is a morphism of degree 0 , then the following diagram is commutative.

$$\begin{tikzcd} X \wedge_{BC} Y \arrow[r, "eq_{BC}"] \arrow[d, "1\wedge_{BC} g"'] & X \wedge Y \arrow[d, "1\wedge g"] \\ X \wedge_{BC} Y' \arrow[r, "eq_{BC}"] & X' \wedge Y \end{tikzcd}$$

3. The morphism $eq_{BC} : X \wedge_{BC} Y \to X \wedge Y$ is an equivalence if any one of the following conditios is satisfied.

   (a) B and C are infinite.

   (b) B is finite, say with d elements and $\xi_r : \Sigma X_r \to X_{r+1}$ is an isomorphism for $r \geq d$.

   (c) C is finite, say with d elmemetns and $\eta_r : \Sigma Y_r \to Y_{r+1}$ is an isomorphism for $r \geq d$.

The handicrafted smash products are commutative for the right choice of B, C at each point. We partition the sets accordingly with the following condition.
Condition Elements number $0,1,2,3$ in A are either four elemnts in B or four elements in C. similarly for elments number $4,5,6,7$ in A and similarly for elementss numver $4r, 4r+1, 4r+2, 4r+3$ for each r. The smash product has the following property regarding commutativity

Theorem 3.3. The equivalence $c : X \wedge Y \to Y \wedge X$ makes the following diagram commutative for each choice of B, C satsifying the condition stated above

$$\begin{tikzcd} X \wedge Y \arrow[r, "c"] & Y \wedge X \\ X \wedge_{BC} Y \arrow[u, "eq_{BC}"] \arrow[r, "c_{BC}"] & Y \wedge_{CB} X \arrow[u, "eq_{CB}"'] \end{tikzcd}$$

The handicrafted smash products have S as a unit if we pick the right product at each point. Say, we partition $A = \phi \cup A$ satisfying the condition we have S as a unit.

Define

$$l : S \wedge Y \to Y$$

to be the composite

$$S \wedge Y \xleftarrow[\text{equivalence}]{eq_{\phi,A}} A \wedge_{\phi A} Y \cong Y (eq_{\phi,A} \text{ is an equivalence})$$

We also have the isomorphisms $S^0 \wedge Y \cong Y$ and $X \wedge S^0 \cong X$ with the obvious component-wise isomorphism.This is also natural for morphisms of degree 0. we noe define

$$r : X \wedge S \to X$$

to be the composite

$$X \wedge S \xleftarrow{eq_{\phi,A}} X \wedge_{A\phi} S \cong X (eq_{\phi,A} \text{ is an equivalence})$$

Since $S \wedge S$ is equivalent to S, we have $[S \wedge S, S \wedge S]_0 \cong [S,S]_0 \cong \mathbb{Z}$. Also we construct the smash product so that the map $c : S \wedge S \to S \wedge S$ has degree 1.

We refer to [A] for details regarding the construction. We will note down some more properties of the smash product especially regarding sphere-spectra.

Let us define sphere-spectra of different stable dimensions.

$$\left(\underline{S}^i\right)_n = \begin{cases} S^{n+1} & n+i \geq 0 \\ \text{pt.} & n+i < 0 \end{cases}$$

Proposition 3.4. We have an equivalence $\underline{S}^i \wedge \underline{S}^j \xrightarrow{e} \underline{S}^{i+j}$ such that the following diagrams are commutative.

$$\begin{tikzcd}
(\underline{S}^i \wedge \underline{S}^j) \wedge \underline{S}^k \arrow[rr, "a"] \arrow[d, "e\wedge 1"] & & \underline{S}^i \wedge (\underline{S}^j \wedge \underline{S}^k) \arrow[d, "1\wedge e"] \\
\underline{S}^{i+j} \wedge \underline{S}^k \arrow[dr, "e"] & & \underline{S}^i \wedge \underline{S}^{j+k} \arrow[dl, "e"'] \\
& \underline{S}^{i+j+k} &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{S}^i \wedge \underline{S}^j \arrow[r, "c"] \arrow[d, "e"] & \underline{S}^j \wedge \underline{S}^i \arrow[d, "e"] \\
\underline{S}^{i+j} \arrow[r, "(-1)^{ij}"] & \underline{S}^{j+i}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{S}^0 \wedge \underline{S}^j \arrow[r, bend left, "e"] \arrow[r, bend right, "l"'] & \underline{S}^j
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{S}^i \wedge \underline{S}^0 \arrow[r, bend left, "e"] \arrow[r, bend right, "r"'] & \underline{S}^i
\end{tikzcd}$$

Proposition 3.5. We have the equialences

$$\gamma_r : X \to \underline{(S)}^r \wedge X \text{of degree r}$$

with the following properties

1. (i) γ_r is natural for maps of X of degree 0 . (This is all we can ask, because we have not yet made $\underline{S}^r \wedge X$ functorial for maps of non-zero degree.).

2. $\gamma_0 = \ell^{-1}$.

3. The following diagram is commutative for each r and s.

$$\begin{tikzcd}
\underline{(S)}^{r+s} \wedge X & (\underline{S}^r \wedge \underline{S}^s) \wedge X \arrow[l, "e\wedge 1"'] \arrow[d, "a"] \\
& \underline{S}^r \wedge (\underline{S}^s \wedge X) \\
X \arrow[uu, "\gamma_{r+s}"] \arrow[r, "\gamma_s"] & \underline{S}^s \wedge X \arrow[u, "\gamma_r"]
\end{tikzcd}$$

With the above proposition at hand we can the original SHC with a new category where the objects are still CW spectra, but the morphisms of degree r are given by $[\underline{S}^r \wedge X, Y]_0$ in the old category.
Composition is as follows. If we have $\underline{S}^r \wedge X \xrightarrow{f} Y$ and $\underline{S}^r \wedge Y \xrightarrow{g} Z$ of degree 0, take their composite to be

$$\underline{(S)}^{s+r} \wedge X \xleftarrow{e\wedge 1} (\underline{S}^s \wedge \underline{S}^r) \wedge X \xrightarrow{a} \underline{S}^s \wedge (\underline{S}^r \wedge X) \xrightarrow{1\wedge f} \underline{S}^s \wedge Y \xrightarrow{g} Z.$$

The composition is associative and $\ell : \underline{S}^0 \wedge X \longrightarrow X$ is an identity map.

Proposition 3.6. The new graded category is isomorphic to the old SHC, under the isomorphism sending

$$\left\{ \begin{array}{c} \underline{S}^r \wedge X \xrightarrow{f} Y \\ \text{(in the new category)} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} X \xrightarrow{\gamma_r} \underline{S}^r \wedge X \xrightarrow{f} Y \\ \text{(in the old category)} \end{array} \right\}$$

It is an easy to show the naturality of γ_r with respect to maps of degree s: the diagram is commutative up toa sighn of $(-1)^{rs}$ if $f \in [X,Y]_s$.

$$\begin{tikzcd}
X \arrow[r, "\gamma_r"'] \arrow[d, "f"] \arrow[dr, phantom, "(-1)^{rs}"] & \underline{S}^r \wedge X \arrow[d, "1\wedge f"] \\
Y \arrow[r, "\gamma_r"] & \underline{S}^r \wedge Y
\end{tikzcd}$$

The smash product is distributive over the wedge-sum.Let $X = \bigvee_\alpha X_\alpha$; let $i_\alpha : X_\alpha \longrightarrow X$ be a typical inclusion. Then the morphism

$$\bigvee_\alpha (X_\alpha \wedge Y) \xrightarrow{i_\alpha \wedge 1} \left(\bigvee_\alpha X_\alpha \right) \wedge Y$$

is an equivalence.

We end this section with a remark on cofiber sequence of smash product.

Proposition 3.7. Let $X \xrightarrow{f} Y \xrightarrow{i} Z$ be a cofibering (it is sufficient to consider morphisms of degree zero). Then

$$W \wedge X \xrightarrow{1 \wedge f} W \times Y \xrightarrow{1 \wedge i} W \wedge Z$$

is also a cofibering.

Proof. [A] It suffices to check for the case in which $f : X \longrightarrow Y$ is the inclusion of a closed subspectrum, $i : Y \longrightarrow Z$ is the projection $Y \longrightarrow Y/X$ and $\bigwedge = \bigwedge_{BC}$. □

※ Duality

If X is a compact subset embedded in S^n, then by Alexander duality theorem, we have

$$\tilde{H}^k(X) \cong \tilde{H}_{n-k-1}(S^n - X).$$

Let K be homotopy equivalent to $S^n - K$ for K a compact set embedded in S^n.We would like to prove that K determines the stable homotopy type of L. The homotopy type of L in general is not determined by K as it dependes on the embedding of K.

Embed S^n as the equatorial sphere in S^{n+1} and embed the suspension ΣK of K in S^{n+1} by joining to the two poles. Then $S^{n+1} - \Sigma K \simeq S^n - K$. So if we have $K \subset S^n$ and $M \subset S^m$ and a homtopy equivalence $f : \Sigma^p K \to \Sigma^q M$, we can embed $\Sigma^p K$ in S^{n+p} and $\Sigma^q M$ in S^{m+q}, since the complements remain homotopy equivalent. So WLOG, we can say we have $K' \subset S^{n'}$ and $M' \subset S^{m'}$ and a homotopy equivalence $f : K' \to M'$. We can even assume f is piecewise linear.

Now suppose $K \subset S^n$ and embed S^n as an equiatorial sphere in S^{n+1} without changin K. Then $S^{n+1} - K = \Sigma(S^n - K)$. Consider the join of to spheres in which $S^n * S^m \simeq S^{m+n+1}$, K and M are embedded, S^n and S^m respectively. We can embed the mapping cylinder M_c of f in the join. In this sphere we have

$$S^{m+n+1} - K = \Sigma^{m+1}(S^n - K)$$

$$S^{m+n+1} - M = \Sigma^{n+1}(S^m - M)$$

and two maps

$$S^{m+n+1} - K \xleftarrow{f} S^{m+n+1} - M_c \xrightarrow{g} S^{m+n+1} - M$$

But the injective maps

$$K \to M_c \leftarrow M$$

indcue isomorphismc of cohomology. The alexander duality isomorphism is natural for inclusion maps and therefore f and g induce isoorphism of homology. We can suspend further to make everything simply connected. So by Whitehead's theorem, f and g are stable homotopy equivalences. Thus we've proved the assignment $K \mapsto L$ is well-defined, up to stable equivalence, for the suspension spectrum of K. The desuspension is made so that degrees are as expected.

Let X be CW spectrum. Consider the set $[W \wedge X, S]_0$. With X fixed this is a contravariant functor of W and this is now a Brown functor. So it is representable, say by X^* and there is a natural isomorphism

$$[W, X^*]_0 \xrightarrow{T} [W \wedge X, S]_0$$

Taking $W = X^*$ and the *id* map we see that there is a map $e : X \wedge X^* \to S$. Since T is natural it carries, $f \to X^*$ into $W \wedge X \xrightarrow{f\wedge 1} X^* \wedge X \xrightarrow{e} S$.We can then extend this isomorphism to maps of degree r

$$[W, X^*]_r \xrightarrow{T} [W \wedge X, S]_r$$

We can think of X^* as the dual. The dual X^* is a contravariant functor of X. If $g : X \to Y$ is a map, then it induces

$$[W, Y^*] \xrightarrow{(1\wedge g)^*} [W, X^*]$$

adn this natural transformation must be induced by a unique map $g^* : Y^* \to X^*$. We have the following commutative map

$$\begin{tikzcd}
Y^* \wedge X \arrow[r, "1\wedge g"] \arrow[d, "g^*\wedge 1"'] & Y^* \wedge Y \arrow[d, "e_Y"] \\
X^* \wedge X \arrow[r, "e_X"'] & S
\end{tikzcd}$$

Let Z be a spectrum , we can make a natural transformation

$$[W, Z \wedge X^*]_r \xrightarrow{T} [W \wedge X, Z]_r$$

as follows: Given $W \xrightarrow{f\wedge 1} Z \wedge X^*$ we take $W \wedge X \xrightarrow{f\wedge 1} Z \wedge X^* \wedge X \xrightarrow{1\wedge e} Z$.
Note that T is an isomorphism if $Z = S^n$.

Proposition 4.1. Suppose we have cofiber sequence $Z_1 \to Z_2 \to Z_3 \to Z_4 \to Z_5$ adn T is an isomorphism for Z_1, z_2, Z_4, Z_5 the it is an isomorphism for Z_3

Proof. The proof is a simple application of five lemma.

□

Proposition 4.2. T is an isomorphism if Z is any finite spectrum.

Proof. We have a cofiber sequence,

$$S \to X \to (X \cup_f D) \to \Sigma S \to \Sigma X$$

We then proceed by induction and the previous remark. □

Proposition 4.3. If W and X are finite spectra , then

$$T : [W, Z \wedge X^*]_r \to [W \wedge X, Z]_r$$

is an isomorphism for any spectrum Z.

Proof. I have to use direct limits. Writing an infinite spectra as direct limit of finte spectra. Not sure how to do it.

□

Lemma 4.4. If X is a finite spectrum then X^* is equivalent to a finite spectrum.

Proof. The proof involves homology theories of a spectra and is postponed till next chapter.

□

Proposition 4.5. Let X be a finite spectrum, Y any spectrum. Then we have an equivalence $(X \wedge Y)^* \xrightarrow{h} X^* \wedge Y^*$ which makes the follwoing diagram commute

$$\begin{tikzcd}
(X \wedge Y)^* \wedge X \wedge Y \arrow[rr, "e_{X \wedge Y}"] \arrow[d, "h \wedge 1"] & & S \\
X^* \wedge Y^* \wedge X \wedge Y \arrow[r, "1 \wedge c \wedge 1"] & X^* \wedge X \wedge Y^* \wedge Y \arrow[ur, "e_X \wedge e_Y"'] &
\end{tikzcd}$$

Proof. By 4.4 we can assume that X^* is a finite spectrum. Then,

$$[W, X^* \wedge Y^*]_r \xrightarrow{T_Y} [W \wedge Y, X^*]_r$$

is an isomorphism for any spectrum W, and so is

$$[W \wedge Y, X^*]_r \xrightarrow{T_X} [W \wedge Y \wedge X, S]_r$$

by the original property of X^* applied to the spectrum $W \wedge Y$. This state of affairs reveals $X^* \wedge Y^*$ as the dual of $Y \wedge X$ with $T_{Y \wedge X} = T_X T_Y$. Writing this equation in terms of maps e, we obtain the desired.

□

※ Homology and Cohomology

We define E-homology and E-cohomology for a given spectrum E and study their properties.

The E-homology is defined as

$$E_n(X) = [S, E \wedge X]_n$$

and E-cohomology is defined as

$$E^n(X) = [X, E]_{-n}$$

These functors satisfy the properties that generalised homology and cohomology functors satisfy. They give an analog for a theoy defined on spectra of the Eilenberg-Steenrod axioms. We record the properties in the proposition below. These are easy to check.

Consider the Eilenberg-Maclane spectrum $H\mathbb{Z}$. Define

$$H_n(X) = [S, H\mathbb{Z} \wedge X]$$

and

$$H^n(X) = [X, H\mathbb{Z}]$$

Proposition 5.1.

1. $E_*(X)$ is a covariant functor of two variables E, X in SHC with values in the category of graded abelian groups. $E^*(X)$ is a covariant fucntor in E and contravariant in X.

2. If we vary E or X along a cofibering, we obtain an exact sequence, That is, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a cofiber sequence, then

$$E_n(X) \xrightarrow{f_*} E_n(Y) \xrightarrow{g_*} E_n(Z)$$

and

$$E^n(X) \xleftarrow{f^*} E^n(Y) \xleftarrow{g^*} E^n(Z)$$

are exact; if $E \xrightarrow{i} F \xrightarrow{j} G$ is a cofiber sequence, then

$$E_n(X) \xrightarrow{i_*} F_n(X) \xrightarrow{j_*} G_n(X)$$

and

$$E^n(X) \xrightarrow{i_*} F^n(X) \xrightarrow{j_*} G^n(X)$$

are exact.

3. There area natural isomorphisms

$$E_n(X) \cong E_{n+1}\left(S^1 \wedge X\right)$$
$$E_n(X) \cong E^{n+1}\left(S^1 \wedge X\right)$$

4.
$$E_n(S) = E^{-n}(S) = \pi_n(E)$$

For a CW complex L we define homology and cohomology to be E_n or E^n applied to the suspension spectrum of the complex.

$$\tilde{E}_n(L) = E_n(\Sigma^\infty L)$$

$$\tilde{E}^n(L) = E^n(\Sigma^\infty L)$$

The following fact holds
$$E_n(X) \cong X_n(E).$$

Proposition 5.2. If X is a finite spectrum $E_n(X^*) \cong E^{-n}(X)$.

Proof. The proof is a simple application of 4.3.

□

Proof of 4.4. Let X be a finite spectrum. Then $[S, X^*] \cong [X, S]$ and the right hadn side is zero if n is negative for large absolute values. But $H_n(X^*) = H^{-n}(X)$ is finitely generated in each dimension and zero for all except for finite number of dimensions. This proves that X^* has only finite stable cells and hence is a finite spectrum □

We now discuss homology and cohomology groups with coefficients.

Moore spectrum Let G be an abelian group. conisder a free resolution $0 \to R \xrightarrow{i} F \to G \to 0$. Take $\vee_\alpha S, \vee_\beta S$ such that π_0 of the two spectra are R and F respectively. take a map $f : \vee_\alpha S \to \vee_\beta S$ inducing i[2]. Form another spectrum $M = \vee_\alpha S \bigcup_f C(\vee_\beta S)$. This is a *Moore spectrum of type G*.
So we have

$$\begin{aligned} \pi_r(M) &= 0 && \text{for } r < 0 \\ \pi_0(M) &= H_0(M) = G \\ H_r(M) &= 0 && \text{for } r > 0 \end{aligned}$$

For any spectrum E, we define the corresponding spectrum with coefficients in G by

$$EG = E \wedge M$$

[2]See here for more details on the induced map f

Proposition 5.3. 1. There exists an exact sequence

$$0 \longrightarrow \pi_n(E) \otimes G \longrightarrow \pi_n(EG) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{n-1}(E)) \longrightarrow 0$$

(This need not split, e.g., take $E = \mathrm{KO}, G = \mathbb{Z}_2$.)

2. More generally, there exists exact sequences

$$0 \longrightarrow E_n(X) \otimes G \longrightarrow (EG)_n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(E_{n-1}(X), G) \longrightarrow 0$$

and (if X is a finite spectrum or G is finitely generated)

$$0 \longrightarrow E^n(E) \otimes G \longrightarrow (EG)^n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}\left(E^{n+1}(X), G\right) \longrightarrow 0$$

Proof. [A, Page 221] for proof □

The Moore spectrum for $\mathbb{Q}$ is same as the Eilenberg-Maclane spectrum for $\mathbb{Q}$. With this fact one can show that the rational stble homotopy is same as rational homology,i.e.

$$\pi_*(X) \otimes \mathbb{Q} \to H_*(X) \otimes \mathbb{Q}.$$

The isomorphism is induced by the map $i : S \to H$ representing a generator of $\pi_0(H) = \mathbb{Z}$.

※ Road to Adams spectral sequence

Inverse Limits Inverse limits that we are gonna consider are over $I = \{1,2,3,\dots\}$. Let $\underline{G}$ be an inverse system of abelian groups indexed over I consisting of abelian groups G_i. We say that $\underline{G}$ satisfies the Mittag-Leffler condition if for each n there exists m such that $Img_{np} = Img_{nm}$ for $p \geq m$, that is $Img_n p$. If $\underline{G}$ satisfies the Mittag-Leffler condition then $\varprojlim^1 \underline{G} = 0$.

Let E^* be a generalised cohomology theory satisfying wedge axiom. Suppose given an increasing sequence of CW pairs (X_n, A_n) and set $X = \bigcup X_n, A = \bigcup A_n$, then there is an exact sequence

$$0 \to \varprojlim_n^1 E^{q-1}(X_n, A_n) \to E^q(X, A) \to \varprojlim_n E^q(X_n, A_n) \to 0.$$

Ring-spectrum

A spectrum E is said to be a *ring-spectrum* if it has given maps

$$\mu : E \wedge E \to e$$

and

$$\eta : S \to E$$

of degree 0 such that the following diagrams commute

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd} \qquad \begin{tikzcd} S \wedge E \arrow[r, "\eta\wedge 1"] \arrow[d] & E \wedge E \arrow[d, "\mu"] \\ E \arrow[r, "1"] & E \\ E \wedge S \arrow[r, "1\wedge\eta"] \arrow[u] & E \wedge E \arrow[u] \end{tikzcd}$$

Let E be a ring spectrum. We say that F is a *module-spectrum* over E if it has given a map $\nu : E \wedge F \to F$ of degree 0 such that the following diagrams commute:

$$\begin{tikzcd} E \wedge E \wedge F \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\nu"] & E \wedge F \arrow[d, "\nu"] \\ E \wedge F \arrow[r, "\nu"] & E \end{tikzcd} \qquad \begin{tikzcd} S \wedge F \arrow[r, "\eta\wedge 1"] \arrow[d, "\cong"] & E \wedge E \arrow[d, "\nu"] \\ F \arrow[r, "1"] & F \end{tikzcd}$$

A ring spectrum E is said to be *commutative* if the following diagram commutes

$$\begin{tikzcd} E \wedge E \arrow[r, "\mu"] \arrow[d, "c"] & E \\ E \wedge E \arrow[ur, "\mu"'] & \end{tikzcd}$$

If E is a ring-spectrum, we can use the product map $\mu : E \wedge E \to E$ to obtain products.

Steenrod Algebra and dual

Let E be a spectrum. Then to e ery element of $E^*(E)$ we can associate a natural transformaation $E^*(X) \to E^*(X)$ defined for all spectra X. So give $X \xrightarrow{f} E$ and $E \xrightarrow{g} E$, we form $X \xrightarrow{gf} E$. This gives a 1-1 correspondence between elements of $E^*(E)$. We can give $E^*(E)$ a ring structure. In our cases we will look at the coalgebra $E_*(E)$.
Take $E = H\mathbb{Z}_p$. Then $A^* = (H\mathbb{Z}_p)^*(H\mathbb{Z}_p)$ is the mod p steenrod algebra. We know that A^* is generated by steenrod squares or powers. $H\mathbb{Z}_p$ is a ring-spectrum.

$E_*(E)$ is a bimodule over $\pi_*(E)$. The left action $\pi_*(E) \otimes E_*(E) \longrightarrow E_*(E)$ is obtained by using the morphism $E \wedge E \wedge E \xrightarrow{\mu\wedge 1} E \wedge E$; the right action $E_*(E) \otimes \pi_*(E) \longrightarrow E_*(E)$ is obtained by using the morphism $E \wedge E \wedge E \xrightarrow{1\wedge\mu} E \wedge E$.
The assumption we make is that $E_*(E)$ is flat as a right module over $\pi_*(E)$.But if E is commutative, which is the usual case it is equivalent to say that $E_*(E)$ is flat as a left module; this is seen by using $c : E \wedge E \longrightarrow E \wedge E$ to interchange the two sides.
The assumption is satisfied for $E = S, H\mathbb{Z}_p$.

Consider the morphism

$$(E \wedge E) \wedge (E \wedge X) \xrightarrow{1\wedge\mu\wedge 1} E \wedge E \wedge X$$

which induces a product map

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \xrightarrow{\simeq} [S, E \wedge E \wedge X]_*$$

which is an isomorphism.

We assume that $E_*(X)$ is projective over $\pi_*(E)$. This hypothesis works true for $X = S$-the case when we need to compute stable homotopy that is $[S,Y]_*$ and when $E = H\mathbb{Z}_p$.

E is a ring spectrum F is a module spectrum over E and we are interested about $F_*(X)$ and $F^*(X)$ given $E_*(X)$. We can get a homomorphism

$$F^*(X) \to \hom_{\pi_*(E)}(E_*(X), \pi_*(F))$$

.

We will be interested in spectra X which satisfy the following conditions.
Condition 1. $F^*(X) \longrightarrow \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), \pi_*(F))$ is an isomorphism for all module-spectra F over E.

Condition 2. E is the direct limit of finite spectra E_α for which $E_*(DE_\alpha)$ is projective over $\pi_*(E)$ and DE_α satisfies 2. Here DE_α means the S-dual of E_α. Condition 2 is satisfied by $E = S, H\mathbb{Z}_p$.

We look at two results that we would want.

Theorem 6.1. Suppose E satisfies Condition 2. Then there is a spectral sequence

$$\mathrm{Ext}^{p,*}_{\pi_*(E)}(E_*(X), \pi_*(F)) \Longrightarrow F^*(X)$$

A special case of this is the following corollary

Corollary 6.2. Suppose E satisfies Condition 13.3 (e.g., E may be one of the examples listed in 13.4). Suppose $E_*(X)$ is projective over $\pi_*(E)$. Then 13.2 holds, i.e.,

$$F^*(X) \longrightarrow \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), \pi_*(F))$$

is an isomorphism for all module-spectra F over E.

Given $E^*(X)$ and $E^*(Y)$, we want to know about $[X,Y]^*$.

A morphism $f : X \to X'$ is E-equivalent if the induced morhpism $f : E_*(X) \to E_*(X')$ is an isomorphism.

Let C be the stable homotopy category. There exists a category F, called the category of ractions and a functor $T : C \to F$ with the following properties(we only consider what would be required later to understand Adams specseq): If $e : X \to Y$ is an $E0$equivalence in C, then $T(e)$ is an actual equivalence in F. The objects of F are the same as objects of C. For the defining properties of the category, refer to [A, Theorem 14.2]. The morphisms in F are denoted as $[X,Y]^E_*$.

Proposition 6.3. The following conditions on Y are equivalent.

1. $f : [X,Y]_* \longrightarrow [X,Y]^E_*$ is an isomorphism for all X.

2. if $E_*(X) = 0$, then $[X,Y]_* = 0$.

If this holds, we say that Y is E-complete. $E = H\mathbb{Z}_P$ is complete. If U is an E module spectrum, then $E_*(X) = 0$, so Y is complete.

Theorem 6.4. For any spectrum Y, there exists an E equivalence $e : Y \to Z$ such that Z is E complete. We have

$$[X,Y^E]_* = [X,Z]_* \xrightarrow{\cong} [X,Y]^E_*$$
$$f \mapsto T(e)^{-1}T(f)$$

X is connective if there exists $n_0 \in \mathbb{Z}$ such that $\pi_r(X) = 0$ for all $r < n_0$.

Proposition 6.5. Suppose that E is a commutative ring-spectrum and $\pi_r(E) = 0$ for $r < 0$; suppose also that Y is connective. Then $[X,Y]^E_*$ depends only on the ring $\pi_0(E)$.

Y is connective , then $[X,Y]^E_*$ depends only on $\pi_0(E)$.
Suppose $\pi_0(E) = \mathbb{Z}_m$ and $\pi_r(Y)$ is finitely generated for all r. Then,

$$Y^E = YI_m,$$

where $I_m = \varprojlim r\mathbb{Z}_{m^r}$. So $[X,Y]^E_* = [X,YI_m]_* = [X,Y]_* \otimes I_m$.

Let E be a commutative ring-spectrum such that $\pi_r(E) = 0$ for $r < 0$, and let $\theta : \mathbb{Z} \longrightarrow \pi_0(E)$ be the unique homomorphism of rings. Let $S \subset \mathbb{Z}$ be the set of n such that $\theta(n)$ is invertible in $\pi_0(E)$. Then S is multiplicatively closed. Let $R \subset Q$ be the localization of $\mathbb{Z}$ at S, i.e., the set of fractions n/m with $m \in S$. Then there exists a unique extension of θ to

$$\theta : R \longrightarrow \pi_0(E).$$

Proposition 6.6. If Y is E complete, then $\pi_r(Y)$ is an R module and more generally $[X,Y]_r$ is an R module for any X.

Convergence We know the usual strong convergence: Serre specseq converges strongly. We say a specseq converges conditionally to $E^{p,0}$ if $\varprojlim_p E^{p,*} = 0$ and $\varprojlim^1_p E^{p,*} = 0$, where $E^{p,*}$ is a filtration of $E^{0,*}$.

Consider the three conditions:

1. $E^{p,q}_\infty \to \varprojlim E^{p,q}_r$ is an isomorphism.
2. $\varprojlim^1_r E^{p,q}_r = 0$
3. Let $F^{p,q}$ be the filtration quotients of $E^{p+q}(X)$, so that we have exact sequences

   $$0 \longrightarrow E^{p,q}_\infty \longrightarrow F^{p,q} \longrightarrow F^{p-1,q+1} \longrightarrow 0$$

   and $F^{-1,q} = 0$. The map $E^n(X) \longrightarrow \varprojlim F^{p,n-p}$ should be isomorphism.

Condition 2 is equivalent to condition 1 and 3. When either one of this holds in addition to conditional convergence, the specseq converges strongly.

※ Adams Spectral sequence

We assume that the following conditions hold true. They are satisfied by S and $H\mathbb{Z}_p$ as that would be the cases that we would be working with generally.

1. E satisfies condition 2.
2. $E_*(E)$ is flat as a right module over $\pi_*(E)$.
3. Y is connective.
4. $\pi_r(E) = 0$ for $r < 0$ and $\mu_* : \pi_0(E) \otimes_{\mathbb{Z}} \pi_0(E) \to \pi_0(E)$ is an isomorphism.
5. $H_r(E)$ is finitely-generated over R for all r.($H_*(E)$ is a ring, so $H_r(E)$ is a module over $H_0(E) = \pi_0(E)$. Let the subring R of the rationals $\mathbb{Q}$ be as we saw here, so that we have a homomorphism $\theta : R \longrightarrow \pi_0(E)$; thus $H_r(E)$ becomes an R-module.)

We will construct a filtration of Y to get an unenrolled exact couple as follows

$$\begin{tikzcd}
Y = Y_0 \arrow[dr] & & Y_1 \arrow[ll] \arrow[dr] & & Y_2 \arrow[ll] \arrow[dr] & & Y_3 \arrow[ll] & \cdots \arrow[l] \\
& W_0 \arrow[ur] & & W_1 \arrow[ur] & & W_2 \arrow[ur] & &
\end{tikzcd}$$

Consider the cofibering

$$\bar{E} \to S \xrightarrow{i} E \to \bar{E}$$

where $E \to \bar{E}$ has degree -1. Let

$$\bar{E}^p = \bar{E} \wedge \bar{E} \wedge \ldots \wedge \bar{E} \ p \text{ factors}$$

Smashing with $\bar{E}^p \wedge Y$, we obtain a cofibering

$$\bar{E}^{p+1} \wedge Y \longrightarrow \bar{E}^p \wedge Y \longrightarrow E \wedge \bar{E}^p \wedge Y \longrightarrow \bar{E}^{p+1} \wedge Y$$

where again the last morphism shown has degree -1. So we may take

$$Y_p = \bar{E}^p \wedge Y, \quad W_p = E \wedge \bar{E}^p \wedge Y.$$

Now we apply the functor $[X,-]_*^E$ and we get

$$E_1^{p*} = [X, E \wedge Y_p]_*$$

The boundary d_1 is induced by the morphism $W_p \to Y_{p+1} \to W_{p+1}$.

Adams Spectral Sequence

There exists a specseq with properties :

1. The E_2 term is given by

$$E_2^{p,*} = \mathrm{Ext}^{p*}_{E_*(E)}\left(E_*(X), E_*(Y)\right), \quad \text{and}$$

2. Consider a decreasing filtration

$$Y \simeq Y_0 \supset Y_1 \supset Y_2 \supset Y_3 \supset \ldots \supset Y_p \supset \ldots$$

   The specseq converges conditionally to $[X,Y]_*^E$.

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