# 1 Signals and Systems

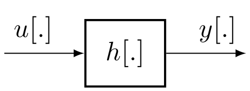
## 1.1 Types of Signals

| | |
|---|---|
| $\ell^1_{\mathbb{R}}(\mathbb{Z})$ | Set of all discrete-time real-valued absolutely summable signals |
| $\ell^2_{\mathbb{C}}(\mathbb{Z})$ | Set of all discrete-time complex-valued square summable signals |
| $\ell^\infty_{\mathbb{R}}(\mathbb{Z})$ | Set of all discrete-time real-valued bounded signals |
| $L^1_{\mathbb{R}}(\mathbb{R})$ | Set of all continuous-time real-valued absolutely summable signals |
| $L^1_{\mathbb{C}}(\mathbb{R})$ | Set of all continuous-time complex-valued square summable signals |
| $L^\infty_{\mathbb{C}}(\mathbb{R})$ | Set of all continuous-time complex-valued bounded signals |

$$\ell^1_{\mathbb{R}} \subset \ell^2_{\mathbb{R}} \subset \ell^\infty_{\mathbb{R}} \qquad \ell^1_{\mathbb{C}} \subset \ell^2_{\mathbb{C}} \subset \ell^\infty_{\mathbb{C}}$$

| type of signal | continuous-time | discrete-time |
|---|---|---|
| absolutely summable | $\int_{-\infty}^{+\infty} \lvert f(x) \rvert \, dx < \infty$ | $\sum_{k=-\infty}^{\infty} \lvert f[k] \rvert < \infty$ |
| square summable | $\int_{-\infty}^{+\infty} \lvert f(x) \rvert^2 \, dx < \infty$ | $\sum_{k=-\infty}^{\infty} \lvert f[k] \rvert^2 < \infty$ |
| bounded: | $\lvert f(x) \rvert < b, \ \forall x \in \mathbb{R}$ | $\lvert f[k] \rvert < b, \ \forall k \in \mathbb{Z}$ |

## 1.2 LTI System




$$y[.] = u[.] * h[.] = \sum_{k \in \mathbb{Z}} u[k] h[. - k]$$

$$y(t) = u(t) * h(t) = \int_{-\infty}^{+\infty} u(\tau) h(t - \tau) d\tau$$

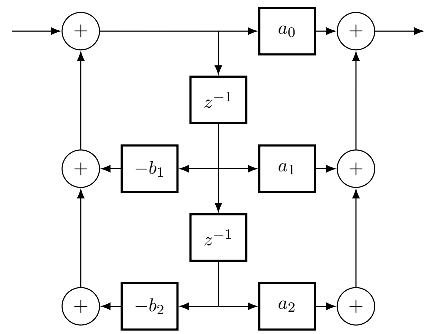
**Theorem 1.1** *An LTI system is* ***BIBO-stable****, if and only if it's impulse response is a stable signal (i.e. absolutely summable or integrable)*

| $f$ | $g$ | $f * g$ |
|---|---|---|
| right-sided | right-sided | right-sided |
| causal | causal | causal |
| anything | finite duration | well defined |
| **bounded** | **absolutely summable** | **bounded** |
| absolutely summable | absolutely summable | absolutely summable |
| square summable | absolutely summable | square summable |
| square summable | square summable | bounded |

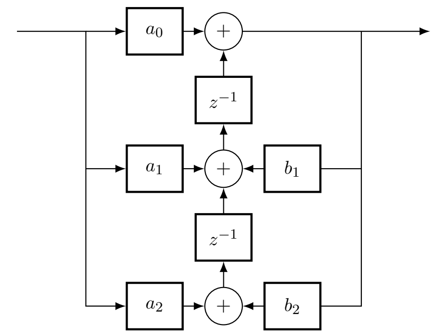
## 1.3 canonical forms

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}} = \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{z^n + b_1 z^{n-1} + \cdots + b_n}$$

**controllable canonical form**




**observable canonical form**




$$x[k] \in \mathbb{R}^n, \ A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times 1}, \ C \in \mathbb{R}^{1 \times n}, \ D \in \mathbb{R}$$

$$A = \begin{bmatrix} -b_1 & -b_2 & -b_3 & \cdots & -b_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} a_1 - a_0 b_1 & \cdots & a_n - a_0 b_n \end{bmatrix}, \quad D = a_0$$

## 1.4 Maison's Gain Formula

$$\frac{Y(z)}{U(z)} = \frac{\sum_{p \in P} T_p(z) \Delta_p(z)}{\Delta(z)}$$

$P$ is the set of all forward paths from node $u$ to node $y$. $T_p(z)$ is the transfer function of the forward path $p$. $\Delta_p(z)$ is the network determinant of that parto of the block diagram that is not touched by the forward path $p$. $\Delta(z)$ is the network determinant.

$$\Delta(z) = 1 - \sum_{s \in S_1)} + \sum_{(s_1, s_2) \in S_2} T_{s_1} T_{s_2} - \sum_{(s_1, s_2, s_3) \in S_3} T_{s_1} T_{s_2} T_{s_3} \pm \cdots$$

$T_s(z)$ denotes the transfer function of a simple loop $s$. $S_1$ is the set of all simple loops. $S_2$ is the set of all unordered pairs $(s_1, s_2)$ of simple loops $s_1, s_2 \in S_1$ that do not touch. $S_3$ is defined as three not touching loops and so on...

## 1.5 $z$-Transform

**Definition 1.1** *The z-transform of a real or complex signal $f[.]$ is the funciton*

$$\mathrm{ROC}(f) \to \mathbb{C} : z \mapsto F(z) := \sum_{k=-\infty}^{\infty} f[k] z^{-k}$$

*with* $\mathrm{ROC}(f) := \left\{ z \in \mathbb{C} : r_1 < |z| < r_2 \right\}$

$$r_2 := \left( \limsup_{\ell \to \infty} \left| f([-\ell]) \right|^{1/\ell} \right)^{-1} \qquad r_1 := \limsup_{k \to \infty} \left| f[k] \right|^{1/k}$$

**Theorem 1.2** *Let $f[.]$ be a real or complex discrete-time signal with rational z-transform. Then $f[.]$ is* ***stable*** $\iff$ $\mathrm{ROC}(f)$ *contains the unit circle.*
*If $f[.]$ is right-sided, then $f[.]$ is* ***stable*** $\iff$ *all poles of $F(z)$ are strictly inside the unit circle.*

| | |
|---|---|
| **Addition** | $\mathrm{ROC}(f+g) \subseteq \mathrm{ROC}(f) \cap \mathrm{ROC}\, g$ |
| **Multiplication** | $H(z) := F(z)G(z) \longrightarrow \mathrm{ROC}(h) \subseteq \mathrm{ROC}(f) \cap \mathrm{ROC}(g)$ |
| **Scalar multipl.** | $\mathrm{ROC}(af) = \mathrm{ROC}(a)$ |
| **Time shift** | $H(z) := z^m F(z) \longrightarrow \mathrm{ROC}(h) = \mathrm{ROC}(f)$ |
| **Time reversal** | $h[k] := f[-k] \longrightarrow \mathrm{ROC}(h) = \left\{ z \in \mathbb{C} : 1/r_2 < \lvert z \rvert < 1/r_1 \right\}$ |

| $f[k]$ | $F(z)$ | |
|---|---|---|
| $f[k]$ | $F(z) = \sum_{k=-\infty}^{\infty} f[k] e^{-k}$ | definition |
| $\alpha f[k] + \beta g[k]$ | $\alpha F(z) + \beta G(z)$ | linearity |
| $f[k-m]$ | $z^m \cdot F(z)$ | time delay |
| $f[k] * g[k]$ | $F(z) \cdot G(z)$ | convolution - multiplicity |

## 1.6 Inverse $z$-Transform

| $F(z)$ | $\mathrm{ROC}(f)$ | $f[.]$ |
|---|---|---|
| $\frac{z}{z-p}$ | $\lvert z \rvert > \lvert p \rvert$ | $f[k] = \begin{cases} 0 & k < 0 \\ p^k & k \geq 0 \end{cases}$ |
| | $\lvert z \rvert < \lvert p \rvert$ | $f[k] = \begin{cases} -p^k & k < 0 \\ 0 & k \geq 0 \end{cases}$ |
| $\frac{Az}{z-p} + \frac{\overline{A}z}{z-\overline{p}}$ | $\lvert z \rvert > \lvert p \rvert$ | $f[k] = \begin{cases} 0 & k < 0 \\ 2\lvert A \rvert \lvert p \rvert^k \cos \Omega k + \varphi & k \geq 0 \end{cases}$ |
| | $\lvert z \rvert < \lvert p \rvert$ | $f[k] = \begin{cases} -2\lvert A \rvert \lvert p \rvert^k \cos \Omega k + \varphi & k < 0 \\ 0 & k \geq 0 \end{cases}$ |

where $A = |A| \cdot e^{i\varphi}$ and $p = |p| \cdot e^{i\Omega}$

## 1.7 Spectrum of Discrete-time Signals

**Theorem 1.3** *The spectrum of a discrete-time signal is the z-transform on the unit circle.*

$$F(e^{i\Omega}) := F(z)\big|_{z=e^{i\Omega}}$$

**Theorem 1.4** *For a stable real-valued discrete-time signal $f[.] \in \mathbb{R}\, \forall\, k \in \mathbb{Z}$:*

$$F(e^{-i\omega}) = \overline{F(e^{i\Omega})}$$

$$F(z) = c \cdot \frac{\prod_{k=1}^{m}(z - z_k)}{\prod_{\ell=1}^{n}(z - p_\ell)} \qquad \left| F(e^{i\Omega}) \right| = |c| \cdot \frac{\prod_{k=1}^{m} \left| e^{i\Omega} - z_k \right|}{\prod_{\ell=1}^{n} \left| e^{i\Omega} - p_\ell \right|}$$

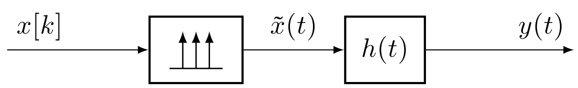
# 2 Discrete Time ⇄ Continuous Time

## 2.1 Laplace Transform

$$F(s) := \int_{-\infty}^{+\infty} f(t)e^{-st} \quad \mathrm{ROC}(f) = \{s \in \mathbb{C} : r_1 < \mathrm{Re}(s) < r_2\}$$

| signal $f$ | $z$-transform | Laplace transform |
|---|---|---|
| right-sided | $\mathrm{ROC}(f) = \{z \in \mathbb{C} : \lvert z\rvert > \rho\}$ | $\mathrm{ROC}(f) = \{s \in \mathbb{C} : \mathrm{Re}(s) > r\}$ |
| left-sided | $\mathrm{ROC}(f) = \{z \in \mathbb{C} : \lvert z\rvert < \rho\}$ | $\mathrm{ROC}(f) = \{s \in \mathbb{C} : \mathrm{Re}(s) > r\}$ |
| stable | $\mathrm{ROC}(f)$ contains unit circle | $\mathrm{ROC}(f)$ contains imaginary axis |

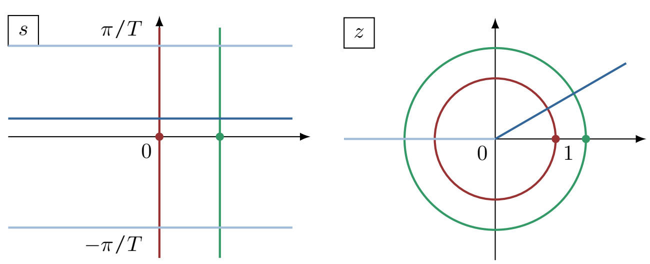
## 2.2 From Discrete-time to Continuous-time




$$\tilde{x}(t) := \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT) \quad y(t) = (\tilde{x} * h)(t) = \sum_{k=-\infty}^{\infty} x[k]h(t - kT)$$

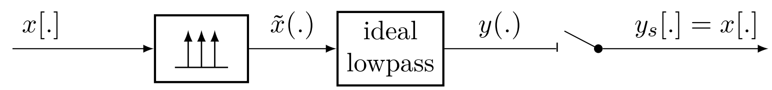
$$\tilde{X}(s) = X(z)\big|_{z=e^{sT}}, \quad \tilde{X}(i\omega) = F(z)\big|_{z=e^{i\omega T}}$$

The spectrum $\tilde{F}(i\omega)$ of a "continuous-time signal" $\tilde{f}(t) := \sum_{k\in\mathbb{Z}} f[k]\delta(t - kT)$ equals the **periodic spectrum** $F(e^{i\Omega})$ of the discrete-time signal $f[.]$ with $\Omega = \omega T$




## 2.3 Ideal Lowpass and the Sampling Theorem

$$H(i\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| \geq \omega_c \end{cases} \qquad h(t) = \frac{\sin(\omega_c t)}{\pi t} = \frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c t}{\pi}\right)$$




The system above is completely transparent and acts like a discrete-time filter with impulse response $\delta[.]$

**Theorem 2.1 (Nyquist-Shannon Sampling Theorem)** *Let $y(.)$ be a signal with spectrum $Y(i\omega)$ satisfying*

$$Y(i\omega) = 0\,\forall\, |\omega| \geq \pi/T \quad \textit{or} \quad Y(i2\pi f) = 0\,\forall\, |f| \geq f_s/2,$$

*Twhere $\omega = 2\pi f$ and $\boldsymbol{f_s := 1/T}$. Then, $y(.)$ can be recovered from the discrete-time signal $y_s[k] = Ty(kT)$ by:*

$$Y(i\omega) = Y_s(e^{i\omega T})\,\forall\,|\omega| < \pi/T, \quad y(t) = \sum_{k\in\mathbb{Z}} y(kT)\,\mathrm{sinc}\left(\frac{t - kT}{T}\right).$$

*The spectra of $y(.)$ and $y_s[.]$ are related by:*

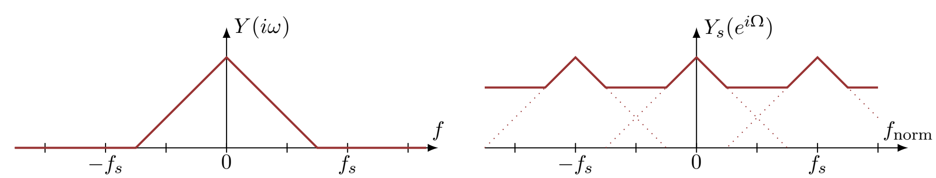
$$Y_s\left(e^{i\omega T}\right) = \sum_{n\in\mathbb{Z}} Y\left(i\left(\omega + \frac{2\pi n}{T}\right)\right) \qquad Y_s\left(e^{i2\pi f/f_s}\right) = \sum_{n\in\mathbb{Z}} Y\left(i2\pi(f + nf_s)\right)$$

Now, we define the digital frequency and the normalized frequency as following:

$$\Omega = \omega T = 2\pi\frac{f}{f_s}, \quad \Omega \in [-\pi, \pi] \text{ or } [0, 2\pi]$$

$$\frac{\omega}{2\pi} = f_{\text{norm}} = \frac{f}{f_s}, \quad f_{\text{norm}} \in \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ or } [0, 1]$$

The spectrum $Y_s(e^{i\Omega})$ or $\tilde{Y}_s(i\omega)$ of a sampled signal is the sum of copies $Y(i\omega)$ that are shifted by $n \cdot 2\pi/T\,\forall\, n \in \mathbb{Z}$.

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