Transversals in Uniform Linear Hypergraphs

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Abstract

The transversal number $\tau(H)$ of a hypergraph H is the minimum number of vertices that intersect every edge of H. A linear hypergraph is one in which every two distinct edges intersect in at most one vertex. A k-uniform hypergraph has all edges of size k. Very few papers give bounds on the transversal number for linear hypergraphs, even though these appear in many applications, as it seems difficult to utilise the linearity in the known techniques. This paper is one of the first that give strong non-trivial bounds on the transversal number for linear hypergraphs, which is better than for non-linear hypergraphs. It is known that $\tau(H) \le (n+m)/(k+1)$ holds for all k-uniform, linear hypergraphs H when $k \in \{2,3\}$ or when $k \ge 4$ and the maximum degree of H is at most two. It has been conjectured (at several conference talks) that $\tau(H) \le (n+m)/(k+1)$ holds for all k-uniform, linear hypergraphs H. We disprove the conjecture for large k, and show that the best possible constant c_k in the bound $\tau(H) \le c_k(n+m)$ has order $\ln(k)/k$ for both linear (which we show in this paper) and non-linear hypergraphs. We show that for those k where the conjecture holds, it is tight for a large number of densities if there exists an affine plane $AG(2,k)$ of order $k \ge 2$. We raise the problem to find the smallest value, $k_{\min}$, of k for which the conjecture fails. We prove a general result, which when applied to a projective plane of order 331 shows that $k_{\min} \le 166$. Even though the conjecture fails for large k, our main result is that it still holds for $k = 4$, implying that $k_{\min} \ge 5$. The case $k = 4$ is much more difficult than the cases $k \in \{2,3\}$, as the conjecture does not hold for general (non-linear) hypergraphs when $k = 4$. Key to our proof is the completely new technique of the deficiency of a hypergraph introduced in this paper.

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1 Introduction

In this paper we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*. The *order* of H is $n(H) = |V|$ and the *size* of H is $m(H) = |E|$. A k-*edge* in H is an edge of size k. The hypergraph H is said to be k-*uniform* if every edge of H is a k-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. For $i \ge 2$, we denote the number of edges in H of size i by $e_i(H)$. The *degree* of a vertex v in H, denoted by $d_H(v)$, is the number of edges of H which contain v. A *degree-k vertex* is a vertex of degree k. The minimum and maximum degrees among the vertices of H is denoted by $\delta(H)$ and $\Delta(H)$, respectively.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. The *neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from the context, is the set of all vertices different from v that are adjacent to v. A vertex in $N(v)$ is a *neighbor* of v. The *neighborhood* of a set S of vertices of H is the set $N_H(S) = \cup_{v \in S} N_H(v)$, and the *boundary* of S is the set $\partial_H(S) = N_H(S) \setminus S$. Thus, $\partial_H(S)$ consists of all vertices of H not in S that have a neighbor in S. If H is clear from context, we simply write $N(S)$ and $\partial(S)$ rather than $N_H(S)$ and $\partial_H(S)$. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of H is a *component* of H. Thus, no edge in H contains vertices from different components. A component of H isomorphic to a hypergraph F we call an F-*component* of H.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A transversal of size $\tau(H)$ is called a $\tau(H)$-transversal. Transversals in hypergraphs are well studied in the literature (see, for example, [7, 10, 11, 12, 18, 19, 20, 21, 22, 23, 31, 32, 37, 39]).

A hypergraph H is called an *intersecting hypergraph* if every two distinct edges of H have a non-empty intersection, while H is called a *linear hypergraph* if every two distinct edges of H intersect in at most one vertex. We say that two edges in H *overlap* if they intersect in at least two vertices. A linear hypergraph therefore has no overlapping edges. Linear hypergraphs are well studied in the literature (see, for example, [4, 9, 13, 14, 28, 30, 35, 36, 38]), as are uniform hypergraphs (see, for example, [7, 8, 9, 14, 15, 23, 25, 32, 34, 35, 36, 38]).

A set S of vertices in a hypergraph H is *independent* (also called strongly independent in the literature) if no two vertices in S belong to a common edge. Independence in hypergraphs is well studied in the literature (see, for example, [2, 26, 31], for recent papers on this topic).

Given a hypergraph H and subsets $X, Y \subseteq V(H)$ of vertices, we let $H(X, Y)$ denote the hypergraph obtained by deleting all vertices in $X \cup Y$ from H and removing all (hyper)edges containing vertices from X and removing the vertices in Y from any remaining edges. If $Y = \emptyset$, we simply denote $H(X, Y)$ by $H - X$; that is, $H - X$ denotes that hypergraph obtained from H by removing the vertices X from H, removing all edges that intersect X and removing all resulting isolated vertices, if any. Further, if $X = \{x\}$, we simply write $H - x$ rather than $H - X$. When we use the definition $H(X, Y)$ we furthermore assume that no edges of size zero are created. That is, there is no edge $e \in E(H)$ such that $V(e) \subseteq Y \setminus X$. In this case we note that if add X to any $\tau(H(X, Y))$-set, then we get a transversal of H, implying that $\tau(H) \leq |X| + \tau(H(X, Y))$. We will often use this fact throughout the paper.

In geometry, a *finite affine plane* is a system of points and lines that satisfy the following rules: (R1) Any two distinct points lie on a unique line. (R2) Each line has at least two points. (R3) Given a point and a line, there is a unique line which contains the point and is parallel to the line, where two lines are called parallel if they are equal or disjoint. (R4) There exist three non-collinear points (points not on a single line). A finite affine plane $AG(2, q)$ of *order* $q \geq 2$ is a collection of q^2 points and $q^2 + q$ lines, such that each line contains q points and each point is contained in $q + 1$ lines.

A *total dominating set*, also called a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Total domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in a recent book on this topic that can be found in [24]. A survey of total domination in graphs can be found in [17]. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

2 Motivation

In this paper we study upper bounds on the transversal number of a linear uniform hypergraph in terms of its order and size, and establish connections between linear uniform hypergraphs with large transversal number and affine planes and projective planes. An affine plane of order k exists if and only if a projective plane of order k exists. Jamison [27] and Brouwer and Schrijver [6] proved that the minimum cardinality of a subset of $AG(2, k)$ which intersects all hyperplanes is $2(k - 1) + 1$.

Theorem 1 ([6, 27]) *If there exists a finite affine plane $AG(2, k)$ of order $k \geq 2$, then the minimum cardinality of a subset of $AG(2, k)$ which intersects all hyperplanes is $2(k-1)+1$.*

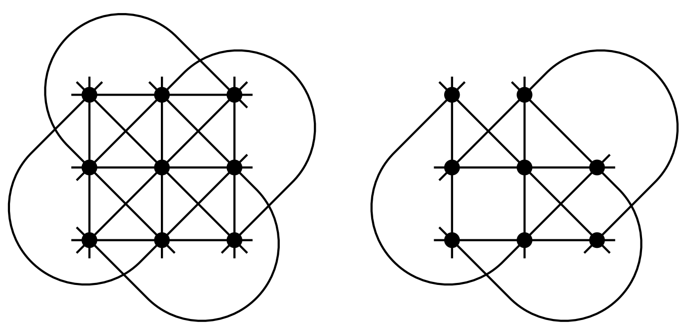
One of the most fundamental results in combinatorics is the result due to Bose [5] that there are $k - 1$ mutually orthogonal Latin squares if and only if there is an affine plane $AG(2, k)$. The prime power conjecture for affine and projective planes, states that there is an affine plane of order k exists only if k is a prime power. Veblen and Wedderburn [40] and Lam [33] established the existence of affine and projective planes of small orders.

The affine plane $AG(2,2)$ of dimension 2 and order 2 is illustrated in Figure 1(a). This is equivalent to a linear 2-uniform 3-regular hypergraph F_4 (the complete graph K_4 on four vertices), where the lines of $AG(2,2)$ correspond to the 2-edges of F_4. Deleting any vertex from F_4 yields the 2-uniform 2-regular hypergraph F_3 (the complete graph K_3) illustrated in Figure 1(b), where the lines correspond to the 2-edges of F_3.

(a) $F_4 = AG(2,2)$ (b) F_3

Figure 1: The affine plane $AG(2,2)$ and the hypergraph F_3

The affine plane $AG(2,3)$ of dimension 2 and order 3 is illustrated in Figure 2(a). This is equivalent to a linear 3-uniform 4-regular hypergraph F_9 on nine vertices, where the lines of $AG(2,3)$ correspond to the 3-edges of F_9. Deleting any vertex from F_9 yields the hypergraph F_8 (see Figure 2(b), where the lines correspond to the 3-edges of F_8). Further, deleting any vertex from F_8 yields a linear 3-uniform hypergraph F_7. It is known (see, for example, [20, 22]) that $\tau(F_7) = 3$, $\tau(F_8) = 4$ and $\tau(F_9) = 5$.



(a) $F_9 = AG(2,3)$ (b) F_8

Figure 2: The affine plane $AG(2,3)$ and the hypergraph F_8

As a consequence of results due to Tuza [39], Chvátal and McDiarmid [10], Henning and Yeo [20, 22], and Dorfling and Henning [12], we have the following result. For $k \geq 2$, let E_k denote the k-uniform hypergraph on k vertices with exactly one edge.

Theorem 2 ([10, 12, 20, 39]) *For $k \in \{2,3\}$ if H is a k-uniform linear connected hypergraph, then*

$$\tau(H) \leq \frac{n+m}{k+1}$$

with equality if and only if H consists of a single edge or H is obtained from the affine plane $AG(2,k)$ of order k by deleting one or two vertices; that is, $H = E_k$ or $H \in \{F_{k^2-2}, F_{k^2-1}\}$.

In fact, if $k \in \{2,3\}$ then $\tau(H) \le (n+m)/(k+1)$ holds for all k-uniform hypergraphs, even when they are not linear. The case when $k \ge 4$ is more complex as then the bound $\tau(H) \le (n+m)/(k+1)$ does not hold in general. As remarked in [12], if $k \ge 4$ and H is a k-uniform hypergraph that is **not** linear, then is not always true that $\tau(H) \le (n+m)/(k+1)$ holds, as may be seen for example by taking $k = 4$ and letting $\overline{F}$ be the complement of the Fano plane F, where the Fano plane is shown in Figure 3 and where its complement $\overline{F}$ is the hypergraph on the same vertex set $V(F)$ and where e is a hyperedge in the complement if and only if $V(F) \setminus e$ is a hyperedge in F.

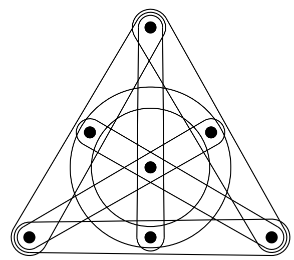
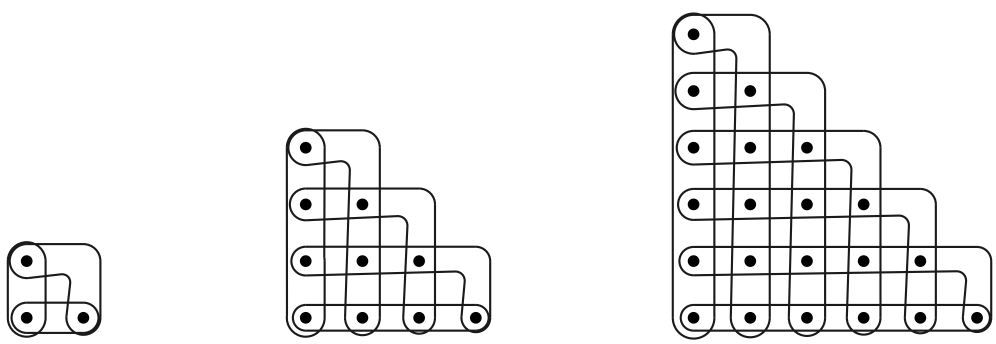


Figure 3: The Fano plane F

A natural question is whether the upper bound in Theorem 2 holds for larger values of k if we impose a linearity constraint. This is indeed the case for all $k \ge 2$ when the maximum degree of a k-uniform linear hypergraph is at most 2. As shown in [12] for $k \ge 2$, there is a unique k-uniform, 2-regular, linear intersecting hypergraph, which we call L_k. The hypergraphs L_2, L_4 and L_6, for example, are illustrated in Figure 4.



(a) L_2 (b) L_4 (c) L_6

Figure 4: The hypergraphs L_2, L_4 and L_6.

Theorem 3 ([12]) *For $k \ge 2$, if H be a k-uniform, linear, connected hypergraph on n vertices with m edges satisfying $\Delta(H) \le 2$, then*

$$\tau(H) \le \frac{n+m}{k+1}$$

with equality if and only if H consists of a single edge or k is even and $H = L_k$.

The authors conjectured in [25] that the bound $\tau(H) \le (n+m)/(k+1)$ also holds for all k-uniform linear hypergraphs in the case when $k = 4$ (which we prove in this paper).

Conjecture 1 ([25]) *For $k = 4$ if H is a k-uniform linear hypergraph on n vertices with m edges, then*

$$\tau(H) \le \frac{n+m}{k+1}.$$

More generally, the following conjecture was posed by the authors at several international conferences in recent years, including in a principal talk at the 8th Slovenian Conference on Graph Theory held in Kranjska Gora in June 2015.

Conjecture 2 *For $k \ge 2$, if H is a k-uniform, linear hypergraph on n vertices with m edges, then*

$$\tau(H) \le \frac{n+m}{k+1}.$$

3 Main Results

We have three immediate aims. Our first aim is to establish the existence of k-uniform, linear hypergraph H on n vertices with m edges satisfying $\tau(H) > (n+m)/(k+1)$. More precisely, we shall prove the following general result, a proof of which is given in Section 4. We remark that this result was motivated by an important paper by Alon et al. [1] on transversal numbers for hypergraphs arising in geometry where each edge in a projective plane is shrunk randomly.

Theorem 4 *For $k \ge 2$, let H be an arbitrary $2k$-uniform $2k$-regular hypergraph on n vertices and let*

$$0 < c < \frac{1}{\ln(4)} \approx 0.72134$$

be arbitrary. Let H' be the k-uniform hypergraph obtained from H by letting $V(H') = V(H)$ and for each edge in H pick k vertices at random from the edge and add these as a hyperedge to H'. We note that $|V(H)| = |E(H)| = |V(H')| = |E(H')| = n$. If

$$5c\ln(k)\ln(n) < k^{1-c\ln(4)},$$

then

$$\tau(H') \le \left(\frac{c\ \ln(k)}{k}\right) n$$

occurs with probability strictly less than 1. Therefore, if $5c\ln(k)\ln(n) < k^{1-c\ln(4)}$, then there exists a k-uniform hypergraph, H^* (where the edges of H^* are subsets of the edges of H), such that $n = |V(H^*)| = |E(H^*)|$ satisfying

$$\tau(H^*) > \left(\frac{c\ \ln(k)}{k}\right) n.$$

Applying Theorem 4 when $k = 2754$ to a projective plane of order $p = 5507$ (which is prime), gives us a counterexample to Conjecture 2. We can in fact prove that from a projective plane of order $p = 331$ one can construct a 166-uniform linear hypergraph H on n vertices with m edges satisfying $\tau(H) > (n+m)/(k+1)$. This result, which we discuss in Remark 3 in Section 4, we state formally as follows.

Theorem 5 *For $k = 166$, there exist k-uniform, linear hypergraph on n vertices with m edges satisfying*

$$\tau(H) > \frac{n+m}{k+1}.$$

Our second aim is to show that for those values of k for which Conjecture 2 holds, the bound is tight for a large number of densities if there exists an affine plane $AG(2,k)$ of order $k \geq 2$. We shall prove the following result, a proof of which is given in Section 5.

Theorem 6 *If the affine plane $AG(2,k)$ of order $k \geq 2$ exists and $\tau(H) \leq (n+m)/(k+1)$ holds for all k-uniform linear hypergraphs, then this upper bound is tight for a number of linear k-uniform hypergraphs H with a wide variety of average degrees.*

Our third aim is to prove Conjecture 1. More precisely, we shall prove the following result, a proof of which is given in Section 7.5.

Theorem 7 *For $k = 4$ if H is a k-uniform linear hypergraph, then*

$$\tau(H) \leq \frac{n+m}{k+1}.$$

By Theorem 2, Conjecture 2 is true for $k \in \{2,3\}$. Theorem 7 implies that the conjecture is also true for $k = 4$. However as remarked earlier, Conjecture 2 fails for large k. We therefore pose the following new problem that we have yet to settle.

Problem 1 ([22]) *Determine the smallest value, $k_{\min}$, of k for which there exists a k-uniform, linear hypergraph H on n vertices with m edges satisfying*

$$\tau(H) > \frac{n+m}{k+1}.$$

As an immediate consequence of Theorem 5 and Theorem 7, we have the following result.

Theorem 8 $5 \leq k_{\min} \leq 166$.

We proceed as follows. In Section 4, we give a proof of Theorem 4. In Section 5, we give a proof of Theorem 6. In Section 6, we present several applications of Theorem 7. In Section 7, we define fifteen special hypergraphs and we introduce the concept of the deficiency of a hypergraph. Thereafter, we prove a key result, namely Theorem 21, about the deficiency of a hypergraph that will enable us to deduce our main result. Finally in Section 7.5, we present a proof of Theorem 7.

4 Proof of Theorem 4

In this section, we give a proof of Theorem 4. We shall need the following well-known lemma.

Lemma 9 *For all $x > 1$, we have $\left(1 - \frac{1}{x}\right)^x < e^{-1} < \left(1 - \frac{1}{x}\right)^{x-1}$.*

We are now in a position to prove Theorem 4.

Proof of Theorem 4. Let H and H' be defined as in the statement of Theorem 4. Let $E(H) = \{e_1, e_2, \ldots, e_n\}$ and let e'_i be the edge in H' obtained by picking k vertices at random from e_i for $i \in [n]$. Let T be a random set of vertices in H' where

$$|T| = \left\lfloor \frac{c\,\ln(k)}{k} n \right\rfloor$$

and let $t_i = |T \cap V(e_i)|$ for each $i \in [n]$. The probability, $\Pr(e'_i \text{ not covered})$, that the edge e'_i is not covered by T (that is, $V(e'_i) \cap T \neq \emptyset$) is given by

$$\Pr(e'_i \text{ not covered}) = \frac{\binom{2k-t_i}{k}}{\binom{2k}{k}}.$$

Given the values t_i for all $i \in [n]$, we note that the probability of an edge e_i being covered by T is independent of an edge e_j being covered by T where $j \in [n] \setminus \{i\}$. Therefore, the probability that T is a transversal of H' is given by

$$\begin{aligned} \Pr(T \text{ is a transversal of } H') &= \prod_{i=1}^{n} \big(1 - \Pr(e'_i \text{ not covered})\big) \\ &= \prod_{i=1}^{n} \left(1 - \frac{\binom{2k-t_i}{k}}{\binom{2k}{k}}\right). \qquad (1) \end{aligned}$$

As every vertex in T belongs to $2k$ edges of H, we note that

$$\sum_{i=1}^{n} t_i = 2k|T|$$

by double counting. Let $s_1, s_2, \ldots, s_n$ be non-negative integers, such that the expression

$$\prod_{i=1}^{n} \left(1 - \frac{\binom{2k-s_i}{k}}{\binom{2k}{k}}\right) \qquad (2)$$

is maximized, where $\sum_{i=1}^{n} s_i = 2k|T|$. We proceed further with the following series of claims.

Claim 1 *$|s_i - s_j| \le 1$ for all $i,j \in [n]$.*

Proof. For the sake of contradiction, suppose that $s_i \le s_j - 2$ for some i and j where $i,j \in [n]$. We will now show that by increasing s_i by one and decreasing s_j by one we increase the value of the maximized expression in (2), a contradiction. Hence it suffices for us to show that Inequality (3) below holds.

$$\left(1 - \frac{\binom{2k-(s_i+1)}{k}}{\binom{2k}{k}}\right)\left(1 - \frac{\binom{2k-(s_j-1)}{k}}{\binom{2k}{k}}\right) > \left(1 - \frac{\binom{2k-s_i}{k}}{\binom{2k}{k}}\right)\left(1 - \frac{\binom{2k-s_j}{k}}{\binom{2k}{k}}\right) \qquad (3)$$

We remark that Inequality (3) is equivalent to the following.

$$\left(\binom{2k}{k} - \binom{2k-s_i-1}{k}\right)\left(\binom{2k}{k} - \binom{2k-s_j+1}{k}\right) > \left(\binom{2k}{k} - \binom{2k-s_i}{k}\right)\left(\binom{2k}{k} - \binom{2k-s_j}{k}\right).$$

Defining $A_i, A_j, B_i, B_j, C_i, C_j$ as follows,

$$\begin{array}{rclcrcl} A_i &=& \binom{2k}{k} - \binom{2k-s_i}{k} & & A_j &=& \binom{2k}{k} - \binom{2k-s_j}{k} \\ B_i &=& \binom{2k-s_i-1}{k-1} & & B_j &=& \binom{2k-s_j+1}{k-1} \\ C_i &=& \binom{2k}{k} - \binom{2k-s_i-1}{k} & & C_j &=& \binom{2k}{k} - \binom{2k-s_j+1}{k}, \end{array}$$

we note that Inequality (3) is equivalent to

$$C_iC_j > A_iA_j. \qquad (4)$$

By Pascal's rule, which states that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, we obtain the following.

$$\begin{array}{rclcl} C_i &=& \binom{2k}{k} - \left(\binom{2k-s_i}{k} - \binom{2k-s_i-1}{k-1}\right) &=& A_i + B_i \\ C_j &=& \binom{2k}{k} - \left(\binom{2k-s_j}{k} + \binom{2k-s_j}{k-1}\right) &=& A_j - B_j. \end{array}$$

Since $s_i \le s_j - 2$, we note that $A_j > C_i = A_i + B_i$ and $B_i \ge B_j$, implying that

$$\begin{array}{rcl} C_iC_j &=& (A_i + B_i)(A_j - B_j) \\ &\ge& (A_i + B_i)(A_j - B_i) \\ &=& A_iA_j + B_i(A_j - (A_i + B_i)) \\ &>& A_iA_j. \end{array}$$

Thus, Inequality (4) holds, producing the desired contradiction. This completes the proof of Claim 1. (□)

Let $\Pr(H)$ denote the probability that $\tau(H') \le \left(\frac{c \ln(k)}{k}\right) n$. Since $\tau(H')$ is an integer, we note that $\Pr(H) = \Pr(\tau(H') \le |T|)$.

Claim 2 $\Pr(H) \leq \binom{n}{|T|} \prod_{i=1}^{n} \left(1 - \frac{\binom{2k-s_i}{k}}{\binom{2k}{k}}\right)$.

Proof. As there are $\binom{n}{|T|}$ possible ways of choosing the set T, Equation (1) implies that the probability that there exists a transversal of H' of size at most $|T|$ is bounded as follows.

$$\begin{aligned}\Pr(\tau(H') \leq |T|) &\leq \binom{n}{|T|} \times \Pr(T \text{ is a transversal of } H') \\ &\leq n^{|T|} \cdot \prod_{i=1}^{n} \left(1 - \frac{\binom{2k-t_i}{k}}{\binom{2k}{k}}\right) \\ &\leq n^{|T|} \cdot \prod_{i=1}^{n} \left(1 - \frac{\binom{2k-s_i}{k}}{\binom{2k}{k}}\right).\end{aligned}$$

This completes the proof of Claim 2. ($\square$)

Let s_{av} denote the average value of the integers $s_1, s_2, \ldots, s_n$, and let

$$s^* = 2c \ln(k).$$

Claim 3 $s_{\text{av}} \leq s^*$.

Proof. Since $\sum_{i=1}^{n} s_i = 2k|T|$, the average value of $s_1, s_2, \ldots, s_n$ is bounded as follows.

$$s_{\text{av}} = \frac{2k|T|}{n} = \frac{2k\lfloor \frac{c \ln(k)}{k} n \rfloor}{n} \leq 2c \ln(k) = s^*.$$

This completed the proof of Claim 3. ($\square$)

Claim 4 $\frac{\binom{2k-s_i}{k}}{\binom{2k}{k}} > \frac{1}{5}\left(\frac{1}{2}\right)^{s_i}$.

Proof. We note that

$$\frac{\binom{2k-s_i}{k}}{\binom{2k}{k}} = \frac{(2k-s_i)!}{k!(k-s_i)!)}\frac{k!k!}{(2k)!} = \frac{k(k-1)\cdots(k-s_i+1)}{(2k)(2k-1)\cdots(2k-s_i+1)} \geq \left(\frac{k-s_i+1}{2k-s_i+1}\right)^{s_i}.$$

Let

$$\theta(s_i) = \frac{2k-2s_i+2}{s_i(s_i-1)}.$$

Since $\theta(s_i)s_i + 1 = \frac{2k-s_i+1}{s_i-1}$, we note by Lemma 9 that the following holds.

$$
\begin{aligned}
\left(\frac{k-s_i+1}{2k-s_i+1}\right)^{s_i} &= \left(\frac{1}{2}\times\frac{2k-2s_i+2}{2k-s_i+1}\right)^{s_i}\\
&= \left(\frac{1}{2}\right)^{s_i}\times\left(1-\frac{s_i-1}{2k-s_i+1}\right)^{s_i}\\
&= \left(\frac{1}{2}\right)^{s_i}\times\left(1-\frac{1}{\frac{2k-s_i+1}{s_i-1}}\right)^{s_i}\\
&= \left(\frac{1}{2}\right)^{s_i}\times\left(\left(1-\frac{1}{\theta(s_i)s_i+1}\right)^{\theta(s_i)s_i}\right)^{\frac{1}{\theta(s_i)}}\\
&> \left(\frac{1}{2}\right)^{s_i}\times\left(e^{-1}\right)^{\frac{1}{\theta(s_i)}}.
\end{aligned}
$$

In order to prove our desired result, it therefore suffices for us to prove that

$$e^{-\frac{1}{\theta(s_i)}} \geq \frac{1}{5}.$$

By Claim 1 and Claim 3, we note that $s_i \leq s^* + 1$ for all $i \in [n]$. As $\theta(s_i)$ is a decreasing function, the function $e^{-1/\theta(s_i)}$ is also a decreasing function in s_i. Hence it suffices for us to show that $e^{-1/\theta(s^*+1)} \geq 1/5$. We note that

$$e^{-\frac{1}{\theta(s^*+1)}} = e^{-\frac{(s^*+1)s^*}{2k-2(s^*+1)+2}} = e^{-\frac{(1+2c\ln(k))2c\ln(k)}{2k-4c\ln(k)}}. \tag{5}$$

Since $c < 1/\ln(4)$,

$$\frac{(1+2c\ln(k))2c\ln(k)}{2k-4c\ln(k)} \leq \frac{\left(1+2\frac{\ln(k)}{\ln(4)}\right)2\ln(k)}{2\ln(4)k-4\ln(k)}.$$

The maximum value of the function on the right-hand-side of the above inequality is approximately 1.5037 obtained at $k = 3.753$, which is always less than $\ln(5) \approx 1.6094$. Thus,

$$\frac{(1+2c\ln(k))2c\ln(k)}{2k-4c\ln(k)} < \ln(5),$$

implying by (5) that

$$e^{-\frac{1}{\theta(s^*+1)}} \geq \frac{1}{5}.$$

This completes the proof of Claim 4. (□)

Claim 5 $\Pr(H) \leq \binom{n}{|T|}\left(\left(1-\frac{1}{5}\left(\frac{1}{2}\right)^{s^*}\right)\right)^{n}$.

Proof. For a non-negative real number s, let

$$f(s) = 1 - \frac{1}{5}\left(\frac{1}{2}\right)^s .$$

We show firstly that

$$f(x-y)f(x+y) \le f(x)^2 \tag{6}$$

holds for all $0 \le y \le x$. This is the case due to the following, where $Y = (1/2)^y$ and $X = (1/2)^x$.

$$\begin{array}{lrcl}
 & 0 & \le & (Y-1)^2 \\
\Updownarrow & & & \\
 & 2Y & \le & Y^2+1 \\
\Updownarrow & & & \\
 & 2X & \le & XY + \frac{X}{Y} \\
\Updownarrow & & & \\
 & -\frac{1}{5}XY - \frac{1}{5}\frac{X}{Y} & \le & -\frac{2}{5}X \\
\Updownarrow & & & \\
 & 1 - \frac{1}{5}\frac{X}{Y} - \frac{1}{5}XY + \frac{1}{25}X^2 & \le & 1 - \frac{2}{5}X + \frac{1}{25}X^2 \\
\Updownarrow & & & \\
 & (1-\frac{1}{5}\frac{X}{Y})(1-\frac{1}{5}XY) & \le & (1-\frac{1}{5}X)^2 \\
\Updownarrow & & & \\
 & \left(1 - \frac{1}{5}\left(\frac{1}{2}\right)^{x-y}\right)\left(1 - \frac{1}{5}\left(\frac{1}{2}\right)^{x+y}\right) & \le & \left(1 - \frac{1}{5}\left(\frac{1}{2}\right)^{x}\right)^2 \\
\Updownarrow & & & \\
 & f(x-y)f(x+y) & \le & f(x)^2
\end{array}$$

Therefore the product $f(s_1)f(s_2)\cdots f(s_n)$, where s_i can be reals and not just integers and $\sum_{i=1}^n s_i = 2k|T|$ is maximized when all s_i have the same value. Furthermore, note that as $f(x)$ is an increasing function the following holds by Lemma 9, Claim 2 and Claim 4.

$$\begin{aligned}
\Pr(H) &\le \binom{n}{|T|}\prod_{i=1}^{n}\left(1 - \frac{\binom{2k-s_i}{k}}{\binom{2k}{k}}\right) \\
&\le \binom{n}{|T|}\prod_{i=1}^{n}\left(1 - \frac{1}{5}\left(\frac{1}{2}\right)^{s_i}\right) \\
&< n^{|T|}\left(1 - \frac{1}{5}\left(\frac{1}{2}\right)^{s^{av}}\right)^n \\
&\le n^{|T|}\left(1 - \frac{1}{5}\left(\frac{1}{2}\right)^{s^*}\right)^n .
\end{aligned}$$

This completes the proof of Claim 5. (□)

We return to the proof of Theorem 4 one final time. Recall that $|T| = \lfloor \frac{c\,\ln(k)}{k} n \rfloor$ and $s^* = 2c\ln(k)$. Further recall that by assumption, $5c\ln(k)\ln(n) < k^{1-c\ln(4)}$. Thus

$$5c\ln(k)\ln(n) < k^{1-c\ln(4)} = \frac{k}{k^{2c\ln(2)}} = \frac{k}{2^{\ln(k^{2c})}},$$

implying that

$$|T|\ln(n) \leq \frac{c\,\ln(k)}{k} n\ln(n) < \frac{n}{5\cdot 2^{2c\ln(k)}} = \frac{n}{5\cdot 2^{s^*}}.$$

Therefore,

$$|T|\ln(n) - \frac{n}{5\cdot 2^{s^*}} < 0.$$

Hence by Claim 5,

$$\begin{aligned}
\Pr(H) &\leq n^{|T|}\left(1-\frac{1}{5}\left(\frac{1}{2}\right)^{s^*}\right)^{n} \\
&= n^{|T|}\left(\left(1-\frac{1}{5\cdot 2^{s^*}}\right)^{5\cdot 2^{s^*}}\right)^{\frac{n}{5\cdot 2^{s^*}}} \\
&< n^{|T|}\left(e^{-1}\right)^{\frac{n}{5\cdot 2^{s^*}}} \\
&= e^{(|T|\ln(n)-\frac{n}{5\cdot 2^{s^*}})} \\
&< 1.
\end{aligned}$$

This completes the proof of Theorem 4. $\square$

We close this section with the following remarks.

Remark 1. If we assume that k is large, then the coefficient 5 of the expression $c\ln(k)\ln(n)$ given in the statement of Theorem 4 can be reduced. In fact the larger k gets, the closer the number can be made to one. For example, if we assume $k \geq 23$, then the 5 can be decreased to a 2, and if $k \geq 54$, then it can be decreased to 1.5.

Remark 2. Let P be a projective plane of order p, where p is an odd prime power. Thus, P has $n = p^2 + p + 1$ points and $m = n$ lines, and each line contains $p + 1$ points. Let H be the hypergraph whose vertices are the points of P and whose edges are the lines of P. We note that H is a $(p + 1)$-regular, $(p + 1)$-uniform, linear hypergraph of order and size $|V(H)| = |E(H)| = p^2 + p + 1$. Letting $k = (p + 1)/2$, we note that $n = |V(H)| = |E(H)| = 4k^2 - 2k + 1$. Applying the result of Theorem 4 on this hypergraph H with $0 < c < \frac{1}{\ln(4)}$, if

$$5c\ln(k)\ln(4k^2 - 2k + 1) < k^{1-c\ln(4)},$$

then there exists a k-uniform hypergraph, H', of order n satisfying

$$\tau(H') > \left(\frac{c\,\ln(k)}{k}\right)n$$

The above holds when $k \geq 2753$ and c is determined such that $(c\ln(k)/k)n = (n{+}m)/(k{+}1)$, which implies that we can use $p = 5507$ (which is a prime) and $k = 2754$.

Remark 3. If we let H be the hypergraph associated with a projective plane of order $p = 331$, then H is a 332-regular, 332-uniform, linear hypergraph with $|V(H)| = |E(H)| = p^2 + p + 1$. Letting $k = 166$ and using Claim 2 in the proof of Theorem 4 we can show that the probability that H' has transversal number less that $(|V(H)| + |E(H)|)/(k + 1)$ is strictly less than one. Therefore, there must exist a linear 166-uniform hypergraph, H^*, where $\tau(H^*) > (|V(H^*)| + |E(H^*)|)/(k + 1)$. This result is stated formally as Theorem 5.

Proposition 1 *For $k = 166$, there exist k-uniform, linear hypergraph on n vertices with m edges satisfying*

$$\tau(H) > \frac{n+m}{k+1}.$$

Remark 4. In Remark 2 and Remark 3, we apply Theorem 4 to projective planes. However, we remark that Theorem 4 can be used on linear hypergraphs H which are not necessarily projective planes as well. Provided the order of H is a polynomial in k, the condition of Theorem 4 will always hold for sufficiently large k.

5 Proof of Theorem 6

In this section, we give a proof of Theorem 6. For this purpose, we shall prove the following result.

Theorem 10 *Let F_{k^2} be the linear, k-uniform, $(k{+}1)$-regular hypergraph of order k^2 which is equivalent to the affine plane $AG(2, k)$ of order k for some $k \geq 2$. Let $e \in E(F_{k^2})$ be an arbitrary edge in F_{k^2} and let $X \subseteq V(e)$ be any non-empty subset of vertices belonging to the edge e. If $H = F_{k^2}(X)$ is the linear, k-uniform hypergraph obtained from F_{k^2} by deleting the vertex set X and all edges intersecting X, then*

$$\tau(H) = \frac{n(H) + m(H)}{k+1}.$$

Proof. Let F_{k^2}, e and X be defined as in the statement of the theorem. For notational simplicity, as k is fixed, denote F_{k^2} by F. By Theorem 1, the transversal number of F is $2(k{-}1){+}1$; that is, $\tau(F) = 2(k{-}1){+}1$. Let $e' \in E(F) \setminus \{e\}$ be an arbitrary edge intersecting e. Since F is equivalent to an affine plane, every edge in $E(F) \setminus \{e, e'\}$ will intersect e or e' (or both e and e'). Thus, $V(e) \cup V(e')$ is a transversal in F of size $2(k - 1) + 1 = 2k - 1$ as F is k-uniform. Let $F(X) = F - X$; that is, $F(X)$ is obtained from F by deleting X and all edges incident with X. If T_X is a transversal in $F(X)$, then $T_X \cup X$ is a transversal in F, which implies that $\tau(F(X)) \geq 2k - 1 - |X|$. However, $(V(e) \cup V(e')) \setminus X$ is a transversal in $F(X)$ of size $2k - 1 - |X|$, and so $\tau(F(X)) \leq 2k - 1 - |X|$. Consequently,

$$\tau(F(X)) = 2k - 1 - |X|.$$

We will now compute the order and size of $F(X)$. The order of $F(X)$ is

$$n(F(X)) = n(F) - |X| = k^2 - |X|.$$

Let x be an arbitrary vertex in X. As $F - x$ is k-regular and has order $k^2 - 1$, the hypergraph $F - x$ contains $k^2 - 1$ edges. Further since F is linear and e is not an edge in $F - x$, no edge in $F - x$ contains more than one vertex from $X \setminus \{x\}$. Therefore, we remove k edges from $F - x$ for every vertex in $X \setminus \{x\}$ we remove when constructing $F(X)$. Thus, the size of $F(X)$ is

$$m(F(X)) = k^2 - 1 - (|X| - 1)k = k^2 + k - 1 - k|X|.$$

Therefore,

$$\tau(F(X)) = 2k - 1 - |X| = \frac{(k^2 - |X|) + (k^2 + k - 1 - k|X|)}{k+1} = \frac{n(F(X)) + m(F(X))}{k+1}.$$

This completes the proof of Theorem 6, noting that $F(X) = F_{k^2}(X)$. $\square$

In particular, when $k = 4$ we know that $AG(2,4)$ exists and therefore, by Theorem 6, we have the following hypergraphs where Theorem 7 is tight.

$F = F_{16}(X)$					
	$\tau(F)$	$n(F)$	$m(F)$	$(n(F) + m(F))/5$	Average degree
$\|X\| = 1$	6	15	15	6	$60/15 = 4$
$\|X\| = 2$	5	14	11	5	$44/14 \approx 3.14\ldots$
$\|X\| = 3$	4	13	7	4	$28/13 \approx 2.15\ldots$
$\|X\| = 4$	3	12	3	3	$12/12 = 1$

We close this section with the following remark.

Remark 5. We remark that if $\tau(H) \leq (n + m)/(k + 1)$ holds for all k-uniform linear hypergraphs for some $k \geq 2$, then the bound cannot be tight for any hypergraph, H, with average degree greater than k as removing any vertex, x, of degree more than k from H and applying the bound on $\tau(H - x)$ gives us a transversal in H of size at most the following,

$$\tau(H - x) + |\{x\}| \leq \frac{(|V(H)| - 1) + (|E(H)| - (k+1))}{k+1} + 1 < \frac{|V(H)| + |E(H)|}{k+1}.$$

Similarly, if $\tau(H) \leq (n + m)/(k + 1)$ holds for all k-uniform linear hypergraphs for some $k \geq 2$, then the bound cannot be tight if the average degree is less than 1 as then there are isolated vertices that can be removed. Therefore, Theorem 10 can be used to show that if $k_{\min} > k$ for some $k \geq 2$ (and the affine plane $AG(2,k)$ exists), then the bound $\tau(H) \leq (n+m)/(k+1)$ is tight for average degree k and average degree 1 and for a number of average degrees in the interval from 1 to k. This is somewhat surprising as there are no similar kinds of bounds which hold for a wide variety of average degrees if we consider non-linear hypergraphs.

6 Applications of Theorem 7

In this section, we present a few applications to serve as motivation for the significance of our result given in Theorem 7.

6.1 Application 1

The following conjecture is posed in [22].

Conjecture 3 ([22]) *If H is a 4-uniform, linear hypergraph on n vertices with m edges, then $\tau(H) \le \frac{n}{4} + \frac{m}{6}$.*

We remark that the linearity constraint in Conjecture 3 is essential. Indeed if H is not linear, then Conjecture 3 is not always true, as may be seen, for example, by taking H to be the complement of the Fano plane, F, shown in Figure 3. The second consequence of our main result proves Conjecture 3.

Theorem 11 *Conjecture 3 is true.*

Proof. Let H be a 4-uniform, linear hypergraph on n vertices with m edges. We show that $\tau(H) \le \frac{n}{4} + \frac{m}{6}$. We proceed by induction on n. If $n = 4$, then H consists of a single edge, and $\tau(H) = 1 < \frac{n}{4} + \frac{m}{6}$. Let $n \ge 5$ and suppose that the result holds for all 4-uniform, linear hypergraphs on fewer than n vertices. Let H be a 4-uniform, linear hypergraph on n vertices with m edges. Suppose that $\Delta(H) \le 6$. In this case, $2m \le 3n$. By Theorem 7, $60\tau(H) \le 12n + 12m = 15n + 10m + 2m - 3n \le 15n + 10m$, or, equivalently, $\tau(H) \le \frac{n}{4} + \frac{m}{6}$. Hence, we may assume that $\Delta(H) \ge 7$, for otherwise the desired result follows from Theorem 7. Let v be a vertex of maximum degree in H, and consider the 4-uniform, linear hypergraph $H' = H - v$ on $n' = n - 1$ vertices with m' edges. We note that $n' = n - 1$ and $m' = m - \Delta(H) \le m - 7$. Every transversal in H' can be extended to a transversal in H by adding to it the vertex v. Hence, applying the inductive hypothesis to H', we have that $\tau(H) \le 1 + \tau(H') \le 1 + \frac{n'}{4} + \frac{m'}{6} \le 1 + \frac{n-1}{4} + \frac{m-7}{6} < \frac{n}{4} + \frac{m}{6}$. $\square$

6.2 Application 2

There has been much interest in determining upper bounds on the transversal number of a 3-regular 4-uniform hypergraph. In particular, as a consequence of more general results we have the Chvátal-McDiarmid bound, the improved Lai-Chang bound, the further improved Thomassé-Yeo bound, and the recent bound given in [25]. These bounds are summarized in Theorem 12. We observe that $\frac{3}{8} < \frac{8}{21} < \frac{7}{18} < \frac{5}{12}$.

Theorem 12 *Let H be a 3-regular, 4-uniform hypergraph on n vertices. Then the following bounds on $\tau(H)$ have been established.*

(a) $\tau(H) \le \frac{5}{12}n \approx 0.41667\,n$ (Chvátal, McDiarmid [10]).
(b) $\tau(H) \le \frac{7}{18}n \approx 0.38888\,n$ (Lai, Chang [32]).
(c) $\tau(H) \le \frac{8}{21}n \approx 0.38095\,n$ (Thomassé, Yeo [37]).
(d) $\tau(H) \le \frac{3}{8}n \approx 0.375\,n$ (Henning, Yeo [25]).

The bound in Theorem 12(d) is best possible, due to the (non-linear) hypergraph, H_8, with $n = 8$ vertices and $\tau(H) = 3$ shown in Figure 5.

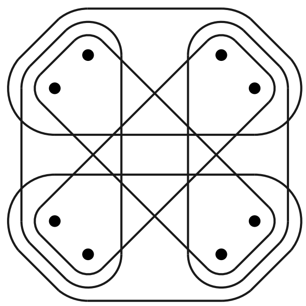


Figure 5: A 3-regular 4-uniform hypergraph, H_8, on n vertices with $\tau(H_8) = \frac{3}{8}n$.

A natural question is whether the upper bound in Theorem 12(d), namely $\tau(H) \le \frac{3}{8}n$, can be improved if we restrict our attention to linear hypergraphs. We answer this question in the affirmative. If H is a 3-regular, 4-uniform, linear hypergraph on n vertices with m edges, then $m = \frac{3}{4}n$, and so, by Theorem 7, $\tau(H) \le \frac{1}{5}(n+m) = \frac{1}{5}(n + \frac{3}{4}n) = \frac{7}{20}n$. Hence, as an immediate corollary of Theorem 7, we have the following result.

Theorem 13 *If H is a 3-regular, 4-uniform, linear hypergraph on n vertices, then $\tau(H) \le \frac{7}{20}n = 0.35n$.*

6.3 Application 3

Lai and Chang [32] established the following upper bound on the transversal number of a 4-uniform hypergraph.

Theorem 14 ([32]) *If H is a 4-uniform hypergraph with n vertices and m edges, then $\tau(H) \le 2(n+m)/9$.*

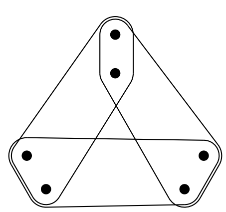


Figure 6: The hypergraph T_4.

The hypergraph T_4, illustrated in Figure 6, shows that the Lai-Chang bound is best possible, even if we restrict the maximum degree to be equal to 2. Our main result, namely Theorem 7, improves this upper bound from $\frac{2}{9}(n+m)$ to $\frac{1}{5}(n+m)$ in the case of linear hypergraphs. As an application of the proof of our main result, we show that the $\frac{1}{5}(n+m)$ bound can be further improved to $\frac{3}{16}(n+m)+\frac{1}{16}$ if we exclude the special hypergraph H_{10}.

For this purpose, we construct a family, $\mathcal{F}$, of 4-uniform, connected, linear hypergraphs with maximum degree $\Delta(H) = 2$ as follows. Let F_0 be the hypergraph with one edge (illustrated in Figure 10(a), but with a different name, H_4). For $i \geq 1$, we now build a hypergraph F_i inductively as follows. Let F_i be obtained from F_{i-1} by adding 12 new vertices, adding three new edges so that each new vertex belongs to exactly one of these added edges, and adding one further edge that contains a vertex in $V(F_{i-1})$ and three additional vertices, one from each of the three newly added edges, in such a way that $\Delta(F_i) = 2$. Let $\mathcal{F}$ be the family of all such hypergraphs, F_i, where $i \geq 0$. A hypergraph, F_6, in the family $\mathcal{F}$ is illustrated in Figure 7.

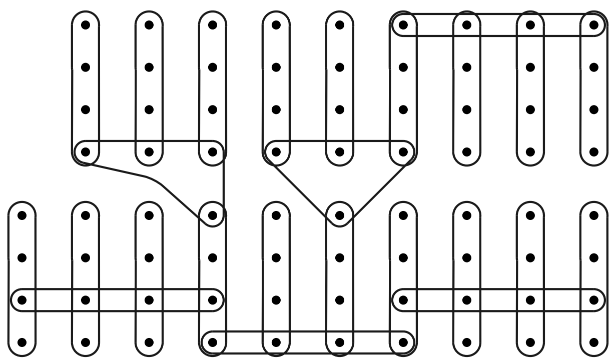


Figure 7: A hypergraph, F_6, in the family $\mathcal{F}$.

We are now in a position to state the following result, where H_{10}, $H_{14,5}$ and $H_{14,6}$ are the 4-uniform, linear hypergraphs shown in Figure 10(b), 10(h) and 10(i), respectively. As observed earlier, $H_4 = F_0$, and so $H_4 \in \mathcal{F}$. A proof of Theorem 15 is given in Section 8.

Theorem 15 *Let $H \neq H_{10}$ be a 4-uniform, connected, linear hypergraph with maximum degree $\Delta(H) \leq 2$ on n vertices with m edges. Then, $\tau(H) \leq \frac{3}{16}(n+m)+\frac{1}{16}$, with equality if and only if $H \in \{H_{14,5}, H_{14,6}\}$ or $H \in \mathcal{F}$.*

6.4 Application 4

The Heawood graph, shown in Figure 8, is the unique 6-cage. The bipartite complement of the Heawood graph is the bipartite graph formed by taking the two partite sets of the Heawood graph and joining a vertex from one partite set to a vertex from the other partite set by an edge whenever they are not joined in the Heawood graph. The bipartite complement of the Heawood graph can also be seen as the incidence bipartite graph of the complement of the Fano plane.

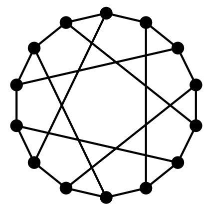


Figure 8: The Heawood graph.

Thomassé and Yeo [37] established the following upper bound on the total domination number of a graph with minimum degree at least 4. Recall that $\delta(G)$ denotes the minimum degree of a graph G.

Theorem 16 ([37]) *If G is a graph of order n with $\delta(G) \ge 4$, then $\gamma_t(G) \le \frac{3}{7}n$.*

The extremal graphs achieving equality in the Thomassé-Yeo bound of Theorem 16 are given by the following result.

Theorem 17 ([22, 24]) *If G is a connected graph of order n with $\delta(G) \ge 4$ that satisfies $\gamma_t(G) = \frac{3}{7}n$, then G is the bipartite complement of the Heawood Graph.*

We remark that every vertex in the bipartite complement of the Heawood Graph belongs to a 4-cycle. It is therefore a natural question to ask whether the Thomassé-Yeo upper bound of $\frac{3}{7}n$ can be improved if we restrict G to contain no 4-cycles. As a consequence of our main result Theorem 7, this question can now be answered in the affirmative. For a graph G, the *open neighborhood hypergraph*, abbreviated ONH, of G is the hypergraph H_G with vertex set $V(H_G) = V(G)$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V\}$ consisting of the open neighborhoods of vertices in G. As first observed in [37] (see also [24]), the transversal number of the ONH of a graph is precisely the total domination number of the graph; that is, for a graph G, we have $\gamma_t(G) = \tau(H_G)$.

As an application of Theorem 7, we have the following result, which significantly improves the upper bound of Theorem 16 when the graph G contains no 4-cycle.

Theorem 18 *If G is a quadrilateral-free graph of order n with $\delta(G) \ge 4$, then $\gamma_t(G) \le \frac{2}{5}n$.*

Proof. Let G be a quadrilateral-free graph of order n with $\delta(G) \ge 4$ and let H_G be the ONH of G. Then, each edge of H_G has size at least 4. Since G is contains no 4-cycle, the hypergraph H_G contains no overlapping edges and is therefore linear. Let H be obtained from H_G by shrinking all edges of H_G, if necessary, to edges of size 4. Then, H is a 4-uniform linear hypergraph with n vertices and n edges; that is, $n(H) = m(H) = n(G) = n$. By Theorem 7 we note that $\tau(H) \le \frac{1}{5}(n(H) + m(H)) = \frac{2}{5}n$. This completes the proof of the theorem since $\gamma_t(G) = \tau(H_G) \le \tau(H)$. $\Box$

That the bound in Theorem 18 is best possible, may be seen by taking, for example, the 4-regular bipartite quadrilateral-free graph G_{30} of order $n = 30$ illustrated in Figure 9 satisfying $\gamma_t(G_{30}) = 12 = \frac{2}{5}n$. We note that the graph G_{30} is the incidence bipartite graph of the linear 4-uniform hypergraph obtained by removing an arbitrary vertex from the affine plane $AG(2, 4)$ of order 4.

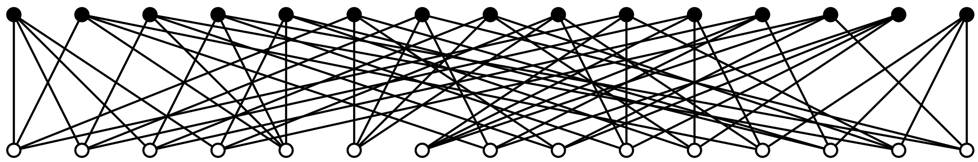


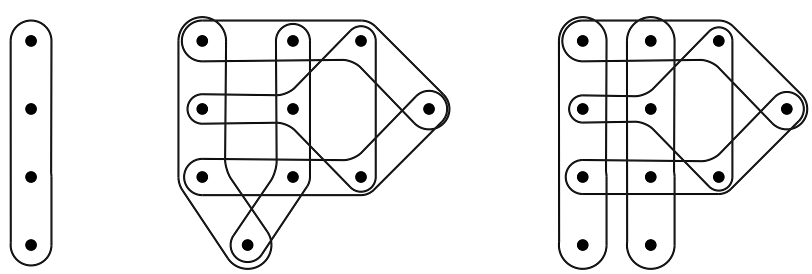
Figure 9: A quadrilateral-free 4-regular graph G_{30} of order $n = 30$ with $\gamma_t(G_{30}) = \frac{2}{5}n$.

7 Key Theorem

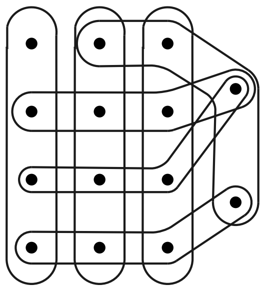
In this section, we prove a key theorem that we will need in proving Theorem 7. First we define a few special hypergraphs. We then introduce the concept of the deficiency of a set of hypergraphs. Thereafter, we prove a key result that will enable us to deduce our main result. Throughout this section, we let $\mathcal{H}_4$ be the class of all 4-uniform, linear hypergraphs with maximum degree at most 3.

7.1 Special Hypergraphs

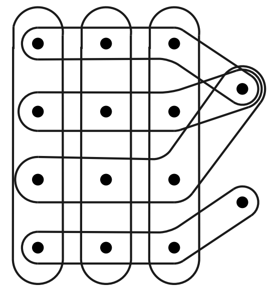
In this section, we define fifteen special hypergraphs, which are shown in Figure 10.



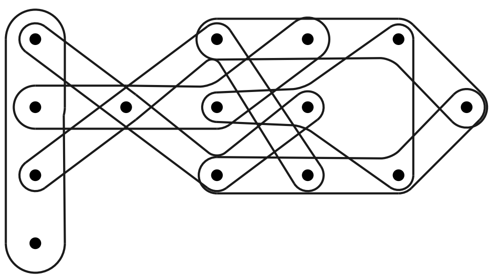
(a) H_4 (b) H_{10} (c) H_{11}



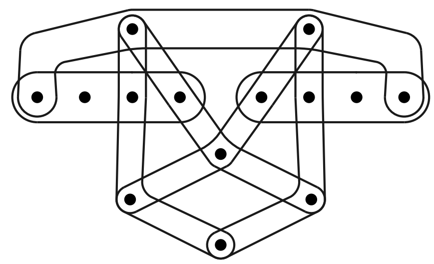
(d) $H_{14,1}$



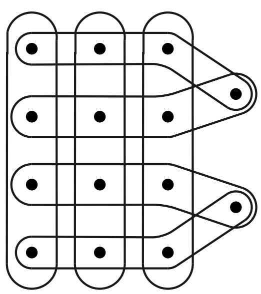
(e) $H_{14,2}$



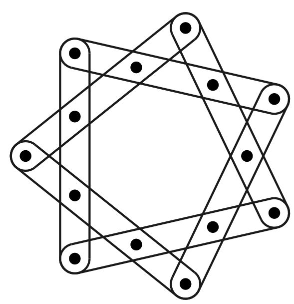
(f) $H_{14,3}$



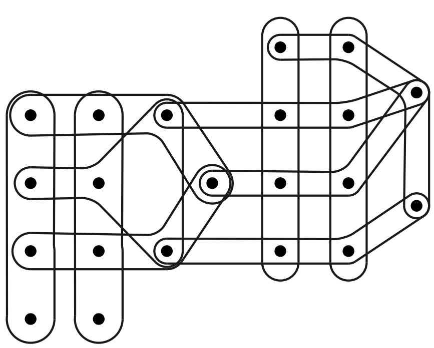
(g) $H_{14,4}$



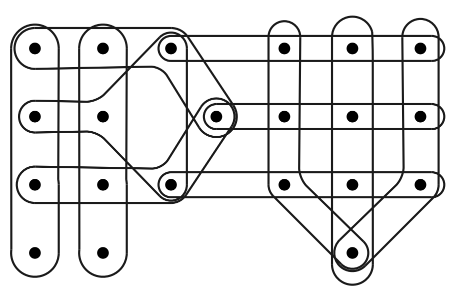
(h) $H_{14,5}$



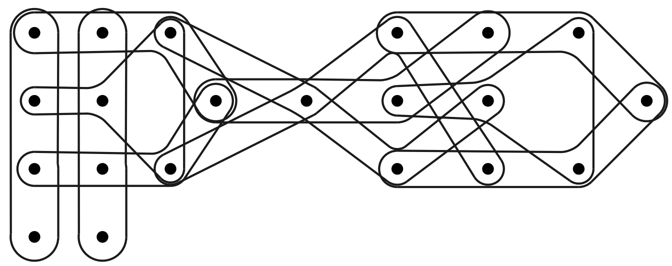
(i) $H_{14,6}$



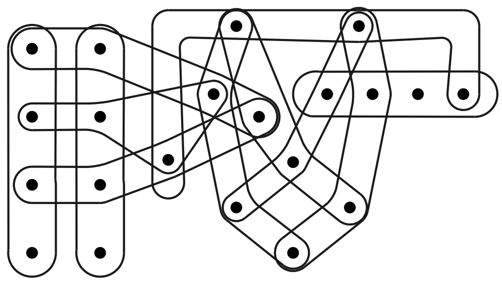
(j) $H_{21,1}$



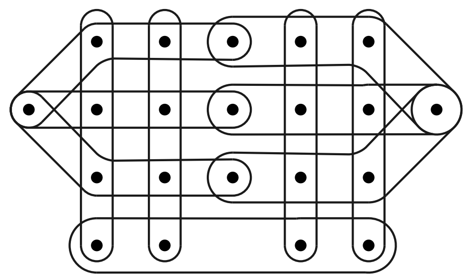
(k) $H_{21,2}$



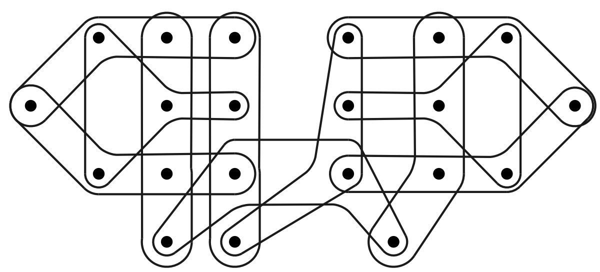
(ℓ) $H_{21,3}$



(m) $H_{21,4}$



(n) $H_{21,5}$



(o) $H_{21,6}$

Figure 10: The fifteen special hypergraphs.

We define

$$\begin{aligned}\mathcal{H}_{14} &= \{H_{14,1}, H_{14,2}, H_{14,3}, H_{14,4}, H_{14,5}, H_{14,6}\} \text{ and}\\ \mathcal{H}_{21} &= \{H_{21,1}, H_{21,2}, H_{21,3}, H_{21,4}, H_{21,5}, H_{21,6}\}.\end{aligned}$$

The following properties of special hypergraphs, which we have verified by computer, will prove to be useful.

Observation 1 *Let H be a special hypergraph of order n_H and size m_H, and let u and v be two arbitrary distinct vertices in H. Then the following hold.*

(a) *If $H = H_4$, then $n_H = 4$, $m_H = 1$ and $\tau(H) = 1$.*
(b) *If $H = H_{10}$, then $n_H = 10$, $m_H = 5$ and $\tau(H) = 3$.*
(c) *If $H = H_{11}$, then $n_H = 11$, $m_H = 5$ and $\tau(H) = 3$.*
(d) *If $H \in \mathcal{H}_{14}$, then $n_H = 14$, $m_H = 7$ and $\tau(H) = 4$.*

(e) *If $H \in \mathcal{H}_{21}$, then $n_{_H} = 21$, $m_{_H} = 11$ and $\tau(H) = 6$.*

(f) *If $H \in \{H_{10}, H_{14,5}, H_{14,6}\}$, then H is 2-regular.*

(g) *Any given vertex in H belongs to some $\tau(H)$-transversal.*

(h) *If $H \in \{H_{10}, H_{14,6}\}$, then there exists a $\tau(H)$-transversal that contains both u and v.*

(i) *If $H \neq H_4$, $T \subset V(H)$ and $|T| = 3$, then there exists a $\tau(H)$-transversal that contains at least two vertices of T, unless $H = H_{11}$ and T is the set of three vertices in H that have no neighbor of degree 1 in H.*

(j) *If $H \neq H_4$, $T \subset V(H)$ and $|T| = 4$, then there exists a $\tau(H)$-transversal that contains at least two vertices of T.*

(k) *If $H \neq H_4$ and T_1 and T_2 are vertex-disjoint subsets of H such that $|T_1| = |T_2| = 2$, then there exists a $\tau(H)$-transversal that contains a vertex from T_1 and a vertex from T_2.*

(ℓ) *If T_1 and T_2 are vertex-disjoint subsets of H such that $|T_1| = 3$ and $|T_2| = 1$ and T_1 contains two vertices that are not adjacent in H, then there exists a $\tau(H)$-transversal that contains a vertex from T_1 and a vertex from T_2.*

(m) *If T_1 and T_2 are vertex-disjoint subsets of H such that $|T_1| = 1$ and $|T_2| = 2$ where T_2 contains two vertices that are not adjacent in H, then there exists a $\tau(H)$-transversal that contains a vertex from T_1 and a vertex from T_2, except if $H = H_{11}$, and one degree-1 vertex in H belongs to T_1 and the other degree-1 vertex to T_2, and the second vertex of T_2 is adjacent to the vertex of T_1.*

(n) *If T_1, T_2 and T_3 are vertex-disjoint subsets of H such that $|T_1| = |T_2| = 3$ and $|T_3| \geq 2$, and T_1, T_2 and T_3 are independent sets in H, then there exists a $\tau(H)$-transversal that contains a vertex from each of T_1, T_2 and T_3.*

(o) *If H is a special subhypergraph of a 4-uniform linear hypergraph F where $\Delta(F) \leq 3$, and if there are three edges of F each of which intersect H in two vertices, then there is a $\tau(H)$-transversal that covers one of these edges and any specified edge of H.*

(p) *If $H = H_{11}$ or if $H \in \mathcal{H}_{14} \cup \mathcal{H}_{21}$, and if v is a vertex of degree 2 in H, then either $H - v$ is connected or $H - v$ is disconnected with exactly two components, one of which consists of an isolated vertex. Further, $H - y$ does not contain two vertex disjoint copies of H_4 that are both intersected by a common edge and such that both copies of H_4 have three vertices of degree 1 and one vertex of degree 2.*

7.2 Known Results and Observations

We shall need the following theorem of Berge [3] about the matching number of a graph, which is sometimes referred to as the Tutte-Berge formulation for the matching number.

Theorem 19 (Tutte-Berge Formula) *For every graph G,*

$$\alpha'(G) = \min_{X \subseteq V(G)} \frac{1}{2} \left(|V(G)| + |X| - \mathrm{oc}(G - X) \right),$$

where $\mathrm{oc}(G - X)$ *denotes the number of odd components of* $G - X$.

We shall also rely heavily on the following well-known theorem due to König [29] and Hall [16] in 1935.

Theorem 20 (Hall's Theorem) *Let G be a bipartite graph with partite sets X and Y. Then X can be matched to a subset of Y if and only if $|N(S)| \geq |S|$ for every nonempty subset S of X.*

7.3 The Deficiency of a Set

Let H be a 4-uniform hypergraph. A set X is a *special H-set* if it consists of subhypergraphs of H with the property that every subhypergraph in X is a special hypergraph and further these special hypergraphs are pairwise vertex disjoint. For notational simplicity, we write $V(X)$ and $E(X)$ to denote the set of all vertices and edges, respectively, in H that belong to a subhypergraph $H' \in X$ in the special H-set X. Let X be an arbitrary special H-set.

A set T of vertices in $V(X)$ is an *X-transversal* if T is a minimum set of vertices that intersects every edge from every subhypergraph in X.

We define $E^*_H(X)$ to be the set of all edges in H that do not belong to a subhypergraph in X but which intersect at least one subhypergraph in X. Hence if $e \in E^*(X)$, then $e \notin E(H')$ for every subhypergraph $H' \in X$ but $V(e) \cap V(H') \neq \emptyset$ for at least one subhypergraph $H' \in X$. If the hypergraph H is clear from context, we simply write $E^*(X)$ rather than $E^*_H(X)$.

We associate with the set X a bipartite graph, which we denote by G_X, with partite sets X and $E^*(X)$, where an edge joins $e \in E^*(X)$ and $H' \in X$ in G_X if and only if the edge e intersects the subhypergraph H' of X in H.

We define a *weak partition* of $X = (X_4, X_{10}, X_{11}, X_{14}, X_{21})$ (where a weak partition is a partition in which some of the sets may be empty) where $X_i \subseteq X$ consists of all subhypergraphs in X of order i, $i \in \{4, 10, 11, 14, 21\}$. Thus, $X = X_4 \cup X_{10} \cup X_{11} \cup X_{14} \cup X_{21}$ and $|X| = |X_4| + |X_{10}| + |X_{11}| + |X_{14}| + |X_{21}|$. As an immediate consequence of Observation 1(a)–(e), we have the following result.

Observation 2 *If X is a special H-set and T is an X-transversal, then*

$$|T| = |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{12}| + 6|X_{21}|.$$

We define the *deficiency* of X in H as

$$\mathrm{def}_H(X) = 10|X_{10}| + 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| - 13|E^*(X)|.$$

We define the *deficiency* of H by

$$\mathrm{def}(H) = \max \mathrm{def}_H(X)$$

where the maximum is taken over all special H-sets X. We note that taking $X = \emptyset$, we have $\mathrm{def}(H) \geq 0$.

7.4 Key Theorem

Recall that $\mathcal{H}_4$ is the class of all 4-uniform linear hypergraphs with maximum degree at most 3. We shall prove the following key result that we will need when proving our main theorem.

Theorem 21 *If $H \in \mathcal{H}_4$, then $45\tau(H) \le 6n(H) + 13m(H) + \mathrm{def}(H)$.*

Proof. For a 4-uniform hypergraph H, let

$$\xi(H) = 45\tau(H) - 6n(H) - 13m(H) - \mathrm{def}(H).$$

We wish to show that if $H \in \mathcal{H}_4$, then $\xi(H) \le 0$. Suppose, to the contrary, that the theorem is false and that $H \in \mathcal{H}_4$ is a counterexample with minimum value of $n(H) + m(H)$. Thus, $\xi(H) > 0$ but every hypergraph $H' \in \mathcal{H}_4$ with $n(H') + m(H') < n(H) + m(H)$ satisfies $\xi(H') \le 0$. We will now prove a number of claims, where the last claim completes the proof of the theorem.

Claim A: The hypergraph H is connected and $\delta(H) \ge 1$.

Proof of Claim A: If H is disconnected, then by the minimality of H we have that the theorem holds for all components of H and therefore also for H, a contradiction. Therefore, H is connected. If H has an isolated vertex, then by the connectivity of H we have that $n(H) = 1$ and $m(H) = 0$, implying that $45\tau(H) = 0 < 6 = 6n(H) + 13m(H) + \mathrm{def}(H)$, a contradiction. Hence, $\delta(H) \ge 1$. ($\Box$)

Claim B: Given a special H-set, X, there is no X-transversal, T, such that $|T| = |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + |X_{21}|$ and T intersects every edge in $E^*(X)$.

Proof of Claim B: Suppose that there does exist an X-transversal, T, such that $|T| = |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + |X_{21}|$ and T intersects every edge in $E^*(X)$. Let H' be obtained from H by removing all vertices in $V(X)$ and removing all edges of H that intersect $V(X)$. Since T is an X-transversal and T intersects every edge in $E^*(X)$, we remark that $H' = H(T, V(X) \setminus T)$. Then, H' is a 4-uniform hypergraph with maximum degree $\Delta(H) \le 3$. Further, $n(H') = n(H) - 4|X_4| - 10|X_{10}| - 11|X_{11}| - 14|X_{14}| - 21|X_{21}|$, $m(H') = m(H) - |X_4| - 5|X_{10}| - 5|X_{11}| - 7|X_{14}| - 11|X_{21}| - |E^*(X)|$, and $\tau(H) \le |T| + \tau(H')$. Let

$$\mathrm{def}(H') = \mathrm{def}_{H'}(X')$$

for some special H'-set, X'. Let $X^* = X \cup X'$. Then, $\mathrm{def}(H') = \mathrm{def}_H(X^*) - \mathrm{def}_H(X)$. By the minimality of $n(H) + m(H)$, H' is not a counterexample to our theorem, and so $45\tau(H') \le 6n(H') + 13m(H') + \mathrm{def}(H')$. Hence,

$$
\begin{array}{rcl}
45\tau(H) & \leq & 45\tau(H') + 45|T| \\
& \leq & (6n(H') + 13m(H') + \mathrm{def}(H'))+ \\
& & \qquad 45(|X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + 6|X_{21}|) \\
& \leq & 6(n(H) - 4|X_4| - 10|X_{10}| - 11|X_{11}| - 14|X_{14}| - 21|X_{21}|)+ \\
& & \quad 13(m(H) - |X_4| - 5|X_{10}| - 5|X_{11}| - 7|X_{14}| - 11|X_{21}| - |E^*(X)|)+ \\
& & \qquad \mathrm{def}_H(X^*) - \mathrm{def}_H(X) + 45(|X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + 6|X_{21}|) \\
& = & 6n(H) + 13m(H) + 8|X_4| + 10|X_{10}| + 4|X_{11}| + 5|X_{14}| + |X_{21}| - 13|E^*(X)|+ \\
& & \quad \mathrm{def}_H(X^*) - \mathrm{def}_H(X) \\
& = & 6n(H) + 13m(H) + \mathrm{def}_H(X^*) \\
& \leq & 6n(H) + 13m(H) + \mathrm{def}(H),
\end{array}
$$

contradicting the fact that H is a counterexample. (□)

As an immediate consequence of Claim A and Claim B, H is not a special hypergraph. We state this formally as follows.

Claim C: H is not a special hypergraph.

Among all special non-empty H-sets, let X be chosen so that

(1) $|E^*(X)| - |X|$ is minimum.
(2) Subject to (1), $|X|$ is maximum.

Claim D: $|E^*(X)| \geq |X| + 1$.

Proof of Claim D: Suppose, to the contrary, that $|E^*(X)| \leq |X|$. Let G_X be the bipartite graph associated with the special H-set X. Suppose there exists a matching, M, in G_X that matches $E^*(X)$ to a subset of X. Then, by Observation 1(g), there exists a minimum X-transversal, T, that intersects every edge in $E^*(X)$, contradicting Claim B. Therefore, no matching in G_X matches $E^*(X)$ to a subset of X. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*(X)$ such that $|N_{G_X}(S)| < |S|$. We now consider the special H-set, $X' = X \setminus N_{G_X}(S)$. Then, $|X'| = |X| - |N_{G_X}(S)|$ and $|E^*(X')| = |E^*(X)| - |S|$. Thus,

$$
\begin{array}{rcl}
|E^*(X')| - |X'| & = & (|E^*(X)| - |S|) - (|X| - |N_{G_X}(S)|) \\
& = & (|E^*(X)| - |X|) + (|N_{G_X}(S)| - |S|) \\
& < & |E^*(X)| - |X|,
\end{array}
$$

contradicting our choice of the special H-set X. (□)

As an immediate consequence of Claim D and by our choice of the special H-set X, we have the following claim.

Claim E: If $X' \neq \emptyset$ is a special H-set, then $|E^*(X')| \geq |X'| + 1$.

Claim F: $\mathrm{def}(H) = 0$.

Proof of Claim F: Let X^* be a special H-set. If $X^* \neq \emptyset$, then by Claim E, $|E^*(X^*)| \geq$

$|X^*|+1$, implying that $\mathrm{def}_H(X^*) \le 10|X^*| - 13|E^*(X^*)| < 0$. However, taking $X^* = \emptyset$, we note that $\mathrm{def}_H(X^*) = 0$. Consequently, $\mathrm{def}(H) = 0$. $(\Box)$

For a hypergraph $H' \in \mathcal{H}_4$, let

$$\Phi(H') = \xi(H') - \xi(H).$$

Claim G: If $H' \in \mathcal{H}_4$ satisfies $n(H') + m(H') < n(H) + m(H)$, then $\Phi(H') < 0$.

Proof of Claim G: Since H is a counterexample with minimum value of $n(H) + m(H)$, and since $n(H') + m(H') < n(H) + m(H)$, we note that $\xi(H') \le 0$. Thus,

$$\begin{aligned} 45\tau(H) &= \xi(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\ &= \xi(H') - \Phi(H') + 6n(H) + 13m(H) + \mathrm{def}(H) \\ &\le -\Phi(H') + 6n(H) + 13m(H) + \mathrm{def}(H), \end{aligned}$$

implying that $\Phi(H') \le 6n(H) + 13m(H) + \mathrm{def}(H) - 45\tau(H) = -\xi(H) < 0$. $(\Box)$

Claim H: $|E^*(X)| \ge |X| + 2$.

Proof of Claim H: Suppose, to the contrary, that $|E^*(X)| \le |X| + 1$. Then, by Claim D, $|E^*(X)| = |X| + 1$. We define an edge $e \in E^*(X)$ to be X-*universal* if there exists a special subhypergraph $F \in X$, such that e intersects F and for every other edge, say f, that intersects F there exists a $\tau(F)$-transversal that covers both e and f. We proceed further with the following series of subclaims.

Claim H.1: If $e \in E^*(X)$, then there exists an X-transversal that intersects every edge in $E^*(X) \setminus \{e\}$. Furthermore, no edge in $E^*(X)$ is X-universal.

Proof of Claim H.1: Let G_X be the bipartite graph associated with the special H-set X and consider the graph $G_X - e$, where e is an arbitrary vertex in $E^*(X)$. We note that the partite sets of $G_X - e$ are $E^*(X) \setminus \{e\}$ and X and these sets have equal cardinalities. Suppose that $G_X - e$ does not have a perfect matching. By Hall's Theorem, there is a nonempty subset $S \subseteq X$ such that in the graph $G_X - e$, we have $|N(S)| < |S|$. (We remark that here $N(S) \subset E^*(X) \setminus \{e\}$.) Since the edge e may possibly intersect a subhypergraph of S in H, we note that in the graph G_X, $|N(S)| \le |S|$. However, in the graph G_X, $|E^*(S)| = |N(S)|$, implying that $|E^*(S)| \le |S|$. By our choice of the special H-set X, $|E^*(X)| - |X| \le |E^*(S)| - |S| \le 0$, contradicting Claim D. Therefore, $G_X - e$ has a perfect matching. By Observation 1(g), this implies the existence of a minimum X-transversal, T, that intersects every edge in $E^*(X) \setminus \{e\}$. This proves the first part of Claim H.1.

For the sake of contradiction, suppose that $e \in E^*(X)$ is a universal edge. Let $F \in X$ be a special subhypergraph of X that is intersected by e, such that for every $f \in E^*(X)$ intersecting F there exists a $\tau(F)$-transversal that covers both e and f. By the above argument there exists a perfect matching, M, in $G_X - e$. Let g be the edge that is matched to F in M. By definition there is a $\tau(F)$-transversal intersecting both e and g. Due to the matching $M \setminus \{gF\}$ and Observation 1(g), we therefore obtain an X-transversal, T, that intersects every edge in $E^*(X)$, a contradiction to Claim B. This proves the second part of Claim H.1. $(\Box)$

Claim H.2: $|X_{10}| = 0$.

Proof of Claim H.2: Suppose, to the contrary, that $|X_{10}| > 0$. Let H' be a copy of H_{10} that belongs to the special H-set X. By Claim E, at least two edges in $E^*(X)$ intersect H' in H. Let e be such an edge of $E^*(X)$ that intersects H'. Let G_X be the bipartite graph associated with X and consider the graph $G_X - e$. We note that the partite sets of $G_X - e$ are $E^*(X) \setminus \{e\}$ and X and these sets have equal cardinalities. Suppose that $G_X - e$ does not have a perfect matching. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*(X) \setminus \{e\}$ such that in the graph $G_X - e$ (and in the graph G_X), we have $|N(S)| < |S|$. We remark that here $N(S) \subset X$. We now consider the special H-set, $X' = X \setminus N(S)$. Then, $X' \neq \emptyset$ and

$$\begin{aligned} |E^*(X')| &\leq |E^*(X)| - |S| \\ &\leq |E^*(X)| - (|N(S)| + 1) \\ &= |X| + 1 - |N(S)| - 1 \\ &= |X'|, \end{aligned}$$

contradicting Claim E. Therefore, $G_X - e$ has a perfect matching, M say. Let e' be the vertex of $E^*(X)$ that is M-matched to H' in $G_X - e$. Let u and v be vertices in the subhypergraph H' that belong to the edges e and e', respectively, in H. By Observation 1(h), there exists a $\tau(H')$-transversal that contains both u and v. If H'' is a subhypergraph in X different from H' and if e'' is the vertex of $E^*(X)$ that is M-matched to H'' in $G_X - e$, then by Observation 1(g) there exists a $\tau(H'')$-transversal that contains a vertex of H'' that belongs to e'' in H. This implies the existence of a minimum X-transversal, T, that intersects every edge in $E^*(X)$, contradicting Claim B. $(\Box)$

Recall that the boundary of a set S of vertices in H, denoted $\partial_H(S)$ or simply $\partial(S)$ if H is clear from context, is the set $N(S) \setminus S$. Abusing notation, we write $\partial(X)$, rather than $\partial(V(X))$, as the boundary of the set $V(X)$. Let X' be a set of new vertices (not in H), where $|X'| = \max(0, 4 - |\partial(X)|)$. We note that $|X' \cup \partial(X)| \geq 4$. Further if $|\partial(X)| \geq 4$, then $X' = \emptyset$. Let H' be the hypergraph obtained from $H - V(X)$ by adding the set X' of new vertices and adding a 4-edge, e, containing four vertices in $X' \cup \partial(X)$. Note that H' may not be linear as the edge e may overlap other edges in H'.

Claim H.3: Either $\mathrm{def}(H') = 0$ or $\mathrm{def}(H') = \mathrm{def}_{H'}(Y)$ where $|Y| = 1$ and e is an edge of the hypergraph in the special H'-set Y.

Proof of Claim H.3: Suppose that $\mathrm{def}(H') > 0$. Let Y be a special H'-set such that $\mathrm{def}(H') = \mathrm{def}_{H'}(Y)$. Since $\mathrm{def}(H') > 0$, we note that in H', $|E^*(Y)| < |Y|$. Suppose that $e \notin E(Y)$. We now consider the special H-set $Q = X \cup Y$. Then in H, $|E^*(Q)| \leq |E^*(X)| + |E^*(Y)| \leq (|X| + 1) + (|Y| - 1) = |X| + |Y| = |Q|$, contradicting Claim E. Therefore, $e \in E(Y)$. Let $e \in E(R)$, where $R \in Y$. Suppose that $|Y| \geq 2$. In this case, we consider the special H-set $Q = X \cup (Y \setminus \{R\})$. Then in H, $|E^*(Q)| \leq |E^*(X)| + |E^*(Y)| \leq (|X| + 1) + (|Y| - 1) = |X| + |Y| = |Q| + 1$. By Claim E, $|E^*(Q)| \geq |Q| + 1$. Consequently, $|E^*(Q)| = |Q| + 1$. However since $|Y| \geq 2$, we note that $|Q| > |X|$, contradicting our choice of the special H-set X. $(\Box)$

Claim H.4: $\Phi(H') \geq -8|X_4| - 5|X_{14}| - 4|X_{11}| - |X_{21}| - 6|X'| + 13|X| - \mathrm{def}(H')$. Furthermore if H' is linear, then both of the following statements hold.

(a): $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - \mathrm{def}(H') \geq 13|X| - 6|X'| - 10$.
(b): If $X' \neq \emptyset$, then $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - 8$.

Proof of Claim H.4: By Claim F, $\mathrm{def}(H) = 0$. Thus,

$$\Phi(H') = 45(\tau(H') - \tau(H)) - 6(n(H') - n(H)) - 13(m(H') - m(H)) - \mathrm{def}(H').$$

We will first show that $\tau(H) \leq \tau(H') + |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + 6|X_{21}|$. Let T' be a minimum transversal in H'. If some vertex of X' belongs to T', then we can simply replace such a vertex in T' with a vertex from $V(e) \cap \partial(X)$ since a vertex in X' belongs to the edge e but no other edge of H'. We may therefore assume that some vertex in $\partial(X)$ belongs to T', and hence T' covers at least one edge from $E^*(X)$. By Claim H.1, we can now obtain a transversal of H of size $\tau(H') + |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + 6|X_{21}|$, which implies that $\tau(H) \leq \tau(H') + |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + 6|X_{21}|$ as desired.

Let $F \in X$. By Claim H.2, $X_{10} = \emptyset$. By Observation 1, if $F \in X_4$, then F contributes 4 to the sum $n(H) - n(H')$, 1 to the sum $m(H) - m(H')$, and at most 1 to the sum $\tau(H) - \tau(H')$ (due to the above bound on $\tau(H)$), and therefore contributes at least $45 \cdot (-1) - 6 \cdot (-4) - 13 \cdot (-1) = -8$ to $\Phi(H')$. Similar arguments show that if $F \in X_{14}$, then F contributes at least -5 to $\Phi(H')$. If $F \in X_{11}$, then F contributes at least -4 to $\Phi(H')$, while if $F \in X_{21}$, then F contributes at least -1 to $\Phi(H')$. The edges in $E^*(X)$ contribute $|E^*(X)| = |X| + 1$ to the sum $m(H) - m(H')$ and therefore $13(|X| + 1)$ to $\Phi(H')$, while the added edge e contributes 1 to the sum $m(H') - m(H)$ and therefore -13 to $\Phi(H')$. If $X' \neq \emptyset$, then each vertex in X' contributes 1 to the sum $n(H') - n(H)$, and therefore the vertices in X' contribute $-6|X'|$ to $\Phi(H')$. This proves the first part of Claim H.4.

Suppose, next, that H' is linear. Then, by Claim G, $\Phi(H') < 0$. Thus, from our previous inequality established in the first part of Claim H.4, $0 > \Phi(H') \geq -8|X_4| - 5|X_{14}| - 4|X_{11}| - |X_{21}| - 6|X'| + 13|X| - \mathrm{def}(H')$. This immediately implies part (a) as $\mathrm{def}(H') \leq 10$ by Claim H.3. If $X' \neq \emptyset$, then the edge e contains a degree-1 vertex and is therefore not part of a H_{10} subhypergraph in H'. Part (b) now again follows from Claim H.3, noting that in this case $\mathrm{def}(H') \leq 8$. ($\Box$)

Claim H.5: Suppose the added edge e overlaps with some other edge in H'.
(a): If e contains a vertex of degree 1 in H', then both of the following statements hold.
  (i): $\Phi(H') \leq 3 - \mathrm{def}(H')$.
  (ii): $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| \geq 13|X| - 6|X'| - 3$.
(b): If e contains no vertex of degree 1 in H', then both of the following statements hold.
  (i): $\Phi(H') \leq 7 - \mathrm{def}(H')$.
  (ii): $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| \geq 13|X| - 6|X'| - 7$.

Proof of Claim H.5: Suppose that the added edge e overlaps with some other edge e' in H'. We will first show the following.

$$\Phi(H') \leq \begin{cases} 3 - \mathrm{def}(H') & \text{if } e \text{ contains a vertex of degree 1 in } H' \\ 7 - \mathrm{def}(H') & \text{if } e \text{ contains no vertex of degree 1 in } H'. \end{cases}$$

Suppose, to the contrary, that $\Phi(H') \geq 4 - \mathrm{def}(H')$ if e contains a vertex of degree 1 in H'

and that $\Phi(H') \ge 8 - \mathrm{def}(H')$ if e contains no vertex of degree 1 in H'. Let $\{x, y\} \subseteq e \cap e'$.

Suppose firstly that $d_{H'}(x) = 3$. Let e'' be the edge incident with x different from e and e', and consider the hypergraph $H^\star = H' - x$. Since $H \in \mathcal{H}_4$, by construction, $H^\star \in \mathcal{H}_4$. Then, $m(H^\star) = m(H') - 3$ and $\tau(H') \le \tau(H^\star) + 1$. Further, if e contains a vertex of degree 1 in H', then $n(H^\star) \le n(H') - 2$, while if e contains no vertex of degree 1 in H', then $n(H^\star) \le n(H') - 1$. Recall that $\Phi(H') = \xi(H') - \xi(H)$ and $\Phi(H^\star) = \xi(H^\star) - \xi(H)$. By Claim G, $\Phi(H^\star) < 0$. Thus,

$$
\begin{aligned}
0 &> \Phi(H^\star) \\
&= \xi(H^\star) - \xi(H) \\
&= \xi(H^\star) + \Phi(H') - \xi(H') \\
&= \Phi(H') - 45(\tau(H') - \tau(H^\star)) + 6(n(H') - n(H^\star)) + \\
&\quad\; 13(m(H') - m(H^\star)) + (\mathrm{def}(H') - \mathrm{def}(H^\star)) \\
&\ge \Phi(H') - 45 + 6(n(H') - n(H^\star)) + 13 \cdot 3 + \mathrm{def}(H') - \mathrm{def}(H^\star) \\
&\ge \Phi(H') - 6 + 6(n(H') - n(H^\star)) + \mathrm{def}(H') - \mathrm{def}(H^\star).
\end{aligned}
$$

Claim H.5.1: If e contains a vertex of degree 1 in H', then $\mathrm{def}(H^\star) \ge 11$.

Proof of Claim H.5.1: If e contains a vertex of degree 1 in H', then $n(H') - n(H^\star) \ge 2$ and, by supposition, $\Phi(H') \ge 4 - \mathrm{def}(H')$. Thus, in this case, the above inequality chain simplifies to $0 > \Phi(H^\star) \ge (4 - \mathrm{def}(H')) - 6 + 6 \cdot 2 + \mathrm{def}(H') - \mathrm{def}(H^\star) = 10 - \mathrm{def}(H^\star)$, and so, $\mathrm{def}(H^\star) > 10$. (□)

Claim H.5.2: If e contains no vertex of degree 1 in H', then $\mathrm{def}(H^\star) \ge 9$.

Proof of Claim H.5.2: If e contains no vertex of degree 1 in H', then $n(H') - n(H^\star) \ge 1$ and, by supposition, $\Phi(H') \ge 8 - \mathrm{def}(H')$. Thus, in this case, our inequality chain simplifies to $0 > \Phi(H^\star) \ge (8 - \mathrm{def}(H')) - 6 + 6 \cdot 1 + \mathrm{def}(H') - \mathrm{def}(H^\star) = 8 - \mathrm{def}(H^\star)$, and so, $\mathrm{def}(H^\star) > 8$. (□)

By Claim H.5.1 and Claim H.5.2, $\mathrm{def}(H^\star) \ge 9$. Let Y be a special $H^\star$-set such that $\mathrm{def}(H^\star) = \mathrm{def}_{H^\star}(Y)$.

Claim H.5.3: $|Y| = 1$.

Proof of Claim H.5.3: Suppose, to the contrary, that $|Y| \ge 2$. If $|E^*(Y)| \ge |Y| - 1$, then $\mathrm{def}_{H^\star}(Y) \le 10|Y| - 13|E^*(Y)| \le 10|Y| - 13(|Y| - 1) = -3|Y| + 13 \le 7$, a contradiction. Hence, $|E^*(Y)| \le |Y| - 2$. We now consider the special H-set $Q = X \cup Y$. Then in H, $|E^*(Q)| \le |E^*(X)| + |E^*(Y)| + |\{e', e''\}| \le (|X| + 1) + (|Y| - 2) + 2 = |X| + |Y| + 1 = |Q| + 1$. By Claim E, $|E^*(Q)| \ge |Q| + 1$. Consequently, $|E^*(Q)| = |Q| + 1$. However $|Q| > |X|$, contradicting our choice of the special H-set X. Therefore, $|Y| = 1$. (□)

By Claim H.5.3, $|Y| = 1$. Let $Y = \{R\}$. If $R \ne H_{10}$, then $\mathrm{def}_{H^\star}(Y) \le 8|Y| - 13|E^*(Y)| \le 8$, a contradiction. Hence, $R = H_{10}$. If $|E^*(Y)| \ge 1$, then $\mathrm{def}_{H^\star}(Y) = 10 - 13|E^*(Y)| \le -3$, a contradiction. Therefore, R is a component of $H^\star$ and $\mathrm{def}_{H^\star}(Y) = 10$. In particular, we note that the edge e therefore contains no vertex of degree 1 in H'. By Observation 1(f),

R is 2-regular.

Recall that $\{x,y\} \subseteq e \cap e'$, implying that $d_{H'}(y) \ge 2$. If $d_{H'}(y) = 2$, then both x and y are removed from H' when constructing $H^\star$, implying that $n(H') - n(H^\star) \ge 2$. In this case, our inequality chain in the proof of Claim H.5.2 simplifies to $0 > \Phi(H^\star) \ge (8 - \mathrm{def}(H')) - 6 + 6 \cdot 2 + \mathrm{def}(H') - \mathrm{def}(H^\star) = 14 - \mathrm{def}(H^\star)$, and so, $\mathrm{def}(H^\star) > 14$, a contradiction. Therefore, $d_{H'}(y) = 3$, implying that $d_{H^\star}(y) = 1$ and $y \notin V(R)$. Let e''' be the edge incident with y different from e and e'. If e''' intersects R, then since R is a component of H', this would imply $y \in V(R)$, a contradiction. Therefore, e''' does not intersects R.

Interchanging the roles of x and y, and considering the hypergraph $H^{\star\star} = H' - y$ (instead of $H^\star = H' - x$), there exists an H_{10}-component, say R', in $H^{\star\star}$. Analogous arguments as employed earlier, show that $d_{H^{\star\star}}(x) = 1$, $x \notin V(R')$ and e'' does not intersect R'. Thus, R and R' are H_{10}-components in the hypergraph, $H' - \{e, e', e'', e'''\}$, obtained from H' by deleting the edges e, e', e'', e'''.

If $R = R'$, then neither e'' nor e''' intersect R. Hence, $|E^*(X \cup R)| \le |E^*(X)| + |\{e'\}| \le (|X| + 1) + 1 = |X \cup R| + 1$. By Claim E, $|E^*(X \cup R)| \ge |X \cup R| + 1$. Consequently, $|E^*(X \cup R)| = |X \cup R| + 1$. However $|X \cup R| > |X|$, contradicting our choice of the special H-set X. Therefore, $R \neq R'$, implying that R and R' are distinct and vertex-disjoint components of $H' - \{e, e', e'', e'''\}$.

As observed earlier, e''' does not intersects R. Suppose that e'' does not intersect R. Then, $|E^*(X \cup R)| \le |E^*(X)| + |\{e'\}| \le (|X| + 1) + 1 = |X \cup R| + 1$, a contradiction. Therefore, e'' intersects R. Analogously, e''' intersects R'. Let $r'' \in e'' \cap V(R)$ and let $r''' \in e''' \cap V(R')$.

If R and R' contain no vertex in $\partial(X)$, then $|E * (R \cup R')| \le |\{e', e'', e'''\}| = 3 = |R \cup R'| + 1$, implying that $E * (R \cup R') = \{e', e'', e'''\}$ and $|E * (R \cup R')| = |R \cup R'| + 1$. Let $r' \in e' \cap V(R) \cup V(R')$. Without loss of generality, we may assume that $r' \in V(R)$. By Observation 1(h), there exists a $\tau(R)$-transversal that contains both r' and r'', and a $\tau(R')$-transversal that contains r'''. Thus, given the special H-set, $\{R, R'\}$, we can find an $\{R, R'\}$-transversal that covers every edge in $E^*(R \cup R'\})$, contradicting Claim B. Therefore, R or R' (or both R and R') contain a vertex in $\partial(X)$. Let z be such a vertex and let e_z be an edge of $E^*(X)$ that contains z. Without loss of generality, we may assume that $r' \in V(R)$. Applying Observation 1(h) and Claim H.1, we can find an $(X \cup \{R, R'\})$-transversal that covers every edge in $E^*(X \cup R \cup R')$, contradicting Claim B. We deduce, therefore, that $d_{H'}(x) = 2$ (recall that in H', $\{x, y\} \subseteq e \cap e'$). Analogously, all vertices in $e \cap e'$ have degree 2 in H'. In particular, $d_{H'}(y) = 2$.

We once again consider the hypergraph $H^\star = H - x$, noting that here both x and y are removed from H' when constructing $H^\star$. Since $H^\star \in \mathcal{H}_4$, as before $\mathrm{def}(H^\star) \ge 0$. Further, $m(H^\star) = m(H') - 2$ and $\tau(H') \le \tau(H^\star) + 1$. If e contains a vertex of degree 1 in H', then $n(H') - n(H^\star) \ge 3$ and, by supposition, $\Phi(H') \ge 4 - \mathrm{def}(H')$. Thus, in this case, our inequality chain simplifies to $0 > \Phi(H^\star) \ge (4 - \mathrm{def}(H')) - 45 + 6 \cdot 3 + 13 \cdot 2 + \mathrm{def}(H') - \mathrm{def}(H^\star) = 3 - \mathrm{def}(H^\star)$, and so, $\mathrm{def}(H^\star) \ge 4$. If e contains no vertex of degree 1 in H', then $n(H') - n(H^\star) \ge 2$ and, by supposition, $\Phi(H') \ge 8 - \mathrm{def}(H')$.

Thus, in this case, our inequality chain simplifies to $0 > \Phi(H^\star) \ge (8 - \mathrm{def}(H')) - 45 + 6 \cdot 2 + 13 \cdot 2 + \mathrm{def}(H') - \mathrm{def}(H^\star) = 1 - \mathrm{def}(H^\star)$, and so, $\mathrm{def}(H^\star) \ge 2$. In both cases, $\mathrm{def}(H^\star) > 0$. Let Y be a special $H^\star$-set such that $\mathrm{def}(H^\star) = \mathrm{def}_{H^\star}(Y)$. Since $\mathrm{def}_{H^\star}(Y) > 0$, we note that, $|E^*(Y)| \le |Y| - 1$. We now consider the special H-set $X \cup Y$. In H, $|E^*(X \cup Y)| \le |E^*(X)| + |E^*(Y)| + |\{e'\}| \le (|X|+1) + (|Y|-1) + 1 = |X| + |Y| + 1 = |X \cup Y| + 1$, contradicting our choice of the special H-set X. This completes the proof of the following.

$$\Phi(H') \le \begin{cases} 3 - \mathrm{def}(H') & \text{if } e \text{ contains a vertex of degree 1 in } H' \\ 7 - \mathrm{def}(H') & \text{if } e \text{ contains no vertex of degree 1 in } H', \end{cases}$$

thereby establishing part (a)(i) and part (b)(i). By Claim H.4, $\Phi(H') \ge -8|X_4| - 5|X_{14}| - 4|X_{11}| - |X_{21}| - 6|X'| + 13|X| - \mathrm{def}(H')$. Thus, if e contains a vertex of degree 1 in H', then

$$3 - \mathrm{def}(H') \ge \Phi(H') \ge -8|X_4| - 5|X_{14}| - 4|X_{11}| - |X_{21}| - 6|X'| + 13|X| - \mathrm{def}(H').$$

This immediately completes the proof of part (a)(ii), as $\mathrm{def}(H')$ cancels out. Part (b) is easily proved analogously. $(\Box)$

Claim H.6: The case $|X| \ge 2$ and $|\partial(X)| \ge 4$ cannot occur.

Proof of Claim H.6: Note that $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| \le 8|X| \le 13|X| - 6|X'| - 10$, as $|X| \ge 2$ and $|X'| = 0$. If H' is linear, then we are obtain a contradiction to Claim H.4(a), and if H' is not linear we obtain a contradiction to Claim H.5. $(\Box)$

Claim H.7: The case $|X| \ge 3$ and $|\partial(X)| = 3$ cannot occur.

Proof of Claim H.7: Note that $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| \le 8|X| \le 13|X| - 6|X'| - 9$, as $|X| \ge 3$ and $|X'| = 1$. If H' is linear, then we are obtain a contradiction to Claim H.4(b), and if H' is not linear we obtain a contradiction to Claim H.5. $(\Box)$

Claim H.8: The case $|X| = 2$ and $|\partial(X)| = 3$ cannot occur.

Proof of Claim H.8: Suppose, to the contrary, that $|X| = 2$ and $|\partial(X)| = 3$. Then, $|E^*(X)| = 3$ and $|X'| = 1$. If H' is not linear, then we obtain a contradiction by Claim H.5(a), as $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| \le 16$ and $13|X| - 6|X'| - 3 = 17$. Therefore, H' is linear.

Claim H.8.1: $|X_4| \le 1$.

Proof of Claim H.8.1: Suppose, to the contrary, that X consists of two (vertex-disjoint) copies of H_4. By Claim H.4(a), $16 = 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - \mathrm{def}(H') = 26 - 6 - \mathrm{def}(H')$, implying that $\mathrm{def}(H') > 4$. If e is a H_4-component in H', then, by the connectivity and linearity of H, we note that $H = H_{11}$, contradicting Claim C. Hence, $H' \ne H_4$, implying by Claim H.3 and the fact that e contains a degree-1 vertex (and so, $Y \ne H_{10}$) that $Y \in X_{14}$ and $\mathrm{def}(H') = \mathrm{def}(Y) = 5$. Since $e \in E(Y)$ and e contains a vertex of degree 1 in H', we note that $H' \ne H_{14,5}$ and $H' \ne H_{14,6}$. Therefore, $H' = H_{14,i}$ for some $i \in [4]$, implying that $H = H_{21,i}$, contradicting Claim C. $(\Box)$

Claim H.8.2: $H' = H_4$, $|X_4| = 1$ and $|X_{14}| = 1$.

Proof of Claim H.8.2: Recall that H' is linear and $|X'| = 1$. By Claim H.4(a), $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - \mathrm{def}(H') = 20 - \mathrm{def}(H')$. By Claim H.8.1,

$|X_4| \le 1$. By Claim H.3 and the fact that e contains a degree-1 vertex, $\mathrm{def}(H') \le 8$. Thus, $8 + 5 \ge 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 20 - \mathrm{def}(H')$, implying that $|X_4| = 1$, $|X_{14}| = 1$ and $\mathrm{def}(H') = 8$, as desired. ($\Box$)

By Claim H.8.2, $H' = H_4$, $|X_4| = 1$ and $|X_{14}| = 1$. Let $E^*(X) = \{e_1, e_2, e_3\}$. Let F_1 and F_2 be the (vertex-disjoint) hypergraphs that belong to X, where $F_1 = H_{14}$ and $F_2 = H_4$. Further, let $E(F_2) = \{f_2\}$. Let $\partial(X) = \{x, y, z\}$.

Claim H.8.3: All three edges in $E^*(X)$ intersect F_1.

Proof of Claim H.8.3: Suppose, to the contrary, that at most two edges in $E^*(X)$ intersect F_1. Therefore, by Claim E, exactly two edges in $E^*(X)$ intersect F_1. Renaming edges in $E^*(X)$ if necessary, we may assume that e_1 and e_2 intersect F_1, and that e_3 does not intersect F_1. By the linearity of H, the edge e_3 contains one vertex of F_2 and all three vertices of $\partial(X)$. This in turn implies by the linearity of H, that both edges e_1 and e_2 intersect F_1 in at least two vertices. Let $T_1 = e_1 \cap V(F_1)$ and $T_2 = e_2 \cap V(F_1)$. Thus, $|T_1| \ge 2$ and $|T_2| \ge 2$. If e_1 and e_2 intersect in a common vertex of F_1, then, by Observation 1(g), we can cover e_1 and e_2 by a $\tau(F_1)$-transversal and we can cover e_3 by a $\tau(F_2)$-transversal, implying that there is an X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Hence, $T_1 \cap T_2 = \emptyset$. By Observation 1(l), there exists a $\tau(F_1)$-transversal that contains one vertex from T_1 and one vertex from T_2. Once again, this implies that there is an X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. ($\Box$)

By Claim H.8.3, all three edges in $E^*(X)$ intersect F_1. If two edges of $E^*(X)$ intersect in a common vertex of F_1, then we can cover these two edges by a $\tau(F_1)$-transversal. If no two edges of $E^*(X)$ intersect in a common vertex of F_1, then by Observation 1(i), we can cover two edges of $E^*(X)$ by a $\tau(F_1)$-transversal. In both cases, two edges of $E^*(X)$ can be covered by a $\tau(F_1)$-transversal. Renaming edges in $E^*(X)$ if necessary, we may assume that e_1 and e_2 can be covered by a $\tau(F_1)$-transversal. If e_3 intersects F_2, then there is an X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Hence, e_3 does not intersect F_2, implying that both e_1 and e_2 intersect F_2.

If $|e_3 \cap V(F_1)| \ge 3$, then by Observation 1(l), we can cover e_1 and e_3 by a $\tau(F_1)$-transversal. Since we can cover e_2 by a $\tau(F_2)$-transversal, there is an X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Hence, $|e_3 \cap V(F_1)| \le 2$.

If $|(e_1 \cup e_2) \cap V(F_1)| = 1$, then both e_1 and e_2 contain one vertex from F_1, one vertex from F_2 and two vertices from $\partial(X)$. However since $|\partial(X)| = 3$, e_1 and e_2 would then overlap, a contradiction. Hence, $|(e_1 \cup e_2) \cap V(F_1)| \ge 2$. Thus, if $|e_3 \cap V(F_1)| = 2$, then by Observation 1(k), there exists a $\tau(F_1)$-transversal that contains a vertex from $e_3 \cap V(F_1)$ and a vertex from $(e_1 \cup e_2) \cap V(F_1)$, implying that we can cover e_3 and one of e_1 and e_2 by a $\tau(F_1)$-transversal and we can cover the remaining edge in $E^*(X)$ by a $\tau(F_2)$-transversal, once again contradicting Claim B. Therefore, $|e_3 \cap V(F_1)| = 1$, implying that the edge e_3 contains one vertex of F_1 and all three vertices in $\partial(X)$, namely x, y, and z. Thus, both e_1 and e_2 contain one vertex from F_2, one vertex from $\partial(X)$ and two vertices from F_1. In particular, we note that $e_1 \cap V(F_1)$ is an independent set in F_1, as is $e_2 \cap V(F_1)$. Further, since e_1 and e_2 do not overlap, $|(e_1 \cup e_2) \cap V(F_1)| \ge 3$. Thus, by Observation 1(l), there exists a $\tau(F_1)$-transversal that contains a vertex from $e_3 \cap V(F_1)$ and a vertex from $(e_1 \cup e_2) \cap V(F_1)$, implying that we can cover e_3 and one of e_1 and e_2 by a $\tau(F_1)$-transversal and we can cover the remaining edge in $E^*(X)$ by a $\tau(F_2)$-transversal, once again contradicting Claim B. This completes the proof of Claim H.8. $(\Box)$

Claim H.9: If $|X| \ge 2$ and $|\partial(X)| = 2$, then $H' \in \mathcal{H}_4$.

Proof of Claim H.9: Let $|X| \ge 2$ and $|\partial(X)| = 2$ and suppose, to the contrary, that $H' \notin \mathcal{H}_4$. Thus, H' is not a linear hypergraph, implying that the edge e overlaps in H' with some other edge, e' say. Since e contains two vertices of degree 1, namely the two vertices in X', we note that in H', $e \cap e' = \partial(X)$. We now consider the hypergraph $H'' = H - e'$. Since H'' has no overlapping edges, $H'' \in \mathcal{H}_4$, and so by Claim G, $\Phi(H'') < 0$. By Claim H.2, $|X_{10}| = 0$. An analogous proof to that of Claim H.4, shows that $\Phi(H'') \ge -8|X_4| - 5|X_{14}| - 4|X_{11}| - |X_{21}| - 6|X'| + 13(|X| + 1) - \mathrm{def}(H'')$, noting that the deleted edge e' contributes 1 to the sum $m(H) - m(H'')$. Thus, since $|X'| = 2$ and $|X| = 2$,

$$\begin{array}{rcl}\Phi(H'') & \ge & -8|X| - 12 + 13(|X| + 1) - \mathrm{def}(H'') \\ & = & 5|X| + 1 - \mathrm{def}(H'') \\ & = & 11 - \mathrm{def}(H'').\end{array}$$

If $\mathrm{def}(H'') \le 11$, then $\Phi(H'') \ge 0$, a contradiction. Hence, $\mathrm{def}(H'') \ge 12$. Let Y be a special H''-set such that $\mathrm{def}(H'') = \mathrm{def}_{H''}(Y)$. If $|Y| = 1$, then $\mathrm{def}_{H''}(Y) \le 10$, a contradiction. Hence, $|Y| \ge 2$. If $|E^*(Y)| \ge |Y| - 1$ in H'', then $\mathrm{def}_{H^\star}(Y) \le 10|Y| - 13|E^*(Y)| \le 10|Y| - 13(|Y| - 1) = -3|Y| + 13 < 12$, a contradiction. Therefore, $|E^*(Y)| \le |Y| - 2$ in H''. We now consider the special H-set $X \cup Y$. Suppose that $e \notin E(Y)$. Then in H, $|E^*(X \cup Y)| \le |E^*(X)| + |E^*(Y)| + |\{e'\}| \le (|X| + 1) + (|Y| - 2) + 1 = |X| + |Y|$, contradicting Claim E. Therefore, $e \in E(Y)$. Let $e \in E(R)$, where $R \in Y$. We consider the special H-set $Q = X \cup (Y \setminus \{R\})$. Then in H, $|E^*(Q)| \le |E^*(X)| + |E^*(Y)| + |\{e'\}| \le (|X|+1)+(|Y|-2)+1 = |X|+|Y| = |Q|+1$. By Claim E, $|E^*(Q)| \ge |Q|+1$. Consequently, $|E^*(Q)| = |Q|+1$. However since $|Y| \ge 2$, we note that $|Q| > |X|$, contradicting our choice of the special H-set X. $(\Box)$

Claim H.10: The case $|X| \ge 4$ and $|\partial(X)| = 2$ cannot occur.

Proof of Claim H.10: Note that $8|X_4|+5|X_{14}|+4|X_{11}|+|X_{21}| \le 8|X| \le 13|X|-6|X'|-8$, as $|X| \ge 4$ and $|X'| = 2$. As H' is linear by Claim H.9, this contradicts Claim H.4(b). $(\Box)$

Claim H.11: The case $|X| = 3$ and $|\partial(X)| = 2$ cannot occur.

Proof of Claim H.11: Suppose, to the contrary, that $|X| = 3$ and $|\partial(X)| = 2$, which implies that $|X'| = 2$. By Claim H.9, $H' \in \mathcal{H}_4$, and so, by Claim H.4(a), we have $24 = 8|X| \ge 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - \mathrm{def}(H') = 39 - 12 - \mathrm{def}(H')$, and so $\mathrm{def}(H') > 3$. As e contains two degree-1 vertices, we must have $Y = H_4$ and $\mathrm{def}(H') = \mathrm{def}(Y) = 8$, by Claim H.3. Therefore, by Claim H.4(a), $8|X_4|+5|X_{14}|+4|X_{11}|+|X_{21}| > 39 - 12 - \mathrm{def}(H') = 19$, implying that $|X_4| \ge 2$. By the connectivity of H, we note therefore that $V(H) = V(X) \cup \partial(X)$. Let $E^*(X) = \{e_1, e_2, e_3, e_4\}$. Let F_1, F_2 and F_3 be the (vertex-disjoint) hypergraphs that belong to X. By Claim H.2, $|X_{10}| = 0$.

Claim H.11.1: $|X_4| = 2$.

Proof of Claim H.11.1: As observed earlier, $|X_4| \ge 2$. Suppose, to the contrary, that $|X_4| \ge 3$, implying that $|X| = |X_4| = 3$. Thus, every hypergraph in X is a copy of H_4. Let $\partial(X) = \{x, y\}$. Thus, F_1, F_2 and F_3 are all copies of H_4. Let $E(F_i) = \{f_i\}$ for $i \in [3]$. Further, $V(H) = V(X) \cup \{x, y\}$ and $E(H) = E(X) \cup E^*(X)$. In particular, $n(H) = 14$ and $m(H) = 6$. Renaming x and y, if necessary, we may assume that $d_H(x) \ge d_H(y)$.

Suppose that every vertex in $V(X)$ has degree at most 2 in H. Then, by the linearity of H, there are three cases to consider. If $d_H(x) = 3$ and $d_H(y) = 2$, then $H = H_{14,1}$. If $d_H(x) = 3$ and $d_H(y) = 1$, then $H = H_{14,2}$. If $d_H(x) = 2$, then $d_H(y) = 2$ and $H = H_{14,5}$. In all three cases, we contradict Claim C. Therefore, some vertex in $V(X)$ has degree 3. We may assume that F_1 contains a vertex, z_1 say, of degree 3 in H and that e_1 and e_4 contain z. By Claim E, at least three edges in $E^*(X)$ intersect $F_2 \cup F_3$. Renaming e_2 and e_3, if necessary, we may assume that e_2 intersects F_2. If e_3 intersects F_3, then in this case letting $z_2 \in e_2 \cap f_2$ and $z_3 \in e_3 \cap f_3$, the set $\{z_1, z_2, z_3\}$ is an X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Hence, e_3 does not intersects F_3, implying by the linearity of H that e_3 contains both x and y, and intersects both F_1 and F_2. This in turn implies that each of e_1, e_2 and e_3 contain exactly one vertex in $\{x, y\}$ and therefore exactly one vertex of F_i for each $i \in [3]$. In this case, letting $z_2 \in e_3 \cap f_2$ and $z_3 \in e_2 \cap f_3$, the set $\{z_1, z_2, z_3\}$ is an X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Therefore, $|X| = 2$. (□)

By Claim H.11.1, $|X_4| = 2$. We may assume that $F_1 \ne H_4$. Thus, both F_2 and F_3 are copies of H_4. Let $E(F_2) = \{f_2\}$ and $E(F_3) = \{f_3\}$.

Claim H.11.2: All four edges in $E^*(X)$ intersect F_1.

Proof of Claim H.11.2: Suppose, to the contrary, that at most three edges in $E^*(X)$ intersect F_1. If two edges in $E^*(X)$ do not intersect F_1, then these two edges would both contain x and y, contradicting the linearity of H. Hence, at least three edges in $E^*(X)$ intersect F_1. Consequently, exactly three edges in $E^*(X)$ intersect F_1. Renaming edges in $E^*(X)$ if necessary, we may assume that e_1, e_2 and e_3 intersect F_1. By the linearity of H, the edge e_4 contains both x and y, and intersects both F_2 and F_3.

Suppose that two of the edges e_1, e_2 and e_3, say e_1 and e_2, can be covered by a minimum transversal, T_1 say, in F_1. If e_3 intersects $F_2 \cup F_3$, say e_3 intersects F_2, then we can extend T_1 to an X-transversal that intersects every edge in $E^*(X)$ by adding to it the vertex in $e_3 \cap f_2$ and the vertex in $e_4 \cap f_3$, contradicting Claim B. Hence, e_3 does not intersect $F_2 \cup F_3$, implying that e_3 contains at most one of x and y and at least three vertices of F_1. Further this implies that both e_1 and e_2 intersect $F_2 \cup F_3$ since by Claim E, at least three edges in $E^*(X)$ intersect $F_2 \cup F_3$. By Observation 1(l), there exists a $\tau(F_1)$-transversal that covers e_3 and one of e_1 and e_2, say e_1. We can then cover e_2 and e_4 from $F_2 \cup F_3$, once again contradicting Claim B.

Therefore, we can only cover at most one of the three edges e_1, e_2 and e_3 by a minimum transversal in F_1. By Observation 1(i), this implies that $F_1 = H_{11}$. Further, for $i \in [3]$ the edge e_i intersects F_1 in exactly one vertex and this vertex has no neighbor of degree 1 in F_1. Let $e_i \cap V(F_1) = \{u_i\}$ for $i \in [3]$. Thus, $U = \{u_1, u_2, u_3\}$ is the set of three vertices in F_1 all of whose neighbors in F_1 have 2 in F_1. Let H' be the hypergraph obtained from H by deleting the eight vertices in $V(F_1) \setminus U$ (and their incident edges), adding a new vertex u', and then adding the edge $f' = \{u', u_1, u_2, u_3\}$. Then, H' has order $n(H') = 14$ and size $m(H') = 7$. We note that $E(H') = \{e_1, e_2, e_3, e_4, f', f_2, f_3\}$. Recall that the edge e_4 contains both x and y, and intersects both F_2 and F_3. Therefore, the edge e_i contains exactly one vertex from each of F_2 and F_3, and exactly one of x and y, for each $i \in [3]$. If every vertex in $F_2 \cup F_3$ has degree at most 2 in H', then by the linearity of H, $H' = H_{14,1}$, implying that $H = H_{21,1}$, contradicting Claim C. Therefore, some vertex in $F_2 \cup F_3$ has degree 3 in H'. We may assume that z_2 is such a vertex and that $z_2 \in V(F_2)$. Further, we may assume that e_1 and e_2 contain the vertex z_2. Let z_1 and z_3 be the vertices of e_3 and e_4, respectively, that belongs to F_1 and F_3. Then, $\{z_1, z_2, z_3\}$ is a $\tau(H')$-transversal that can be extended to a $\tau(H)$-transversal by adding to it two additional vertices of $V(H) \setminus V(H')$. This produces an X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Therefore, all four edges in $E^*(X)$ intersect F_1. (□)

By Claim H.11.2, all four edges in $E^*(X)$ intersect F_1.

Claim H.11.3: All four edges in $E^*(X)$ intersect $F_2 \cup F_3$.

Proof of Claim H.11.2: Suppose, to the contrary, that at most three edges in $E^*(X)$ intersect $F_2 \cup F_3$. By Claim E, at least three edges in $E^*(X)$ intersect $F_2 \cup F_3$. Therefore, exactly three edges in $E^*(X)$ intersect $F_2 \cup F_3$. We may assume that e_1 does not intersect $F_2 \cup F_3$. Thus, e_2, e_3, e_4 all intersect $F_2 \cup F_3$. We note that $|e_1 \cap V(F_1)| \ge 2$. Further since $\Delta(H) \le 3$, we note that $|(e_2 \cup e_3 \cup e_4) \cap V(F_1)| \ge 2$. Thus, by Observation 1(k), there is a $\tau(F_1)$-transversal, T_1 say, that covers e_1 and covers one of the edges e_2, e_3, e_4, say e_2. Renaming F_2 and F_3, if necessary, we may assume that e_3 intersects F_2. If e_4 intersects F_3, then we can extend T_1 to an X-transversal that intersects every edge in $E^*(X)$ by adding to it the vertex in $e_3 \cap f_2$ and the vertex in $e_4 \cap f_3$, contradicting Claim B. Therefore, e_4 does not intersect F_3, implying that e_4 intersects F_2. If e_3 intersects F_3, then analogously we can extend T_1 to an X-transversal that intersects every edge in $E^*(X)$, a contradiction. Therefore, neither e_3 nor e_4 intersects F_3, implying that e_2 is the only possible edge in $E^*(X)$ that intersect F_3, contradicting Claim E. (□)

By Claim H.11.3, all four edges in $E^*(X)$ intersect $F_2 \cup F_3$.

Claim H.11.4: At least three edges in $E^*(X)$ intersect F_3.

Proof of Claim H.11.4: Suppose, to the contrary, that at most two edges in $E^*(X)$ intersect F_3. Then, by Claim E, exactly two edges in $E^*(X)$ intersect F_3. We may assume that e_1 and e_2 intersect F_3, and therefore e_3 and e_4 do not intersect F_3. By the linearity of H, this implies that at least one of e_3 and e_4 intersects F_1 in at least two vertices. Renaming e_3 and e_4 if necessary, we may assume that $|e_3 \cap V(F_1)| \ge 2$.

If $|(e_1 \cup e_2) \cap V(F_1)| \ge 2$, then by Observation 1(k), there is a $\tau(F_1)$-transversal, T_1 say, that covers e_3 and covers one of the edges e_1 and e_2, say e_1. We can now extend T_1 to an X-transversal that intersects every edge in $E^*(X)$ by adding to it the vertex in $e_2 \cap f_3$ and the vertex in $e_4 \cap f_2$, contradicting Claim B. Therefore, $|(e_1 \cup e_2) \cap V(F_1)| = 1$. By the linearity of H, this implies that both e_1 and e_2 intersect F_2.

Since $\Delta(H) \le 3$, we note that $|(e_1 \cup e_2 \cup e_4) \cap V(F_1)| \ge 2$. Thus, considering the sets $|e_3 \cap V(F_1)| \ge 2$ and the set $|(e_1 \cup e_2 \cup e_4) \cap V(F_1)| \ge 2$, by Observation 1(k), there is a $\tau(F_1)$-transversal, T_1 say, that covers e_3 and covers one of the edges e_1, e_2, e_4. If T_1 covers e_1 (and therefore also e_2), then we can extend T_1 to an X-transversal that intersects every edge in $E^*(X)$ by adding to it the vertex in $e_4 \cap f_2$ and any vertex in f_3, contradicting Claim B. Therefore, T_1 covers e_4. Since both e_1 and e_2 intersect both F_2 and F_3, we can once again extend T_1 to an X-transversal that intersects every edge in $E^*(X)$, a contradiction. $(\Box)$

By Claim H.11.4, at least three edges in $E^*(X)$ intersect F_3. Analogously, at least three edges in $E^*(X)$ intersect F_2. Recall that by Claim H.11.2, all four edges in $E^*(X)$ intersect F_1. On the one hand, if two edges of $E^*(X)$ intersect F_1 in a common vertex, then there is a $\tau(F_1)$-transversal that covers two edges of $E^*(X)$. On the other hand, if no two edges of $E^*(X)$ intersect F_1 in a common vertex, then by Observation 1(k), we can once again find a $\tau(F_1)$-transversal that covers two edges of $E^*(X)$ (by considering, for example, the set $|(e_1 \cup e_2) \cap V(F_1)| \ge 2$ and the set $|(e_1 \cup e_2) \cap V(F_1)| \ge 2$). In both cases, there is a $\tau(F_1)$-transversal, T_1, that covers two edges of $E^*(X)$. Renaming edges in $E^*(X)$ if necessary, we may assume that T_1 covers e_1 and e_2. Recall that by Claim H.11.3, all four edges in $E^*(X)$ intersect $F_2 \cup F_3$. In particular, both e_3 and e_4 intersect $F_2 \cup F_3$. Renaming F_2 and F_3 if necessary, we may assume that e_3 intersects F_2. If e_4 intersects F_3, then we can extend T_1 to an X-transversal that intersects every edge in $E^*(X)$ by adding to it the vertex in $e_3 \cap f_2$ and the vertex in $e_4 \cap f_3$, contradicting Claim B. Therefore, e_4 does not intersect F_3, implying that e_4 intersects F_2. If e_3 intersects F_3, then analogously we can extend T_1 to an X-transversal that intersects every edge in $E^*(X)$, a contradiction. Therefore, neither e_3 nor e_4 intersects F_3, implying that e_1 and e_2 are the only possible edge in $E^*(X)$ that intersect F_3, contradicting Claim H.11.4. This completes the proof of Claim H.11. $(\Box)$

Claim H.12: The case $|X| = 2$ and $|\partial(X)| = 2$ cannot occur.

Proof of Claim H.12: Suppose, to the contrary, that $|X| = 2$ and $|\partial(X)| = 2$. We note that $|E^*(X)| = 3$ and $|X'| = 2$. Let $E^*(X) = \{e_1, e_2, e_3\}$. By Claim H.9, $H' \in \mathcal{H}_4$, and so by Claim G, $\Phi(H') < 0$. Since at most one of the edges in $E^*(X)$ can contain both vertices in $\partial(X)$, we note that

$$\sum_{i=1}^{3} |e_i \cap V(F_1)| + \sum_{i=1}^{3} |e_i \cap V(F_2)| \ge 8. \tag{7}$$

Suppose that at most two edges in $E^*(X)$ intersect F_1. Then, by Claim E, exactly two edges in $E^*(X)$ intersect F_1. We may assume that both e_1 and e_2 intersect F_1, and therefore that e_3 does not intersect F_1. Thus, $|e_3 \cap V(F_1)| = 0$ and $|e_3 \cap V(F_2)| \ge 2$ If $|e_1 \cap V(F_1)| + |e_2 \cap V(F_1)| \ge 4$, then by Observation 1, we can find a $\tau(F_1)$-transversal, T_1, that covers both e_1 and e_2. In this case, we can extend T_1 to an X-transversal that intersects every edge in $E^*(X)$ by adding to it a $\tau(F_2)$-transversal that covers e_3, a contradiction.

Therefore, $|e_1 \cap V(F_1)| + |e_2 \cap V(F_1)| \leq 3$, implying by Inequality (7) that

$$\sum_{i=1}^{3} |e_i \cap V(F_2)| \geq 5. \tag{8}$$

Therefore, by Inequality (8), we note that $(|e_1 \cap V(F_2)|+|e_3 \cap V(F_2)|)+(|e_2 \cap V(F_2)|+|e_3 \cap V(F_2)|) \geq 5+|e_3 \cap V(F_2)| \geq 7$. Renaming e_1 and e_2 if necessary, we may assume that $|e_1 \cap V(F_2)|+|e_3 \cap V(F_2)| \geq 4$. By Observation 1, we can find a $\tau(F_2)$-transversal, T_2, that covers both e_1 and e_3. In this case, we can extend T_3 to an X-transversal that intersects every edge in $E^*(X)$ by adding to it a $\tau(F_1)$-transversal that covers e_2, a contradiction. Hence, all three edges in $E^*(X)$ intersect F_1. Analogously, all three edges in $E^*(X)$ intersect F_2. Renaming F_1 and F_2 if necessary, we may assume by Inequality (7) that $\sum_{i=1}^{3} |e_i \cap V(F_1)| \geq 4$. By Observation 1, we can find a $\tau(F_1)$-transversal, T_1, that covers two of the edges in $E^*(X)$. The third edge in $E^*(X)$ can be covered by a $\tau(F_2)$-transversal noting that all three edges in $E^*(X)$ intersect F_2. This produces an X-transversal that intersects every edge in $E^*(X)$, a contradiction. $(\Box)$

We proceed further with some additional notation. We associate with the set X a bipartite multigraph, which we denote by M_X, with partite sets X and $E^*(X)$ as follows. If an edge $e \in E^*(X)$ intersects a subhypergraph $H' \in X$ in H in k vertices, then we add k multiple edges joining $e \in E^*(X)$ and $H' \in X$ in M_X. Two multiple edges (also called parallel edges in the literature) joining two vertices in M_X we call *double edges*, while three multiple edges joining two vertices in M_X we call *triple edges*. An edge that is not a multiple edge we call a *single edge*. By supposition in our proof of Claim H, $|E^*(X)| = |X| + 1$. We say that a pair (e_1, e_2) of edges in $E^*(X)$ form an $E^*(X)$-*pair* if they intersect a common subhypergraph, F, of X in H, and, further, there exists a $\tau(F)$-transversal that covers both e_1 and e_2 in H. We call F a subhypergraph of X associated with the $E^*(X)$-pair, (e_1, e_2).

Claim H.13: $|E(M_X)| \geq 4|E^*(X)| - 3|\partial(X)| = 4(|X| + 1) - 3|\partial(X)|$. Furthermore, the following holds.
(a) M_X does not contain triple edges.
(b) No $F \in X$ has double edges to three distinct vertices of $E^*(X)$ in M_X.
(c) Every $e \in E^*(X)$ has degree at least $4 - |\partial(X)|$ in M_X.
(d) Every $F \in X$ has degree at least 2 in M_X.

Proof of Claim H.13: Consider the bipartite multigraph, M'_X, which is identical to M_X, except we add a new vertex b to M_X and for each vertex $e \in E^*(X)$ we add r multiple edges joining e and b in M'_X if e intersects $\partial(X)$ in r vertices in H. In M'_X, every vertex in $E^*(X)$ has degree 4. Further, since $\Delta(H) \leq 3$, the vertex b has degree at most $3|\partial(X)|$. Therefore, $M_X = M'_X - b$ contains at least $4|E^*(X)| - 3|\partial(X)|$ edges. This proves the first part of Claim H.13.

To prove part (a), for the sake of contradiction, suppose that M_X does contain triple edges that join vertices $e \in E^*(X)$ and $F \in X$. By Observation 1(ℓ), we note that e is an X-universal edge, a contradiction to Claim H.1. This proves part (a).

To prove part (b), for the sake of contradiction, suppose that some $F \in X$ has double edges to three distinct vertices, say e_1, e_2, e_3, in $E^*(X)$ in M_X. By Observation 1(o), we note that at least one of e_1, e_2 or e_3 is an X-universal edge, a contradiction to Claim H.1. This proves part (b).

By the construction of M_X, every vertex $e \in E^*(X)$ has degree $4 - |V(e) \cap \partial(X)| \geq 4 - |\partial(X)|$ in M_X. By Claim E (used on F), we note that at least two edges in $E^*(X)$ intersect F in H, and therefore the degree of F in M_X is at least 2. This proves part (c) and part (d). $(\Box)$

Claim H.14: The case $|X| \geq 5$ and $|\partial(X)| = 1$ cannot occur.

Proof of Claim H.14: Suppose, to the contrary, that $|X| \geq 5$ and $|\partial(X)| = 1$. We note that in this case $|X'| = 3$ and H' is linear. Let $\partial(X) = \{y\}$.

Claim H.14.1: $|X| = 5$ and $X = X_4$.

Proof of Claim H.14.1: By Claim H.4(b), $8|X| \geq 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - 8 = 8|X| + 5|X| - 26$, and so $5|X| < 26$, implying that $|X| \leq 5$. This clearly implies that $|X| = 5$. By supposition, $|X| \geq 5$. Consequently, $|X| = 5$, implying that $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 8|X| - 1$, which in turn implies that $X = X_4$. $(\Box)$

By Claim H.14.1, $X = X_4$ and $|X_4| = 5$. Let $X = \{x_1, x_2, \ldots, x_5\}$ and let $E^*(X) = \{e_1, e_2, \ldots, e_6\}$. We now consider the bipartite multigraph M_X defined earlier. Since $X = X_4$ and H is linear, the multigraph M_X is in this case a graph. By Claim H.13(c), since here $|\partial(X)| = 1$, $d_{M_X}(e) \geq 3$ for each $e \in E^*(X)$. By Claim H.13(d), $d_{M_X}(F) \geq 2$ for each $F \in X$. Also by Claim H.13, we note that $\sum_{F \in X} d_{M_X}(F) = |E(M_X)| \geq 4(|X| + 1) - 3|\partial(X)| = 24 - 3 = 21$.

Claim H.14.2: Suppose the edges e_i and e_j form an $E^*(X)$-pair and that x_ℓ is the vertex in X corresponding to a copy of H_4 of X that contains a vertex covering both e_1 and e_2. Then there exists a vertex $x_k \in X \setminus \{x_\ell\}$ in M_X such that e_i and e_j are both neighbors of x_k in M_X and one of the following holds.
(a) $d_{M_X}(x_k) = 2$.
(b) $d_{M_X}(x_k) = 3$ and $N_{M_X}(x_{k'}) \subseteq N_{M_X}(x_k)$ for some vertex $x_{k'} \in X \setminus \{x_k, x_\ell\}$.

Proof of Claim H.14.2: For notational convenience, we may assume that $i = 1$ and $j = 2$. Thus, (e_1, e_2) is an $E^*(X)$-pair. Further, we may assume that x_1 is the vertex in X corresponding to a copy of H_4 of X that contains a vertex covering both e_1 and e_2. We now consider the bipartite graph $M'_X = M_X - \{e_1, e_2, x_1\}$ with partite sets $X' = X \setminus \{x_1\}$ and $E'_X = E^*(X) \setminus \{e_1, e_2\}$. If there exists a matching in M'_X that matches E'_X to X', then there exists a minimum X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Therefore, no matching in M'_X matches E'_X to X'. By Hall's Theorem, there is a nonempty subset $S \subseteq E'_X$ such that in M'_X, $|N(S)| < |S|$. If $S = E'_X$, then let $x_k \in X' \setminus N_{M_X}(S)$. In this case, the vertices e_1 and e_2 are the only possible neighbors of x_k in M_X, implying that $N_{M_X}(x_k) = \{e_1, e_2\}$ and $d_{M_X}(x_k) = 2$. Thus, Part (a) holds, as desired. Hence, we may assume that $|S| \leq 3$ and that no vertex in $X' \setminus \{x_1\}$ has degree 2 with e_1 and e_2 as its neighbors.

Since $|S| \le 3$, we note that $|N_{M'_X}(S)| \le 2$. Since every vertex in $E^*(X)$ has degree at least 3 in M_X, we note that $|N_{M'_X}(S)| \ge 2$. Consequently, $|N_{M'_X}(S)| = 2$, implying that $|S| = 3$. Renaming vertices if necessary, we may assume that $S = \{e_3, e_4, e_5\}$ and that $N_{M'_X}(S) = \{x_2, x_3\}$. Since $d_{M_X}(e) \ge 3$ for each $e \in E^*(X)$, this implies that for each vertex $e \in S$, we have $N_{M_X}(e) = \{x_1, x_2, x_3\}$. We note that the only possible neighbors of x_4 and x_5 in M_X are e_1, e_2 and e_6, and so $d_{M_X}(x_4) \le 3$ and $d_{M_X}(x_5) \le 3$.

We show next that x_4 or x_5 dominate both e_1 and e_2 in M_X. Suppose, to the contrary, that neither x_4 nor x_5 dominate both e_1 and e_2 in M_X. Since both x_4 and x_5 have degree at least 2 in M_X, this implies that $d_{M_X}(x_4) = d_{M_X}(x_5) = 2$ and that e_6 is adjacent to both x_4 and x_5. Renaming e_1 and e_2, if necessary, we may assume that e_2 is adjacent to x_5, and so $N_{M_X}(x_5) = \{e_2, e_6\}$. Since $\sum_{x \in X} d_{M_X}(x) \ge 21$, the degree sequence of vertices of X in M_X is therefore either $2, 2, 5, 6, 6$ or $2, 2, 6, 6, 6$. If the degree sequence is given by $2, 2, 6, 6, 6$, then e_6 would be adjacent to every vertex of X in M_X, which is not possible since $d_{M_X}(e) \le 4$ for each $e \in E^*(X)$. Thus, the degree sequence of vertices of X in M_X is $2, 2, 5, 6, 6$. In particular, this implies that $d_{M_X}(e_6) = 4$, which in turn implies that x_2 and x_3 are both adjacent to e_1 and e_2 in M_X. Thus, $M_X - \{e_6, x_4, x_5\} = K_{3,5}$. By assumption, e_2 is adjacent to x_5. Thus, since $d_{M_X}(e_2) = 4$, we note that $N_{M_X}(e_2) = \{x_1, x_2, x_3, x_5\}$ and therefore $N_{M_X}(x_4) = \{e_1, e_6\}$. The graph M_X is therefore determined.

We note that the two vertices of X of degree 6 in M_X both give rise to at least two $E^*(X)$-pairs, while the vertex of X of degree 5 in M_X gives rise to at least one $E^*(X)$-pairs. Further since there are no overlapping edges, these five $E^*(X)$-pairs are distinct. Therefore, there are at least five distinct $E^*(X)$-pairs. Recall that $N_{M_X}(x_4) = \{e_1, e_6\}$ and $N_{M_X}(x_5) = \{e_2, e_6\}$. Suppose that there is a vertex in the copy of H_4 in H corresponding to the vertex x_4 that contains a vertex covering both e_1 and e_6. Then the structure of M_X implies that there exists a minimum X-transversal that intersects every edge in $E^*(X)$, contradicting Claim B. Analogously, if there is a vertex in the copy of H_4 in H corresponding to the vertex x_5 that contains a vertex covering both e_2 and e_6, we produce a contradiction. Therefore, since e_6 has only two neighbors in M_X different from x_4 and x_5, at most two $E^*(X)$-pairs contain e_6. Since there are at least five $E^*(X)$-pairs, there exists an $E^*(X)$-pair, (e_{i_1}, e_{i_2}) say, that is not the pair (e_1, e_2) and such that $e_6 \notin \{e_{i_1}, e_{i_2}\}$.

Renaming i_1 and i_2, if necessary, we may assume that $e_{i_1} \notin \{e_1, e_2\}$. Thus, $e_{i_1} \in \{e_3, e_4, e_5\}$. Let x_j be the vertex in X corresponding to a copy of H_4 of X that contains a vertex covering both e_{i_1} and e_{i_2}. Since e_{i_1} is adjacent to neither x_4 nor x_5, we note that $j \in \{1, 2, 3\}$. We now proceed analogously as we did with the $E^*(X)$-pair (e_1, e_2). We consider the bipartite graph $M''_X = M_X - \{e_{i_1}, e_{i_2}, x_j\}$ with partite sets $X'' = X \setminus \{x_j\}$ and $E''_X = E^*(X) \setminus \{e_{i_1}, e_{i_2}\}$. No matching in M''_X matches E''_X to X''. By Hall's Theorem, there is a nonempty subset $R \subseteq E''_X$ such that in M''_X, $|N(R)| < |R|$. If $R = E'_X$, then let $x' \in X \setminus N_{M_X}(R)$. The only possible neighbors of x' are the vertices e_{i_1} and e_{i_2}. Since $d_{M_X}(x') = 2$, we note that $x' \in \{x_4, x_5\}$. However, x' is not adjacent to e_{i_1}, and so $d_{M_X}(x') \le 1$, a contradiction. Therefore, $|R| = 3$ and in M''_X, $|N(R)| = 2$. We note that both vertices in $N(R)$ are adjacent to all three vertices in R, implying that the two vertices of X not in $R \cup \{x_j\}$ are the two vertices of X of degree 2 in M_X, namely x_4 and x_5. Further, $e_6 \notin R$ and e_6 is the only common neighbor of x_4 and x_5. If $i_2 = 1$, then x_5 is adjacent to neither e_{i_1} nor e_{i_2}, implying that $d_{M_X}(x_5) = 1$. If $i_2 \neq 1$, then x_4 is adjacent to neither e_{i_1} nor e_{i_2}, implying that $d_{M_X}(x_4) = 1$. Both cases produce a contradiction. Therefore, at least one of x_4 and x_5 dominate both e_1 and e_2 in M_X. Renaming x_4 and x_5, if necessary, we may assume that x_4 dominate both e_1 and e_2 in M_X.

By our earlier assumption, no vertex in $X' \setminus \{x_1\}$ has degree 2 with e_1 and e_2 as its neighbors. Hence, $N_{M_X}(x_4) = \{e_1, e_2, e_6\}$. As observed earlier, the only possible neighbors of x_5 in M_X are e_1, e_2 and e_6, and so $N_{M_X}(x_5) \subseteq N_{M_X}(x_4)$. Taking $x_k = x_4$ and $x_{k'} = x_5$, Part (b) holds. This completes the proof of Claim H.14.2. $(\square)$

We now consider the degree sequence of vertices of X in M_X. Let this degree sequence, in nondecreasing order, be given by $s\colon d_1, d_2, d_3, d_4, d_5$, and so $2 \leq d_1 \leq d_2 \leq \cdots \leq d_5 \leq 6$. As observed earlier, $\sum_{i=1}^{5} d_i \geq 21$. By Claim H.14.2, $d_1 = 2$ or $d_1 = d_2 = 3$.

Claim H.14.3: There are at most three $E^*(X)$-pairs.

Proof of Claim H.14.3: Suppose, to the contrary, that there are at least four $E^*(X)$-pairs. Renaming vertices of $E^*(X)$, if necessary, we may assume that (e_1, e_2) and (e_3, e_4) are $E^*(X)$-pairs. Let x_{12} (respectively, x_{34}) be the vertex in X corresponding to a copy of H_4 of X that contains a vertex covering both e_1 and e_2 (respectively, e_3 and e_4). Possibly, $x_{12} = x_{34}$. Let $x_1 \in X \setminus \{x_{12}$ be adjacent to both e_1 and e_2 in M_X and such that either $d_{M_X}(x_1) = 2$ or $d_{M_X}(x_1) = 3$ and there exists a vertex $x_{1'} \in X \setminus \{x_1, x_{12}\}$ with $N_{M_X}(x_{1'}) \subseteq N_{M_X}(x_1)$. Further, let $x_2 \in X \setminus \{x_{34}$ be adjacent to both e_3 and e_4 in M_X and such that either $d_{M_X}(x_2) = 2$ or $d_{M_X}(x_2) = 3$ and there exists a vertex $x_{2'} \in X \setminus \{x_2, x_{34}\}$ with $N_{M_X}(x_{2'}) \subseteq N_{M_X}(x_2)$. We note that x_1 and x_2 exists by Claim H.14.2. Further, $x_{12} \notin \{x_1, x_2\}$, $x_{34} \notin \{x_1, x_2\}$ and $x_1 \neq x_2$. We may assume that $d_{M_X}(x_1) \leq d_{M_X}(x_2)$.

Suppose that $d_{M_X}(x_1) = 2$, and so $N_{M_X}(x_1) = \{e_1, e_2\}$. Then, $d_1 = 2$. Since $d_2 \leq 3$ and $\sum_{i=1}^{5} d_i \geq 21$, this implies that $d_3 \geq 4$, and so x_1 and x_2 are the only vertices in X of degree at most 3 in M_X. If $d_{M_X}(x_2) = 3$, then by Claim H.14.2, $N_{M_X}(x_1) \subseteq N_{M_X}(x_2)$, a contradiction since at least one of e_1 and e_2 is not adjacent to x_2. Thus, $d_{M_X}(x_2) = 2$, and so $d_2 = 2$ and $N_{M_X}(x_2) = \{e_3, e_4\}$. Let (e_i, e_j) be an $E^*(X)$-pair different from (e_1, e_2) and (e_3, e_4). By Claim H.14.2, there exists a vertex in X of degree at most 3 adjacent to both e_i and e_j. The vertices x_1 and x_2 are the only two vertices in X of degree at most 3 in M_X. However, neither x_1 nor x_2 is adjacent to both e_i and e_j, a contradiction. Therefore, $d_{M_X}(x_1) = d_{M_X}(x_2) = 3$. Since $\sum_{i=1}^{5} d_i \geq 21$, we note that $d_3 \geq 3$. This implies that $N_{M_X}(x_{1'}) = N_{M_X}(x_1)$ and $N_{M_X}(x_{2'}) = N_{M_X}(x_2)$.

Suppose $d_3 = 3$. Then, s is given by $3, 3, 3, 6, 6$. Let x_3 be the vertex of $X \setminus \{x_1, x_2\}$ of degree 3 in M_X. Since $N_{M_X}(x_1) \neq N_{M_X}(x_2)$, we note that $x_{1'} = x_3$ and $x_{2'} = x_3$. But then $\{e_1, e_2, e_3, e_4\} \subseteq N_{M_X}(x_3)$, and so $d_{M_X}(x_3) \geq 4$, a contradiction. Therefore, $d_3 \geq 4$. Thus, x_1 and x_2 are the only vertices in X of degree at most 3 in M_X. This implies that $x_{1'} = x_2$ (and $x_{2'} = x_1$). But then $\{e_1, e_2, e_3, e_4\} \subseteq N_{M_X}(x_2)$, and so $d_{M_X}(x_2) \geq 4$, a contradiction. $(\square)$

Claim H.14.4: The degree sequence s is given by $3, 3, 5, 5, 5$.

Proof of Claim H.14.4: We show firstly that $d_1 = 3$. Suppose, to the contrary, that

$d_1 = 2$. If $d_2 = 2$, then s is given by $2,2,5,6,6$ or $2,2,6,6,6$, implying that there are at least five $E^*(X)$-pairs. If $d_2 = 3$, then s is given by $2,3,4,6,6$ or $2,3,5,5,6$ or $2,3,5,6,6$ or $2,3,6,6,6$, implying that there are at least four $E^*(X)$-pairs. In both cases, we contradict Claim H.14.3. Therefore, $d_1 = d_2 = 3$.

We show next that $d_6 \le 5$. Suppose, to the contrary, that $d_6 = 6$. If $d_3 = 3$, then s is given by $3,3,3,6,6$, implying that there are at least four $E^*(X)$-pairs, a contradiction. Hence, $d_3 \ge 4$. Let x_1 and x_2 be the two vertices of degree 3 in M_X. By Claim H.14.2, $N_{M_X}(x_1) = N_{M_X}(x_2)$. We may assume that e_1, e_2 and e_3 are the three neighbors of x_1 and x_2. By Claim H.14.2, if (e_i, e_j) is an $E^*(X)$-pair, then $1 \le i, j \le 3$. We may assume that x_4 is a vertex of degree 6 in M_X. Then, x_4 gives rise to two $E^*(X)$-pairs that comprise of four distinct vertices. At least one such pair is distinct from (e_1, e_2), (e_1, e_3) and (e_2, e_3), a contradiction. Therefore, $d_3 \le 5$. Since $\sum_{i=1}^{5} d_i \ge 21$, the degree sequence s is given by $3,3,5,5,5$. ($\Box$)

Renaming vertices in X if necessary, we may assume that $d_{M_X}(x_i) = d_i$. By Claim H.14.4, x_1 and x_2 have degree 3 in M_X, while x_3, x_4 and x_5 have degree 3 in M_X. By Claim H.14.2, $N_{M_X}(x_1) = N_{M_X}(x_2)$. We may assume that e_1, e_2 and e_3 are the three neighbors of x_1 and x_2. Further, we may assume that e_1 is not adjacent to x_3, e_2 is not adjacent to x_4, and e_3 is not adjacent to x_5. By Claim H.14.2, the three $E^*(X)$-pairs are (e_1, e_2), (e_1, e_3), and (e_2, e_3). The graph M_X is completely determined.

Let F_i be the copy of $H_4 \in X_4$ associated with the vertex x_i in M_X for $i \in [5]$. We note that e_4, e_5 and e_6 all have degree 3 in M_X and are all adjacent in M_X to x_3, x_4 and x_5. Thus, in H, each of the edges e_4, e_5 and e_6 contain the vertex y and one vertex from each copy of H_4 associated with x_3, x_4 and x_5. In H, each of the edges e_1, e_2 and e_3 contain one vertex from each copy of H_4 associated with x_3, x_4 and x_5. The copy of H_4 associated with x_3 contains a vertex covering both e_2 and e_3. The copy of H_4 associated with x_4 contains a vertex covering both e_1 and e_3. The copy of H_4 associated with x_5 contains a vertex covering both e_1 and e_2. By the linearity of H, and since (e_1, e_2), (e_1, e_3), and (e_2, e_3) are the only three $E^*(X)$-pairs, the graph H is now completely determined, and $H = H_{21,2}$. By Observation 1, $\xi(H) = 0$, contradicting the fact that H is a counterexample to the theorem. This completes the proof of Claim H.14. ($\Box$)

Claim H.15: The case $|X| = 4$, and $|\partial(X)| = 1$ cannot occur.

Proof of Claim H.15: By Claim H.13, there are no triple edges in M_X. By Claim H.4(b) we note that $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - 8 = 26$, as $|X| = 4$ and $|X'| = 3$. This implies that $|X_4| \ge 3$. Since $|\partial(X)| = 1$, by Claim H.13(c), each vertex of $E^*(X)$ has degree at least 3 in M_X. By Claim H.13(d), each vertex of X has degree at least 2 in M_X. We now prove the following subclaims.

Claim H.15.1: There does not exist a double edge between $e \in E^*(X)$ and $F \in X$ in M_X, such that e belongs to a $E^*(X)$-pair associated with F.

Proof of Claim H.15.1: For the sake of contradiction, suppose that there is a double edge between $e \in E^*(X)$ and $F \in X$ in M_X, such that e belongs to a $E^*(X)$-pair, say (e, e'), associated with F. As there is a double edge incident with F in M_X we note that

$F \notin X_4$, implying that $|X_4| = 3$. We now consider the bipartite multigraph M_X^*, where M_X^* is defined as follows. Let M_X^* be obtained from M_X by removing the vertex $e \in E^*(X)$ and removing all edges (e'', F), where (e, e'') is not a $E^*(X)$-pair associated with F. We note that (e', F) is an edge in M_X^* and, therefore, $d_{M_X^*}(F) \ge 1$.

First suppose that there is a perfect matching, M, in M_X^*. Let (e^*, F) be an edge of the matching incident with F. By the definition of M_X^*, we note that (e^*, e) is a $E^*(X)$-pair associated with F. By Observation 1(g), we can find a $\tau(X)$-transversal covering $E^*(X)$, contradicting Claim B. Therefore, there is no perfect matching in M_X^*. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*(X) \setminus \{e\}$ such that $|N_{M_X^*}(S)| < |S|$. Let $X^* = X \setminus N_{M_X^*}(S)$.

We will show that $F \in X^*$. For the sake of contradiction, suppose that $F \notin X^*$, and so $F \in N_{M_X^*}(S)$. Since $F \notin X^*$, the neighbors of a vertex of X^* in M_X remain unchanged in M_X^*. We note that no vertex in X^* has a neighbor that belongs to S in M_X^*, and therefore also no neighbor that belongs to S in M_X. This implies that in M_X the vertices in X^* are adjacent to at most $|E^*(X) \setminus S| = |E^*(X)| - |S| = |X| + 1 - |S| \le |X^*|$ vertices in $E^*(X)$. Thus, $|E^*(X^*)| \le |X^*|$ in H, contradicting Claim D. Therefore, $F \in X^*$. Since all three subhypergraphs in $X \setminus \{F\}$ belong to X_4, all edges in M_X from S to vertices in $N_{M_X^*}(S)$ are therefore single edges.

If $|S| = 4$, then the vertices in X^* have no edge to $S = E^*(X) \setminus \{e\}$ in M_X^*. In particular, the vertex $F \in X^*$ is isolated in M_X^*, contradicting our earlier observation that $d_{M_X^*}(F) \ge 1$. Therefore, $|S| \le 3$.

Suppose that $|S| = 3$, and let $S = \{e_1, e_2, e_3\}$. As observed earlier, all edges in M_X from S to vertices in $N_{M_X^*}(S)$ are single edges. Since each vertex of S has degree at least 3 in M_X and $|N_{M_X^*}(S)| \le 2$, each vertex of S therefore has F as a neighbor in M_X, but not in M_X^* since $F \in X^*$. Since $\Delta(H) \le 3$, this implies that $|(V(e_1) \cup V(e_2) \cup V(e_3)) \cap V(F)| \ge 2$. By Observation 1(k), either (e, e_1) or (e, e_2) or (e, e_3) is a $E^*(X)$-pair associated with F. Renaming edges in $E^*(X)$ if necessary, we may assume that (e, e_1) is a $E^*(X)$-pair associated with F. However, since the edge (e_1, F) was removed from M_X when constructing M_X^*, this implies that (e, e_1) is not a $E^*(X)$-pair associated with F, a contradiction. Therefore, $|S| \le 2$.

Suppose that $|S| = 2$, and let $S = \{e_1, e_2\}$. We note that there is no edge in M_X^* joining e_1 and F. If e_1 has a double edge to F in M_X, then (e_1, e) would be a $E^*(X)$-pair associated with F, by Observation 1(k), and therefore the edge (e_1, F) would still exist in M_X^*, a contradiction. Hence, either e_1 has a single edge to F in M_X or is not adjacent to F in M_X. This implies that the degree of e_1 is at most 2 in M_X since it can only be adjacent to the vertex in $N_{M_X^*}(S)$ and to F. This contradicts Claim H.13. Therefore, $|S| = 1$. However if $|S| = 1$, we analogously get a contradiction as the vertex in S has degree at most 1 in M_X. This completes the proof of Claim H.15.1. ($\Box$)

Claim H.15.2: There does not exist a double edge in M_X.

Proof of Claim H.15.2: Suppose, to the contrary, that there is a double edge between $e \in E^*(X)$ and $F \in X$ in M_X. By Claim H.15.1, for every edge (e', F) in M_X, the pair

(e, e') is not a $E^*(X)$-pair associated with F. If there is a double edge between some vertex $e' \in E^*(X) \setminus \{e\}$ and F in M_X, then (e, e') would be a $E^*(X)$-pair associated with F, a contradiction. Therefore, there is no other double edge incident to F, except for the double edge that joins it to $e \in E^*(X)$. By Claim D, the set $N_{M_X}(F)$ contains at least two vertices. If the set $N_{M_X}(F)$ contains four or more vertices, then by by Observation 1(k), we would get a $E^*(X)$-pair associated with F containing e, a contradiction. Therefore, the set $N_{M_X}(F)$ contains at most three vertices. That is, $|N_{M_X}(F)| \in \{2, 3\}$.

As there is a double edge incident with F in M_X, we note that $F \notin X_4$, implying that $|X_4| = 3$ and $X \setminus \{F\} = X_4$. By Claim H.13, $|E(M_X)| \ge 4(|X|+1) - 3|\partial(X)| = 4 \times 5 - 3 = 17$.

We first consider the case when $|N_{M_X}(F)| = 2$, and let $N_{M_X}(F) = \{e, e'\}$. That is $d_{M_X}(F) = 3$, since the double edge between e and F counts 2 to the degree of F in M_X. Since $X \setminus \{F\} = X_4$, we have no double edges incident with a vertex in $X \setminus \{F\}$ in M_X. Further, as $|E(M_X)| \ge 17$, the degree-sequence of the four vertices of X in M_X is either $(3, 4, 5, 5)$ or $(3, 5, 5, 5)$. Let F_1 and F_2 be two vertices of degree 5 in M_X that belong to X, and let F_3 be the remaining vertex in $X \setminus \{F\}$. We note that in H, both F_1 and F_2 belong to X_4, and have an associated $E^*(X)$-pair in M_X. Since H is linear, there are in fact distinct $E^*(X)$-pairs associated with F_1 and F_2. Renaming F_1 and F_2, if necessary, we may assume that the $E^*(X)$-pair, say (e_1, e_2), associated with F_1 is distinct from $\{e, e'\}$. Thus, there is a vertex in F_1 that covers both e_1 and e_2, and we can cover a vertex in $\{e, e'\} \setminus \{e_1, e_2\}$ using a $\tau(F)$-transversal. Let e_3 be such a vertex covered from F. The vertex F_3 in M_X has degree at least 4 in M_X, and can be used to cover a vertex in $E^*(X) \setminus \{e_1, e_2, e_3\}$, say e_4. Finally, the vertex F_2 in M_X which has degree 5 in M_X, can be used to cover the vertex in $E^*(X) \setminus \{e_1, e_2, e_3, e_4\}$. We thereby obtain a contradiction to Claim B.

We next consider the case when $|N_{M_X}(F)| = 3$. Let $N_{M_X}(F) = \{e_1, e_2, e_3\}$, where $e = e_1$, and let $E^*(X) \setminus \{e_1, e_2, e_3\} = \{e_4, e_5\}$. Since there is no $\tau(F)$-transversal covering e_1 and a vertex in $V(e_2) \cup V(e_3)$, we note that e_2 and e_3 intersect F in the same vertex (by Observation 1(k)). Therefore, (e_2, e_3) is a $E^*(X)$-pair associated with F. Since $d_{M_X}(e_1) \ge 3$, and since there are no triple edges in M_X, there is a vertex F' in $X \setminus \{F\}$ adjacent to e_1 in M_X. Since neither e_4 nor e_5 has an edge to F, but they do have degree at least 3 in M_X, both e_4 and e_5 must be adjacent in M_X to all three vertices in $X \setminus \{F\}$. We can therefore cover the edge e_1 from F', cover both edges e_2 and e_3 from F, and cover the remaining two edges, e_4 and e_5, from $X \setminus \{F, F'\}$, thereby obtaining a $\tau(X)$-transversal covering $E^*(X)$, contradicting Claim B. This completes the proof of Claim H.15.2. ($\Box$)

We now return to the proof of Claim H.15. By Claim H.15.2, there does not exist a double edge in M_X, implying that M_X is therefore a graph. As observed earlier, $|X_4| \ge 3$. By Claim H.13, $|E(M_X)| \ge 4(|X| + 1) - 3|\partial(X)| = 17$. Recall that each vertex of X has degree at least 2 in M_X. If some vertex of X has degree 2 in M_X, then the degree sequence of X in M_X is $(2, 5, 5, 5)$, implying that there is a unique vertex of degree 2 in M_X. Since $|X_4| \ge 3$, there exist two distinct vertices $F, F' \in X_4$ of degree 5 in M_X. Both F and F' are associated with $E^*(X)$-pairs. Further, since H is linear, the $E^*(X)$-pairs associated with F and F' are distinct. Renaming F and F', if necessary, we may assume that F has an associated $E^*(X)$-pair, say (e_1, e_2), such that the neighborhood of the degree-2 vertex in

X is not the set $\{e_1, e_2\}$. If every vertex of X has degree at least 3 in M_X, then we must still have a vertex, F, in X of degree 5 in the graph M_X, and such a vertex is necessarily associated with an $E^*(X)$-pair, say (e_1, e_2). In both cases, we have therefore determined a vertex $F \in X$ of degree 5 in M_X with an associated $E^*(X)$-pair, (e_1, e_2), such that the neighborhood of the degree-2 vertex in X, if it exists, is not the set $\{e_1, e_2\}$.

We now consider the bipartite graph $M'_X = M_X - \{e_1, e_2, F\}$ with partite sets $X' = X \setminus \{F\}$ and $E'_X = E^*(X) \setminus \{e_1, e_2\}$. We note that $|E'_X| = 3$, and that there is no matching in M'_X that matches E'_X to X', by Claim B. By Hall's Theorem, there is a nonempty subset $S \subseteq E'_X$ such that in M'_X, $|N(S)| < |S|$. Since every vertex in $E^*(X)$ has degree at least 3 in M_X, we note that $|N_{M'_X}(S)| \ge 2$. Consequently, $|S| = 3$ and $|N_{M'_X}(S)| = 2$. Thus, $S = E'_X$. Let F'' be the vertex in X that is adjacent to no vertex of S in M_X. Therefore, $d_{M_X}(F'') = 2$ and $N_{M_X}(F'') = \{e_1, e_2\}$. However, this is a contradiction to our choice of the $E^*(X)$-pair (e_1, e_2), which was chosen so that the neighborhood of the degree-2 vertex in X, if it exists, is not the set $\{e_1, e_2\}$. This completes the proof of Claim H.15. $(\Box)$

Claim H.16: The case $|X| = 3$ and $|\partial(X)| = 1$ cannot occur.

Proof of Claim H.16: Suppose, to the contrary, that $|X| = 3$ and $|\partial(X)| = 1$. By Claim H.13, there are no triple edges in M_X. Also, by Claim H.13(b), every vertex $F \in X$ has double edges to at most two vertices in $E^*(X)$ in M_X. Since $|\partial(X)| = 1$, by Claim H.13(c), each vertex of $E^*(X)$ has degree at least 3 in M_X. By Claim H.13(d), each vertex of X has degree at least 2 in M_X.

Suppose that some vertex $F \in X$ has double edges to two distinct vertices, say $\{e_1, e_2\}$, of $E^*(X)$ in M_X. Let $E^*(X) \setminus \{e_1, e_2\} = \{e_3, e_4\}$. If there exists a perfect matching in $M_X - \{e_1, e_2, F\}$, then we obtain a X-transversal covering all edges of $E^*(X)$ (by Observation 1(g) and 1(k)), contradicting Claim B. Therefore, there does not exist a perfect matching in $M_X - \{e_1, e_2, F\}$. By Hall's Theorem, there is a nonempty subset $S \subseteq \{e_3, e_4\}$ such that $|N_{M_X}(S) \setminus \{F\}| < |S|$. If $|S| = 1$, say $S = \{e_3\}$, then e_3 has degree at most 1 in M_X, a contradiction. Therefore, $|S| = 2$ and $S = \{e_3, e_4\}$. Let $F' \in X \setminus \{F\}$ be defined such that $N_{M_X}(S) \setminus \{F\} = \{F'\}$. Since both e_3 and e_4 have degree at least 3 in M_X, we must have an edge from e_3 to F and two edges from e_3 to F'. Analogously, there is one edge from e_4 to F and two edges from e_4 to F'. Therefore, by Observation 1(k), there exists a $\tau(F')$-transversal covering both e_3 and e_4. Since e_1 and e_2 can be covered by a $\tau(F)$-transversal, we can therefore cover all edges of $E^*(X)$ with a X-transversal, a contradiction to Claim B. Hence, every $F \in X$ has double edges to at most one vertex of $E^*(X)$ in M_X.

By Claim H.13, $|E(M_X)| \ge 4(|X| + 1) - 3|\partial(X)| = 16 - 3 = 13$. By the Pigeonhole Principle, there is therefore a vertex F of X of degree at least 5 in M_X. By Observation 1 this implies that there exists a $E^*(X)$-pair, say (e_1, e_2), associated with F. Let $E^*(X) \setminus \{e_1, e_2\} = \{e_3, e_4\}$. If there exists a perfect matching in $M_X - \{e_1, e_2, F\}$, then, as before, we obtain a X-transversal covering all edges of $E^*(X)$. Therefore, there is no perfect matching in $M_X - \{e_1, e_2, F\}$. By Hall's Theorem, there is a nonempty subset $S \subseteq \{e_1, e_2\}$ such that $|N_{M_X}(S) \setminus \{F\}| < |S|$. If $|S| = 1$, say $S = \{e_3\}$, then e_3 has degree at most 2 in M_X, a contradiction. Therefore, $|S| = 2$ and $S = \{e_3, e_4\}$. Let $F' \in X \setminus \{F\}$ be defined such that $N_{M_X}(S) \setminus \{F\} = \{F'\}$.

Suppose firstly that the $E^*(X)$-pair, say (e_1, e_2), can be chosen so that either e_1 or e_2 has a double edge to F in M_X. In this case, both e_3 and e_4 have at most one edge to F. Since both e_3 and e_4 have degree at least 3 in M_X, this implies that both e_3 and e_4 have double edges to F', contradicting the fact that every vertex in X has double edges to at most one vertex of $E^*(X)$ in M_X. Therefore, all $E^*(X)$-pairs, (e_1, e_2), have only single edges to F. Renaming e_3 and e_4, if necessary, we may assume that e_3 has a single edge to F' in M_X. Thus, e_3 has a double edge to F in M_X, which in turn implies that e_4 has a single edge to F and a double edge to F' in M_X. However, this implies that F has a double edge to e_3, and single edges to e_1, e_2 and e_4. Therefore, by Observation 1(k), there exists a $E^*(X)$-pair containing e_3 and one of e_1, e_2 and e_4, contradicting the fact that all $E^*(X)$-pairs have only single edges to F. This completes the proof of Claim H.16. (□)

Claim H.17: The case $|X| = 2$ and $|\partial(X)| = 1$ cannot occur.

Proof of Claim H.17: Suppose, to the contrary, that $|X| = 2$ and $|\partial(X)| = 1$. By Claim H.13, there are no triple edges in M_X, and $|E(M_X)| \geq 4(|X| + 1) - 3|\partial(X)| = 12 - 3 = 9$. By the Pigeonhole Principle, there is therefore a vertex F of X of degree at least 5 in M_X. By Observation 1 this implies that there exists a $E^*(X)$-pair, say (e_1, e_2), associated with F. Let $E^*(X) = \{e_1, e_2, e_3\}$ and $X = \{F, F'\}$. If there is an edge from e_3 to F' in M_X we obtain a contradiction to Claim B analogously to the above cases. If there is no edge from e_3 to F' in M_X we obtain a contradiction to e_3 having degree at least three in M_X. This contradiction completes the proof of Claim H.17. (□)

Claim H.18: If $|\partial(X)| = 0$, then $|X| \leq 6$ and the following holds.

(a) If $|X| = 6$, then $X = X_4$.
(b) If $|X| = 5$, then $|X_4| \geq 3$.
(c) All vertices in $E^*(X)$ have degree 4 in M_X and $|E(M_X)| = 4|X| + 4$.
(d) M_X does not contain triple edges.
(e) No $F \in X$ has double edges to three distinct vertices of $E^*(X)$ in M_X.

Proof of Claim H.18: Suppose that $\partial(X) = \emptyset$, and so $V(H) = V(X)$. Further, H' is obtained from $H - V(X)$ by adding the set X' of four new vertices and adding a 4-edge, e, containing these four vertices. Thus, $H' = H_4$ and $|X'| = 4$. By Claim H.4(b), we have $8|X| \geq 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - 8 = 8|X| + 5|X| - 32$. In particular, $5|X| \leq 32$, and so $|X| \leq 6$. Furthermore if $|X| = 6$, then we must have $X = X_4$, while if $|X| = 5$, then $|X_4| \geq 3$. The last three statements in Claim H.18 follows directly from Claim H.13. (□)

Claim H.19: If $|\partial(X)| = 0$, then there does not exist a double edge between $e \in E^*(X)$ and $F \in X$ in M_X, such that e belongs to a $E^*(X)$-pair associated with F.

Proof of Claim H.19: For the sake of contradiction, suppose that there is a double edge between $e \in E^*(X)$ and $F \in X$ in M_X, such that e belongs to a $E^*(X)$-pair, say (e, e'), associated with F. We assume, further, that e and F are chosen so that the number of $E^*(X)$-pairs, (e, e''), associated with F, where $e'' \in E^*(X) \setminus \{e, e'\}$, is maximized. As there is a double edge incident with F in M_X, we note that $F \notin X_4$.

We now consider the bipartite multigraph M^*_X, where M^*_X is defined as follows. Let M^*_X be obtained from M_X by removing the vertex $e \in E^*(X)$ and removing all edges (e'', F), where (e, e'') is not a $E^*(X)$-pair associated with F. We note that (e', F) is an edge in M^*_X and, therefore, $d_{M^*_X}(F) \geq 1$. Also due to Observation 1(k), we note that at most two edges (and no double edges) have been removed from F to $E^*(X) \setminus \{e\}$ when constructing M^*_X.

First suppose that there is a perfect matching, M, in M^*_X. Let (e^*, F) be an edge of the matching incident with F. By the definition of M^*_X, we note that (e^*, e) is a $E^*(X)$-pair associated with F. By Observation 1(g), we can find a $\tau(X)$-transversal covering $E^*(X)$, contradicting Claim B. Therefore, there is no perfect matching in M^*_X. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*(X) \setminus \{e\}$ such that $|N_{M^*_X}(S)| < |S|$. Let $X^* = X \setminus N_{M^*_X}(S)$.

We will show that $F \in X^*$. For the sake of contradiction, suppose that $F \notin X^*$, and so $F \in N_{M^*_X}(S)$. Since $F \notin X^*$, the neighbors of a vertex of X^* in M_X remain unchanged in M^*_X. We note that no vertex in X^* has a neighbor that belongs to S in M^*_X, and therefore also no neighbor that belongs to S in M_X. This implies that in M_X the vertices in X^* are adjacent to at most $|E^*(X) \setminus S| = |E^*(X)| - |S| = |X| + 1 - |S| \leq |X^*|$ vertices in $E^*(X)$. Thus, $|E^*(X^*)| \leq |X^*|$ in H, contradicting Claim D. Therefore, $F \in X^*$. By Claim H.18, $|X| \leq 6$. As observed earlier, $F \notin X_4$, which implies that $|X| \neq 6$, by Claim H.18. Thus, $|X| \leq 5$.

Suppose that $|X| = 5$. If $|S| = 5$, then since F is adjacent to e' in M^*_X, this would imply that $F \in N_{M^*_X}(S)$, contradicting the fact that $F \in X^*$. Hence, $|S| \leq 4$. Suppose that $|S| = 4$ and $S = \{e_1, e_2, e_3, e_4\}$. We note that in this case, $E^*(X) \setminus S = \{e, e'\}$ and that (e, e') is the only $E^*(X)$-pair associated with F. As observed earlier, at most two edges (and no double edges) were removed from F to $E^*(X) \setminus \{e\}$ when constructing M^*_X. Thus, at least two vertices in S did not have any edges to F in M_X. Renaming vertices of S, if necessary, we may assume that neither e_1 nor e_2 has an edge to F in M_X. By Claim H.18(c), all vertices in $E^*(X)$ have degree 4 in M_X, implying that both e_1 and e_2 have a double edge to a vertex in $N_{M^*_X}(S)$. Since $|X_4| \geq 3$ and $F \notin X_4$, e_1 and e_2 both have double edges to the same vertex $F' \in N_{M^*_X}(S)$. Thus, $X \setminus \{F, F'\} = X_4$. Furthermore, as e_3 and e_4 have degree 4 in M_X, and therefore degree at least 3 in M^*_X, both e_3 and e_4 are adjacent to F' in M^*_X. Therefore, by Observation 1(k) and 1(ℓ), there is a $E^*(X)$-pair containing one vertex from $\{e_1, e_2\}$ and one vertex from $\{e_3, e_4\}$ associated with F'. Renaming vertices of S, if necessary, we may assume that (e_1, e_3) is a $E^*(X)$-pair associated with F'. If we had used this $E^*(X)$-pair instead of (e, e'), and F' instead of F, then we note that e_1 is a double edge to F' and both (e_1, e_2) and (e_1, e_3) are $E^*(X)$-pairs associated with F', while e is a double edge to F but there is only one $E^*(X)$-pair of the form (e, e''), where $e'' \in E^*(X) \setminus \{e, e'\}$, (namely the pair (e, e')) that is associated with F. This contradicts our choice of e and F. Therefore, $|S| \leq 3$. Suppose that $|S| = 3$ and $S = \{e_1, e_2, e_3\}$. Since at most two edges (and no double edges) were removed from F to $E^*(X) \setminus \{e\}$ when constructing M^*_X, at least one vertex in S, say e_1, has the same degree in M^*_X as in M_X. Thus, e_1 has degree 4 in M^*_X, implying that e_1 has double edges to both vertices in $N_{M^*_X}(S)$. However, this implies that no vertex in $N_{M^*_X}(S)$ belongs to X_4. As observed earlier, $F \notin X_4$. This implies that $|X_4| \leq 2$, a contradiction. Therefore, $|S| \leq 2$. However in this case the vertices in S cannot have degree 4 in M_X, a contradiction. This completes the case when $|X| = 5$.

Suppose that $|X| = 4$. If $|S| = 4$, then since F is adjacent to e' in M_X^*, this would imply that $F \in N_{M_X^*}(S)$, contradicting the fact that $F \in X^*$. Hence, $|S| \le 3$. Suppose that $|S| = 3$ and $S = \{e_1, e_2, e_3\}$. We note that in this case, $E^*(X) \setminus S = \{e, e'\}$. Since at most two edges (and no double edges) were removed from F to $E^*(X) \setminus \{e\}$ when constructing M_X^*, at least one vertex in S, say e_3, has the same degree in M_X^* as in M_X. Thus, e_3 has degree 4 in M_X^*, implying that e_3 has double edges to both vertices in $N_{M_X^*}(S) = \{F_1, F_2\}$. Since both e_1 and e_2 have degree 4 in M_X, and degree at least 3 in M_X^*, both e_1 and e_2 have a double edge to F_1 or F_2. They cannot have a double edge to the same vertex in $\{F_1, F_2\}$, for otherwise such a vertex would then have double edges to all three vertices in S, contradicting Claim H.18(e). Renaming e_1 and e_2, if necessary, we may assume that e_1 has a double edge to F_1 and e_2 has a double edge to F_2 in M_X^* (and in M_X). We can now cover $\{e, e'\}$ from F, $\{e_1, e_3\}$ from F_1, and e_2 from F_2, contradicting Claim B. Therefore, $|S| \le 2$. However in this case the vertices in S cannot have degree 4 in M_X, a contradiction. This completes the case when $|X| = 4$.

Consider the case when $|X| = 3$. If $|S| = 3$, then we contradict the fact that $F \in X^*$. Hence, $|S| \le 2$. However in this case the vertices in S cannot have degree 4 in M_X, a contradiction. This completes the case when $|X| = 3$. Analogously, we get a contradiction when $|X| = 2$. This completes the proof of Claim H.19. ($\Box$)

Claim H.20: The case $|X| \ge 6$ and $|\partial(X)| = 0$ cannot occur.

Proof of Claim H.20: Suppose, to the contrary, that $|X| \ge 6$ and $|\partial(X)| = 0$. By Claim H.18, $|X| = 6$, $X = X_4$ and $|E(M_X)| = 4|X| + 4 = 28$. Since $X = X_4$, the multigraph M_X is a graph. By the Pigeonhole Principle, there is a vertex F of X of degree at least 5 in M_X. By Observation 1 this implies that there exists a $E^*(X)$-pair, say (e_1, e_2), associated with F. Among all such $E^*(X)$-pairs, we choose the pair (e_1, e_2) so that the number of neighbors of e_1 and e_2 in M_X of degree less than 4 is a minimum. Since $F \in X_4$, we note that in H, the edges e_1 and e_2 intersects F in the same vertex.

We now consider the bipartite multigraph $M'_X = M_X - \{e_1, e_2, F\}$ with partite sets $X' = X \setminus \{F\}$ and $E'_X = E^*(X) \setminus \{e_1, e_2\}$. If there exists a perfect matching in M'_X, then we can find a $\tau(X)$-transversal covering $E^*(X)$, contradicting Claim B. Therefore, there is no perfect matching in M'_X. By Hall's Theorem, there is a nonempty subset $S \subseteq E'_X$ such that in M'_X, $|N(S)| < |S|$. By Claim H.18(c), all vertices in $E^*(X)$ have degree 4 in M_X. Thus, each vertex of E'_X has degree at least 3 in M'_X, implying that in M'_X, $|S| > |N(S)| \ge 3$. Hence, $|S| \in \{4, 5\}$.

Suppose that $|S| = 5$. If $|N_{M_X}(S)| \le 3$, then let $X^* = X \setminus N_{M_X}(S)$ and note that $|E^*(X^*)| \le |\{e_1, e_2\}| \le |X^*|$, a contradiction to Claim D. Therefore, $|N_{M_X}(S)| = 4$ and let $F(e_1, e_2)$ denote the vertex in $X \setminus (N_{M_X}(S) \cup \{F\})$. The degree of every vertex of X is at least 2 in M_X, implying that the vertex $F(e_1, e_2)$ is adjacent in M_X to both e_1 and e_2 (and to no other vertex of $E^*(X)$), and therefore has degree 2 in M_X. Hence, in this case when $|S| = 5$, the vertices e_1 and e_2 have a neighbor of degree 2. Before completing our proof of this case when $|S| = 5$, we first consider the case when $|S| = 4$.

Suppose that $|S| = 4$. Since every vertex of E'_X has degree at least 3 in M'_X, we note that in M'_X, $|N(S)| = 3$. Further, in M'_X, every vertex of S is adjacent to every vertex of $N(S)$. This implies that in M_X, the vertex F is adjacent to all four vertices of S, and therefore F has degree at least 6 in M_X. Let $S = \{e_3, e_4, e_5, e_6\}$ and let $N_{M'_X}(S) = \{F_1, F_2, F_3\}$. Let $X \setminus \{F_1, F_2, F_3\} = \{F, F_5, F_6\}$. We note that F_5 (respectively, F_6) has degree 2 or 3 in M_X, and is adjacent to at least one of e_1 and e_2. Hence, in this case when $|S| = 4$, there are two vertices of degree less than 4 adjacent to e_1 or e_2. Recall that in H, the edges e_1 and e_2 intersects F in the same vertex, implying that in H, two edges in $\{e_3, e_4, e_5, e_6\}$ intersect F in the same vertex. Renaming vertices of S, if necessary, we may assume that in H, the edges e_3 and e_4 both intersect F in the same vertex, and therefore form a $E^*(X)$-pair associated with F. If we had considered the $E^*(X)$-pair (e_3, e_4) instead of the $E^*(X)$-pair (e_1, e_2), then we note that every neighbor of e_3 and e_4 has degree at least 4 in M_X. This contradicts our choice of the $E^*(X)$-pair, (e_1, e_2). Therefore, $|S| = 4$ is not possible.

We now return to the case when $|S| = 5$. Given a $E^*(X)$-pair, (e_1, e_2), we showed that there exists a vertex $F(e_1, e_2)$ in $X \setminus \{F\}$ that has degree 2 in M_X, and has e_1 and e_2 as its neighbors. Since $|S| \le 4$ is not possible, this implies that for every $E^*(X)$-pair, (e, e'), associated with some vertex $F' \in X$, there exists a vertex $F(e, e')$ in $X \setminus \{F'\}$ that has degree 2 in M_X, and has e and e' as its neighbors. We note that if a vertex $F^* \in X$ has degree $4 + i$ in M_X, then it is associated with at least i $E^*(X)$-pairs. Thus, since the vertex $F(e_1, e_2)$ has degree 2 in M_X, and since $|E(M_X)| = 28$, there are at least six $E^*(X)$-pairs, which are all distinct. To each one we associated a vertex of degree 2 in M_X that belongs to X. However this implies that there are only 12 edges in M_X, a contradiction to $|E(M_X)| = 28$. This completes the proof of Claim H.20. (□)

Claim H.21: The case $|X| = 5$ and $|\partial(X)| = 0$ cannot occur.

Proof of Claim H.21: Suppose, to the contrary, that $|X| = 5$ and $|\partial(X)| = 0$. By Claim H.18, $|X_4| \ge 3$ and $|E(M_X)| = 4|X| + 4 = 24$. Also, by Claim H.18, M_X does not contain triple edges. By Claim H.19 and Observation 1(k), no $F \in X$ has double edges to two distinct vertices of $E^*(X)$ in M_X. By Claim H.18(c), all vertices in $E^*(X)$ have degree 4 in M_X.

Since $|E(M_X)| = 24$ and $|X| = 5$, there is a vertex F of X of degree at least 5 in M_X. By Observation 1 this implies that there exists a $E^*(X)$-pair, say (e_1, e_2), associated with F. We now consider the bipartite multigraph $M'_X = M_X - \{e_1, e_2, F\}$ with partite sets $X' = X \setminus \{F\}$ and $E'_X = E^*(X) \setminus \{e_1, e_2\}$. If there exists a perfect matching in M'_X, then we can find a $\tau(X)$-transversal covering $E^*(X)$, contradicting Claim B. Therefore, there is no perfect matching in M'_X. By Hall's Theorem, there is a nonempty subset $S \subseteq E'_X$ such that in M'_X, $|N(S)| < |S|$. We note that $1 \le |S| \le 4$.

If $|S| = 1$, then the vertex in S has degree at most 2 in M_X, a contradiction. If $|S| = 2$, then the two vertices in S must have double edges to F (and to the vertex in $N_{M'_X}(S)$), contradicting the fact that no vertex in X has double edges to two distinct vertices of $E^*(X)$ in M_X. Therefore, $|S| \ge 3$. Suppose that $|S| = 3$, and let $S = \{e_3, e_4, e_5\}$. For each $i \in [3]$, the vertex $e_i \in S$ has degree 4 in M_X and $N_{M_X}(e_i) \subseteq \{F\} \cup N_{M'_X}(S)$, implying that $|N_{M'_X}(S)| = 2$ and that $e_i \in S$ is adjacent to a double edge. Let $N_{M'_X}(S) = \{F_1, F_2\}$.

Since no vertex in X is adjacent to two double edges, we may assume, renaming vertices if necessary, that e_3 has a double edge to F, e_4 has a double edge to F_1, and e_5 has a double edge to F_2. However, this implies that none of F, F_1 or F_2 belong to X_4, contradicting the fact that $|X_4| \geq 3$. Therefore, $|S| \geq 4$.

We have shown that $|S| = 4$. Let $S = \{e_3, e_4, e_5, e_6\}$. If $|N_{M'_X}(S)| \leq 2$, then let $X^* = X \setminus (N_{M_X}(S) \cup \{F\})$ and note that $|E^*(X^*)| \leq |\{e_1, e_2\}| \leq |X^*|$, a contradiction to Claim D. Therefore, $|N_{M'_X}(S)| = 3$. Let $N_{M'_X}(S) = \{F_1, F_2, F_3\}$ and let $X \setminus N_{M'_X}(S) = \{F, F(e_1, e_2)\}$. We note that $N_{M_X}(F(e_1, e_2)) \subseteq \{e_1, e_2\}$. By Claim E, $|N_{M_X}(F(e_1, e_2))| \geq 2$, implying that $N_{M_X}(F(e_1, e_2)) = \{e_1, e_2\}$. Since no vertex of X has double edges in M_X to two distinct vertices in $E^*(X)$, we note that $d_{M_X}(F(e_1, e_2)) \leq 3$. Thus, for each $E^*(X)$-pair, (e', e'') say, there is a vertex $F' \in X$ such that $N_{M_X}(F(e', e'')) = \{e', e''\}$ and $d_{M_X}(F(e', e'')) \leq 3$.

We show next that if $F' \in X$ has degree $4 + i$ in M_X, then it is associated with at least i $E^*(X)$-pairs. We may assume that $F' \notin X_4$, for otherwise this property clearly holds if $F' \in X_4$. If there are three neighbors $\{f_1, f_2, f_3\}$ of F' in M_X, no two of which form a $E^*(X)$-pair associated with F', then every vertex in $N_{M_X}(F') \setminus \{f_1, f_2, f_3\}$ forms a $E^*(X)$-pair with one of the vertices in $\{f_1, f_2, f_3\}$, by Observation 1(j). This gives us at least $|N_{M_X}(F')| - 3$ distinct $E^*(X)$-pairs associated with F'. Since no vertex in X has double edges to two distinct vertices in $E^*(X)$, we note that $|N_{M_X}(F')| \geq d_{M_X}(F') - 1 = 3 + i$ and the result follows. Therefore, we may assume that every set of three neighbors of F' in M_X yields at least one $E^*(X)$-pair associated with F'. If every two neighbors of F' in M_X form a $E^*(X)$-pair associated with F', then we clearly have enough $E^*(X)$-pairs, so we assume that there are two neighbors, f_1 and f_2 say, of F' in M_X which do not form a $E^*(X)$-pair associated with F'. However, since every set of three neighbors of F' in M_X contains a $E^*(X)$-pair, for every $f' \in N_{M_X}(F') \setminus \{f_1, f_2\}$ at least one of (f', f_1) and (f', f_2) is a $E^*(X)$-pair associated with F', yielding at least $d_{M_X}(F') - 2 = 2 + i$ $E^*(X)$-pairs associated with F'. Therefore, we have shown that if $F' \in X$ has degree $4 + i$ in M_X, then it is associated with at least i $E^*(X)$-pairs. Recall that $d_{M_X}(F(e_1, e_2)) \leq 3$ and $|E(M_X)| = 24$. Thus, there are at least five $E^*(X)$-pairs.

If (e, e') is an arbitrary $E^*(X)$-pair, then, by the linearity of H, (e, e') can be associated with at most one vertex from X_4. Hence, since $|X \setminus X_4| \leq 2$, the $E^*(X)$-pair can be associated with at most three vertices of X. Since there are at least five $E^*(X)$-pairs, there are therefore at least $\lceil \frac{5}{3} \rceil = 2$ distinct $E^*(X)$-pairs associated with distinct vertices of X. Let (e', e'') and (f', f'') be two such $E^*(X)$-pairs. Thus, there exist two vertices in X, say F' and F'', of degree at most 3 in M_X, where F' has neighborhood $\{e', e''\}$ and F'' has neighborhood $\{f', f''\}$, in M_X.

Suppose that both F' and F'' have degree 3 in M_X. Then, both F' and F'' are incident with double edges in M_X and therefore do not belong to X_4. Since $|X_4| \geq 3$, this implies that $X \setminus \{F', F''\} = X_4$. Hence, no vertex in $X \setminus \{F', F''\}$ is incident with a double edge in M_X, and therefore has degree at most $|E^*(X)| = 6$ in M_X. Since $|E(M_X)| = 24$, the degree sequence of the vertices in X must be $(3, 3, 6, 6, 6)$ in M_X. However in this case, each of the three vertices of X of degree 6 belongs to X_4, and is therefore intersected by six edges in H, which gives rise to at least two associated $E^*(X)$-pairs. Further, by the linearity of H, the $E^*(X)$-pairs associated with two distinct vertices of X that belong to X_4 are distinct. Hence, there are at least six distinct $E^*(X)$-pairs associated with distinct vertices of X. However, as observed earlier, for each $E^*(X)$-pair, (e, f), there is a vertex in X whose neighborhood is precisely $\{e, f\}$, implying that $|X| \geq 6$, a contradiction. Therefore, at least one of F' and F'' has degree 2 in M_X.

Since $|E(M_X)| = 24$, and since the sum of the degrees of F' and F'' in M_X is at most 5, the sum of the degrees of the three vertices of $X \setminus \{F', F''\}$ in M_X is at least 19. As observed earlier, if $F' \in X$ has degree $4 + i$ in M_X, then it is associated with at least i $E^*(X)$-pairs. Hence there are at least seven $E^*(X)$-pairs. Since each $E^*(X)$-pair can be associated with at most three vertices of X, there are therefore at least $\lceil \frac{7}{3} \rceil = 3$ distinct $E^*(X)$-pairs associated with distinct vertices of X. Thus, there exist at least three vertices in X of degree at most 3 in M_X. Since $|E(M_X)| = 24$, the sum of the remaining two vertices of X in M_X is at least 15, implying that X has a vertex of degree at least 8 in M_X. Such a vertex has triple edges to a vertex of $E^*(X)$ or double edges to two distinct vertices in $E^*(X)$, a contradiction. This completes the proof of Claim H.21. $(\square)$

Claim H.22: The case $|X| = 4$ and $|\partial(X)| = 0$ cannot occur.

Proof of Claim H.22: Suppose, to the contrary, that $|X| = 4$ and $|\partial(X)| = 0$. By Claim H.18, $|E(M_X)| = 4|X| + 4 = 20$. Also, by Claim H.18, M_X does not contain triple edges. By Claim H.19 and Observation 1(k), no $F \in X$ has double edges to two distinct vertices of $E^*(X)$ in M_X. By Claim H.18(c), all vertices in $E^*(X)$ have degree 4 in M_X.

Since $|E(M_X)| = 20$ and $|X| = 4$, there is a vertex F of X of degree at least 5 in M_X. By Observation 1 this implies that there exists a $E^*(X)$-pair associated with F. Among all such $E^*(X)$-pairs, we choose the pair (e_1, e_2) to maximize the number of edges between $\{e_1, e_2\}$ and F in M_X. Since F has double edges to at most one vertex of $E^*(X)$ in M_X, we note that there are at most three edges between $\{e_1, e_2\}$ and F in M_X.

We now consider the bipartite multigraph $M'_X = M_X - \{e_1, e_2, F\}$ with partite sets $X' = X \setminus \{F\}$ and $E'_X = E^*(X) \setminus \{e_1, e_2\}$. If there exists a perfect matching in M'_X, then we can find a $\tau(X)$-transversal covering $E^*(X)$, contradicting Claim B. Therefore, there is no perfect matching in M'_X. By Hall's Theorem, there is a nonempty subset $S \subseteq E'_X$ such that in M'_X, $|N(S)| < |S|$. We note that $1 \leq |S| \leq 3$.

If $|S| = 1$, then the vertex in S has degree at most 2 in M_X, a contradiction. If $|S| = 2$, then the two vertices in S must have double edges to F (and to the vertex in $N_{M'_X}(S)$), contradicting the fact that no vertex in X has double edges to two distinct vertices of $E^*(X)$ in M_X. Therefore, $|S| = 3$.

Let $S = \{e_3, e_4, e_5\}$. For each $i \in [3]$, the vertex $e_i \in S$ has degree 4 in M_X and $N_{M_X}(e_i) \subseteq \{F\} \cup N_{M'_X}(S)$, implying that $|N_{M'_X}(S)| = 2$ and that $e_i \in S$ is adjacent to a double edge. Let $N_{M'_X}(S) = \{F_1, F_2\}$. Since no vertex in X is adjacent to two double edges, we may assume, renaming vertices if necessary, that e_3 has a double edge to F, e_4 has a double edge to F_1, and e_5 has a double edge to F_2. Furthermore, e_4 and e_5 also have (single) edges to F. Thus, F has degree 6 in M_X, with a single edge to each of e_1, e_2, e_4, e_5 and a double edge to e_3. By Observation 1 this implies that there exists a $E^*(X)$-pair, (e_3, e), associated with F, for some $e \in \{e_1, e_2, e_4, e_5\}$. The number of edges between $\{e_1, e_2\}$ and

F in M_X is 2, while the number of edges between $\{e_3, e\}$ and F in M_X is 3. This contradicts our choice of the $E^*(X)$-pair $\{e_1, e_2\}$, and completes the proof of Claim H.22. $(\square)$

Claim H.23: The case $|\partial(X)| = 0$ cannot occur.

Proof of Claim H.23: Suppose, to the contrary, that $|\partial(X)| = 0$. By Claim H.20, Claim H.21 and Claim H.22 we must have $|X| \le 3$. By Claim H.18, M_X does not contain triple edges. By Claim H.19 and Observation 1(k), no $F \in X$ has double edges to two distinct vertices of $E^*(X)$ in M_X. Since every vertex of $E^*(X)$ has degree 4 in M_X, we note that $|X| \le 2$ is therefore impossible. Therefore, $|X| = 3$. Let $E^*(X) = \{e_1, e_2, e_3, e_4\}$. Since every vertex of $E^*(X) = \{e_1, e_2, e_3, e_4\}$ has a double edge to a vertex of X in M_X, by the Pigeonhole Principle, some vertex in X has double edges to two distinct vertices in $E^*(X)$, a contradiction. This completes the proof of Claim H.23. $(\square)$

Claim H.24: $|X| = 1$.

Proof of Claim H.24: By Claim H.10, Claim H.14, Claim H.15, and Claim H.23, the case $|X| \ge 4$ cannot occur. By Claim H.11, Claim H.16, Claim H.17, and Claim H.23, the case $|X| = 3$ cannot occur. By Claim H.6, Claim H.8, Claim H.12, Claim H.17, and Claim H.23, the case $|X| = 2$ cannot occur. Therefore, $|X| = 1$. $(\square)$

Throughout the remaining subclaims of Claim H, we implicitly use the fact that $|X| = 1$, and we let e_1 and e_2 be the two edges of $E^*(X)$ that intersect X in H.

Claim H.25: $|\partial(X)| \ge 4$.

Proof of Claim H.25: Suppose, to the contrary, that $|\partial(X)| \le 3$. Recall that e_1 and e_2 were the edges intersecting X in H. If $|V(e_1) \cap V(X)| = 1$, then $\partial(X) \subseteq V(e_1)$ and therefore $|V(e_2) \cap V(X)| \ge 3$ as H is linear. Therefore, by Observation 1(ℓ), we can cover $E^*(X)$ with a $\tau(X)$-transversal, a contradiction to Claim B. If $|V(e_1) \cap V(X)| \ge 3$, we analogously get a contradiction using Observation 1(ℓ) and Claim B. If $|V(e_1) \cap V(X)| = 2$, then $|V(e_2) \cap V(X)| \ge 2$, as $|\partial(X)| \le 3$. We now obtain a contradiction to Claim B using Observation 1(k). Therefore, $|\partial(X)| \ge 3$. $(\square)$

Claim H.26: If H' is linear, then one of the following hold.

(a) $\mathrm{def}(H') \in \{8, 10\}$.

(b) If $\mathrm{def}(H') = 8$, then $X = X_4$.

Proof of Claim H.26: By Claim H.4(a), we note that $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| > 13|X| - 6|X'| - \mathrm{def}(H')$. As $|X| = 1$ and $|X'| = 0$, we note that $\mathrm{def}(H') \ge 6$. By Claim H.3, $\mathrm{def}(H') = \mathrm{def}_{H'}(Y)$, where $|Y| = 1$ and e is an edge of the hypergraph in the special H'-set Y. Thus, either Y is an H_4-component of H', in which case $\mathrm{def}(H') = 8$, or Y is an H_{10}-component of H', in which case $\mathrm{def}(H') = 10$. This proves Part (a).

Suppose that $\mathrm{def}(H') = 8$, and so Y is an H_4-component of H' consisting of the edge e. We now let $H^* = H - V(X) - V(e)$. Equivalently, H^* is obtained from H' by deleting the H_4-component Y. If $\mathrm{def}(H^*) > 0$ and Y^* is a special H^*-set such that $\mathrm{def}(H^*) =$

$\mathrm{def}_{H^*}(Y^*) > 0$, then $\mathrm{def}(H') \ge \mathrm{def}_{H'}(Y \cup Y^*) > 8$, a contradiction. Hence, $\mathrm{def}(H^*) = 0$. By Observation 1(e), we can choose a $\tau(X)$-set, T_X, to contain a vertex of e_1. Every $\tau(H^*)$-set can be extended to a transversal of H by adding to the set T_X and an arbitrary vertex of e_2, implying that $\tau(H) \le \tau(H^*) + \tau(X) + 1$. We note that $n(H^*) = n(H) - n(X) - 4$ and $m(H^*) = m(H) - m(H) - 2$. We show that $X = X_4$. Suppose, to the contrary, $X \ne X_4$. Thus, $X = X_i$ for some $i \in \{11, 14, 21\}$. If $X = X_{11}$, then $45\tau(X) = 6n(X) + 13m(X) + 4$. If $X = X_{14}$, then $45\tau(X) = 6n(X) + 13m(X) + 5$. If $X = X_{21}$, then $45\tau(X) = 6n(X) + 13m(X) + 1$. In all cases, $45\tau(X) \le 6n(X) + 13m(X) + 5$. Thus,

$$
\begin{array}{rcl}
\Phi(H^*) & = & \xi(H^*) - \xi(H) \\
& = & 45\tau(H^*) - 6n(H^*) - 13m(H^*) - \mathrm{def}(H^*) \\
& & \qquad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\
& \ge & 45(\tau(H) - \tau(X) - 1) - 6(n(H) - n(X) - 4) \\
& & \qquad -13(m(H) - m(H) - 2) \\
& & \qquad\quad -45\tau(H) + 6n(H) + 13m(H) \\
& = & -45\tau(X) + 6n(X) + 13m(X) - 45 + 24 + 26 \\
& \ge & -5 - 45 + 24 + 26 \\
& = & 0,
\end{array}
$$

a contradiction to Claim G. This proves Part (b), and completes the proof of Claim H.26. (□)

Claim H.27: $\mathrm{def}(H') \le 8$.

Proof of Claim H.27: By Claim H.3, either $\mathrm{def}(H') = 0$ or $\mathrm{def}(H') = \mathrm{def}_{H'}(Y)$ where $|Y| = 1$ and e is an edge of the hypergraph in the special H'-set Y. For the sake of contradiction suppose that $\mathrm{def}(H') > 8$, which implies that $\mathrm{def}(H') = \mathrm{def}_{H'}(Y)$ where $|Y| = 1$ and Y is an H_{10}-component, F, of H' and e is an edge of F. By Claim H.25, $|\partial(X)| \ge 4$.

Suppose that $|\partial(X)| = 4$. If $|V(e_1) \cap V(X)| = 1$ and $|V(e_2) \cap V(X)| = 1$, then the linearity of H implies that $|\partial(X)| \ge 5$, a contradiction. Renaming the edges e_1 and e_2, we may assume that $|V(e_2) \cap V(X)| \ge 2$. By Claim B, there is no $\tau(X)$-transversal covering both e_1 and e_2. Therefore, by Observation 1(k) and 1(ℓ), we note that $|V(e_1) \cap V(X)| = 1$ and $|V(e_2) \cap V(X)| = 2$. Thus, by Observation 1(m), we have $X = X_{11}$. Let $T_1 = V(e_1) \cap V(X)$ and let $T_2 = V(e_2) \cap V(X)$, and so T_1 and T_2 are vertex-disjoint subsets of X such that $|T_1| = 1$ and $|T_2| = 2$ where T_2 contains two vertices that are not adjacent in X. Further by Observation 1(m), one degree-1 vertex in $X = X_{11}$ belongs to T_1 and the other degree-1 vertex to T_2, and the second vertex of T_2 is adjacent to the vertex of T_1. As $|\partial(X)| = 4$, the edges e_1 and e_2 intersect in a vertex in $\partial(X)$. As every edge in H_{10} is equivalent, the above observations imply that we get that $H = H_{21,6}$, contradicting Claim C. Hence, $|\partial(X)| \ge 5$.

Since e is an edge of F, we note that $|\partial(X) \cap V(F)| \ge 4$. Suppose that $|\partial(X) \cap V(F)| \ge 5$. Let $z \in (\partial(X) \cap V(F)) \setminus V(e)$ be arbitrary. Removing from F the edge e and all edges incident to z, we are left with only two edges. Further, these two remaining edges intersect in a vertex, say z'. Let H'' be obtained from H' by removing the two vertices z and z' (and all edges incident with z and z') and removing the edge e; that is, $H'' = H' - \{z, z'\} - e$. Equivalently, H'' is obtained from H' by deleting the H_{10}-component F. If $\mathrm{def}(H'') > 0$ and Y'' is a special H''-set such that $\mathrm{def}(H'') = \mathrm{def}_{H''}(Y'') > 0$, then $\mathrm{def}(H') \ge \mathrm{def}_{H'}(Y \cup Y'') > \mathrm{def}(H')$, a contradiction. Hence, $\mathrm{def}(H'') = 0$. Every $\tau(H'')$-transversal can be extended to a transversal of H by adding to a $\tau(X)$-transversal the two vertices $\{z, z'\}$, implying that $\tau(H) \le \tau(H'') + \tau(X) + 2$. We note that $45\tau(X) = 6n(X) + 13m(X) + \mathrm{def}(X) \le 6n(X) + 13m(X) + 10$. Thus,

$$\begin{aligned}
\Phi(H'') &= \xi(H'') - \xi(H) \\
&= 45\tau(H'') - 6n(H'') - 13m(H'') - \mathrm{def}(H'') \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\
&\ge 45(\tau(H) - \tau(X) - 2) - 6(n(H) - n(X) - 10) \\
&\qquad -13(m(H) - m(X) - 6) \\
&\qquad\qquad -45\tau(H) + 6n(H) + 13m(H) \\
&= -45\tau(X) + 6n(X) + 13m(X) - 90 + 60 + 78 \\
&\ge -10 - 90 + 60 + 78 \\
&> 0,
\end{aligned}$$

a contradiction to Claim G. Hence, $|\partial(X) \cap V(F)| = 4$, implying that $\partial(X) \cap V(F) = V(e)$. Let $w \in \partial(X) \setminus V(F)$ be arbitrary. Assume we had picked the edge e to contain three vertices from $V(F)$ and the vertex w. In this case, H' would be linear. Further, $\mathrm{def}(H') = \mathrm{def}_{H'}(Y')$, where $|Y'| = 1$ and e is an edge of the hypergraph in the special H'-set Y'. However, Y is neither an H_4-component nor an H_{10}-component of H', implying that $\mathrm{def}(H') \notin \{8, 10\}$, contradicting Claim H.26. This completes the proof of Claim H.27. (□)

Claim H.28: H' is not linear.

Proof of Claim H.28: Suppose, to the contrary, that H' is linear. By Claim H.26 and Claim H.27, $X = X_4$ and $\mathrm{def}(H') = 8$. As $X = X_4$ and H is linear, we note that $|\partial(X)| \ge 5$. By Claim H.3, the edge e is an isolated edge in H' and therefore there are at least four isolated vertices in $\partial(X)$ in $H' - e$. Suppose that there is a non-isolated vertex in $\partial(X)$ in $H - V(X)$. In this case, choosing the edge e to contain this vertex together with three isolated vertices of $H - V(X)$ that belong to $\partial(X)$, yields a new linear hypergraph H' in which the newly chosen edge e does not belong to a H_4- or H_{10}-component, contradicting Claim H.3 and Claim H.26. Therefore, all vertices in $\partial(X)$ are isolated vertices in $H - V(X)$, implying that $|V(H)| \in \{9, 10\}$, $|E(H)| = 3$ and $\tau(H) = 2$, which implies that the theorem holds for H, a contradiction. Thus, $\xi(H) = 45\tau(H) - 6n(H) - 13m(H) - \mathrm{def}(H) \le 90 - 54 - 39 = -3$, contradicting the fact that $\xi(H) > 0$. (□)

Claim H.29: The following holds.
(a) $X = X_4$.
(b) The edge e contains no vertex of degree 1 in H'.
(c) Every edge in $H' - e$ intersects $\partial(X)$ in at most two vertices.

Proof of Claim H.29: By Claim H.28, H' is not linear, implying that the edge e overlaps some other edge in H'. If the edge e contains a degree-1 vertex in H', then, by Claim H.5(a), $8 \ge 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| \ge 13|X| - 6|X'| - 3 = 10$, a contradiction. Therefore, the edge e does not contain a degree-1 vertex in H'. Hence, by Claim H.5(b), $8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| \ge 13|X| - 6|X'| - 7 = 6$, which implies that $X = X_4$. This completes the proof of Part (a) and Part (b). To prove Part (c), suppose to the contrary that some edge, say f, in $H' - e$ intersects $\partial(X)$ in at least three vertices. In this case, the edge f must intersect e_1 or e_2 in at least two vertices, a contradiction to H being linear. (□)

Claim H.30: There exists two distinct edges in $H' - e$ that both intersect $\partial(X)$ in exactly two vertices.

Proof of Claim H.30: By Claim H.29, $X = X_4$. The linearity of H implies that $|\partial(X)| \ge 5$. Suppose that at most one edge in $H' - e$ intersects $\partial(X)$ in more than one vertex. By Claim H.29, such an edge intersects $\partial(X)$ in exactly two vertices, implying that the edge e could have be chosen so that H' is linear, contradicting Claim H.28. Therefore there are at least two distinct edges in $H' - e$ that intersects $\partial(X)$ in more than one vertex. By Claim H.29, they each intersect $\partial(X)$ in exactly two vertices, as claimed. (□)

By Claim H.30, there exists two distinct edges in $H' - e$ that both intersect $\partial(X)$ in exactly two vertices. Let f_1 and f_2 be two such edges in $H' - e$. Thus, f_1 and f_2 are edges in $H - V(X)$ and $|V(f_1) \cap \partial(X)| = |V(f_2) \cap \partial(X)| = 2$.

Claim H.31: The following holds.

(a) Every vertex in $\partial(X) \cap (V(f_1) \cup V(f_2))$ has degree 1 in $H' - e$.

(b) Every vertex in $\partial(X)$ belongs to at most one of f_1 and f_2.

Proof of Claim H.31: Suppose, to the contrary, that $x \in \partial(X) \cap (V(f_1) \cup V(f_2))$ and $d_{H'-e}(x) \ne 1$. If $d_{H'-e}(x) = 0$, then we could have chosen the edge e to contain the vertex x, implying that e would contain a vertex of degree 1 in $H' - e$, contradicting Claim H.29(b). Therefore, $d_{H'-e}(x) = 2$ noting that $\Delta(H) \le 3$ and $x \in \partial(X)$. Renaming f_1 and f_2, if necessary, we may assume that $x \in V(f_1)$. Let y be the vertex in $V(f_1) \cap \partial(X)$ different from x. Let H^* be obtained from $H' - e - f_1$ by adding two new vertices, say s_1 and s_2, and the edge $s = \{x, y, s_1, s_2\}$.

We show that $\tau(H) \le \tau(H^*) + 1$. Let T^* be a minimum transversal in H^*. As T^* covers the edge $\{x, y, s_1, s_2\}$ and s_1 and s_2 have degree 1 in H^*, we can choose T^* to contain x or y, implying that T^* covers the edge f_1. Further since $\{x, y\} \subset \partial(X)$, the set T^* covers at least one of e_1 and e_2 in H, say e_1. We can therefore add to T^* a vertex from $X \cap V(e_2)$ in order to cover the edge in X and the edge e_2, thereby getting a transversal for H. This proves that $\tau(H) \le |T^*| + 1 = \tau(H^*) + 1$. We note that H^* is linear, and so $H^* \in \mathcal{H}_4$. Further, $n(H) = n(H^*) + 2$ and $m(H) = m(H^*) + 3$. By Claim G, we have $\Phi(H^*) < 0$.

Thus,

$$\begin{array}{rcl} 0 > \Phi(H^*) & = & \xi(H^*) - \xi(H) \\ & = & 45\tau(H^*) - 6n(H^*) - 13m(H^*) - \mathrm{def}(H^*) \\ & & \quad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\ & \geq & 45(\tau(H) - 1) - 6(n(H) - 2) - 13(m(H) - 3) \\ & & \quad -\mathrm{def}(H^*) - 45\tau(H) + 6n(H) + 13m(H) \\ & = & -45 + 12 + 39 - \mathrm{def}(H^*) \\ & = & 6 - \mathrm{def}(H^*), \end{array}$$

and so $\mathrm{def}(H^*) \geq 7$. Let $Y \subseteq H^*$ be a special H^*-set such that $\mathrm{def}_{H^*}(Y) = \mathrm{def}(H^*) \geq 7$. If $|E^*_{H^*}(Y)| \geq |Y|$, then $\mathrm{def}_{H^*}(Y) \leq 10|Y| - 13|E^*_{H^*}(Y)| \leq 10|Y| - 13|Y| < 0$, a contradiction. Hence, $|E^*_{H^*}(Y)| \leq |Y| - 1$. If $s \notin E(Y)$, then

$$|E^*_H(X \cup Y)| \leq |E^*_{H^*}(Y)| + |\{e_1, e_2, f_1\}| \leq (|Y| - 1) + 3 = |X \cup Y| + 1,$$

contradicting the maximality of $|X|$. Therefore, $s \in E(Y)$. Thus, since the edge s contains two degree-1 vertices, namely s_1 and s_2, and at least one vertex of degree 2, namely x, in H^*, we note that $|Y| \geq 2$ and that s is a H_4-component in Y. If $|E^*_{H^*}(Y)| = |Y| - 1$, then $\mathrm{def}_{H^*}(Y) \leq 10(|Y| - 1) + 8 - 13|E^*_{H^*}(Y)| = 10|Y| - 2 - 13(|Y| - 1) = -3|Y| + 11 \leq 5$, a contradiction. Hence, $|E^*_{H^*}(Y)| \leq |Y| - 2$. Therefore,

$$|E^*_H(X \cup (Y \setminus \{s\})| \leq |E^*_{H^*}(Y)| + |\{e_1, e_2, f_1\}| \leq (|Y| - 2) + 3 = |X \cup (Y \setminus \{s\})| + 1,$$

contradicting the maximality of $|X|$. This completes the proof of Part (a). Part (b) follows directly from Part (a). (□)

Recall that $|V(f_1) \cap \partial(X)| = |V(f_2) \cap \partial(X)| = 2$. By Claim H.31, every vertex in $\partial(X)$ belongs to at most one of f_1 and f_2. We now choose the edge e to contain the four vertices in $\partial(X) \cap (V(f_1) \cup V(f_2))$. We next define a new hypergraph H_f as follows. Let H'' be constructed from H by removing the four vertices in X and the four vertices in e and removing the edge in $E(X)$ and the four edges e_1, e_2, f_1, f_2. We now define the edge f as follows. If f_1 and f_2 have no common neighbor in $V(H') \setminus \partial(X)$, then $|(V(f_1) \cup V(f_2)) \setminus V(e)| = 4$ and, in this case, we let f contain these four vertices. If f_1 and f_2 have a common neighbor in $V(H') \setminus \partial(X)$, then $|(V(f_1) \cup V(f_2)) \setminus V(e)| = 3$ and, in this case, we let f contain these three vertices as well as a vertex from $\partial(X) \setminus V(e)$. Let H_f be obtained from H'' by adding to it the edge f; that is, $H_f = H'' \cup f$.

Claim H.31: $\tau(H) \leq \tau(H_f) + 2$.

Proof of Claim H.31: Let T_f be a minimum transversal in H_f. In order to cover the edge f, there is a vertex z in T_f that belongs to f. Suppose that $z \in V(f_1) \cup V(f_2)$. Thus, at least one of the edges in $\{f_1, f_2\}$, say f_2, is covered by T_f. Let w_1 be any vertex in $V(e) \cap V(f_2)$. We note that w_1 covers the edge f_2 and at least one of the vertices in $\{e_1, e_2\}$, say e_1. Let w_2 be the vertex in $V(X) \cap V(e_2)$. We note that w_2 covers the edge in X and the edge e_2. Hence, $T_f \cup \{w_1, w_2\}$ is a transversal in H, which implies that $\tau(H) \leq \tau(H_f) + 2$. Suppose that $z \notin V(f_1) \cup V(f_2)$. This implies that $|(V(f_1) \cup V(f_2)) \setminus V(e)| = 3$ and $z \in \partial(X) \setminus V(e)$.

We note that z covers at least one of the vertices in $\{e_1, e_2\}$, say e_1. Let z_2 be the vertex in $V(X) \cap V(e_2)$. We note that w_2 covers the edge in X and the edge e_2. Let z_1 be the vertex common to f_1 and f_2. Hence, $T_f \cup \{z_1, z_2\}$ is a transversal in H, which implies that $\tau(H) \leq \tau(H_f) + 2$. $(\Box)$

Claim H.32: If Y is a special H_f-set and $f \notin E(Y)$, then $|E^*_{H_f}(Y)| \geq |Y| + 1$.

Proof of Claim H.32: Suppose, to the contrary, that $|E^*_{H_f}(Y)| \leq |Y|$. We now prove the following claims.

Claim H.32.1: $f \in E^*_{H_f}(Y)$.

Proof of Claim H.32.1: If $f \notin E^*_{H_f}(Y)$, then $|E^*_H(X \cup Y)| \leq |E^*_{H_f}(Y)| + |\{e_1, e_2\}| \leq |Y| + 2 = |X \cup Y| + 1$, contradicting the maximality of $|X|$. $(\Box)$

Claim H.32.2: $|E^*_{H_f}(Y)| = |Y|$.

Proof of Claim H.32.2: If $|E^*_{H_f}(Y)| < |Y|$, then $|E^*_H(X \cup Y)| \leq |E^*_{H_f}(Y)| - |\{f\}| + |\{f_1, f_2, e_1, e_2\}| \leq (|Y| - 1) - 1 + 4 = |X \cup Y| + 1$, contradicting the maximality of $|X|$. $(\Box)$

We construct a bipartite graph G_Y, with partite sets $Y \cup Y_{10}$ (that is, there are two copies of every element in Y_{10}) and $E^*_{H_f}(Y)$, where an edge joins $g \in E^*_{H_f}(Y)$ and $F \in Y \cup Y_{10}$ in G_Y if and only if the edge g intersects the subhypergraph F of Y in H_f.

Claim H.32.3: There is no matching in G_Y saturating every vertex in $E^*_{H_f}(Y)$.

Proof of Claim H.32.3: Suppose, to the contrary, that there is a matching, M, in G_Y saturating every vertex in $E^*_{H_f}(Y)$. In this case, by Observation 1(g) and 1(h), using the matching M we can find a $\tau(Y)$-transversal in H_f covering all edges in $E(Y) \cup E^*_{H_f}(Y)$. Let $H^* = H_f - V(Y)$; that is, H^* is obtained from H_f by removing all vertices in Y and all edges in $E(Y) \cup E^*_{H_f}(Y)$.

We show that $\mathrm{def}(H^*) = 0$. Suppose, to the contrary, that $\mathrm{def}(H^*) > 0$. Let Y^* be a special H^*-set with $\mathrm{def}_{H^*}(Y^*) > 0$. This implies that $|E^*_{H^*}(Y^*)| < |Y^*|$. By Claim H.32.2, $|E^*_{H_f}(Y)| = |Y|$. Thus, $|E^*_{H_f}(Y \cup Y^*)| = |E^*_{H_f}(Y)| + |E^*_{H^*}(Y^*)| < |Y| + |Y^*|$. Hence we have shown the existence of a special H_f-set, $Y \cup Y^*$, such that $f \notin E(Y \cup Y^*)$ and $|E^*_{H_f}(Y \cup Y^*)| < |Y| + |Y^*|$, contradicting Claim H.32.2. Hence, $\mathrm{def}(H^*) = 0$.

As $f \in E^*_{H_f}(Y)$, we note that $f \notin E(H^*)$. Therefore, H^* is linear. Recall that for any given special hypergraph, F, we have $\mathrm{def}(F) = 45\tau(F) - 6|V(F)| - 13|E(F)|$. Thus, $\mathrm{def}(H_{10}) = 10$, $\mathrm{def}(H_4) = 8$, $\mathrm{def}(H_{14}) = 5$, $\mathrm{def}(H_{11}) = 4$, and $\mathrm{def}(H_{21}) = 1$. In particular, $\mathrm{def}(F) \leq 10$ for any given special hypergraph, F. By Claim H.31, $\tau(H) \leq \tau(H_f) + 2$. Thus, since $\tau(H_f) \leq \tau(H^*) + \tau(Y)$, we have $\tau(H) \leq \tau(H^*) + \tau(Y) + 2$. We note that $n(H) = n(H^*) + n(Y) + 8$ and $m(H) = m(H^*) + m(Y) + |E^*_{H_f}(Y)| + 4$. Thus,

$$\begin{array}{rcl}
\Phi(H^*) & = & \xi(H^*) - \xi(H) \\
& = & 45\tau(H^*) - 6n(H^*) - 13m(H^*) - \mathrm{def}(H^*) \\
& & \qquad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\
& \ge & 45(\tau(H) - \tau(Y) - 2) - 6(n(H) - n(Y) - 8) \\
& & \qquad -13(m(H) - m(Y) - |E^*_{H_f}(Y)| - 4) \\
& & \qquad\quad -45\tau(H) + 6n(H) + 13m(H) \\
& = & 10 + 13|E^*_{H_f}(Y)| - 45\tau(Y) + 6n(Y) + 13m(Y) \\
& = & 10 + 13|Y| - \displaystyle\sum_{F \in Y} \mathrm{def}(F) \\
& = & 10 + \displaystyle\sum_{F \in Y} (13 - \mathrm{def}(F)) \\
& > & 10,
\end{array}$$

a contradiction to Claim G. This completes the proof of Claim H.32.3. (□)

By Claim H.32.3, there is no matching in G_Y saturating every vertex in $E^*_{H_f}(Y)$. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*_{H_f}(Y)$ such that $|N_{G_Y}(S)| < |S|$. We now consider the bipartite graph G'_Y, with partite sets Y and $E^*_{H_f}(Y)$, where an edge joins $g \in E^*_{H_f}(Y)$ and $F \in Y$ in G_Y if and only if the edge g intersects the subhypergraph F of Y in H_f. Thus, G'_Y is obtained from G_Y by deleting all duplicated vertices associated with copies of H_{10} in Y. We note that $|N_{G_{Y'}}(S)| \le |N_{G_Y}(S)| < |S|$. We now consider the special H-set, $Y' = Y \setminus N_{G_{Y'}}(S)$. Recall that by Claim H.32.2, $|E^*_{H_f}(Y)| = |Y|$. Thus, $|E^*_{H_f}(Y')| = |E^*_{H_f}(Y)| - |S| = |Y| - |S| = |Y'| + |N_{G_{Y'}}(S)| - |S| < |Y'|$, contradicting Claim H.32.2. This completes the proof of Claim H.32. (□)

Claim H.33: Let Y be a special H_f-set. If $f \notin E(Y)$ and $f \in E^*_{H_f}(Y)$ and $|Y_{10}| > 0$, then $|E^*_{H_f}(Y)| \ge |Y| + 2$.

Proof of Claim H.33: Suppose, to the contrary, that $|E^*_{H_f}(Y)| \le |Y| + 1$. By Claim H.32, $|E^*_{H_f}(Y)| \ge |Y| + 1$. Consequently, $|E^*_{H_f}(Y)| = |Y| + 1$. Let G_Y be the bipartite graph as defined in the proof of Claim H.32.

Claim H.33.1: There is no matching in G_Y saturating every vertex in $E^*_{H_f}(Y)$.

Proof of Claim H.33.1: Suppose, to the contrary, that there is a matching, M, in G_Y saturating every vertex in $E^*_{H_f}(Y)$. We proceed now analogously as in the proof of Claim H.32.3. If Y^* is a special H^*-set with $\mathrm{def}_{H^*}(Y^*) > 0$, then $|E^*_{H^*}(Y^*)| < |Y^*|$ and $|E^*_{H_f}(Y \cup Y^*)| = |E^*_{H_f}(Y)| + |E^*_{H^*}(Y^*)| \le (|Y| + 1) + (|Y^*| - 1) = |Y| + |Y^*|$, contradicting Claim H.32. Hence, $\mathrm{def}(H^*) = 0$. Proceeding now exactly as in the proof of Claim H.32.3, we show that $\Phi(H^*) > 0$, contradicting Claim G. (□)

By Claim H.32.3, there is no matching in G_Y saturating every vertex in $E^*_{H_f}(Y)$. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*_{H_f}(Y)$ such that $|N_{G_Y}(S)| < |S|$. If

$S = E^*_{H_f}(Y)$, then $N_{G_Y}(S) = Y \cup Y_{10}$. However, by assumption $|Y_{10}| > 0$, and so $|N_{G_Y}(S)| \ge |Y| + 1 = |E^*_{H_f}(Y)| = |S|$, a contradiction. Hence, S is a proper subset of $E^*_{H_f}(Y)$. Let G'_Y be the bipartite graph defined in the proof of Claim H.32. We note that $|N_{G_{Y'}}(S)| \le |N_{G_Y}(S)| < |S|$. We now consider the special H-set, $Y' = Y \setminus N_{G_{Y'}}(S)$. If $Y' = \emptyset$, then, $|N_{G_{Y'}}(S)| = |Y| = |E^*_{H_f}(Y)| - 1 \ge |S|$, a contradiction. Hence, $Y' \neq \emptyset$. Further, $|E^*_{H_f}(Y')| = |E^*_{H_f}(Y)| - |S| = (|Y| + 1) - |S| = |Y'| + 1 + |N_{G_{Y'}}(S)| - |S| < |Y'| + 1$, and so $|E^*_{H_f}(Y')| \le |Y'|$, contradicting Claim H.32. This completes the proof of Claim H.33. $(\Box)$

Claim H.34: If Y is a special H_f-set and $f \in E(Y)$, then $|E^*_{H_f}(Y)| \ge |Y| - 1$.

Proof of Claim H.34: Suppose, to the contrary, that $|E^*_{H_f}(Y)| < |Y| - 1$. This implies that $|Y| \ge 2$. Let $F \in Y$ be the special subhypergraph containing f and let $Y' = Y \setminus F$. In this case, $|E^*_{H_f}(Y')| \le |E^*_{H_f}(Y)| < |Y| - 1 = |Y'|$. Therefore, Y' is a special H_f-set and $f \notin Y'$ satisfying $|E^*_{H_f}(Y')| \le |Y'|$, contradicting Claim H.32. $(\Box)$

Claim H.35: H_f is not linear.

Proof of Claim H.35: Suppose, to the contrary, that H_f is linear. Thus, by Claim G and Claim H.31, we have

$$\begin{aligned}
0 > \Phi(H_f) &= \xi(H_f) - \xi(H) \\
&= 45\tau(H_f) - 6n(H_f) - 13m(H_f) - \mathrm{def}(H_f) \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\
&\ge 45(\tau(H) - 2) - 6(n(H) - 8) - 13(m(H) - 4) \\
&\qquad\quad -45\tau(H) + 6n(H) + 13m(H) \\
&= -45 \cdot 2 + 6 \cdot 8 + 13 \cdot 4 - \mathrm{def}(H_f) \\
&= 10 - \mathrm{def}(H_f),
\end{aligned}$$

and so, $\mathrm{def}(H_f) \ge 11$. Let Y be a special H_f-set such that $\mathrm{def}(H_f) = \mathrm{def}_{H_f}(Y)$. As $\mathrm{def}_{H_f}(Y) \ge 11$, we note that $|E^*_{H_f}(Y)| \le |Y| - 2$. This is a contradiction to Claim H.32, Claim H.33 and Claim H.34, no matter whether $f \notin E(Y)$ or $f \in E(Y)$. $(\Box)$

We now return to the proof of Claim H one final time. By Claim H.35, there exists an edge g in H_f that overlaps f. Let $\{u, v\} \subseteq V(f) \cap V(g)$. Let H^*_f be the hypergraph obtained from H_f by removing the edges f and g, adding two new vertices z_1 and z_2, and adding a new edge $e_{fg} = \{u, v, z_1, z_2\}$. Since $H_f - f$ is linear, we note that H^*_f is linear. Let T^*_f be a minimum transversal of H^*_f. If z_1 or z_2 belong to T^*_f, we can replace it with u or v, implying that T^*_f is a transversal of H_f. Thus, by Claim H.31, $\tau(T^*_f) \ge \tau(H_f) \ge \tau(H) - 2$. Thus, by Claim G, we have

$$
\begin{aligned}
0 > \Phi(H_f^*) &= \xi(H_f^*) - \xi(H) \\
&= 45\tau(H_f^*) - 6n(H_f^*) - 13m(H_f^*) - \mathrm{def}(H_f^*) \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\
&\geq 45(\tau(H) - 2) - 6(n(H) - 6) - 13(m(H) - 5) \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) \\
&= -45 \cdot 2 + 6 \cdot 6 + 13 \cdot 5 - \mathrm{def}(H_f^*) \\
&= 11 - \mathrm{def}(H_f^*),
\end{aligned}
$$

and so, $\mathrm{def}(H_f^*) \geq 12$. Let Y be a special H_f^*-set such that $\mathrm{def}(H_f^*) = \mathrm{def}_{H_f^*}(Y)$. As $\mathrm{def}_{H_f^*}(Y) \geq 12$ we note that $|E^*_{H_f^*}(Y)| \leq |Y| - 2$. If $e_{fg} \notin E(Y)$, then $|E^*_{H_f}(Y)| \leq |Y| - 2 + |\{f, g\}| = |Y|$, contradicting Claim H.32. Therefore, $e_{fg} \in E(Y)$. Let $F_{fg} \in Y$ be the special subhypergraph containing the edge e_{fg}. Since e_{fg} contains at least two degree-1 vertices in H_f^*, we note that $F_{fg} \in Y_4$. Let $Y_{fg} = Y \setminus \{F_{fg}\}$, and note that $f \notin E(Y_{fg})$. Further,

$$
|E^*_{H_f}(Y_{fg})| \leq |E^*_{H_f^*}(Y)| + |\{f, g\}| = |E^*_{H_f^*}(Y)| + 2.
$$

As observed earlier, $|E^*_{H_f^*}(Y)| \leq |Y| - 2$. Suppose that $|E^*_{H_f^*}(Y)| = |Y| - 2$. In this case, since $\mathrm{def}_{H_f^*}(Y) \geq 12$, we note that $|Y_{10}| > 0$. Since $F_{fg} \in Y_4$, this implies that $|(Y_{fg})_{10}| > 0$. If $f \in E^*_{H_f}(Y_{fg})$, then $|E^*_{H_f}(Y_{fg})| \leq |E^*_{H_f^*}(Y)| + 2 = |Y| = |Y_{fg}| + 1$, contradicting Claim H.33. If $f \notin E^*_{H_f}(Y_{fg})$, then $|E^*_{H_f}(Y_{fg})| \leq |E^*_{H_f^*}(Y)| + |\{g\}| = |E^*_{H_f^*}(Y)| + 1 = |Y| - 1 = |Y_{fg}|$, contradicting Claim H.32. Therefore, $|E^*_{H_f^*}(Y)| \leq |Y| - 3$, implying that $|E^*_{H_f}(Y_{fg})| \leq |E^*_{H_f^*}(Y)| + 2 \leq |Y| - 1 = |Y_{fg}|$, once again contradicting Claim H.32. This completes the proof of Claim H. (□)

Claim I: If Y be a special H-set, then the following holds.
(a) $|E^*(Y)| \geq |Y| + 2$.
(b) If $|Y_{10}| \geq 2$, then $|E^*(Y)| \geq |Y| + 3$.

Proof of Claim I: Part (a) follows immediately from our choice of the H-special set X and by Claim H. To prove Part (b), suppose, to the contrary, that $|Y_{10}| \geq 2$ and $|E^*(Y)| \leq |Y| + 2$. By Part (a), $|E^*_H(Y)| \geq |Y| + 2$. Consequently, $|E^*(Y)| = |Y| + 2$. We construct a bipartite graph G'_Y, with partite sets $Y \cup Y_{10}$ (that is, there are two copies of every element in Y_{10}) and $E^*(Y)$, where an edge joins $e \in E^*(Y)$ and $F \in Y \cup Y_{10}$ in G'_Y if and only if the edge e intersects the subhypergraph F of Y in H.

Suppose that there is a matching, M, in G'_Y saturating every vertex in $E^*(Y)$. By Observation 1(g) and 1(h), using the matching M we can find a $\tau(Y)$-transversal in H covering all edges in $E(Y) \cup E^*(Y)$, contradicting Claim B. Hence, there is no matching in G'_Y saturating every vertex in $E^*(Y)$.

By Hall's Theorem, there is a nonempty subset $S \subseteq E^*(Y)$ such that $|N_{G'_Y}(S)| < |S|$. If $S = E^*(Y)$, then $N_{G'_Y}(S) = Y \cup Y_{10}$. However, by assumption $|Y_{10}| \geq 2$, and so $|N_{G'_Y}(S)| =$

$|Y| + |Y_{10}| \ge |Y| + 2 = |E^*(Y)| = |S|$, a contradiction. Hence, S is a proper subset of $E^*(Y)$. We now consider the special H-set, Y', obtained from Y by deleting every element in Y that is a neighbor of S in H. We note that at most $|N_{G'_Y}(S)|$ such elements have been deleted from Y, possibly fewer since some elements $F \in Y_{10}$ may get deleted twice, and so $|Y'| \ge |Y| - |N_{G'_Y}(S)|$. Therefore, $|E^*(Y')| \le |E^*(Y)| - |S| < (|Y|+2) - |N_{G'_Y}(S)| \le |Y'|+2$. Thus, $|E^*(Y')| - |Y'| < 2$. We therefore have a contradiction to the choice of the special H-set X which was chosen so that $|E^*(X)| - |X|$ is minimum. ($\Box$)

In particular, Claim I implies that if $|X_{10}| \ge 2$, then $|E^*(X)| \ge |X| + 3$.

Claim J: There is no H_{10}-subhypergraph in H.

Proof of Claim J: Suppose, to the contrary, that F is a H_{10}-subhypergraph in H. By Claim C, F is not a component of H, and so there exists an edge $e \in E^*(F)$. We choose such an edge e so that $|V(e) \cap V(F)|$ is maximum possible. Since H is linear, we note that $|V(e) \cap V(F)| \le 2$. Let $x \in V(e) \cap V(F)$ be arbitrary. Let $E(F) = \{e_1, e_2, e_3, e_4, e_5\}$, where e_1 and e_2 are the two edge of F that contain x. We note that $d_H(x) = 3$, and e, e_1, e_2 are the three edges of H that contains x. Let $H' = H - x$. Let $E' = \{e_3, e_4, e_5\}$ and note that $E' \subseteq E(H')$ and every pair of edges in E' intersect in H'. Every $\tau(H')$-transversal can be extended to a transversal of H by adding to it the vertex x, and so $\tau(H) \le \tau(H') + 1$. By Claim G,

$$\begin{aligned}
0 > \Phi(H') &= \xi(H') - \xi(H) \\
&= 45\tau(H') - 6n(H') - 13m(H') - \mathrm{def}(H') \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\
&\ge 45(\tau(H) - 1) - 6(n(H) - 1) - 13(m(H) - 3) \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) \\
&= -45 + 6 \cdot 1 + 13 \cdot 3 - \mathrm{def}(H') \\
&= -\mathrm{def}(H'),
\end{aligned}$$

and so, $\mathrm{def}(H') > 0$. Let Y be a special H'-set such that $\mathrm{def}(H') = \mathrm{def}_{H'}(Y)$. By Claim I(a), $|E^*_H(Y)| \ge |Y| + 2$.

Claim J.1: $|Y| = 2$, $|E^*_{H'}(Y)| = 1$, and $0 < \mathrm{def}_{H'}(Y) \le 5$.

Proof of Claim J.1: If $|E^*_{H'}(Y)| \le |Y| - 2$, then $|E^*_H(Y)| \le |E^*_{H'}(Y)| + |\{e, e_1, e_2\}| \le |Y| + 1$, a contradiction. Therefore, $|E^*_{H'}(Y)| \ge |Y| - 1$. As $\mathrm{def}_{H'}(Y) \ge 1$, we note that $|E^*_{H'}(Y)| \le |Y| - 1$. Consequently, $|E^*_{H'}(Y)| = |Y| - 1$.

If $|Y| \ge 5$, then $\mathrm{def}(H') \le 10|Y| - 13|E^*_{H'}(Y)| = 10|Y| - 13(|Y| - 1) = -3|Y| + 13 < 0$, a contradiction. Hence, $|Y| \le 4$. Suppose that $|Y| \ge 3$. If $|Y_{10}| \le 1$, then $\mathrm{def}(H') \le 10 + 8(|Y| - 1) - 13|E^*_{H'}(Y)| = 8|Y| + 2 - 13(|Y| - 1) = -5|Y| + 15 \le 0$, a contradiction. Hence, $|Y_{10}| \ge 2$. However, $|E^*_H(Y)| \le |E^*_{H'}(Y)| + |\{e, e_1, e_2\}| = |Y| + 2$, contradicting Claim I(b). Hence, $|Y| \le 2$.

Suppose that $|Y| = 1$, and let $Y = \{F_1\}$. By Claim I(a), $|Y| + 2 \le |E^*_H(Y)| \le |E^*_{H'}(Y)| + |\{e, e_1, e_2\}| = |Y| + 2$, implying that $|E^*_H(Y)| = |Y| + 2$ and $E^*(F_1) = \{e, e_1, e_2\}$. Therefore,

$E' \subseteq E(F_1)$, which implies that $V(F) \setminus \{x\} \subseteq V(F_1)$. Therefore, the edges e_1 and e_2 intersect F_1 in three vertices, while the edge e intersects F_1 in at least one vertex. If e intersects F_1 in at least two vertices, then by Observation 1(n) we can cover $E^*(F_1)$ with a $\tau(F_1)$-transversal in H, a contradiction to Claim B. Therefore, we may assume that the edge e intersects F_1 in only one vertex.

By Observation 1(ℓ), there exists a $\tau(F_1)$-transversal, T_1, of F_1 covering both e_1 and e_2. Let $H_1 = H - T_1$. We will now show that $\mathrm{def}(H_1) \le 8$. Suppose, to the contrary, that $\mathrm{def}(H_1) \ge 9$ and let Y_1 be a special H_1-set such that $\mathrm{def}(H_1) = \mathrm{def}_{H_1}(Y_1)$. If the edge e does not belong to any special subhypergraph in Y_1, then $\mathrm{def}_{H'}(Y_1 \cup F_1) > \mathrm{def}_{H'}(F_1) = \mathrm{def}(H')$, a contradiction. Therefore, $e \in E(F_e)$ for some special subhypergraph in Y_1. As the edge e has at least two vertices of degree 1 in H_1 (namely, the vertex x and the vertex in $V(F_1) \cup V(e)$), the subhypergraph F_e is an H_4-component. However since $\mathrm{def}_{H_1}(Y_1) \ge 9$, the special set $\{F_1\} \cup (Y_1 \setminus \{F_e\})$ has higher deficiency than F_1 in H', a contradiction. Therefore, $\mathrm{def}(H_1) \le 8$, as desired.

As there is an edge, e_1 (and e_2), that intersects F_1 in three vertices we note that the deficiency of F_1 is at most 5. Note that we have removed the edges e_1 and e_2 as well as all edges in F_1 to get from H to H_1 as well as all vertices in F_1 except the vertex in $V(e) \cap V(F_1)$. Therefore, $n(H_1) = n(H) - (n(F_1) - 1)$, $m(H_1) = m(H) - m(F_1) - |\{e_1, e_2\}|$, and $\tau(H) \le \tau(H_1) + \tau(F_1)$. As observed earlier, $\mathrm{def}(F_1) \le 5$. Thus, $45\tau(F_1) \le 6n(F_1) + 13m(F_1) + 5$, and therefore

$$
\begin{aligned}
0 > \Phi(H_1) &= \xi(H_1) - \xi(H) \\
&= 45\tau(H_1) - 6n(H_1) - 13m(H_1) - \mathrm{def}(H_1) \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) + \mathrm{def}(H) \\
&\ge 45(\tau(H) - \tau(F_1)) - 6(n(H) - n(F_1) + 1) - 13(m(H) - m(F_1) - 2) - 8 \\
&\qquad -45\tau(H) + 6n(H) + 13m(H) \\
&= -45\tau(F_1) + 6 \cdot (n(F_1) - 1) + 13 \cdot (m(F_1) + 2) - 8 \\
&\ge -5 - 6 \cdot 1 + 13 \cdot 2 - 8 \\
&> 0,
\end{aligned}
$$

a contradiction. Therefore, $|Y| = 2$ and $|E^*_{H'}(Y)| = |Y| - 1 = 1$. As observed earlier, if $|Y_{10}| \ge 2$, then $|E^*_H(Y)| \le |Y| + 2$, contradicting Claim I(b). Hence, $|Y_{10}| \le 1$, implying that $\mathrm{def}(H') = \mathrm{def}_{H'}(Y) \le 10 + 8 - 13|E^*_{H'}(Y)| = 18 - 13 = 5$. This completes the proof of Claim J.1. ($\Box$)

By Claim J.1, we have $|Y| = 2$, $|E^*_{H'}(Y)| = 1$, and $0 < \mathrm{def}_{H'}(Y) \le 5$. Let $Y = \{F_1, F_2\}$. Since F_1 and F_2 are vertex disjoint and every pair of edges in E' intersect in H', we note that $E(F_1) \cap E' = \emptyset$ or $E(F_2) \cap E' = \emptyset$. Renaming F_1 and F_2 if necessary, we may assume that $E(F_2) \cap E' = \emptyset$. If neither e_1 nor e_2 intersects F_2 in H, then $E^*_H(F_2) \subseteq \{e\} \cup E^*_{H'}(Y)$, implying that $|E^*_H(F_2)| \le 1 + |E^*_{H'}(Y)| = 2$. However, by Claim I(a), $|E^*_H(F_2)| \ge |F_2| + 2 = 3$, a contradiction. Therefore, e_1 or e_2 intersects F_2 in H. Renaming e_1 and e_2 if necessary, we may assume that e_1 intersects F_2 in H.

Claim J.2: $F_2 \notin Y_{10}$.

Proof of Claim J.2: Suppose, to the contrary, that $F_2 \in Y_{10}$. Let $z \in V(e_1) \cap V(F_2)$ be arbitrary. The vertex z is incident with two edges from F and two edges from F_2 in H and these four edges are distinct, contradicting the fact that the maximum degree in H is three. Therefore, $F_2 \notin Y_{10}$. $(\Box)$

Claim J.3: $F_2 \notin Y_4$.

Proof of Claim J.3: Suppose, to the contrary, that $F_2 \in Y_4$. Since H is linear and $F_2 \in Y_4$, every edge in $E^*_H(F_2)$ intersects F_2 in exactly one vertex. Let $V(e_1) \cap V(F_2) = \{z_1\}$. The vertex z_1 is incident with two edges from F and the one edge from F_2 in H, and so $d_H(z) = 3$.

Suppose that both e and e_2 intersects F_2 in H. Let $V(e_2) \cap V(F_2) = \{z_2\}$. Since x is the only vertex common to both e_1 and e_2, we note that $z_1 \neq z_2$. Let f be the edge of F_2. By the linearity of H and since F is a H_{10}-subhypergraph in H, we note that $V(f) \cap V(F) = \{z_1, z_2\}$. Thus, the edge $f \in E^*_H(F)$ and $|V(f) \cap V(F)| = 2$, implying by our choice of the edge e that $|V(e) \cap V(F)| = 2$. This in turn implies by the linearity of H and the structure of F, that the edge e does not intersect F_2, a contradiction.

As observed above, at least one of e and e_2 does not intersect F_2 in H. Thus, $|E^*_H(F_2)| \le 2 + |E^*_{H'}(Y)| = 3$, implying that there exists a vertex q in F_2 that has degree 1 in H. If we had chosen to use the vertex z_1 instead of x when constructing H', the vertex q would become isolated and therefore deleted from H', implying that $n(H') \le n(H) - 2$. Thus, $0 > \Phi(H') \ge -45 + 6 \cdot 2 + 13 \cdot 3 - \mathrm{def}(H') = 6 - \mathrm{def}(H')$. Hence, $\mathrm{def}(H') > 6$, which is a contradiction to Claim J.1. Therefore, $F_2 \notin F_4$. $(\Box)$

Claim J.4: $F_1 \notin Y_{10}$.

Proof of Claim J.4: Suppose, to the contrary, that $F_1 \in Y_{10}$.

Suppose that $|E(F_1) \cap E'| = 0$. By Claim I(a), at least two of the edges in $\{e, e_1, e_2\}$ intersect F_1 in H, implying that there is a vertex $q \in V(F_2) \cap V(F)$. The vertex q is incident with two edges from F and two edges from F_2 in H and these four edges are distinct, contradicting the fact that the maximum degree in H is three. Therefore, $|E(F_1) \cap E'| \ge 1$. If $|E(F_1) \cap E'| = 1$, then $|E^*_{H'}(Y)| \ge 2$, as all edges in E' intersect each other, contradicting Claim J.1. Therefore, $|E(F_1) \cap E'| \ge 2$. If $|E(F_1) \cap E'| = 3$ (and so, $E' \subset E(F_1)$), then, since F_1 and F_2 are vertex disjoint, the edges e_1 and e_2 do not intersect F_2, implying that $E^*_H(F_2) \subseteq \{e\} \cup E^*_{H'}(Y)$ and hence that $|E^*_H(F_2)| \le 1 + |E^*_{H'}(Y)| = 2$, contradicting Claim I(a). Therefore, $|E(F_1) \cap E'| = 2$.

Renaming the edges in E' if necessary, we may assume that $e_3, e_4 \in E(F_1)$, which implies that $E^*_{H'}(Y) = \{e_5\}$. As e_5 intersects both e_3 and e_4, the edge e_5 intersects F_1 in at least two vertices; that is, $e_5 \in E^*_H(F_1)$ and $|V(e_5) \cap V(F_1)| \ge 2$ where recall that F_1 is a H_{10}-subhypergraph in H. By the maximality of $|V(e) \cap V(F)|$, we may therefore assume that $|V(e) \cap V(F)| \ge 2$. Let $\{x, y\} \subseteq V(e) \cap V(F)$, where recall that $H' = H - x$. By the linearity of H and since F is a H_{10}-subhypergraph in H, we note that $V(e) \cap V(F) = \{x, y\}$ and $y \notin V(e_1) \cup V(e_2)$, which implies that y belongs to two of the edges in $\{e_3, e_4, e_5\}$. Therefore, $y \in V(F_1)$ and all four edges e, e_1, e_2, e_5 intersect F_1.

Recall that $E^*_{H'}(Y) = \{e_5\}$. If the edge e_5 does not intersect F_2, then neither do the edges e_1 and e_2, implying that $|E^*_H(F_2)| \le |\{e\}| = 1$, contradicting Claim I(a). Therefore, $V(e_5) \cap V(F_2) \neq \emptyset$. This implies that one of the edges in $\{e_1, e_2, e_3, e_4\}$ intersects F_2 as every vertex of e_5 intersects one of the edges in $\{e_1, e_2, e_3, e_4\}$. Since F_1 and F_2 are vertex disjoint and $e_3, e_4 \in E(F_1)$, the edges e_3 and e_4 do not intersect F_2. Thus, the common vertex of e_1 and e_5 belongs to F_2 or the common vertex of e_2 and e_5 belongs to F_2. Renaming e_1 and e_2 if necessary, we may assume that the common vertex of e_1 and e_5 belongs to F_2, and therefore we can cover e_1 and e_5 with a $\tau(F_2)$-transversal. As F_1 is a H_{10} subhypergraph in H we can cover e and e_2 with a $\tau(F_1)$-transversal by Observation 1(h). Therefore, we can cover $E^*(Y)$ in H with a $\tau(Y)$-transversal in H, contradicting Claim B. This completes the proof of Claim J.4. ($\Box$)

We now return to the proof of Claim J. Recall that $Y = \{F_1, F_2\}$ and that by Claim J.1, $|E^*_{H'}(Y)| = 1$. By Claims J.2 and J.3, $F_2 \notin Y_4 \cup Y_{10}$. By Claim J.4, $F_1 \notin F_{10}$. Thus, $\mathrm{def}(H') = \mathrm{def}_{H'}(Y) \le 8 + 5 - 13 = 0$, a contradiction to $\mathrm{def}(Y) > 0$. This completes the proof of Claim J. ($\Box$)

Recall that the boundary of a set Z of vertices in a hypergraph H is the set $N_H(Z) \setminus Z$, denoted $\partial_H(Z)$ or simply $\partial(Z)$ if H is clear from context.

Claim K: Let $Z \subseteq V(H)$ be an arbitrary nonempty set of vertices and let $H' = H - Z$. Then either $|E^*_{H'}(Y)| \ge |Y|$ for all special H'-sets Y in H' (and therefore $\mathrm{def}(H') = 0$) or there exists a transversal T' in H', such that

$$45|T'| \le 6n(H') + 13m(H') + \mathrm{def}(H') \quad \text{and} \quad T' \cap \partial(Z) \neq \emptyset.$$

Proof of Claim K: Suppose that $|E^*_{H'}(Y)| < |Y|$ for some special H'-set, Y, in H', as otherwise the claim holds. Among all such special H'-sets Y, let Y be chosen so that

(1) $|Y| - |E^*_{H'}(Y)|$ is maximum.
(2) Subject to (1), $|Y|$ is minimum.

If $\partial(Z) \cap V(Y) = \emptyset$, then $|E^*_H(Y)| < |Y|$, a contradiction to Claim I(a). Therefore, $\partial(Z) \cap V(Y) \neq \emptyset$ and let q be any vertex in $\partial(Z) \cap V(Y)$. We now consider the bipartite graph, G_Y, with partite sets Y and $E^*_{H'}(Y) \cup \{q\}$, where an edge joins $e \in E^*_{H'}(Y)$ and $F \in Y$ in G_Y if and only if the edge e intersects the subhypergraph F of Y in H' and q is joined to $F \in Y$ in G_Y if and only if $q \in V(F)$.

Claim K.1: If there is a matching in G_Y saturating every vertex in $E^*_{H'}(Y) \cup \{q\}$, then Claim K holds.

Proof of Claim K.1: Suppose that there exists a matching in G_Y that saturates every vertex in $E^*_{H'}(Y) \cup \{q\}$. This implies that there exists a $\tau(Y)$-transversal, T_Y, in H' covering $E^*_{H'}(Y)$ and with $q \in T_Y$. Let H_Y be obtained from H' by deleting the vertices $V(Y)$ and edges $E(Y) \cup E^*_{H'}(Y)$; that is, $H_Y = H' - T_Y$. Suppose that $\mathrm{def}(H_Y) > 0$, and let Y' be a special H_Y-set in H_Y such that $\mathrm{def}_{H_Y}(Y') = \mathrm{def}(H_Y) > 0$. Since $|E^*_{H_Y}(Y')| \le |Y'| - 1$, we note that $|E^*_{H'}(Y \cup Y')| = |E^*_{H'}(Y)| + |E^*_{H_Y}(Y')| < |Y| + |Y'|$ and

$$|Y \cup Y'| - |E^*_{H'}(Y \cup Y')| = |Y| + |Y'| - (|E^*_{H'}(Y)| + |E^*_{H_Y}(Y')|) \geq |Y| - |E^*_{H'}(Y)| + 1,$$

contradicting the choice of the special H'-set Y. Therefore, $\mathrm{def}(H_Y) = 0$. Since H is a counterexample with minimum value of $n(H) + m(H)$, and since $n(H_Y) + m(H_Y) < n(H)+m(H)$, we note that $45\tau(H_Y) \leq 6n(H_Y)+13m(H_Y)+\mathrm{def}(H_Y) = 6n(H_Y)+13m(H_Y)$. As $45|T_Y| = 6n(Y) + 13m(Y) + 13|E^*_{H'}(Y)| + \mathrm{def}_{H'}(Y)$ we get the following,

$$\begin{array}{rcl} 45|T'| & \leq & 45|T_Y| + 45\tau(H_Y) \\ & \leq & 6n(Y) + 13m(Y) + 13|E^*_{H'}(Y)| + \mathrm{def}_{H'}(Y) + 6n(H_Y) + 13m(H_Y) \\ & \leq & 6n(H') + 13m(H') + \mathrm{def}_{H'}(Y). \end{array}$$

As q belonged to T_Y this proves Claim K in this case. (□)

By Claim K.1, we may consider the case when there is no perfect matching in G_Y saturating every vertex in $E^*_{H'}(Y) \cup \{q\}$. By Hall's Theorem, there is therefore a nonempty subset $S \subseteq E^*_{H'}(Y) \cup \{q\}$ such that $|N_{G_Y}(S)| < |S|$. We now consider the special H-set $Y' = Y \setminus N_{G_Y}(S)$. Since $|Y'| = |Y| - |N_{G_Y}(S)|$ and $|E^*_{H'}(Y')| \leq |(E^*_{H'}(Y) \cup \{q\}) \setminus S| = |E^*_{H'}(Y)| + 1 - |S| \leq |E^*_{H'}(Y)| - |N_{G_Y}(S)|$, we note that $|Y'| - |E^*_{H'}(Y')| \geq |Y| - |E^*_{H'}(Y)|$ and $|Y'| < |Y|$, contradicting our choice of Y. This completes the proof of Claim K. (□)

Now let f be the function defined in Table 1.

i	1	2	3	4	≥ 5
$f(i)$	39	33	27	23	22

Table 1. The function f.

Claim L: Let $Z \subseteq V(H)$ be an arbitrary nonempty set of vertices that intersects at least two edges of H, and let $H' = H - Z$. If $\mathrm{def}(H') \leq 21$ and $|\partial(Z)| \geq 1$, then there exists a transversal, T', in H', such that $T' \cap \partial(Z) \neq \emptyset$ and the following holds.

(a) $45|T'| \leq 6n(H') + 13m(H') + f(|\partial(Z)|)$.

(b) If $|\partial(Z)| \geq 5$ and H' does not contain two intersecting edges e and f, such that

  (i) $\partial(Z) \subseteq (V(e) \cup V(f)) \setminus (V(e) \cap V(f))$,

  (ii) e contains three degree-1 vertices, and

  (iii) $|\partial(Z) \cap V(e)|, |\partial(Z) \cap V(f)| \geq 2$,

then $45|T'| \leq 6n(H') + 13m(H') + f(|\partial(Z)|) - 1 = 6n(H') + 13m(H') + 21$.

Proof of Claim L: Suppose, to the contrary, that there exists a set $Z \subseteq V(H)$, such that $H' = H - Z$ with $\mathrm{def}(H') \leq 21$ and $|\partial(Z)| \geq 1$, but there exists no transversal T' in H' satisfying (a) or (b) in the statement of the claim. Among all such sets, let Z be chosen so that $|Z|$ is as large as possible. Let T' be a smallest possible transversal in H' containing a vertex from $\partial(Z)$.

Claim L.1: For any special H'-set, Y, in H' we have $|E^*_{H'}(Y)| \geq |Y|$. Furthermore $\mathrm{def}(H') = 0$.

Proof of Claim L.1: If $|E^*_{H'}(Y)| < |Y|$ for some special H'-set, Y, in H', then by Claim K there exists a transversal T' in H', such that $45|T'| \leq 6n(H') + 13m(H') + \mathrm{def}(H')$ and $T' \cap \partial(Z) \neq \emptyset$. Since $\mathrm{def}(H') \leq 21$, this implies that $45|T'| \leq 6n(H') + 13m(H') + 21 < 6n(H') + 13m(H') + f(|\partial(Z)|)$, contradicting our definition of Z. Therefore $|E^*_{H'}(Y)| \geq |Y|$ for all H'-sets Y, which furthermore implies that $\mathrm{def}(H') = 0$. ($\Box$)

By supposition, $|\partial(Z)| \geq 1$. Let Z' be a set of new vertices (not in H), where $|Z'| = \max(0, 4 - |\partial(Z)|)$. We note that $|Z' \cup \partial(Z)| \geq 4$ and $0 \leq |Z'| \leq 3$. Further if $|\partial(Z)| \geq 4$, then $Z' = \emptyset$. Let H_e be the hypergraph obtained from H' by adding the set Z' of new vertices and adding a 4-edge, e, containing four vertices in $Z' \cup \partial(Z)$. Note that H_e may not be linear as the edge e may overlap other edges in H_e. Every $\tau(H_e)$-transversal is a transversal of H'. Further, if $Z' = \emptyset$, then in order to cover the edge e every $\tau(H_e)$-transversal contains a vertex of $\partial(Z)$, while if $Z' \neq \emptyset$, then the edge e contains at least one vertex of $\partial(Z)$ and we can choose a $\tau(H_e)$-transversal to contain such a vertex of $\partial(Z)$. Therefore, $|T'| \leq \tau(H_e)$, where recall that T' is a smallest possible transversal in H' containing a vertex from $\partial(Z)$.

Claim L.2: H_e is not linear and $|\partial(Z)| \geq 2$.

Proof of Claim L.2: Suppose, to the contrary, that H_e is linear. By construction, $n(H_e) \leq n(H)$ and $m(H_e) = m(H') + 1 < m(H)$. Therefore, $n(H_e) + m(H_e) < n(H) + m(H)$. Since H is a counterexample with minimum value of $n(H) + m(H)$, we note that $45\tau(H_e) \leq 6n(H_e) + 13m(H_e) + \mathrm{def}(H_e)$.

Claim L.2.1: $\mathrm{def}(H_e) \geq 9$.

Proof of Claim L.2.1: Suppose, to the contrary, that $\mathrm{def}(H_e) \leq 8$. Recall that $m(H_e) = m(H') + 1$. If $Z' = \emptyset$, then $n(H_e) = n(H')$ and either $|\partial(Z)| = 4$ and $f(|\partial(Z)|) = 23$ or $|\partial(Z)| \geq 5$ and $f(|\partial(Z)|) = 22$. In this case,

$$\begin{array}{rcl} 45|T'| & \leq & 45\tau(H_e) \\ & \leq & 6n(H_e) + 13m(H_e) + \mathrm{def}(H_e) \\ & = & 6n(H') + 13m(H')| + 13 + 8 \\ & \leq & 6n(H') + 13m(H') + f(|\partial(Z)|) - 1, \end{array}$$

a contradiction. If $|Z'| = 1$, then $n(H_e) = n(H') + 1$, $|\partial(Z)| = 3$ and $f(|\partial(Z)|) = 27$, implying that $45|T'| \leq 6n(H') + 13m(H')| + 6 + 13 + 8 = 6n(H') + 13m(H') + f(|\partial(Z)|)$, a contradiction. If $|Z'| = 2$, then $n(H_e) = n(H') + 2$, $|\partial(Z)| = 2$ and $f(|\partial(Z)|) = 33$, implying that $45|T'| \leq 6n(H') + 13m(H')| + 12 + 13 + 8 = 6n(H') + 13m(H') + f(|\partial(Z)|)$, a contradiction. If $|Z'| = 3$, then $n(H_e) = n(H') + 3$, $|\partial(Z)| = 1$ and $f(|\partial(Z)|) = 39$, implying that $45|T'| \leq 6n(H') + 13m(H')| + 18 + 13 + 8 = 6n(H') + 13m(H') + f(|\partial(Z)|)$, a contradiction. This completes the proof of Claim L.2.1. ($\Box$)

By Claim L.2.1, $\mathrm{def}(H_e) \geq 9$. Let Y_e be a special H_e-set in H_e such that $\mathrm{def}_{H_e}(Y_e) = \mathrm{def}(Y_e) \geq 9$. Suppose that $|Y_e| \geq 2$. In this case $|E^*_{H_e}(Y_e)| \leq |Y_e| - 2$, for otherwise

$\text{def}(H_e) < 9$. Let Y'_e be equal to Y_e, except we remove any special subhypergraph from Y_e if it contains the edge e. We note that $|E_{H'}(Y'_e)| \le |E^*_{H_e}(Y_e)| \le |Y_e| - 2 \le |Y'_e| - 1$, contradicting Claim L.1. Therefore, $|Y_e| = 1$. Since $\text{def}(H_e) \ge 9$ and $|Y_e| = 1$, the special H_e-set consists of one H_{10}-component in H_e and $\text{def}(H_e) = 10$. By Claim J, this H_{10}-component, Y_e, in H_e contains the edge e.

If $|\partial(Z)| \le 3$, then $Z' \ne \emptyset$ and the edge e would contain a vertex of degree 1 in H_e, implying that Y_e would contain a vertex of degree 1. However, H_{10} is 2-regular, a contradiction. Hence, $|\partial(Z)| \ge 4$. Thus, $Z' = \emptyset$ and $n(H_e) = n(H')$. If $|\partial(Z)| = 4$, then $f(|\partial(Z)|) = 23$, implying that $45|T'| \le 45\tau(H_e) \le 6n(H_e)+13m(H_e)+\text{def}(H_e) = 6n(H')+13m(H')|+13+10 = 6n(H') + 13m(H') + f(|\partial(Z)|)$, a contradiction. Hence, $|\partial(Z)| \ge 5$, and so $f(|\partial(Z)|) = 22$.

If there is a vertex $v \in \partial(Z) \setminus V(Y_e)$, then we can change e by removing from it an arbitrary vertex and adding to it the vertex v instead. With this new choice of the added edge e, we note that H_e is once again linear but now the component in H_e containing the edge e contains at least 11 vertices, implying that there would be no H_{10}-component in H_e containing the edge e, contradicting our earlier arguments. Therefore, $\partial(Z) \subseteq V(Y_e)$. Since H is connected, this implies that $H' = Y_e - e$. Let $u \in \partial(Z) \setminus V(e)$ be arbitrary. We note that u exists since $|\partial(Z)| \ge 5$ and $|V(e)| = 4$. We note further that there are exactly two edges in $H' - u$ and these two edges intersect in a vertex, say w. Hence, $\{u, w\}$ is a transversal in H' containing a vertex from $\partial(Z)$, implying that $45|T'| \le 90 < 60+52+21 = 6n(H') + 13m(H') + f(|\partial(Z)|) - 1$, a contradiction. Therefore, H_e cannot be linear. This implies that $|\partial(Z)| \ge 2$, which completes the proof of Claim L.2. (□)

Claim L.3: For any non-empty special H'-set, Y, in H' we have $|E^*_{H'}(Y)| \ge |Y| + 1$ or $|\partial(Z) \setminus V(Y)| \le 1$.

Proof of Claim L.3: Suppose, to the contrary, that there is a non-empty special H'-set, Y in H' where $|E^*_{H'}(Y)| \le |Y|$ and $|\partial(Z) \setminus V(Y)| \ge 2$. Assume that $|Y|$ is minimum possible with the above property. By Claim L.1, $|E^*_{H'}(Y)| = |Y|$. By Claim E, we note that $\partial(Z) \cap V(Y) \ne \emptyset$. Let $Z^* = Z \cup V(Y)$ and let $H^* = H - Z^*$. Further, let $Q^* = \partial(Z^*)$, and so $Q^* = N_H(Z^*) \setminus Z^*$. Since $|\partial(Z) \setminus V(Y)| \ge 2$, we have $|Q^*| \ge 2$.

Claim L.3.1: $\text{def}(H^*) \le 21$.

Proof of Claim L.3.1: Suppose, to the contrary, that $\text{def}(H^*) \ge 22$. Let Y^* be a special H^*-set with $\text{def}_{H^*}(Y^*) \ge 22$. By Claim J, there is no H_{10}-subhypergraph in H, implying that $\text{def}_{H^*}(Y^*) \le 8|Y^*| - 13|E^*_{H^*}(Y^*)|$. If $|E^*_{H^*}(Y^*)| \ge |Y^*| - 2$, then $22 \le \text{def}_{H^*}(Y^*) \le 8|Y^*| - 13|Y^*| + 26 = -5|Y^*| + 26$, implying that $|Y^*| = 0$, a contradiction. Thus, $|E^*_{H^*}(Y^*)| \le |Y^*| - 3$. Hence, $|E^*_{H'}(Y \cup Y^*)| \le |E^*_{H'}(Y)| + |E^*_{H^*}(Y^*)| \le |Y| + |Y^*| - 3 < |Y \cup Y^*|$, contradicting Claim L.1. (□)

Let T^* be a minimum transversal in H^* that contains a vertex from Q^*. By the maximality of $|Z|$, we note that $45|T^*| \le 6n(H^*) + 13m(H^*) + f(|Q^*|)$.

Claim L.3.2: $|T'| \le |T^*| + \tau(Y)$.

Proof of Claim L.3.2: By construction of T^*, either $\partial(Z) \cap T^* \ne \emptyset$ or T^* intersects an edge in $E^*_{H'}(Y)$. We now consider the bipartite graph, G_Y, with partite sets Y and $E^*_{H'}(Y)\cup\{q\}$, where an edge joins $e\in E^*_{H'}(Y)$ and $F\in Y$ in G_Y if and only if the edge e intersects the subhypergraph F of Y in H' and q is joined to $F\in Y$ in G_Y if and only if $\partial(Z)\cap V(F)\neq\emptyset$. We now construct G'_Y from G_Y by removing either q if $\partial(Z)\cap T^*\neq\emptyset$ or by removing some $e'\in E^*_{H'}(Y)$ where T^* intersects the edge e'. We note that since $|E^*_{H'}(Y)|=|Y|$, the partite sets in G'_Y have the same size.

Suppose there is no perfect matching in G'_Y. By Hall's Theorem, there is a nonempty subset $S\subseteq V(G'_Y)\setminus Y$ such that $|N_{G'_Y}(S)|<|S|$. We now consider the special H'-set, $Y'=Y\setminus N_{G'_Y}(S)$. Then, $|Y'|=|Y|-|N_{G'_Y}(S)|$. Since possibly $q\in S$, we therefore have $|E^*_{H'}(Y')|\leq|E^*_{H'}(Y)|-(|S|-1)\leq|Y|-|N_{G'_Y}(S)|=|Y'|$. Thus, Y' is a non-empty special H'-set such that $|E^*_{H'}(Y')|\leq|Y'|$ and $|Y'|<|Y|$. Further, since $Y'\subset Y$ and $|\partial(Z)\setminus V(Y)|\geq 2$, we note that $|\partial(Z)\setminus V(Y')|\geq 2$. This contradicts our choice of Y. Therefore, there is a perfect matching in G'_Y, implying that we can find a $\tau(Y)$-transversal that together with T^* intersects all the edges in $E^*_{H'}(Y)$ and intersects $\partial(Z)$. This implies that $|T'|\leq|T^*|+\tau(Y)$, as desired. □

As observed earlier, $45|T^*|\leq 6n(H^*)+13m(H^*)+f(|Q^*|)$. Since Y is a non-empty special H'-set, we note that $45\tau(Y)=6n(Y)+13m(Y)+\mathrm{def}_{H'}(Y)\leq 6n(Y)+13m(Y)+\mathrm{def}_{H'}(Y)+13|E^*_{H'}(Y)|$. By Claim L.3.2, $|T'|\leq|T^*|+\tau(Y)$. Therefore,

$$\begin{aligned}45|T'| &\leq 45|T^*|+45\tau(Y)\\ &\leq 6n(H^*)+13m(H^*)+f(|Q^*|)+6n(Y)+13m(Y)+13|E^*_{H'}(Y)|+\mathrm{def}_{H'}(Y)\\ &= 6n(H')+13m(H')+f(|Q^*|)+\mathrm{def}_{H'}(Y).\end{aligned}$$

As observed earlier, $|E^*_{H'}(Y)|=|Y|$, implying that $\mathrm{def}_{H'}(Y)<0$. In fact, by Claim J, $\mathrm{def}_{H'}(Y)=8|Y_4|+5|Y_{14}|+4|Y_{11}|+|Y_{21}|-13|Y|=-5|Y_4|-8|Y_{14}|-9|Y_{11}|-12|Y_{21}|$. Since $|Q^*|\geq 2$, we note that $f(|Q^*|)\leq 33$.

Suppose that $|Y|\geq 2$. In this case, $\mathrm{def}_{H'}(Y)\leq -10$, implying that $f(|Q^*|)+\mathrm{def}_{H'}(Y)\leq 33-10=22\leq f(|\partial(Z)|)$. By supposition, the transversal T' in H' does not satisfy (a) or (b) in the statement of the claim. Hence we immediately obtain a contradiction unless $f(|Q^*|)=33$, $\mathrm{def}_{H'}(Y)=-10$ and $|\partial(Z)|\geq 5$. This in turn implies that $|Y|=2$, $|Y_4|=2$ and $|Q^*|=2$. However if $|Y|=|Y_4|=2$ and $|E_{H'}(Y)|=2$, then by the linearity of H the two edges intersecting Y both have two vertices not in Y, implying that $|Q^*|\geq 3$, a contradiction to $|Q^*|=2$. Therefore, $|Y|=1$. Let $E^*_{H'}(Y)=\{f\}$. Recall that by supposition, $|\partial(Z)\setminus V(Y)|\geq 2$.

Suppose that $|V(f)\cap V(Y)|\geq 2$. By the linearity of H we note that $Y_4=\emptyset$. Thus, $\mathrm{def}_{H'}(Y)=-8|Y_{14}|-9|Y_{11}|-12|Y_{21}|\leq -8$, and so $45\tau(Y)=6n(Y)+13m(Y)+\mathrm{def}_{H'}(Y)\leq 6n(Y)+13m(Y)-8$. By Observation 1(k) we can find a $\tau(Y)$-transversal, T_Y, that contains a vertex in $\partial(Z)$ and a vertex in f. Let H'' be the hypergraph obtained from H' by removing the vertices $V(Y)$ and edges $E(Y)\cup E^*_{H'}(Y)$; that is, $H''=H'-T_Y$. If $\mathrm{def}(H'')>0$, then letting Y'' be a special H''-set with $\mathrm{def}_{H''}(Y'')>0$ we note that $|E^*_{H'}(Y\cup Y'')|\leq|E^*_{H'}(Y)|+|E^*_{H''}(Y'')|\leq 1+(|Y''|-1)<|Y''\cup Y|$, contradicting Claim L.1. Therefore, $\mathrm{def}(H'')=0$, and so $45\tau(H'')\leq 6n(H'')+13m(H'')+\mathrm{def}(H'')=6n(H'')+13m(H'')$.

Hence,

$$\begin{aligned} 45|T'| &\le 45|T_Y| + 45\tau(H'') \\ &\le (6n(Y)+13m(Y)-8) + (6n(H'')+13m(H'')) \\ &= 6n(H') + 13(m(H')-1) - 8 \\ &< 6n(H') + 13m(H') - 8, \end{aligned}$$

contradicting the supposition that the transversal T' in H' does not satisfy (a) or (b) in the statement of the claim. Therefore, $|V(f)\cap V(Y)| = 1$, implying that $|Q^*| \ge 3$. Recall that

$$45|T'| \le 6n(H') + 13m(H') + f(|Q^*|) + \mathrm{def}_{H'}(Y).$$

If $Y_4 = \emptyset$, then $\mathrm{def}_{H'}(Y) \le -8$, and so $f(|Q^*|) + \mathrm{def}_{H'}(Y) \le 27 - 8 = 19$. If $Y_4 \ne \emptyset$ and $|Q^*| \ge 4$, then $f(|Q^*|)+\mathrm{def}_{H'}(Y) \le 23-5 = 18$. In both cases, $45|T'| \le 6n(H')+13m(H')+f(|Q^*|)+\mathrm{def}_{H'}(Y) \le 6n(H')+13m(H')+f(|\partial(Z)|)-1$, a contradiction. Therefore, $|Y| = |Y_4| = 1$ and $|Q^*| = 3$, implying that $f(|Q^*|)+\mathrm{def}_{H'}(Y) = 27-5 = 22$ and $45|T'| \le 6n(H')+13m(H')+22 \le 6n(H')+13m(H')+f(|\partial(Z)|)$. This proves part (a) of Claim L.

In order to prove part (b) of Claim L, we consider next the case when $|\partial(Z)| \ge 5$. Since $|Q^*| = 3$, we note that $\partial(Z) \subseteq V(Y) \cup V(f)$. Let $E(Y) = \{e'\}$. By our earlier observations, e' and f intersect in exactly one vertex, say v'. Further, the edge e' contains three degree-1 vertices in H'. If v' belongs to $\partial(Z)$, then we can find a $\tau(Y)$-transversal that contains a vertex in $\partial(Z)$ and a vertex in f, which analogously to our previous arguments gives us a contradiction. Therefore, $V(Y)\cap V(f)\cap\partial(Z) = \emptyset$, implying that $\partial(Z) \subseteq (V(e')\cup V(f))\setminus\{v'\}$. Since $|\partial(Z)| \ge 5$, this implies that $|\partial(Z)\cap V(e')| \ge 2$ and $|\partial(Z)\cap V(f)| \ge 2$. Hence, $|\partial(Z)| \ge 5$ and H contains two intersecting edges e and f such that (i), (ii) and (iii) hold in the statement of part (b) in Claim L. Hence, from our earlier observations, part (b) of Claim L holds. This completes the proof of Claim L.3. (□)

We now return to the proof of Claim L. By Claim L.2 there exists an edge f in H' that overlaps the edge e in H_e. Let $\{u,v\} \subseteq V(e)\cap V(f)$ be arbitrary. Let H_{ef} be obtained from $H-e-f$ by adding two new vertices z_1 and z_2 and a new edge $g = \{u,v,z_1,z_2\}$. Since H is linear, so too is H_{ef} is linear. We show that $\mathrm{def}(H_{ef}) \le 9$. Suppose, to the contrary, that $\mathrm{def}(H_{ef}) > 9$ and let Y_{ef} be a special H_{ef}-set with $\mathrm{def}_{H_{ef}}(Y_{ef}) > 9$. By Claim J, $(Y_{ef})_{10} = \emptyset$, implying that $|E^*_{H_{ef}}(Y_{ef})| \le |Y_{ef}| - 2$. If $g \notin E(Y_{ef})$, then

$$|E_{H'}(Y_{ef})| \le |E_{H_{ef}}(Y_{ef})| + |\{f\}| \le (|Y_{ef}|-2) + 1 < |Y_{ef}|,$$

contradicting Claim L.1. Therefore, $g \in E(Y_{ef})$. Since g has two degree-1 vertices in H_{ef}, we note that g is the edge of a H_4-hypergraph, R_g, in Y_{ef}. In this case,

$$|E^*_{H'}(Y_{ef}\setminus\{R_g\})| \le |E^*_{H_{ef}}(Y_{ef})| + |\{f\}| \le (|Y_{ef}|-2) + 1 = |Y_{ef}\setminus\{R_g\}|.$$

As $\{u,v\} \subseteq \partial(Z)$ and u and v do not belong to $Y_{ef}\setminus\{R_g\}$ (as they belong to $V(R_g)$) we obtain a contradiction to Claim L.3. Therefore, $\mathrm{def}(H_{ef}) \le 9$. Thus, by the minimality of

$n(H) + m(H)$,

$$\begin{array}{rcl} 45|T'| \le 45\tau(H_{ef}) & \le & 6n(H_{ef}) + 13m(H_{ef}) + \mathrm{def}(H_{ef}) \\ & \le & 6(n(H') + 2) + 13m(H') + \mathrm{def}(H_{ef}) \\ & = & 6n(H') + 13m(H') + 12 + \mathrm{def}(H_{ef}) \\ & \le & 6n(H') + 13m(H') + 21 \\ & \le & 6n(H') + 13m(H') + f(|\partial(Z)|) - 1, \end{array}$$

a contradiction. This completes the proof of Claim L. ($\Box$)

We call a component of a 4-uniform, linear hypergraph that contains two vertex disjoint copies of H_4 that are both intersected by a common edge and such that each copy of H_4 has three vertices of degree 1 and one vertex of degree 2 a *double-H_4-component*. We call these two copies of H_4 the *H_4-pair* of the double-H_4-component, and the edge that intersects them the *linking edge*. We note that a double-H_4-component contains at least ten vertices, namely eight vertices from the H_4-pair and at least two additional vertices that belong to the linking edge.

Claim M: If x is an arbitrary vertex of H of degree 3, then one of the following holds.

(a) $\mathrm{def}(H-x) = 8$ and the hypergraph $H-x$ contains an H_4-component that is intersected by all three edges incident with x.

(b) $\mathrm{def}(H-x) = 3$ and the hypergraph $H-x$ contains a double-H_4-component. Further, the H_4-pair in this component is intersected by all three edges incident with x.

Proof of Claim M: Let x be an arbitrary vertex of H of degree 3 and let e_1, e_2 and e_3 be the three edges incident with x. By the linearity of H, the vertex x is the only common vertex in e_i and e_j for $1 \le i < j \le 3$. Let $H' = H - x$. We note that $n(H') = n(H) - 1$ and $m(H') = m(H) - 3$. Every transversal in H' can be extended to a transversal in H by adding to it the vertex x. Hence, applying the inductive hypothesis to H', we have that $45\tau(H) \le 45(\tau(H')+1) \le 6n(H')+13m(H')+\mathrm{def}(H')+45 \le 6n(H)+13m(H)+\mathrm{def}(H')$. If $\mathrm{def}(H') = 0$, then $45\tau(H) \le 6n(H)+13m(H)$, implying that $\xi(H) \le 0$, contradicting the fact that H is a counterexample to the theorem. Hence, $\mathrm{def}(H') > 0$. Let X be a special H'-set satisfying $\mathrm{def}(H') = \mathrm{def}_{H'}(X)$.

Claim M.1: $|E^*_{H'}(X)| = |X| - 1$. Further, all three edges incident with the vertex x intersect X.

Proof of Claim M.1: Since $\mathrm{def}(H') > 0$, we note that $|E^*_{H'}(X)| \le |X| - 1$. By Claim I and since $E^*_H(X) \setminus E^*_{H'}(X) \subseteq \{e_1, e_2, e_3\}$, we note that $|X| + 2 \le |E^*_H(X)| \le |E^*_{H'}(X)| + 3$, implying that $|E^*_{H'}(X)| \ge |X| - 1$. Consequently, $|E^*_{H'}(X)| = |X| - 1$ and all three edges incident with the vertex x intersect X. ($\Box$)

Claim M.2: $|X| = 1$ or $|X| = 2$ and $X = X_4$.

Proof of Claim M.2: By Claim M.1, $|E^*_{H'}(X)| = |X| - 1$ and all three edges incident with the vertex x intersect X. By Claim J, there is no H_{10}-subhypergraph in H. Thus,

$$\begin{aligned}
\mathrm{def}_{H'}(X) &= 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| - 13|E^*_{H'}(X)| \\
&= 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| - 13(|X| - 1) \\
&= 13 - 5|X_4| - 8|X_{14}| - 9|X_{11}| - 12|X_{21}|.
\end{aligned}$$

Since $\mathrm{def}_{H'}(X) > 0$, either $|X| = 1$ or $|X| = 2$ and $X = X_4$. (□)

Claim M.3: If $|X| = 2$, then $H - x$ contains a double-H_4-component and $\mathrm{def}(H') = 3$.

Proof of Claim M.3: Suppose that $|X| = 2$. By Claim M.2, $X = X_4$, and so $\mathrm{def}_{H'}(X) = 3$. By Claim M.1, $|E^*_{H'}(X)| = 1$. Let e be the edge in $E^*_{H'}(X)$. If e intersects only one of the copies of H_4 in X, then the other copy of H_4 is an H_4-component in $H - x$ implying that $\mathrm{def}(H') \geq 8$, contradicting the fact that $\mathrm{def}(H') = \mathrm{def}_{H'}(X) = 3$. Therefore, e intersects both copies of H_4 in X. By the linearity of H the edge e intersects each copy in one vertex, implying that $H - x$ contains a double-H_4-component. (□)

Claim M.4: If $|X| = 1$, then $H - x$ contains an H_4-component and $\mathrm{def}(H') = 8$.

Proof of Claim M.4: Suppose that $|X| = 1$, and let R be the special subhypergraph in X. By Claim M.1 we note that $|E^*_{H'}(X)| = 0$, and so R is a component of $H - x$. Suppose, to the contrary, that $|X_4| = 0$. Among all degree-3 vertices, we choose the vertex x so that $X = X_j$ where j is a maximum; that is, X is chosen to contain a special subhypergraph of maximum possible size j. By supposition, $j \neq 4$, and so $j \in \{11, 14, 21\}$.

Suppose that each edge incident with x intersects R in at most two vertices. By Claim M.1, all three edges incident with x intersect R in at least one vertex, implying that in this case at least three neighbors of x in H belong to R. Since every special subhypergraph different from H_4 contains at most two vertices of degree 1, we can choose a neighbor y of x in R such that $d_{H'}(y) \geq 2$. Since every neighbor of x in R has degree at most 2 in R since H has maximum degree 3, this implies that $d_{H'}(y) = 2$, and so $d_H(y) = 3$. This implies by Observation 1(p) that $H' - y$ is connected or is disconnected with exactly two components, one of which consists of an isolated vertex. In both cases, since all three edges incident with x intersect R we note that either $H - y$ is connected and has cardinality at least $n(R) + 3$ or $H - y$ is disconnected with two components, one of which consists of an isolated vertex with the other component of cardinality at least $n(R) + 2$. Analogously as with the vertex x, there is a special $(H - y)$-set, Y, with $\mathrm{def}(H - y) = \mathrm{def}_{H-y}(Y)$ satisfying $|Y| = 1$ or $|Y| = 2$ and $Y = Y_4$.

If $|Y| = 1$, then by our earlier observations, Y consists of a special subhypergraph of cardinality at least $n(R) + 2$, contradicting the maximality of R. Hence, $|Y| = 2$ and $Y = Y_4$. Analogously as with the vertex x (see Claim M.3), $H - y$ contains a double-H_4-component. Let R_1 and R_2 be the H_4-pair of this double-H_4-component, and let h^* be the linking edge that intersects them. We note that $Y = \{R_1, R_2\}$. By Observation 1(p), $R - y$ does not contain a double-H_4-component. Further, $R - y$ does not contain a component of order 4.

Suppose that the edge of R_1 belongs to R. If the edge h^* does not belong to R, then R_1 is a component in $R - y$ (of order 4), a contradiction. Hence, the edge h^* also belongs to

R. If the edge in R_2 belongs to R, then $R - y$ would contain a double-H_4-component, a contradiction. Hence, the edge in R_2 does not belong to R and must therefore contain the vertex x. However in this case R_2 is intersected by at least two edges in $R - y$, namely h^* and the edge incident with x that intersects R_2. Therefore, the pair R_1 and R_2 is not the H_4-pair of a double-H_4-component in $H - y$, a contradiction. Therefore, the edge in R_1 does not belong to R. Analogously, the edge in R_2 does not belong to R.

Since the H_4-pair, R_1 and R_2, in this double-H_4-component is intersected by all three edges incident with y, we note that both R_1 and R_2 intersect $V(R)$. However as they do not belong to R, they must both contain the vertex x, which implies that they are not vertex disjoint, a contradiction. Therefore, at least one edge incident with x intersects R in three vertices.

Renaming edges if necessary, we may assume that the edge e_1 intersects R in three vertices. From the structure of special subhypergraphs, this implies that we can choose a vertex $y \in V(e_1) \cap V(R)$ so that $R - y$ is connected (of cardinality $n(R) - 1$). If at least one neighbor of x does not belong to R, then analogously as before $H - y$ is a connected hypergraph of cardinality at least $n(R) + 1$, contradicting the maximality of R. Hence, letting $T_i = V(e_i) \cap V(R)$, we note that $|T_i| = 3$ for all $i \in [3]$. Further, by the linearity of H, the sets T_1, T_2 and T_3 are vertex disjoint independent sets in R. By Observation 1(n) there exists a $\tau(R)$-transversal, T^* say, that contains a vertex from each of the sets T_i for $i \in [3]$. Since H is connected and all neighbors of x belong to R, we note that $H' = R$, $V(H) = V(R) \cup \{x\}$ and $E(H) = E(R) \cup \{e_1, e_2, e_3\}$. Thus, T^* is a transversal of H, implying that $45\tau(H) \le 45|T^*| = 45\tau(H') \le 6n(H') + 13m(H') + \mathrm{def}(H') = 6n(H) + 13m(H) + \mathrm{def}(H') - 45$. By Claim M.2, $\mathrm{def}(H') \le 8$, and so $45\tau(H) < 6n(H) + 13m(H)$, a contradiction. Therefore, $X = X_4$, implying that $R = H_4$ is an H_4-component of $H - x$ and $\mathrm{def}(H') = 13 - 5|X_4| = 8$. (□)

By Claims M.2, M.3 and M.4, $H - x$ contains an H_4-component or a double-H_4-component. This completes the proof of Claim M. (□)

Claim N: No edge in H contains two degree-1 vertices of H.

Proof of Claim N: Suppose, to the contrary, that $e = \{v_1, v_2, v_3, v_4\}$ is an edge in H, where $1 = d_H(v_1) = d_H(v_2) \le d_H(v_3) \le d_H(v_4)$. Suppose that $d_H(v_4) = 3$. Let H' be obtained from $H - v_4$ by deleting all resulting isolated vertices (including v_1 and v_2). By Claim M, $\mathrm{def}(H') \le 8$. We note that $n(H') \le n(H) - 3$ and $m(H') = m(H) - 3$. Applying the inductive hypothesis to H', we have that $45\tau(H) \le 45(\tau(H')+1) \le 6n(H')+13m(H')+\mathrm{def}(H') + 45 \le 6n(H) - 3 \times 6 + 13m(H) - 3 \times 13 + \mathrm{def}(H') + 45 \le 6n(H) + 13m(H) - 18 - 39 + 8 + 45 < 6n(H) + 13m(H)$, a contradiction. Therefore, $d_H(v_4) \le 2$, implying that at most two edges of H intersect the edge e. Taking Y to the special H-set consisting only of the special subhypergraph H_4 with e as its edge, we have $|E^*(Y)| \le 2 = |Y| + 1$, contradicting Claim I(a). (□)

Claim O: Let H' be a 4-uniform, linear hypergraph with no H_{10}-subhypergraph satisfying $\mathrm{def}(H') > 0$. If Y is a special H'-set satisfying $\mathrm{def}(H') = \mathrm{def}_{H'}(Y)$, then $|E^*_{H'}(Y)| = |Y| - i$

for some $i \geq 1$ and

$$\mathrm{def}(H') \leq 13i - 5|Y_4| - 8|Y_{14}| - 9|Y_{11}| - 12|Y_{21}|.$$

In particular, $\mathrm{def}(H') \leq 13i - 5|Y|$. Further, if $\mathrm{def}(H') > 8j$ for some $j \geq 0$, then $|E^*_{H'}(Y)| \leq |Y| - (j+1)$.

Proof of Claim O: Since $\mathrm{def}_{H'}(Y) > 0$, we note that $|E^*_{H'}(Y)| = |Y| - i$ for some $i \geq 1$. Thus, $\mathrm{def}_{H'}(Y) = 8|Y_4| + 5|Y_{14}| + 4|Y_{11}| + |Y_{21}| - 13|E^*_{H'}(Y)| = 8|Y_4| + 5|Y_{14}| + 4|Y_{11}| + |Y_{21}| - 13(|Y| - i) = 13i - 5|Y_4| - 8|Y_{14}| - 9|Y_{11}| - 12|Y_{21}|$, and so $\mathrm{def}_{H'}(Y) \leq 13i - 5|Y|$.

Further, let $\mathrm{def}(H') > 8j$ for some $j \geq 0$ and suppose, to the contrary, that $|E^*_{H'}(Y)| \geq |Y| - j$. Hence, $|E^*_{H'}(Y)| = |Y| - k$ where $k \leq j$. Therefore, $\mathrm{def}(H') \leq 13k - 5|Y| \leq 13k - 5k = 8k \leq 8j$, a contradiction. Therefore, $|E^*_{H'}(Y)| \leq |Y| - (j+1)$. $(\Box)$

We show next that the removal of any specified vertex of degree 3 from H produces an H_4-component. In what follows, we use the following notation for simplicity. If e and f are intersecting edges of H, then we denote the vertex in this intersection by (ef); that is, $e \cap f = \{(ef)\}$.

Claim P: If x is an arbitrary vertex of H of degree 3, then $H - x$ contains an H_4-component.

Proof of Claim P: Let x be an arbitrary vertex of H of degree 3. Let e_1, e_2 and e_3 be the three edges incident with x and let $H' = H - x$. By the linearity of H, the vertex x is the only common vertex in e_i and e_j for $1 \leq i < j \leq 3$. Suppose, to the contrary, that H' does not contain an H_4-component. By Claim M, H' therefore contains a double-H_4-component, say C', and $\mathrm{def}(H') = 3$. Let f_1 and f_2 be the two edges belonging to the H_4-pair in C' and let h be the linking edge in H' that intersects f_1 and f_2. By definition of a double-H_4-component, we note that f_i has three vertices of degree 1 and one vertex of degree 2 in H' for $i \in [2]$. We proceed further with the following series of subclaims.

Claim P.1: $2 \leq |f_i \cap N_H(x)| \leq 3$ for $i \in [2]$.

Proof of Claim P.1: By the linearity of H, $|f_i \cap e_j| \leq 1$ for every $j \in [3]$, and so $|f_i \cap N_H(x)| \leq 3$. If $|f_i \cap N_H(x)| \leq 1$, then the edge f_i contains at least two vertices of degree 1 in H, contradicting Claim N. $(\Box)$

Claim P.2: Any two edges amongst e_1, e_2 and e_3 can be covered by two vertices one of which belongs to f_1 and the other to f_2.

Proof of Claim P.2: Renaming edges if necessary, it suffices to consider the two edges e_1 and e_2. By Claim P.1, the edge f_1 intersects at least one of e_1 and e_2, say e_1. If f_2 intersects e_2, then we can cover e_1 by the vertex (e_1f_1) and we can cover e_2 by the vertex (e_2f_2). If f_2 does not intersect e_2, then by Claim M and Claim P.1, the edge f_1 intersects e_2 and the edge f_2 intersects e_1, and we can therefore cover e_1 by the vertex (e_1f_2) and we can cover e_2 by the vertex (e_2f_1). $(\Box)$

Let $h \setminus (f_1 \cup f_2) = \{a, b\}$, and let $h \cap f_1 = \{c\}$ and $h \cap f_2 = \{d\}$. Thus, $h = \{a, b, c, d\}$.

By definition of a double-H_4-component, we note that $d_H(c) = d_{H'}(c) = 2$ and $d_H(d) = d_{H'}(d) = 2$.

Claim P.3: $d_{H'}(a) \ge 2$ and $d_{H'}(b) \ge 2$.

Proof of Claim P.3: Suppose, to the contrary, that $d_{H'}(a) = 1$ or $d_{H'}(b) = 1$. Interchanging the names of a and b if necessary, we may assume that $d_{H'}(a) = 1$. Let H'' be obtained from $H - \{x, c, d\}$ by deleting all resulting isolated vertices (including the vertex a, as well as three vertices of degree 1 in each of f_1 and f_2). We note that $n(H'') \le n(H') - 10$. Further, the edges e_1, e_2, e_3, f_1, f_2 and h are deleted from H when constructing H'', and so $m(H'') = m(H) - 6$. Applying the inductive hypothesis to H'', we have that $45\tau(H) \le 45(\tau(H'') + 3) \le 6n(H'') + 13m(H'') + \mathrm{def}(H'') + 135 \le 6n(H) - 6 \times 10 + 13m(H) - 13 \times 6 + \mathrm{def}(H'') + 135 = 6n(H) + 13m(H) + \mathrm{def}(H'') - 3$. If $\mathrm{def}(H'') \le 3$, then $45\tau(H) \le 6n(H) + 13m(H)$, a contradiction. Hence, $\mathrm{def}(H'') \ge 4$. Let Y'' be a special H''-set satisfying $\mathrm{def}_{H''}(Y) = \mathrm{def}(H'') > 0$. By Claim O, $|E^*_{H''}(Y'')| \le |Y''| - 1$. Let Y_i be the H_4-subhypergraph of H with $E(Y_i) = \{f_i\}$ for $i \in [2]$, and consider the special H-set $Y = Y'' \cup \{Y_1, Y_2\}$. We note that $|E^*_H(Y)| \le |E^*_{H''}(Y'')| + |\{e_1, e_2, e_3, h\}| \le (|Y''| - 1) + 4 = |Y''| + 3 = |Y| + 1$, contradicting Claim I(a). (□)

As observed earlier, the edge f_i intersects the edge e_j in at most one vertex for every $i \in [2]$ and $j \in [3]$. For $j \in [3]$, let s_j be a vertex different from x that belongs to e_j but not to $f_1 \cup f_2$.

Claim P.4: The following holds.
(a) $d_H(s_i) \ge 2$ for all $i \in [3]$.
(b) If $d_H(s_i) = 3$ for some $i \in [3]$, then $e_i \cap f_1 = \emptyset$ or $e_i \cap f_2 = \emptyset$.
(c) $d_H(s_i) = 2$ for at least one $i \in [3]$.

Proof of Claim P.4: (a) Suppose, to the contrary, that the vertex s_i has degree 1 in H for some $i \in [3]$, and so e_i is the only edge in H containing s_i. Letting $H'' = H - \{x, s_i\}$, we note that $n(H'') = n(H) - 2$, $m(H') = m(H) - 3$ and $\mathrm{def}(H'') = \mathrm{def}(H') = 3$. Every transversal in H'' can be extended to a transversal in H by adding to it the vertex x. Hence, applying the inductive hypothesis to H'', we have that $45\tau(H) \le 45(\tau(H'')+1) \le 6n(H'') + 13m(H'') + \mathrm{def}(H'') + 45 \le 6n(H) + 3m(H) - 6 \cdot 2 - 13 \cdot 3 + 3 + 45 = 6n(H) + 13m(H) - 3$, a contradiction. Therefore, $d_H(s_i) \ge 2$ for all $i \in [3]$.

(b) Let $d_H(s_i) = 3$ for some $i \in [3]$, and suppose, to the contrary, both edges f_1 and f_2 intersect e_i. Renaming indices if necessary, we may assume $i = 1$. By Claim M, $H - s_1$ either contains an H_4-component that is intersected by the edge e_1 or a double-H_4-component in which the H_4-pair is intersected by the edge e_1. However such a component of $H - s_1$ would contain the vertex x and the four edges e_2, e_3, f_1, f_2, which is not possible.

(c) Suppose that $d_H(s_1) = d_H(s_2) = d_H(s_3) = 3$. By Claim P.1, $e_i \cap f_1 \neq \emptyset$ and $e_i \cap f_2 \neq \emptyset$ for some $i \in [3]$. This, however, contradicts part (b) above. (□)

We show next that the edge h contains no neighbor of x.

Claim P.5: $N_H(x) \cap V(h) = \emptyset$.

Proof of Claim P.5: Suppose, to the contrary, that $y \in N_H(x) \cap V(h)$.

Claim P.5.1: $y \notin f_1 \cup f_2$.

Proof of Claim P.5: Suppose, to the contrary, that $y \in f_1 \cup f_2$. Renaming edges if necessary, we may assume that $y \in f_1 \cap e_3$. Since y belongs to the three edges e_3, f_1, h, we note that $d_H(y) = 3$. Further since $f_1 \cap h = \{y\}$, we note that the edge f_1 contains a vertex of degree 1 in H that is not a neighbor of x. By Claim N, no edge contains two degree-1 vertices, implying that in this case f_1 intersects all three edges e_1, e_2 and e_3. Let $v_i = f_1 \cap e_i$ for $i \in [3]$. By Claim M, $H - y$ contains an H_4-component or a double-H_4-component.

If $H - y$ contains an H_4-component R, then by Claim M the component R intersects all three edges incident with y. In particular, R intersects the edge f_1 and therefore contains the vertex v_1 or v_2. In both cases, the component R contains the vertex x and both edges e_1 and e_2, a contradiction.

Hence, $H - y$ contains a double-H_4-component R'. By Claim M, all three edges incident with y intersect the H_4-pair in R'. In particular, the edge f_1 intersects the H_4-pair in R', implying that e_1 or e_2 belongs to the H_4-pair in R'. In both cases, the component R' contains the vertex x and both edges e_1 and e_2. Renaming e_1 and e_2 if necessary, we may assume that e_1 belongs to the H_4-pair in R', and so e_1 contains three vertices of degree 1 and one vertex, namely x, of degree 2 in $H - y$. Let $e_1 = \{x, v_1, u, v\}$. We note that neither u nor v belong to f_1 or e_3, and at most one of u and v belong to h. Thus, at least one of u and v, say v, has degree 1 in H. Thus, v is isolated in $H' = H - x$. Hence, letting $H'' = H - \{x, y\}$, we note that $n(H'') = n(H) - 2$, $m(H') = m(H) - 3$ and $\mathrm{def}(H'') = \mathrm{def}(H') = 3$. Every transversal in H'' can be extended to a transversal in H by adding to it the vertex x. Hence, applying the inductive hypothesis to H'', we have that $45\tau(H) \le 45(\tau(H'')+1) \le 6n(H'') + 13m(H'') + \mathrm{def}(H'') + 45 \le 6n(H) - 6 \cdot 2 + 13m(H) - 13 \cdot 3 + \mathrm{def}(H') + 45 = 6n(H) + 13m(H) - 3$, a contradiction. (□)

By Claim P.5.1, $y \notin f_1 \cup f_2$. Thus, $y \in \{a, d\}$. Recall that $y \in N_H(x)$, and so y is incident with e_1, e_2 or e_3. Thus, by Claim P.3, $d_H(y) = d_{H'}(y) + 1 \ge 3$. Consequently, $d_H(y) = 3$. Renaming edges if necessary, we may assume that y is incident with e_1. Let g be the third edge (different from e_1 and h) incident with y. By Claim P.2, the edges e_2 and e_3 can be covered by two vertices one of which belongs to f_1 and the other to f_2. Thus, $e_2 \cap f_i \ne \emptyset$ and $e_3 \cap f_{3-i} \ne \emptyset$ for some $i \in [2]$. Let $Z = f_1 \cup f_2 \cup \{x, y\}$ and let $H^* = H - Z$. We note that $n(H^*) = n(H) - 10$. The edges $e_1, e_2, e_3, f_1, f_2, g, h$ are deleted from H when constructing H^*, and so $m(H^*) = m(H) - 7$.

Every transversal in H^* can be extended to a transversal in H by adding to it the vertices y, $(e_2 f_i)$ and $(e_3 f_{3-i})$. Hence, applying the inductive hypothesis to H^*, we have that $45\tau(H) \le 45(\tau(H^*)+3) \le 6n(H^*) + 13m(H^*) + \mathrm{def}(H^*) + 45 \le 6n(H) + 13m(H) - 6 \cdot 10 - 13 \cdot 7 + \mathrm{def}(H^*) + 45 \times 3 = 6n(H) + 13m(H) + \mathrm{def}(H^*) + 16$. If $\mathrm{def}(H^*) \le 16$, then $45\tau(H) \le 6n(H) + 13m(H)$, a contradiction. Hence, $\mathrm{def}(H^*) \ge 17$. Recall that $H' = H - x$ and note that H^* is obtained from H' by deleting the edges f_1, f_2, g, h and the resulting isolated vertex y. We now consider the hypergraph H'' obtained from H' by deleting the edges f_1, f_2, h; that is, $H'' = H' - f_1 - f_2 - h$. We note that $\mathrm{def}(H'') \ge \mathrm{def}(H^*) - 13 \ge 4$

since the edge g, which contributes at most 13 to $\mathrm{def}(H^*)$ has been added back to H^* (along with the vertex y). Let Y'' be a special H''-set satisfying $\mathrm{def}_{H''}(Y'') = \mathrm{def}(H'')$. Let Y_i be the H_4-subhypergraph of H with $E(Y_i) = \{f_i\}$ for $i \in [2]$, and consider the special H'-set $Y = Y'' \cup \{Y_1, Y_2\}$. We note that each of Y_1 and Y_2 contribute 8 to $\mathrm{def}_{H'}(Y)$, and the edge h contributes -13 to $\mathrm{def}_{H'}(Y)$, implying that $\mathrm{def}_{H'}(Y) = \mathrm{def}_{H''}(Y'') + 8 + 8 - 13 \geq 4 + 8 + 8 - 13 = 7$, contradicting the fact that $\mathrm{def}(H') = 3$. This completes the proof of Claim P.5. (□)

For $i \in [3]$, let s_i be a vertex in $e_i \setminus \{x\}$ that does not belong to $f_1 \cup f_2$. By Claim P.5, the edge h contains no neighbor of x in H. In particular, $h \cap \{s_1, s_2, s_3\} = \emptyset$. Further, we note by Claim P.5 that every vertex that belongs to the edge h has the same degree in H and in H', that is, $d_H(v) = d_{H'}(v)$ for all $v \in \{a, b, c, d\} = V(h)$. Recall that $d_H(c) = d_H(d) = 2$ and by Claim P.3, $d_H(a) \geq 2$ and $d_H(b) \geq 2$.

Claim P.6: $d_H(a) = 2$ and $d_H(b) = 2$.

Proof of Claim P.6: Suppose, to the contrary, that $d_H(a) = 3$ or $d_H(b) = 3$. Interchanging the names of a and b if necessary, we may assume that $d_H(a) = 3$. Let h, h_1 and h_2 be the three edges incident with a. By Claim M, $\mathrm{def}(H - a) = 3$ or $\mathrm{def}(H - a) = 8$. We consider the two possibilities in turn, and show that neither case can occur.

Claim P.6.1: The case $\mathrm{def}(H - a) = 3$ cannot occur.

Proof of Claim P.6.1: Suppose that $\mathrm{def}(H - a) = 3$. By Claim M, $H - a$ therefore contains a double-H_4-component, say C^*. Let g_1 and g_2 be the two edges belonging to the H_4-pair in C^* and let h^* be the linking edge in $H - a$ that intersects g_1 and g_2. By definition of a double-H_4-component, we note that g_i has three vertices of degree 1 and one vertex of degree 2 in $H - a$ for $i \in [2]$. By Claim M, the edges g_1 and g_2 combined are intersected by all three edges h, h_1, h_2 incident with a. Analogously as in the proof of Claim P.1, $2 \leq |g_i \cap N_H(a)| \leq 3$ for $i \in [2]$. By Claim P.1 and P.5, we note that the edges g_1 and g_2 are distinct from the edges f_1 and f_2. Further, we note that the vertices c and d both have degree 1 in $H - a$ and belong to the edges f_1 and f_2, respectively, in $H - a$. Thus, neither g_1 nor g_2 contain the vertex c or d, while at least one of g_1 and g_2 contains the vertex b. Renaming g_1 and g_2 if necessary, we may assume that the vertex b belongs to the edge g_1. Thus, b has degree 1 in $H - a$.

Let $Z = f_1 \cup f_2 \cup g_1 \cup g_2 \cup \{x, a\}$ and let $H^* = H - Z$. We note that $n(H^*) = n(H) - 18$. The edges $e_1, e_2, e_3, f_1, f_2, g_1, g_2, h, h_1, h_2, h^*$ are deleted from H when constructing H^*, and so $m(H^*) = m(H) - 11$.

We show firstly that $\mathrm{def}(H^*) \leq 21$. Suppose, to the contrary, that $\mathrm{def}(H^*) \geq 22$. Let Y^* be a special H^*-set satisfying $\mathrm{def}_{H^*}(Y^*) = \mathrm{def}(H^*)$. By Claim O, $|E^*_{H^*}(Y^*)| \leq |Y^*| - 3$. Let Y_1, Y_2, Y_3, Y_4 be the H_4-subhypergraph of H with edge set f_1, f_2, g_1, g_2, respectively, and consider the special H-set $Y = Y^* \cup \{Y_1, Y_2, Y_3, Y_4\}$. We note that $|E^*_H(Y)| \leq |E^*_{H^*}(Y^*)| + |\{e_1, e_2, e_3, h, h_1, h_2, h^*\}| \leq (|Y^*| - 3) + 7 = |Y^*| + 4 = |Y|$, contradicting Claim I(a). Therefore, $\mathrm{def}(H^*) \leq 21$.

We show next that every transversal in H^* that contains a vertex in $\partial(Z)$ can be extended to a transversal in H by adding to it five vertices. We note that if $\partial(Z)$ contains a vertex from some edge e of H, then $e \in \{e_1, e_2, e_3, h_1, h_2, h^*\}$. Let T^* be a transversal in H^* that contains a vertex in $\partial(Z)$. Suppose that T^* contains a vertex in e_1, e_2 or e_3. Renaming edges if necessary, we may assume T^* contains a vertex in e_1. By Claim P.2, the edges e_2 and e_3 can be covered by two vertices one of which belongs to f_1 and the other to f_2. Thus, $e_2 \cap f_i \neq \emptyset$ and $e_3 \cap f_{3-i} \neq \emptyset$ for some $i \in [2]$. In this case, $T^* \cup \{(e_2f_i), (e_3f_{3-i}), a, (g_1h^*), (g_2h^*)\}$ is a transversal in H of size $|T^*| + 5$. Suppose that T^* contains a vertex in h_1 or h_2, say in h_1. We note that $h_2 \cap g_j \neq \emptyset$ for some $j \in [2]$. In this case, $T^* \cup \{x, c, d, (g_jh_2), (g_{3-j}h^*)\}$ is a transversal in H of size $|T^*| + 5$. Suppose that T^* contains a vertex in h^*. We note that $h_1 \cap g_i \neq \emptyset$ and $h_2 \cap g_{3-i} \neq \emptyset$ for some $i \in [2]$. In this case, $T^* \cup \{x, c, d, (g_ih_1), (g_{3-i}h_2)\}$ is a transversal in H of size $|T^*| + 5$. In all cases, T^* can be extended to a transversal in H by adding to it five vertices.

For $i \in [2]$, let θ_i be a vertex in h_i that does not belong to $g_1 \cup g_2$. Let θ_3 and θ_4 be the two vertices in h^* that do not belong to g_1 or g_2. Analogously as in Claim P.5, we note that $N_H(a) \cap h^* = \emptyset$. In particular, the vertices θ_1, θ_2, θ_3 and θ_4 are distinct, implying that $|\partial(Z)| \geq 4$. Recall that $\mathrm{def}(H^*) \leq 21$. By Claim L, there exists a transversal, T^*, in H^*, that contains a vertex in $\partial(Z)$ such that $45|T^*| \leq 6n(H^*) + 13m(H^*) + f(|\partial(Z)|) \leq 6n(H^*) + 13m(H^*) + 23$. As observed earlier, T^* can be extended to a transversal in H by adding to it five vertices. Hence, $45\tau(H) \leq 45(|T^*|+5) \leq (6n(H^*)+13m(H^*)+23)+225 = 6n(H)+13m(H)-6\cdot 18-13\cdot 11+23+225 = 6n(H)+13m(H)-3$, a contradiction. Hence the case $\mathrm{def}(H-a) = 3$ cannot occur. (□)

Claim P.6.2: The case $\mathrm{def}(H - a) = 8$ cannot occur.

Proof of Claim P.6.2: Suppose that $\mathrm{def}(H-a) = 8$. By Claim M, $H-a$ therefore contains an H_4-component. Let g be the edge belonging in this H_4-component. By Claim M, the edge g is intersected by all three edges h, h_1, h_2 incident with the vertex a. We note that the edge g is distinct from f_1 and f_2, and contains the vertex b which therefore has degree 1 in $H-a$. For $i \in [2]$, let z_{i1} and z_{i2} be the two vertices in $h_i \setminus \{a\}$ that do not belong to the edge g, and let z_{i3} be the vertex (h_ig) that is common to h_i and g. Thus, $h_i = \{a, z_{i1}, z_{i2}, z_{i3}\}$.

Let $Z = f_1 \cup f_2 \cup g \cup \{x, a\}$ and let $H^* = H - Z$. We note that $n(H^*) = n(H) - 14$. The edges $e_1, e_2, e_3, f_1, f_2, g, h, h_1, h_2$ are deleted from H when constructing H^*, and so $m(H^*) = m(H) - 9$.

We show firstly that $\mathrm{def}(H^*) \leq 21$. Suppose, to the contrary, that $\mathrm{def}(H^*) \geq 22$. Let Y^* be a special H^*-set satisfying $\mathrm{def}_{H^*}(Y^*) = \mathrm{def}(H^*)$. By Claim O, $|E^*_{H^*}(Y^*)| \leq |Y^*| - 3$. Let R_1, R_2, R_3 be the H_4-subhypergraphs of H with edge set f_1, f_2, g, respectively, and consider the special H-set $Y = Y^* \cup \{R_1, R_2, R_3\}$. We note that $|E^*_H(Y)| \leq |E^*_{H^*}(Y^*)| + |\{e_1, e_2, e_3, h_1, h_2\}| \leq (|Y^*| - 3) + 6 = |Y^*| + 3 = |Y|$, contradicting Claim I(a). Therefore, $\mathrm{def}(H^*) \leq 21$.

We show next that every transversal in H^* that contains a vertex in $\partial(Z)$ can be extended to a transversal in H by adding to it four vertices. We note that if $\partial(Z)$ contains a vertex from some edge e of H, then $e \in \{e_1, e_2, e_3, h_1, h_2\}$. Let T^* be a transversal in H^* that contains a vertex in $\partial(Z)$. Suppose that T^* contains a vertex in e_1, e_2 or e_3. Renaming edges if necessary, we may assume T^* contains a vertex in e_1. By Claim P.2, $e_2 \cap f_i \neq \emptyset$

and $e_3 \cap f_{3-i} \neq \emptyset$ for some $i \in [2]$, implying that $T^* \cup \{(e_2 f_i), (e_3 f_{3-i}), a, b\}$ is a transversal in H of size $|T^*|+4$. If T^* contains a vertex in h_1 or h_2, say in h_1, then $T^* \cup \{(gh_2), x, c, d\}$ is a transversal in H of size $|T^*|+4$. In all cases, T^* can be extended to a transversal in H by adding to it four vertices.

Recall that s_i is a vertex in $e_i \setminus \{x\}$ that does not belong to f_1 or f_2. Thus, s_1, s_2, s_3 are distinct vertices in $\partial(Z)$. Recall that $\mathrm{def}(H^*) \le 21$. As observed earlier, T^* can be extended to a transversal in H by adding to it four vertices, and so $\tau(H) \le |T^*|+4$. If H^* contains two intersecting edges e and f, such that $\partial(Z) \subseteq (V(e) \cup V(f)) \setminus (V(e) \cap V(f))$, then $|\partial(Z)| \le 6$. Hence if $|\partial(Z)| \ge 7$, then by Claim L(b) there exists a transversal, T^*, in H^*, that contains a vertex in $\partial(Z)$ such that $45|T^*| \le 6n(H^*)+13m(H^*)+21$. Thus, $45\tau(H) \le 45(|T^*|+4) \le (6n(H^*)+13m(H^*)+21)+180 = 6n(H)+13m(H)-6\cdot 14-13\cdot 9+21+180 = 6n(H)+13m(H)$, a contradiction. Hence, $|\partial(Z)| \le 6$. Renaming vertices if necessary, we may assume that $s_1 = z_{11}$.

If $d_H(s_1) = 2$, then the vertex s_1 is isolated in H^*. In this case, adding s_1 to the set Z we note that $n(H^*) = n(H) - 15$. Further z_{12}, z_{21}, z_{22} are distinct vertices in $\partial(Z)$, and so $|\partial(Z)| \ge 3$ and $f(|\partial(Z)|) \le 27$. By Claim L, there exists a transversal, T^*, in H^*, that contains a vertex in $\partial(Z)$ such that $45|T^*| \le 6n(H^*) + 13m(H^*) + f(|\partial(Z)|) \le 6n(H) + 13m(H) + 27 - 6 \cdot 15 - 13 \cdot 9 = 6n(H) + 13m(H) - 180$, implying that $45\tau(H) \le 45(|T^*| + 4) \le 6n(H) + 13m(H)$, a contradiction. Hence, $d_H(s_1) = 3$.

Since $z_{11}, z_{12}, z_{21}, z_{22}$ are distinct vertices in $\partial(Z)$, we note that $|\partial(Z)| \ge 4$. If $|\partial(Z)| = 4$, then $\{s_1, s_2, s_3\} \subset \{z_{11}, z_{12}, z_{21}, z_{22}\}$, implying analogously as above that $d_H(s_i) = 3$ for all $i \in [3]$, contradicting Claim P.5. Therefore, $|\partial(Z)| \ge 5$, and so $f(|\partial(Z)|) = 22$. If the statement of Claim L(b) holds, then $45|T^*| \le 6n(H^*) + 13m(H^*) + 21$, implying as before that $45\tau(H) \le 45(|T^*| + 4) \le 6n(H) + 13m(H)$, a contradiction. Hence, H^* contains two intersecting edges e and f having properties (i), (ii) and (iii) in Claim L(b). Recall that (ef) denotes the vertex in the intersection of e and f. Thus, $\partial(Z) \subseteq (V(e) \cup V(f)) \setminus \{(ef)\}$, e contains three degree-1 vertices, and $|\partial(Z) \cap V(e)|, |\partial(Z) \cap V(f)| \ge 2$.

Renaming vertices if necessary, we may assume that $s_1 = z_{11}$. If $z_{11} \in e$, let $z \in \partial(Z) \cap V(f)$, while if $z_{11} \in f$, let $z \in \partial(Z) \cap V(e)$. Let $Z^\bullet = Z \cup (e \setminus \{(ef)\}) \cup \{z\}$ and let $H^\bullet = H - Z^\bullet$. We note that $n(H^\bullet) = n(H) - 14 - 4 = n(H) - 18$. The edges $e, e_1, e_2, e_3, f, f_1, f_2, g, h, h_1, h_2$ are deleted from H when constructing $H^\bullet$, and so $m(H^\bullet) = m(H) - 11$. By Claim P.2, $e_2 \cap f_i \neq \emptyset$ and $e_3 \cap f_{3-i} \neq \emptyset$ for some $i \in [2]$. Every transversal in $H^\bullet$ can be extended to a transversal in H by adding to it the five vertices in the set $\{b, (e_2 f_i), (e_3 f_{3-i}), s_1, z\}$, and so $\tau(H) \le \tau(H^\bullet) + 5$. Applying the inductive hypothesis to $H^\bullet$, we have that $45\tau(H) \le 45(\tau(H^\bullet) + 5) \le 6n(H^\bullet) + 13m(H^\bullet) + \mathrm{def}(H^\bullet) + 225 \le 6n(H) + 13m(H) + \mathrm{def}(H^\bullet) - 6 \times 18 - 13 \times 11 + 225 \le 6n(H) + 13m(H) + \mathrm{def}(H^\bullet) - 26$. If $\mathrm{def}(H^\bullet) \le 26$, then $45\tau(H) \le 6n(H) + 13m(H)$, a contradiction. Hence, $\mathrm{def}(H^\bullet) \ge 27$. Let $Y^\bullet$ be a special $H^\bullet$-set satisfying $\mathrm{def}(H^\bullet) = \mathrm{def}_{H^\bullet}(Y^\bullet)$. By Claim O, $|E^*_{H^\bullet}(Y^\bullet)| \le |Y^*| - 4$. Let Y_1, Y_2, Y_3 be the H_4-subhypergraphs of H with edge set f_1, f_2, g, respectively, and consider the special H-set $Y = Y^\bullet \cup \{Y_1, Y_2, Y_3\}$. We note that $|E^*_H(Y)| \le |E^*_{H^\bullet}(Y^\bullet)| + |\{e, e_1, e_2, e_3, f, h, h_1, h_2\}| \le (|Y^\bullet| - 4) + 8 = |Y^\bullet| + 4 = |Y| + 1$, contradicting Claim I(a). Hence the case $\mathrm{def}(H - a) = 8$ cannot occur. ($\Box$)

Since neither the case $\text{def}(H - a) = 3$ nor the case $\text{def}(H - a) = 8$ can occur, this completes the proof of Claim P.6. ($\Box$)

Recall that s_i is a vertex in $V(e_i) \setminus \{x\}$ that does not belong to f_1 or to f_2, and that by Claim P.4, $d_H(s_i) \ge 2$.

Claim P.7: If $d_H(s_i) = 3$ and $d_H(s_j) = 2$ for some i and j where $1 \le i, j \le 3$, then the vertices s_i and s_j do not belong to a common edge.

Proof of Claim P.7: Renaming edges e_1, e_2, e_3 if necessary, let $d_H(s_1) = 3$ and $d_H(s_2) = 2$ and suppose, to the contrary, that there is an edge g containing both s_1 and s_2. Let $g^\bullet$ be the third edge containing s_1 different from e_1 and g. By Claim M, $\text{def}(H - s_1) = 3$ or $\text{def}(H - s_1) = 8$. We consider the two possibilities in turn, and show that neither case can occur.

Claim P.7.1: The case $\text{def}(H - s_1) = 3$ cannot occur.

Proof of Claim P.7.1: Suppose that $\text{def}(H - s_1) = 3$. By Claim M, $H - s_1$ therefore contains a double-H_4-component, say F. Let q_1 and q_2 be the two edges belonging to the H_4-pair in F and let $q^\bullet$ be the linking edge in $H - s_1$ that intersects q_1 and q_2. Recall that $V(h) = \{a, b, c, d\}$, where the edges f_1 and h intersect in the vertex c and where the edges f_2 and h intersect in the vertex d. By Claim P.6 and our earlier observations, $d_H(v) = 2$ for all $v \in V(h)$. Let $q^\bullet \cap q_1 = \{c_q\}$ and $q^\bullet \cap q_2 = \{d_q\}$.

Let $Z = (f_1 \cup f_2 \cup q_1 \cup q_2 \cup \{x, s_1, s_2\}) \setminus \{c, d, c_q, d_q\}$ and let $H^* = H - Z$. We note that $n(H^*) = n(H) - 15$. The edges $e_1, e_2, e_3, f_1, f_2, g, g^\bullet, q_1, q_2$ are deleted from H when constructing H^*, and so $m(H^*) = m(H) - 9$.

Claim P.7.1.1: $\text{def}(H^*) \le 21$.

Proof of Claim P.7.1.1: Suppose, to the contrary, that $\text{def}(H^*) \ge 22$. Let Y^* be a special H^*-set satisfying $\text{def}_{H^*}(Y^*) = \text{def}(H^*)$. By Claim O, $|E^*_{H^*}(Y^*)| \le |Y^*| - 3$. Let F_1, F_2, F_3, F_4 be the H_4-subhypergraphs of H with edge set f_1, f_2, q_1, q_2, respectively. Let Y^{**} be the special H^*-set obtained from Y^* by removing all special subhypergraphs in Y^*, if any, that contain the edge h or the edge $q^\bullet$, and consider the special H-set $Y = Y^{**} \cup \{F_1, F_2, F_3, F_4\}$.

Since we remove at most two special subhypergraphs from Y^* when constructing Y, we note that $|Y| \ge |Y^*| - 2$. Further, we note that $|E^*_{H^*}(Y^{**})| \le |E^*_{H^*}(Y^*)| \le |Y^*| - 3$, and $|E^*_H(Y)| \le |E^*_{H^*}(Y^{**})| + |\{e_1, e_2, e_3, g, g^\bullet, h, q^\bullet\}| \le (|Y^*| - 3) + 7 = |Y^*| + 4$. If $|Y| \ge |Y^*| - 1$ or if $|E^*_{H^*}(Y^*)| \le |Y^*| - 4$, then $|E^*_H(Y)| \le |Y| + 1$, contradicting Claim I(a). Therefore, $|Y| = |Y^*| - 2$ and $|E^*_{H^*}(Y^*)| = |Y^*| - 3$. In particular, since $|E^*_{H^*}(Y^*)| \ge 0$, we note that $|Y^*| \ge 3$. By Claim O, $22 \le \text{def}(H^*) \le 13 \cdot 3 - 5|Y^*_4| - 8|Y^*_{14}| - 9|Y^*_{11}| - 12|Y^*_{21}|$, implying that $|Y^*| = |Y^*_4| = 3$ and $|E^*_{H^*}(Y^*)| = 0$. Let $Y^* = \{R_1, R_2, R_3\}$, where R_1 and R_2 are the special subhypergraphs in Y^* containing the edges h and $q^\bullet$, respectively. We note that all three subhypergraph R_1, R_2 and R_3 are H_4-components in H^*. In particular, $E(R_1) = \{h\}$ and $E(R_2) = \{q^\bullet\}$. Let $E(R_3) = \{\theta\}$.

If q_1 or q_2 intersects the edge h, then $h = q^\bullet$, a contradiction. Hence, neither q_1 nor q_2

intersect h. By Claim P.6, $d_H(a) = 2$ and $d_H(b) = 2$, implying by the linearity of H that both edges g and $g^\bullet$ intersect h. Analogously, both edges e_2 and e_3 intersect the edge $q^\bullet$. Thus, no vertex in $N_H[x]$ belongs to the edge θ, and no vertex in $N_H[s_1]$ belongs to the edge θ. Thus, no edge in H intersects the edge θ, implying that R_3 is an H_4-component in H, a contradiction. This completes the proof of Claim P.7.1.1. (□)

Claim P.7.1.2: Every transversal in H^* that contains a vertex in $\partial(Z)$ can be extended to a transversal in H by adding to it four vertices.

Proof of Claim P.7.1.2: We note that if $\partial(Z)$ contains a vertex from some edge e of H, then $e \in \{e_2, e_3, f_1, f_2, g, g^\bullet, q_1, q_2\}$. Let T^* be a transversal in H^* that contains a vertex in $\partial(Z)$. By Claim P.2, any two edges amongst e_1, e_2 and e_3 (incident with x) can be covered by two vertices one of which belongs to f_1 and the other to f_2. Analogously, any two edges amongst e_1, g and $g^\bullet$ (incident with s_1) can be covered by two vertices one of which belongs to q_1 and the other to q_2. If T^* contains a vertex in f_1 or f_2, say in f_1, then $T^* \cup \{(f_2h), (gq_i), (g^\bullet q_{3-i}), x\}$ is a transversal in H of size $|T^*| + 4$ for some $i \in [2]$. If T^* contains a vertex in q_1 or q_2, say in q_1, then $T^* \cup \{(e_2f_i), (e_3f_{3-i}), s_1, v_2\}$ is a transversal in H of size $|T^*| + 4$ for some $i \in [2]$, where v_2 is an arbitrary vertex in q_2. If T^* contains a vertex in e_2 or e_3, say in e_2, then $T^* \cup \{(e_1f_i), (e_3f_{3-i}), (gq_j), (g^\bullet q_{3-j})\}$ is a transversal in H of size $|T^*| + 4$ for some $i \in [2]$ and $j \in [2]$. If T^* contains a vertex in g or $g^\bullet$, say in g, then $T^* \cup \{(g^\bullet q_i), (e_1q_{3-i}), (gq_j), (e_2f_j), (e_3f_{3-j})\}$ is a transversal in H of size $|T^*| + 4$ for some $i \in [2]$ and $j \in [2]$. In all cases, T^* can be extended to a transversal in H by adding to it four vertices. This completes the proof of Claim P.7.1.2. (□)

By Claim P.5, the vertex s_3 does not belong to the edge h, implying that $\{a, b, s_3\} \subseteq \partial(Z)$, and so $|\partial(Z)| \ge 3$ and $f(|\partial(Z)|) \le 27$. By Claim P.7.1.1, $\mathrm{def}(H^*) \le 21$. By Claim L, there exists a transversal, T^*, in H^*, that contains a vertex in $\partial(Z)$ such that $45|T^*| \le 6n(H^*) + 13m(H^*) + f(|\partial(Z)|) \le 6n(H^*) + 13m(H^*) + 27$. By Claim P.7.1.2, T^* can be extended to a transversal in H by adding to it four vertices. Hence, $45\tau(H) \le 45(|T^*|+4) \le (6n(H^*)+13m(H^*)+27)+180 = 6n(H)+13m(H)-6\cdot15-13\cdot9+27+180 = 6n(H)+13m(H)$, a contradiction. Hence the case $\mathrm{def}(H - s_1) = 3$ cannot occur. (□)

Claim P.7.2: The case $\mathrm{def}(H - s_1) = 8$ cannot occur.

Proof of Claim P.7.2: Suppose that $\mathrm{def}(H-s_1) = 8$. By Claim M, $H-a$ therefore contains an H_4-component. Let q be the edge belonging in this H_4-component. By Claim M, the edge q is intersected by all three edges $e_1, g, g^\bullet$ incident with the vertex s_1. Let $Z = f_1 \cup f_2 \cup q \cup \{x, s_1, s_2\}$ and let $H^* = H - Z$. We note that $n(H^*) = n(H) - 15$. The edges $e_1, e_2, e_3, f_1, f_2, g, g^\bullet, h, q$ are deleted from H when constructing H^*, and so $m(H^*) = m(H) - 9$.

We show firstly that $\mathrm{def}(H^*) \le 21$. Suppose, to the contrary, that $\mathrm{def}(H^*) \ge 22$. Let Y^* be a special H^*-set satisfying $\mathrm{def}_{H^*}(Y^*) = \mathrm{def}(H^*)$. By Claim O, $|E^*_{H^*}(Y^*)| \le |Y^*| - 3$. Let F_1, F_2, F_3 be the H_4-subhypergraphs of H with edge set f_1, f_2, q, respectively, and consider the special H-set $Y = Y^* \cup \{F_1, F_2, F_3\}$. We note that $|E^*_H(Y)| \le |E^*_{H^*}(Y^*)| + |\{e_1, e_2, e_3, g, g^\bullet, h\}| \le (|Y^*| - 3) + 6 = |Y^*| + 3 = |Y|$, contradicting Claim I(a). Therefore, $\mathrm{def}(H^*) \le 21$.

We show next that every transversal in H^* that contains a vertex in $\partial(Z)$ can be extended to a transversal in H by adding to it four vertices. We note that if $\partial(Z)$ contains a vertex from some edge e of H, then $e \in \{e_2, e_3, g, g^\bullet, h\}$. Let T^* be a transversal in H^* that contains a vertex in $\partial(Z)$. If T^* contains a vertex in e_2 or e_3, say in e_2, then $T^* \cup \{(e_3 f_i), (h f_{3-i}), s_1, w\}$ is a transversal in H of size $|T^*| + 4$ for some $i \in [2]$ where w is an arbitrary vertex in the edge q. If T^* contains a vertex in h, then $T^* \cup \{(e_2 f_i), (e_3 f_{3-i}), s_1, w\}$ is a transversal in H of size $|T^*| + 4$ for some $i \in [2]$ where w is an arbitrary vertex in the edge q. If T^* contains a vertex in g or $g^\bullet$, say in g, then $T^* \cup \{(g^\bullet q), c, d, x\}$ is a transversal in H of size $|T^*| + 4$. In all cases, T^* can be extended to a transversal in H by adding to it four vertices.

By Claim P.5, the vertex s_3 does not belong to the edge h, implying that $\{a, b, s_3\} \subseteq \partial(Z)$, and so $|\partial(Z)| \ge 3$ and $f(|\partial(Z)|) \le 27$. An identical argument as in the last paragraph of the proof of Claim P.7.1 yields the contradiction $45\tau(H) \le 6n(H) + 13m(H)$. Hence the case $\mathrm{def}(H - s_1) = 8$ cannot occur. ($\square$)

Since neither the case $\mathrm{def}(H - s_1) = 3$ not the case $\mathrm{def}(H - s_1) = 8$ can occur, this completes the proof of Claim P.7. ($\square$)

Claim P.8: There is no edge containing all of s_1, s_2 and s_3.

Proof of Claim P.8: Suppose, to the contrary, that there is an edge, g say, in H containing all of s_1, s_2 and s_3. By Claim P.4, $d_H(s_i) \ge 2$ for all $i \in [3]$, and $d_H(s_i) = 2$ for at least one $i \in [3]$. Thus, by Claim P.7, $d_H(s_1) = d_H(s_2) = d_H(s_3) = 2$. Let $Z = f_1 \cup f_2 \cup \{x, s_1, s_2, s_3\}$ and let $H^* = H - Z$. We note that $n(H^*) = n(H) - 12$. The edges $e_1, e_2, e_3, f_1, f_2, g, h$ are deleted from H when constructing H^*, and so $m(H^*) = m(H) - 7$.

If $\mathrm{def}(H^*) \ge 22$, then let Y^* be a special H^*-set satisfying $\mathrm{def}_{H^*}(Y^*) = \mathrm{def}(H^*)$. By Claim O, $|E^*_{H^*}(Y^*)| \le |Y^*| - 3$. Let F_1, F_2 be the H_4-subhypergraphs of H with edge set f_1, f_2, respectively, and consider the special H-set $Y = Y^* \cup \{F_1, F_2\}$. We note that $|E^*_H(Y)| \le |E^*_{H^*}(Y^*)| + |\{e_1, e_2, e_3, g, h\}| \le (|Y^*| - 3) + 5 = |Y^*| + 2 = |Y|$, contradicting Claim I(a). Therefore, $\mathrm{def}(H^*) \le 21$.

We show next that every transversal in H^* that contains a vertex in $\partial(Z)$ can be extended to a transversal in H by adding to it three vertices. We note that if $\partial(Z)$ contains a vertex from some edge e of H, then $e \in \{e_1, e_2, e_3, g, h\}$. Let T^* be a transversal in H^* that contains a vertex in $\partial(Z)$. If T^* contains a vertex in e_1, e_2 or e_3, say in e_1, then $T^* \cup \{s_2, (e_3 f_i), (h f_{3-i})\}$ is a transversal in H of size $|T^*| + 3$ for some $i \in [2]$. If T^* contains a vertex in h, then $T^* \cup \{s_1, (e_2 f_i), (e_3 f_{3-i})\}$ is a transversal in H of size $|T^*| + 3$ for some $i \in [2]$. If T^* contains a vertex in g, then $T^* \cup \{c, d, x\}$ is a transversal in H of size $|T^*| + 3$. In all cases, T^* can be extended to a transversal in H by adding to it three vertices.

As observed earlier, $\mathrm{def}(H^*) \le 21$. By Claim L, there exists a transversal, T^*, in H^*, that contains a vertex in $\partial(Z)$ such that $45|T^*| \le 6n(H^*) + 13m(H^*) + f(|\partial(Z)|)$. As observed earlier, T^* can be extended to a transversal in H by adding to it three vertices. Hence, $45\tau(H) \le 45(|T^*| + 3) \le (6n(H^*) + 13m(H^*) + f(|\partial(Z)|)) + 135 = 6n(H) + 13m(H) + f(|\partial(Z)|) - 6 \cdot 12 - 13 \cdot 7 + 135 = 6n(H) + 13m(H) + f(|\partial(Z)|)) - 28$. If $|\partial(Z)| \ge 3$, then $f(|\partial(Z)|) \le 27$ and $45\tau(H) \le 6n(H) + 13m(H)$, a contradiction. Hence, $|\partial(Z)| \le 2$. Since

$\{a,b\} \subseteq \partial(Z)$, we note that $|\partial(Z)| \ge 2$. Consequently, $|\partial(Z)| = 2$ and $\partial(Z) = \{a,b\}$. This implies that f_i intersects each of the edges e_1, e_2, e_3 for $i \in [2]$ and that either a or b belong to the edge g. Renaming the vertices a and b if necessary, we may assume that $g = \{a, s_1, s_2, s_3\}$. This implies that the only edges in H that intersect the edges f_1, f_2 and g are e_1, e_2, e_3, h. We now let F_1, F_2, F_3 be the H_4-subhypergraphs of H with edge set f_1, f_2, g, respectively, and consider the special H-set $Y = \{F_1, F_2, F_3\}$. We note that $|E^*_H(Y)| \le |\{e_1, e_2, e_3, h\}| = 4 = |Y|+1$, contradicting Claim I(a). This completes the proof of Claim P.8. $_{(\Box)}$

Claim P.9: If $d_H(s_i) = d_H(s_j) = 3$ for some i and j where $1 \le i,j \le 3$ and $i \ne j$, then the vertices s_i and s_j do not belong to a common edge.

Proof of Claim P.9: Renaming edges e_1, e_2, e_3 if necessary, let $d_H(s_1) = d_H(s_2) = 3$ and suppose, to the contrary, that there is an edge g containing both s_1 and s_2. By Claim P.8, the edge g does not contain the vertex s_3. By Claim P.4, we may assume, renaming the edges f_1 and f_2 if necessary, that $e_1 \cap f_1 \ne \emptyset$, $e_1 \cap f_2 = \emptyset$, and that $e_2 \cap f_1 = \emptyset$ and $e_2 \cap f_2 \ne \emptyset$. Thus, e_1 intersects f_1 but not f_2, and e_2 intersects f_2 but not f_1. By Claim P.1, this implies that e_3 intersects both f_1 and f_2; that is, $e_3 \cap f_1 \ne \emptyset$ and $e_3 \cap f_2 \ne \emptyset$. For $i \in [2]$, let w_i be the vertex in e_i different from $(e_i f_i)$, x and s_i, and so $V(e_i) = \{x, s_i, w_i, (e_i f_i)\}$. We note that the vertex $(e_i f_i)$ has degree 2 in H for $i \in [2]$.

Claim P.9.1: There is no edge containing both w_1 and w_2.

Proof of Claim P.9.1: Suppose, to the contrary, that there is an edge f' that contains both w_1 and w_2. We now consider the hypergraph $H - s_1$. By Claim M, either f' belongs to an H_4-component or to a double-H_4-component. Since f' is intersected by the edge e_2 in $H - s_1$, the edge f' cannot belong to an H_4-component. Hence, f' belongs to a double-H_4-component. Let f'' be the second edge belonging to the H_4-pair in this double-H_4-component. We note that e_2 is the linking edge in $H - s_1$ that intersects f' and f''. Analogously as in Claim P.5, $N_H(s_1) \cap V(e_2) = \emptyset$. However, $s_2 \in N_H(s_1) \cap V(e_2)$, a contradiction. $_{(\Box)}$

Claim P.9.2: There is no edge containing s_3 and one of w_1 or w_2.

Proof of Claim P.9.2: Suppose, to the contrary, that there is an edge f' that contains s_3 and one of w_1 or w_2. By symmetry, we may assume that $w_1 \in V(f')$. We now consider the hypergraph $H - s_1$. By Claim M, either f' belongs to an H_4-component or to a double-H_4-component. Since f' is intersected by the edge e_3 in $H - s_1$, the edge f' cannot belong to an H_4-component. Hence, f' belongs to a double-H_4-component. Let f'' be the second edge belonging to the H_4-pair in this double-H_4-component. We note that e_3 is the linking edge in $H - s_1$ that intersects f' and f''. Analogously as in Claim P.5, $N_H(s_1) \cap V(e_3) = \emptyset$. However, $x \in N_H(s_1) \cap V(e_3)$, a contradiction. $_{(\Box)}$

By Claim P.9.1 and P.9.2, $\{w_1, w_2, s_3\}$ is an independent set. Recall that $h = \{a, b, c, d\}$, where the edges f_1 and h intersect in the vertex c and where the edges f_2 and h intersect in the vertex d. Further, $d_H(c) = d_{H'}(c) = 2$ and $d_H(d) = d_{H'}(d) = 2$. In particular, we note that $\{d, w_1, w_2, s_3\}$ is an independent set. Let $Z = (f_1 \cup f_2 \cup \{x\}) \setminus \{c, d\}$ and let

$H^\bullet$ be obtained from $H - Z$ by adding to it the edge $f^\bullet = \{d, w_1, w_2, s_3\}$. We note that $n(H^\bullet) = n(H) - 7$ and $m(H^\bullet) = m(H) - 4$. Further since $\{d, w_1, w_2, s_3\}$ is an independent set in H, the hypergraph $H^\bullet$ is linear.

We show next that $\tau(H) \le \tau(H^\bullet) + 2$. Let $T^\bullet$ be a $\tau(H^\bullet)$-transversal. In order to cover the edge $f^\bullet$, the transversal $T^\bullet$ contains at least one vertex in $f^\bullet$. If $w_1 \in T^\bullet$, let $T = T^\bullet \cup \{(e_2f_2), (e_3f_1)\}$. If $w_2 \in T^\bullet$, let $T = T^\bullet \cup \{(e_1f_1), (e_3f_2)\}$. If $s_3 \in T^\bullet$, let $T = T^\bullet \cup \{(e_1f_1), (e_2f_2)\}$. If $d \in T^\bullet$, let $T = T^\bullet \cup \{x, c\}$. In all cases, T is a transversal in H of size $|T^\bullet| + 2$, and so $\tau(H) \le |T^\bullet| + 2 = \tau(H^\bullet) + 2$.

Applying the inductive hypothesis to $H^\bullet$, we have that $45\tau(H) \le 45(\tau(H^\bullet) + 2) \le 6n(H^\bullet) + 13m(H^\bullet) + \mathrm{def}(H^\bullet) + 90 \le 6n(H) + 13m(H) + \mathrm{def}(H^\bullet) - 7 \times 6 - 4 \times 13 + 90 = 6n(H) + 13m(H) + \mathrm{def}(H^\bullet) - 4$. If $\mathrm{def}(H^\bullet) \le 4$, then $45\tau(H) \le 6n(H) + 13m(H)$, a contradiction. Hence, $\mathrm{def}(H^\bullet) \ge 5$. Let $Y^\bullet$ be a special $H^\bullet$-set satisfying $\mathrm{def}_{H^\bullet}(Y^\bullet) = \mathrm{def}(H^\bullet)$. By Claim O, $|E^*_{H^\bullet}(Y^\bullet)| \le |Y^\bullet| - 1$.

Claim P.9.3: $f^\bullet \in Y^\bullet$.

Proof of Claim P.9.3: Suppose, to the contrary, that $f^\bullet \notin Y^\bullet$. If $f^\bullet \notin E^*_{H^\bullet}(Y^\bullet)$, then $h \notin Y^\bullet$, implying that $|E^*_H(Y^\bullet)| \le |E^*_{H^\bullet}(Y^\bullet)| + |\{e_1, e_2\}| \le (|Y^\bullet| - 1) + 2 = |Y^\bullet| + 1$, contradicting Claim I. Hence, $f^\bullet \in E^*_{H^\bullet}(Y^\bullet)$. Since $H - Z = H^\bullet - f^\bullet$, this implies that $\mathrm{def}(H - Z) = \mathrm{def}(H^\bullet - f^\bullet) = \mathrm{def}(H^\bullet) + 13 \ge 18$. Let Y be a special $(H - Z)$-set satisfying $\mathrm{def}_{H-Z}(Y) = \mathrm{def}(H - Z)$. If $h \notin Y$, then neither f_1 nor f_2 intersect any special subhypergraphs in Y, implying that $\mathrm{def}_{H-x}(Y) = \mathrm{def}_{H-Z}(Y) \ge 18$, contradicting Claim M. Hence, $h \in Y$. Let R_1 be the special subhypergraph in Y that contains the edge h. We note that both c and d have degree 1 in $H - Z$, implying that the subhypergraph R_1 is an H_4-component in $H - Z$ and therefore contributes 8 to the deficiency of Y in $H - Z$. Since neither f_1 nor f_2 intersect any special subhypergraph in Y different from R_1, this implies that $\mathrm{def}_{H-x}(Y \setminus \{R_1\}) \ge 18 - 8 = 10$, once again contradicting Claim M. $(\Box)$

By Claim P.9.3, $f^\bullet \in Y^\bullet$. Let R_1 be the special subhypergraph in $Y^\bullet$ that contains the edge $f^\bullet$. Suppose that $|Y^\bullet| \ge 2$. Let $|E^*_{H^\bullet}(Y^\bullet)| = |Y^\bullet| - i$ for some $i \ge 1$. By Claim O and by our earlier observations, $5 \le \mathrm{def}(H^\bullet) \le 13i - 5|Y^\bullet| \le 13i - 10$, implying that $i \ge 2$; that is, $|E^*_{H^\bullet}(Y^\bullet)| \le |Y^\bullet| - 2$. Let R_2 and R_3 be the H_4-subhypergraphs of H with edge set f_1 and f_2, respectively. Let $Y^* = (Y^\bullet \setminus \{R_1\}) \cup \{R_2, R_3\}$, and so $|Y^*| = |Y^\bullet| + 1$. Thus, $|E^*_H(Y^*)| \le |E^*_{H^\bullet}(Y^\bullet)| + |\{e_1, e_2, e_3, h\}| \le (|Y^\bullet| - 2) + 4 = |Y^*| + 1$, contradicting Claim I. Hence, $|Y^\bullet| = 1$; that is, $Y^\bullet = \{R_1\}$.

Recall that $\mathrm{def}(H^\bullet) \ge 5$ and $|E^*_{H^\bullet}(Y^\bullet)| \le |Y^\bullet| - 1$. Thus since $|Y^\bullet| = 1$, we note that $|E^*_{H^\bullet}(Y^\bullet)| = 0$, implying that R_1 is a component in $H^\bullet$. Since the vertex s_3 belongs to R_1 and is contained in both edges $f^\bullet$ and h, the component R_1 is not an H_4-component in $H^\bullet$, implying that R_1 is an H_{14}-component in $H^\bullet$ and $\mathrm{def}(H^\bullet) = 5$.

We now consider the hypergraph $H - s_1$. By Claim M, either $H - s_1$ contains an H_4-component or a double-H_4-component. We note that in either case, the vertex w_1 belongs to such a component. If $H - s_1$ contains an H_4-component, then this component contains a vertex of degree 1 in H different from w_1. Suppose that $H - s_1$ contains a double-H_4-component. In this case, let q_1 and q_2 be the two edges belonging to the H_4-pair in this component and let $q^\bullet$ be the linking edge in $H - s_1$ that intersects q_1 and q_2. Renaming q_1 and q_2 if necessary, we may assume that $w_1 \in V(q_1)$. Thus, $q_1 \cap V(e_1) = \emptyset$, implying that q_2 contains a vertex of degree 1 in H. In both cases, there is a vertex of degree 1 in H that belongs to the component of $H - s_1$ that contains w_1, implying that the component R_1 of $H^\bullet$ contains a vertex of degree 1 in H. Further, the component R_1 of $H^\bullet$ contains the vertex c which has degree 1 in $H^\bullet$ and degree 2 in H. Thus, R_1 has at least two vertices of degree 1 in $H^\bullet$, implying that R_1 is an $H_{14,4}$-component. We now consider the edge h in R_1 that contains the vertex c of degree 1 in $H^\bullet$. All three edges intersecting the edge h in $R_1 \cong H_{14,4}$ have at least one vertex of degree 3. Since $f^\bullet$ is one of the three edges that intersect h in R_1, this implies in particular that the edge $f^\bullet$ contains at least one vertex of degree 3 in R_1. However, no vertex of $f^\bullet$ has degree 3 in $H^\bullet$, a contradiction. This completes the proof of Claim P.9. (□)

Claim P.10: Every edge contains at most one vertex from the set $\{s_1, s_2, s_3\}$.

Proof of Claim P.10: Suppose, to the contrary, that there is an edge, g say, in H containing s_i and s_j where $1 \le i < j \le 3$. Renaming the edges e_1, e_2, e_3 if necessary, we may assume that $\{s_1, s_2\} \subset V(g)$. Let $g = \{s_1, s_2, u, v\}$. By Claim P.8, the edge g does not contain the vertex s_3, and so $s_3 \notin \{u, v\}$. By Claim P.4, Claim P.7 and Claim P.9, $d_H(s_1) = d_H(s_2) = 2$. Let $Z = f_1 \cup f_2 \cup \{s_1, s_2, x\}$ and let $H^* = H - Z$. We note that $n(H^*) = n(H) - 11$. The edges $e_1, e_2, e_3, f_1, f_2, g, h$ are deleted from H when constructing H^*, and so $m(H^*) = m(H) - 7$.

We show firstly that $\mathrm{def}(H^*) \le 21$. Suppose, to the contrary, that $\mathrm{def}(H^*) \ge 22$. Let Y^* be a special H^*-set satisfying $\mathrm{def}_{H^*}(Y^*) = \mathrm{def}(H^*)$. By Claim O, $|E^*_{H^*}(Y^*)| \le |Y^*| - 3$. Let R_1 and R_2 be the H_4-subhypergraphs of H with edges f_1 and f_2, respectively, and consider the special H-set $Y = Y^* \cup \{R_1, R_2\}$. We note that $|E^*_H(Y)| \le |E^*_{H^*}(Y^*)| + |\{e_1, e_2, e_3, g, h\}| \le (|Y^*| - 3) + 5 = |Y^*| + 2 = |Y|$, contradicting Claim I(a). Therefore, $\mathrm{def}(H^*) \le 21$.

We show next that every transversal in H^* that contains a vertex in $\partial(Z)$ can be extended to a transversal in H by adding to it three vertices. We note that if $\partial(Z)$ contains a vertex from some edge e of H, then $e \in \{e_1, e_2, e_3, g, h\}$. Let T^* be a transversal in H^* that contains a vertex in $\partial(Z)$. If T^* contains a vertex in e_1 or in e_2, say in e_1 by symmetry, then $T^* \cup \{(e_2 g), (e_3 f_i), (h f_{3-i})\}$ is a transversal in H for some $i \in [2]$. If T^* contains a vertex in g, then $T^* \cup \{c, d, x\}$ is a transversal in H. If T^* contains a vertex in h, then $T^* \cup \{(e_1 g), (e_2 f_i), (e_3 f_{3-i})\}$ is a transversal in H for some $i \in [2]$. In all cases, T^* can be extended to a transversal in H by adding to it three vertices.

By Claim L, there exists a transversal, T^*, in H^*, that contains a vertex in $\partial(Z)$ such that $45|T^*| \le 6n(H^*) + 13m(H^*) + f(|\partial(Z)|) = 6n(H) + 13m(H) + f(|\partial(Z)|) - 11 \cdot 6 - 7 \cdot 13 = 6n(H) + 13m(H) + f(|\partial(Z)|) - 157$. As observed earlier, T^* can be extended to a transversal in H by adding to it three vertices. Hence, $45\tau(H) \le 45(|T^*| + 3) \le 6n(H) + 13m(H) + f(|\partial(Z)|) - 22$. If $|\partial(Z)| \ge 5$, then $f(|\partial(Z)|) = 22$, implying that $45\tau(H) \le 6n(H) + 13m(H)$, a contradiction. Hence, $|\partial(Z)| \le 4$.

We note that $\{a, b\} \subseteq \partial(Z)$, $\{u, v\} \subseteq \partial(Z)$, and $s_3 \in \partial(Z)$. By Claim P.5, the edge h contains no neighbor of x. In particular, $s_3 \notin \{a, b\}$, and so $\{a, b, s_3\} \subseteq \partial(Z)$. As observed earlier, $s_3 \notin \{u, v\}$. By the linearity of H, we note that $\{a, b\} \neq \{u, v\}$. Since $|\partial(Z)| \le 4$, this implies that $|\{a,b\} \cap \{u,v\}| = 1$ and $|\partial(Z)| = 4$. Renaming vertices if necessary, we may assume that $b = v$. By Claim P.6, $d_H(b) = 2$. Since $b = v$, the vertex b is therefore incident only with the edge g and h in H, implying that $d_{H^*}(b) = 0$.

We now add the vertex b to the set Z. With this addition of b to Z, we note that $n(H^*) = n(H) - 12$, $m(H^*) = m(H) - 7$, and $45|T^*| \le 6n(H) + 13m(H) + f(|\partial(Z)|) - 157 - 6 = 6n(H) + 13m(H) + f(|\partial(Z)|) - 163$. Further since $\{a, s_3, u\} \subseteq \partial(Z)$, we note that $|\partial(Z)| \ge 3$ and $f(|\partial(Z)|) \le 27$. Thus, $45\tau(H) \le 45(|T^*| + 3) \le 6n(H) + 13m(H) + f(|\partial(Z)|) - 28 < 6n(H) + 13m(H)$, a contradiction. This completes the proof of Claim P.10. (□)

We now continue with our proof of Claim P. By Claim P.10, $\{s_1, s_2, s_3\}$ is an independent set. Recall that $h = \{a,b,c,d\}$, where the edges f_1 and h intersect in the vertex c and where the edges f_2 and h intersect in the vertex d. Further, $d_H(c) = d_{H'}(c) = 2$ and $d_H(d) = d_{H'}(d) = 2$. In particular, we note that $\{d, s_1, s_2, s_3\}$ is an independent set. Let $Z = (f_1 \cup f_2 \cup \{x\}) \setminus \{c,d\}$ and let $H^\bullet$ be obtained from $H - Z$ by adding to it the edge $f^\bullet = \{d, s_1, s_2, s_3\}$. We note that $n(H^\bullet) = n(H) - 7$ and $m(H^\bullet) = m(H) - 4$. Further since $\{d, s_1, s_2, s_3\}$ is an independent set in H, the hypergraph $H^\bullet$ is linear. By Claim P.1, at least one of the edges e_1, e_2, e_3 intersects both f_1 and f_2. Renaming edges if necessary, we may assume that e_3 intersects both f_1 and f_2; that is, $e_3 \cap f_1 \ne \emptyset$ and $e_3 \cap f_2 \ne \emptyset$. Let $Y^\bullet$ be a special $H^\bullet$-set satisfying $\mathrm{def}_{H^\bullet}(Y^\bullet) = \mathrm{def}(H^\bullet)$.

Claim P.11: The following properties hold in the hypergraph $H^\bullet$.
(a) $\tau(H) \le \tau(H^\bullet) + 2$.
(b) $\mathrm{def}(H^\bullet) \ge 5$ and $|E^*_{H^\bullet}(Y^\bullet)| \le |Y^\bullet| - 1$.
(c) There is no H_{10}-subhypergraph in $H^\bullet$.
(d) $f^\bullet \in Y^\bullet$.
(e) $|Y^\bullet| = 1$.

Proof of Claim P.11: (a) Let $T^\bullet$ be a $\tau(H^\bullet)$-transversal. In order to cover the edge $f^\bullet$, the transversal $T^\bullet$ contains at least one vertex in $f^\bullet$. Suppose that $s_j \in T^\bullet$ for some $j \in [3]$. Renaming edges e_1, e_2, e_3 if necessary, we may assume that $s_1 \in T^\bullet$. In this case, $T^\bullet \cup \{(e_2 f_i), (e_3 f_{3-1})\}$ is a transversal of H for some $i \in [2]$. If $d \in T^\bullet$, then $T^\bullet \cup \{x, c\}$ is a transversal of H. In both cases, we produce a transversal in H of size $|T^\bullet| + 2$, and so $\tau(H) \le |T^\bullet| + 2 = \tau(H^\bullet) + 2$.

(b) Applying the inductive hypothesis to $H^\bullet$, we have that $45\tau(H) \le 45(\tau(H^\bullet) + 2) \le 6n(H^\bullet) + 13m(H^\bullet) + \mathrm{def}(H^\bullet) + 90 \le 6n(H) + 13m(H) + \mathrm{def}(H^\bullet) - 7 \times 6 - 4 \times 13 + 90 = 6n(H) + 13m(H) + \mathrm{def}(H^\bullet) - 4$. If $\mathrm{def}(H^\bullet) \le 4$, then $45\tau(H) \le 6n(H) + 13m(H)$, a contradiction. Hence, $\mathrm{def}(H^\bullet) \ge 5$. By Claim O, $|E^*_{H^\bullet}(Y^\bullet)| \le |Y^\bullet| - 1$.

(c) Suppose, to the contrary, that there is a H_{10}-subhypergraph, say F, in $H^\bullet$. By Claim J, F is not a subhypergraph of H, implying that the added edge $f^\bullet$ is an edge of F and therefore the vertex d belongs to F. We note that every vertex of F has degree 2 in F and therefore degree at least 2 in $H^\bullet$. Since the vertex d has degree 2 in $H^\bullet$ and is contained in the edges $f^\bullet$ and h in $H^\bullet$, the edge h must belong to F. This implies that the vertex c belongs to F. However the vertex c has degree 1 in $H^\bullet$, and therefore degree 1 in F, a contradiction.

(d) Suppose, to the contrary, that $f^\bullet \notin Y^\bullet$. Recall that the edge e_3 intersects both f_1 and f_2. If $f^\bullet \notin E^*_{H^\bullet}(Y^\bullet)$, then $h \notin Y^\bullet$, implying that $|E^*_H(Y^\bullet)| \le |E^*_{H^\bullet}(Y^\bullet)| + |\{e_1, e_2\}| \le (|Y^\bullet| - 1) + 2 = |Y^\bullet| + 1$, contradicting Claim I. Hence, $f^\bullet \in E^*_{H^\bullet}(Y^\bullet)$. If $h \notin Y$, then neither f_1 nor f_2 intersect any special subhypergraphs in $^\bullet$, implying that $|E^*_H(Y^\bullet)| \le |E^*_{H^\bullet}(Y^\bullet)| + |\{e_1, e_2, e_3\}| - |\{f^\bullet| \le (|Y^\bullet| - 1) + 3 - 1 = |Y^\bullet| + 1$, contradicting Claim I. Hence, $h \in Y$. Let R_1 be the special subhypergraph in Y that contains the edge h. We note that both c and d have degree 1 in R_1, implying that the subhypergraph R_1 is an H_4-component in $H^\bullet$. By Claim P.5 and Claim P.6, both vertices a and b have degree 2 in $H^\bullet$. By the linearity of H, the edge different from h that contains a and the edge different from h that contains b are distinct. These two edges, together with the edge $f^\bullet$, all intersect the H_4-component R_1, implying that $|E^*_H(Y^\bullet)| \ge 3$. Thus, by Claim O, $|E^*_H(Y^\bullet)| \le |Y^\bullet| - 3$. Hence, $|E^*_H(Y^\bullet)| \le |E^*_{H^\bullet}(Y^\bullet)| + |\{e_1, e_2, e_3, f_1, f_2\}| - |\{f^\bullet| \le (|Y^\bullet| - 3) + 5 - 1 = |Y^\bullet| + 1$, contradicting Claim I.

(e) By Part (d), $f^\bullet \in Y^\bullet$. Let R_1 be the special subhypergraph in $Y^\bullet$ that contains the edge $f^\bullet$. If $|Y^\bullet| \ge 3$, then by Claim O and by our earlier observations, $|E^*_{H^\bullet}(Y^\bullet)| \le |Y^\bullet| - 2$. Let R_2 and R_3 be the H_4-subhypergraphs of H with edge set f_1 and f_2, respectively. Let $Y^* = (Y^\bullet \setminus \{R_1\}) \cup \{R_2, R_3\}$, and so $|Y^*| = |Y^\bullet| + 1$. Thus, $|E^*_H(Y^*)| \le |E^*_{H^\bullet}(Y^\bullet)| + |\{e_1, e_2, e_3, h\}| \le (|Y^\bullet| - 2) + 4 = |Y^*| + 1$, contradicting Claim I. Hence, $|Y^\bullet| \le 2$. Suppose that $|Y^\bullet| = 2$. Let R_2 be the special subhypergraph in $Y^\bullet$ different from R_1, where recall that $f^\bullet \in E(R_1)$. As observed earlier, $\mathrm{def}(H^\bullet) \ge 5$ and there is no H_{10}-subhypergraph in $H^\bullet$. Thus, $5 \le \mathrm{def}(H^\bullet) \le 8|Y^\bullet| - 13|E^*_{H^\bullet}(Y^\bullet)| = 16 - 13|E^*_{H^\bullet}(Y^\bullet)|$, implying that $E^*_{H^\bullet}(Y^\bullet) = \emptyset$. Letting $Y^* = \{R_2\}$, this in turn implies that $E^*_H(Y^*) = \emptyset$ and therefore that $\mathrm{def}(H) > 0$, a contradiction. Hence, $|Y^\bullet| = 1$. This completes the proof of Claim P.11. $(\Box)$

By Claim P.11(d), $f^\bullet \in Y^\bullet$. By Claim P.11(e), $|Y^\bullet| = 1$. Let R_1 be the special subhypergraph in $Y^\bullet$ that contains the edge $f^\bullet$; that is, $Y^\bullet = \{R_1\}$. By Claim P.11, we note that $5 \le \mathrm{def}(H^\bullet) \le 8|Y^\bullet| - 13|E^*_{H^\bullet}(Y^\bullet)| = 8 - 13|E^*_{H^\bullet}(Y^\bullet)|$, implying that $E^*_{H^\bullet}(Y^\bullet) = \emptyset$; that is, R_1 is a component in $H^\bullet$. By our way in which $H^\bullet$ is constructed, this implies that $R_1 = H^\bullet$. Thus, $n(H) = n(H^\bullet) + 7 = 14 + 7 = 21$. Since the vertex d belongs to R_1 and is contained in both edges $f^\bullet$ and h, the component R_1 is not an H_4-component in $H^\bullet$, implying that R_1 is an H_{14}-component in $H^\bullet$ and $\mathrm{def}(H^\bullet) = 5$. Since R_1 contains the edge h, the component R_1 has at least one vertex of degree 1, namely the vertex c which has degree 1 in $H^\bullet$. This implies that $R_1 \notin \{H_{14,5}, H_{14,6}\}$; that is, $R_1 \cong H_{14,i}$ for some $i \in [4]$.

Suppose that $R_1 = H_{14,1}$. In this case, h is the edge in R_1 that contains the (unique) vertex of degree 1 in R_1. Further, the edge h is intersected by three edges in R_1, one of which is the edge $f^\bullet$. The structure of $H_{14,1}$ implies that we can choose a $\tau(H^\bullet)$-transversal, $T^\bullet$, to contain three vertices of $f^\bullet$, one of which is the vertex d (that belongs to both $f^\bullet$ and h). Renaming the edges e_1, e_2, e_3 if necessary, we may assume that $\{e_2, e_3\} \subset T^\bullet$. If the edges e_1 and f_1 intersect, let $T = T^\bullet \cup \{(e_1 f_1)\}$. If the edges e_1 and f_1 do not intersect, then the edges e_1 and f_2 intersect and we let $T = (T^\bullet \setminus \{d\}) \cup \{c, (e_1 f_2)\}$. In both cases, the set T is a transversal in H of size $|T^\bullet| + 1$, and so $\tau(H) \le \tau(H^\bullet) + 1$. Thus, $45\tau(H) \le 45(\tau(H^\bullet) + 1) \le 6n(H^\bullet) + 13m(H^\bullet) + \mathrm{def}(H^\bullet) + 45 \le 6n(H) + 13m(H) + 5 + 45 - 7 \times 6 - 4 \times 13 < 6n(H) + 13m(H)$, a contradiction.

Suppose that $R_1 = H_{14,3}$. In this case, h is the edge in R_1 that contains the (unique) vertex of degree 1 in R_1. Further, the edge h is intersected by three edges in R_1, one of which is the edge $f^\bullet$. Using analogous arguments as in the previous case, due to the structure of $H_{14,3}$ we can choose a $\tau(H^\bullet)$-transversal, $T^\bullet$, to contain three vertices of $f^\bullet$, one of which is the vertex d, implying as before that $\tau(H) \le \tau(H^\bullet) + 1$, producing a contradiction.

Suppose that $R_1 = H_{14,4}$. In this case, R_1 contains two vertices of degree 1, and h is one of the two edges in R_1 that contain a vertex of degree 1 in R_1. Further, the edge h is intersected by three edges in R_1, one of which is the edge $f^\bullet$. Using analogous arguments as in the previous two cases, due to the structure of $H_{14,4}$ we can choose a $\tau(H^\bullet)$-transversal, $T^\bullet$, to contain at least three vertices of $f^\bullet$, one of which is the vertex d, implying as before that $\tau(H) \le \tau(H^\bullet) + 1$, producing a contradiction.

By the above, we must have $R_1 = H_{14,2}$. In this case, h is the edge in R_1 that contains the (unique) vertex of degree 1 in R_1. Further, the edge h is intersected by three edges in R_1, one of which is the edge $f^\bullet$. We note that in this case, each vertex s_i has degree 2 in R_1 and therefore degree 2 in H.

Suppose that f_1 or f_2 (or both f_1 and f_2) does not intersect one of the edges e_1, e_2, e_3. Renaming vertices and edges, if necessary, we may assume that f_1 does not intersect the edge e_2. By Claim P.1, the edge f_1 therefore intersects both e_1 and e_3. Let x_2 be the vertex in the edge e_2 different from x, s_2 and $(e_2 f_2)$. We note that we could have chosen s_2 to be the vertex x_2, implying by our earlier observations that $d_H(x_2) = 2$ and $\{s_1, x_2, s_3\}$ is an independent set. If the vertex x_2 belongs to the set $V(R_1)$, then x_2 would be adjacent to s_1 or s_3 in H or x_2 and s_2 would be adjacent in $H - e_2$, a contradiction. Hence, $x_2 \notin V(R_1)$. Interchanging the roles of x_2 and s_2 in our earlier arguments and letting $f^\bullet = \{d, s_1, x_2, s_3\}$, the special subhypergraph, say $R_1^\bullet$, in $Y^\bullet$ that contains the edge $f^\bullet$ is a $H_{14,2}$-component of $H^\bullet$ that does not contain the vertex s_2. This, however, is a contradiction since the vertex of degree 3 in R_1 is also the degree 3 vertex in $R_1^\bullet$, implying that $s_2 \in V(R_1^\bullet)$, a contradiction.

Therefore, the edge f_i intersects all three edges e_1, e_2, e_3 for $i \in [2]$. The hypergraph H is now determined and $H = H_{21,5}$, contradicting Claim C. $_{(\Box)}$

Claim Q: No edge in H contains two degree-3 vertices.

Proof of Claim Q: We first need the following subclaim.

Claim Q.1: There does not exist three degree-3 vertices in H that are pairwise adjacent.

Proof of Claim Q.1: For the sake of contradiction, suppose to the contrary that x_1, x_2 and x_3 are pairwise adjacent degree-3 vertices in H. First consider the case when some edge, e, in H contains all three vertices. Let $V(e) = \{x_1, x_2, x_3, x_4\}$. By Claim P, the hypergraph $H - x_i$ contains an H_4-component, R_i, for all $i \in [3]$. Further, let f_i be the edge in R_i. By Claim M, the edge f_i is intersected by all three edges incident with x_i. Since every vertex in f_i has degree at most 2 in H and $d(x_1) = d(x_2) = d(x_3) = 3$, we note that $x_4 \in V(f_i)$ for all $i \in [3]$, which implies that $f_1 = f_2 = f_3$, which is impossible as the two edges different from e containing x_1 intersect f_1 and therefore R_1 is not an H_4-component in $H - x_2$.

Therefore there exists three distinct edges e_{12}, e_{13} and e_{23}, such that e_{ij} contains x_i and x_j for all $1 \le i < j \le 3$. Let R_i be the H_4-component in $H - x_i$ for $i \in [3]$, and let h_i be the edge in R_i. Note that $V(h_i) \cap \{x_1, x_2, x_3\} = \emptyset$ for all $i \in [3]$ as no vertex in h_i has degree 3. If $h = h_1 = h_2$, then the edge h intersects all edges containing x_1 (as $h = h_1$) and all edges intersecting x_2 (as $h = h_2$), but then h does not belong to an H_4 component in $H - x_1$ (or in $H - x_2$), a contradiction. Hence, h_1, h_2 and h_3 are distinct edges. Furthermore, if $V(h_1) \cap V(h_2) \neq \emptyset$ and $y \in V(h_1) \cap V(h_2)$, then y is contained in the three edges e_{12}, h_1 and h_2, and so $d(y) = 3$, a contradiction to all vertices in h_1 having degree at most two. Therefore, h_1, h_2 and h_3 are non-intersecting edges.

For each $i \in [3]$, let f_i be the edge incident with x_i, different from e_{12}, e_{13} and e_{23}, which exists as $d(x_i) = 3$ and two of the three edges e_{12}, e_{13} and e_{23} contain x_i. By the above definitions, we note that h_1 intersects the three edges e_{12}, e_{13} and f_1, which all contain x_1. Also, h_2 intersects the three edges e_{12}, e_{23} and f_2, which all contain x_2. And finally, h_3 intersects the three edges e_{13}, e_{23} and f_3, which all contain x_3.

Let $Z = V(e_{12}) \cup V(e_{13}) \cup V(e_{23}) \cup V(h_1) \cup V(h_2) \cup V(h_3)$ and note that $|Z| = 15$, as $|V(e_{12}) \cup V(e_{13}) \cup V(e_{23})| = 9$ and each h_i contains two vertices not in $V(e_{12}) \cup V(e_{13}) \cup V(e_{23})$. Also note that the only edges in H intersecting Z are $e_{12}, e_{13}, e_{23}, h_1, h_2, h_3, f_1, f_2, f_3$. Let $H' = H \setminus Z$ and note that $n(H') = n(H) - 15$ and $m(H') = m(H) - 9$. Further, we note that $|V(f_i) \setminus Z| = 2$ for $i \in [3]$. Moreover since H is linear, $V(f_1) \setminus Z$ and $V(f_2) \setminus Z$ can have at most one vertex in common, implying that $|\partial(Z)| \ge 3$.

We wish to use Claim L, so first we need to show that $\mathrm{def}(H') \le 21$. Suppose to the contrary that $\mathrm{def}(H') \ge 22$. Let Y' be a special H'-sets in H' with $\mathrm{def}_{H'}(Y') = \mathrm{def}(H') \ge 22$. By Claim O, $|E^*_{H'}(Y')| \le |Y'| - 3$. This implies that $|E^*_{H}(Y')| \le |E^*_{H'}(Y')| + |\{f_1, f_2, f_3\}| \le (|Y'| - 3) + 3 = |Y'|$, a contradiction to Claim I(a). Therefore, $\mathrm{def}(H') \le 21$.

As observed earlier, $|\partial(Z)| \ge 3$, and so $f(|\partial(Z)|) \le 27$. By Claim L, there exists a transversal, T', in H', that contains a vertex in $\partial(Z)$ such that $45|T'| \le 6n(H') + 13m(H') + f(|\partial(Z)|) \le 6(n(H) - 15) + 13(m(H) - 9) + 27 = 6n(H) + 13m(H) - 180$. Since T' contains a vertex in $\partial(Z)$, the transversal T' contains a vertex from $V(f_1)$, $V(f_2)$ or $V(f_3)$. Without loss of generality, T' contains a vertex from $V(f_1)$. With this assumption, the set $T = T' \cup \{x_3, (e_{12}h_1), (f_2h_2), w\}$ where w is an arbitrary vertex from $V(h_3)$, is a transversal in H of size $|T'| + 4$. Hence, $45\tau(H) \le 45|T| = 45(|T'| + 4) \le 6n(H) + 13m(H) - 180 + 4 \cdot 45 = 6n(H) + 13m(H)$, a contradiction. This proves Claim Q.1 (□)

We now return to the proof of Claim Q. Suppose, to the contrary, that there exists an edge $e = \{x_1, x_2, x_3, x_4\}$, such that $d(x_1) = d(x_2) = 3$ in H. Let e, f_1 and f_2 be the three edges in H containing x_1 and let e, g_1 and g_2 be the three edges in H containing x_2. Let R_i be the H_4-component in $H - x_i$ and let h_i be the edge of R_i for $i \in [2]$, and so $E(R_i) = \{h_i\}$. Renaming vertices if necessary, we may assume that $x_3 \in V(h_1)$ and $x_4 \in V(h_2)$, as h_i intersects all three edges incident with x_i for $i \in [2]$ and h_1 and h_2 are non-intersecting edges. Let $Z' = V(h_1) \cup V(h_2) \cup \{x_1, x_2\}$ and let

$$
\begin{aligned}
Q_1 &= V(g_1) \setminus \{(g_1h_2), x_2\}, & Q_2 &= V(g_2) \setminus \{(g_2h_2), x_2\},\\
Q_3 &= V(f_1) \setminus \{(f_1h_1), x_1\}, & Q_4 &= V(f_2) \setminus \{(f_2h_1), x_1\}.
\end{aligned}
$$

Let $Q' = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ and note that $Q' = \partial(Z') = N_H(Z') \setminus Z'$. Let $H' = H - Z'$. We note that $n(H') = n(H) - 10$ and $m(H') = m(H) - |\{e, f_1, f_2, g_1, g_2, h_1, h_2\}| = m(H) - 7$. Let $R_1 = Q_1 \cup Q_2$ and $R_2 = Q_3 \cup Q_4$. As $|R_1| = |R_2| = 4$ and $Q' = R_1 \cup R_2$ we note that $4 \le |Q'| \le 8$.

Claim Q.2: If there is a transversal, T', in H' containing a vertex from Q', then $\tau(H) \le |T'| + 3$. Furthermore, if T' contains a vertex from three of the sets Q_1, Q_2, Q_3, Q_4, then $\tau(H) \le |T'| + 2$.

Proof of Claim Q.2: Let T' be a transversal in H' containing a vertex from Q'. Renaming vertices if necessary, we may assume that T' intersects Q_1, and therefore covers the edge g_1. With this assumption, the set $T = T' \cup \{x_1, x_3, (g_2h_2)\}$ is a transversal in H of size $|T'| + 3$, where we note that the vertex x_3 covers h_1, the vertex x_1 covers e, f_1 and f_2, and the vertex (g_2h_2) covers g_2 and h_2. Hence, $\tau(H) \le |T| = |T'| + 3$. Furthermore, suppose that the transversal T' of H' contains a vertex from three of the sets Q_1, Q_2, Q_3, Q_4. Without loss of generality, we may assume that T' intersects Q_1, Q_2 and Q_3 and therefore covers the edges g_1, g_2 and f_1. In this case, the set $T = T' \cup \{(eh_2), (f_2h_1)\}$ is a transversal in H of size $|T'| + 2$, implying that $\tau(H) \le |T| = |T'| + 2$. This completes the proof of Claim Q.2 ($\square$)

Claim Q.3: Every vertex in $R_1 \cap R_2$ has degree 2 in H.

Proof of Claim Q.3: Every vertex in $R_1 \cap R_2$ is intersected by one of g_1 or g_2 and one of f_1 or f_2 and therefore has degree at least 2. By Claim Q.1, such a vertex in $R_1 \cap R_2$ cannot have degree 3, which completes the proof of Claim Q.3 ($\square$)

Claim Q.4: $|Q'| \ge 5$.

Proof of Claim Q.4: For the sake of contradiction, suppose that $|Q'| \le 4$. As observed earlier, $|Q'| \ge 4$. Consequently, $|Q'| = 4$ and $Q' = R_1 = R_2$, implying by Claim Q.3 that every vertex in Q has degree 2. The hypergraph H is now determined and by linearity we note that H is isomorphic to $H_{14,4}$, a contradiction. This proves Claim Q.4 ($\square$)

Claim Q.5: $|E^*_{H'}(Y)| \ge |Y|$ for all special H'-sets Y in H' and therefore $\mathrm{def}(H') = 0$.

Proof of Claim Q.5: We first prove that $\mathrm{def}(H') \le 16$. Suppose to the contrary that $\mathrm{def}(H') \ge 17$. Let Y be a special H'-sets with $\mathrm{def}_{H'}(Y) = \mathrm{def}(H') \ge 17$. By Claim O, $|E^*_{H'}(Y)| \le |Y| - 3$. Let R_1 and R_2 be the H_4-subhypergraphs of H with edges h_1 and h_2, respectively. Let $Y^* = Y \cup \{R_1, R_2\}$, and so $|Y^*| = |Y| + 2$. Thus, $|E^*_H(Y^*)| \le |E^*_{H'}(Y)| + |\{e, f_1, f_2, g_1, g_2\}| \le |Y| - 3 + 5 = |Y^*|$, contradicting Claim I(a). Therefore $\mathrm{def}(H') \le 16$.

For the sake of contradiction suppose that $|E^*_{H'}(Y)| < |Y|$ for some special H'-sets Y in H'. By Claim K, there exists a transversal T' in H', such that $45|T'| \le 6n(H') + 13m(H') + \mathrm{def}(H')$ and $T' \cap Q' \ne \emptyset$. By Claim Q.2 we note that $\tau(H) \le |T'| + 3$. Recall that $n(H') = n(H) - 10$ and $m(H') = m(H) - 7$. Thus, $45\tau(H) \le 45(|T'| + 3) \le 6n(H') + 13m(H') + \mathrm{def}(H') + 135 \le 6(n(H) - 10) + 13(m(H) - 7) + 16 + 135 = 6n(H) + 13m(H)$, a contradiction. Therefore, $|E^*_{H'}(Y)| \ge |Y|$ for all special H'-sets Y in H', and so $\mathrm{def}(H') = 0$ which proves Claim Q.5 ($\square$)

Claim Q.6: $|Q'| = 8$.

Proof of Claim Q.6: For the sake of contradiction suppose that $|Q'| \neq 8$. By Claim Q.4 and the fact that $|Q'| \le 8$ we obtain $5 \le |Q'| \le 7$. Therefore, $R = R_1 \cap R_2$ is non-empty and by Claim Q.3 every vertex in R has degree 2 in H and therefore degree zero in H'.

Let $Z'' = Z' \cup R$, let $Q'' = Q' \setminus R$ and note that $Q'' = \partial(Z'')$. Let $H'' = H - Z''$. By Claim Q.5 we note that $\mathrm{def}(H') = 0$ and therefore $\mathrm{def}(H'') = 0$ also holds (as we only removed isolated vertices). Note that $n(H'') = n(H) - 10 - |R|$ and $m(H'') = m(H') = m(H) - 7$. Further, we note that $|Q'| = 8 - |R|$ and $|Q''| = |Q'| - |R| = 8 - 2|R|$. By Claim L, there exists a transversal, T'', in H'' containing a vertex from Q'', such that

$$\begin{aligned}
45|T''| &\le 6n(H'') + m(H'') + f(|Q''|) \\
&\le 6(n(H) - 10 - |R|) + 13(m(H) - 7) + f(8 - 2|R|) \\
&= 6n(H) + 13m(H) - 3 \cdot 45 - 16 - 6|R| + f(8 - 2|R|).
\end{aligned}$$

If $|R| = 1$, then $f(8-2|R|) - 16 - 6|R| = 22 - 22 = 0$. If $|R| = 2$, then $f(8-2|R|) - 16 - 6|R| = 23 - 28 = -6$. If $|R| = 3$, then $f(8 - 2|R|) - 16 - 6|R| = 33 - 34 = -1$. So in all cases, $f(8 - 2|R|) - 16 - 6|R| \le 0$, implying that $45|T''| \le 6n(H) + 13m(H) - 3 \cdot 45$. Thus by Claim Q.2, $45\tau(H) \le 45(|T'| + 3) \le 6n(H) + 13m(H)$, a contradiction. This completes the proof of Claim Q.6 ($\Box$)

By Claim Q.6 we note that Q_1, Q_2, Q_3 and Q_4 are vertex disjoint. We will now define the following four different hypergraphs.

Let H_1^* be the hypergraph obtained from H' by adding three vertices x_{23}, x_{24} and x_{34} and three hyperedges $h_2^* = Q_2 \cup \{x_{23}, x_{24}\}$, $h_3^* = Q_3 \cup \{x_{23}, x_{34}\}$ and $h_4^* = Q_4 \cup \{x_{24}, x_{34}\}$.

Analogously let H_2^* be the obtained from H' by adding three vertices y_{13}, y_{14} and y_{34} and three hyperedges $e_1^* = Q_1 \cup \{y_{13}, y_{14}\}$, $e_3^* = Q_3 \cup \{y_{13}, y_{34}\}$ and $e_4^* = Q_4 \cup \{y_{14}, y_{34}\}$.

Let H_3^* be the obtained from H' by adding three vertices z_{13}, z_{14} and z_{34} and three hyperedges $f_1^* = Q_1 \cup \{z_{12}, z_{14}\}$, $f_2^* = Q_3 \cup \{z_{12}, z_{24}\}$ and $f_4^* = Q_4 \cup \{z_{14}, z_{24}\}$.

Finally let H_4^* be the obtained from H' by adding three vertices w_{12}, w_{13} and w_{23} and three hyperedges $g_1^* = Q_1 \cup \{w_{12}, w_{13}\}$, $g_2^* = Q_2 \cup \{w_{12}, w_{23}\}$ and $g_3^* = Q_3 \cup \{w_{13}, w_{23}\}$.

Claim Q.7: $\tau(H) \le \tau(H_i^*) + 2$, for $i \in [4]$.

Proof of Claim Q.7: Let T^* be a minimum transversal in H_1^*. If $T^* \cap \{x_{23}, x_{24}, x_{34}\} = \emptyset$, then T^* covers Q_2, Q_3 and Q_4, and by Claim Q.2, $\tau(H) \le |T^*| + 2$. We may therefore, without loss of generality, assume that $x_{23} \in T^*$. We may also assume that $T^* \cap \{x_{24}, x_{34}\} = \emptyset$, since otherwise we could have picked a vertex from Q_4 instead of this vertex (or vertices). Therefore, Q_4 is covered by $T^* \setminus \{x_{23}\}$. By Claim Q.2, $\tau(H) \le |T^* \setminus \{x_{23}\}| + 3 = |T^*| + 2$, which completes the proof for H_1^*. Analogously, the claim holds for H_i^*, with $i = 2, 3, 4$. ($\Box$)

Claim Q.8: $\mathrm{def}(H_i^*) \ge 5$, for $i \in [4]$.

Proof of Claim Q.8: For the sake of contradiction, suppose to the contrary that $\mathrm{def}(H_1^*) \le 4$ and let T^* be a minimum transversal in H_1^*. Applying the inductive hypothesis to H'', we have by Claim Q.7 that $45\tau(H) \le 45(|T^*|+2) \le 6n(H_1^*) + 13m(H_1^*) + \mathrm{def}(H_1^*) + 90 \le 6(n(H)-7)+13(m(H)-4)+4+90 = 6n(H)+13m(H)$, a contradiction. Hence, $\mathrm{def}(H_1^*) \ge 5$. Analogously, $\mathrm{def}(H_i^*) \ge 5$, for $i = 2,3,4$. $_{(\Box)}$

Let Y_i^* be a special H_i^*-set satisfying $\mathrm{def}(H_i^*) = \mathrm{def}_{H_i^*}(Y_i^*)$ for $i \in [4]$.

Claim Q.9: $|Y_1^*| = 1$ and Y_1^* is an H_{14}-component in H_1^* containing the edges h_2^*, h_3^*, h_4^*.

Proof of Claim Q.9: If none of h_2^*, h_3^* or h_4^* belong to $E(Y_1^*)$, then in this case $|E^*(Y_1^*)| \le |Y_1^*| - 1$ in H_1^*, which implies that $|E^*(Y_1^*)| \le |Y_1^*| - 1$ in H', contradicting Claim Q.5. Therefore, without loss of generality, we may assume that $h_2^* \in E(Y_1^*)$.

Suppose that there is an H_4-component, R^*, in Y_1^* that contains the edge h_2^*. Thus $E(R^*) = \{h_2^*\}$. In this case, we note that $h_3^*, h_4^* \in E^*(Y_1^*)$ in H_1^*. As $\mathrm{def}(H_1^*) \ge 5$ and $|E^*(Y_1^*)| \ge 2$, we have $|E^*(Y_1^*)| \le |Y_1^*| - 2$ in H_1^*. Let $Y_1' = Y_1^* \setminus \{R_1\}$. Since $h_3^*, h_4^* \in E^*(Y_1^*)$, we note that $|E^*(Y_1')| \le |Y_1^*| - 2 - 2 = |Y_1'| - 3$ in H', contradicting Claim Q.5. Therefore, R^* is not an H_4-component in Y_1^*. Analogously, there is no H_4-component in Y_1^* that contains the edge h_3^* or h_4^*.

If two of the three edges h_2^*, h_3^*, h_4^* belong to $E(Y_1^*)$ in H_1^*, then these two edges, which intersect, both contain a degree-1 vertex in Y_1^*. However, there is no special hypergraph with two intersecting edges both containing a degree-1 vertex, a contradiction. Therefore, we must have all three edges h_2^*, h_3^*, h_4^* belonging to $E(Y_1^*)$.

Now consider the case when $|Y_1^*| \ge 2$. Let $Y_1^* = \{R_1^*, R_2^*, \ldots, R_\ell^*\}$, where $\ell \ge 2$. Renaming the elements of Y_1^*, we may assume that $\{h_2^*, h_3^*, h_4^*\} \subseteq E(R_1^*)$. As $\mathrm{def}(H_1^*) \ge 5$, we must have $|E^*(Y_1^*)| \le |Y_1^*| - 2$ in H_1^*. This implies that in H we have $|E^*(Y_1^* \setminus \{R_1^*\})| \le |Y_1^*| - 2 + |\{g_1\}| = |Y_1^* \setminus \{R_1^*\}|$, contradicting Claim I(a). Therefore, $|Y_1^*| = 1$ and $E^*(Y_1^*) = \emptyset$ in H^*. As all three edges h_2^*, h_3^*, h_4^* belong to $E(Y_1^*)$, we note that Y_1^* is not an H_4-component. As $\mathrm{def}(H_1^*) \ge 5$ we must therefore have that Y_1^* is an H_{14}-component, completing the proof of Claim Q.9. $_{(\Box)}$

We note that an analogous claim to Claim Q.9 also holds for H_2^*, H_3^* and H_4^*.

Claim Q.10: Both vertices in at least two of the sets Q_1, Q_2, Q_3, Q_4 have degree 1 in H'.

Proof of Claim Q.10: For the sake of contradiction, suppose to the contrary that Q_2, Q_3 and Q_4 contain a vertex which does not have degree 1 in H'. If it is an isolated vertex, then without loss of generality assume that $y \in Q_2$ is isolated in H'. Consider the hypergraph $H'' = H - y$ and let $Z'' = Z' \cup \{y\}$. By Claim Q.5 we note that $\mathrm{def}(H'') = \mathrm{def}(H') = 0$. Furthermore, $H'' = H - Z''$. Let $Q'' = N(Z'') \setminus Z''$, and so $Q'' = \partial(Z'')$. By Claim Q.6, we note that $|Q''| = |Q' \setminus \{y\}| = 7$. By Claim L, there exists a transversal, T'', in H'' containing a vertex from Q'', such that the following holds.

$$
\begin{array}{rcl}
45|T''| & \le & 6n(H'') + m(H'') + f(|Q''|) \\
& \le & 6(n(H) - 11) + 13(m(H) - 7) + f(7) \\
& = & 6n(H) + 13m(H) - 66 - 91 + 22 \\
& = & 6n(H) + 13m(H) - 3 \times 45.
\end{array}
$$

By Claim Q.2, this implies that $45\tau(H) \le 45(|T''| + 3) \le 6n(H) + 13m(H)$, a contradiction. Therefore, Q_2, Q_3 and Q_4 do not contain isolated vertices in H' and they therefore all contain a vertex of degree at least 2 (in fact, equal to 2), which implies that they have degree 3 in H_1^*. As all these vertices belong to Y_1^*, we get a contradiction to the fact that no H_{14}-component contains three degree-3 vertices. This completes the proof of Claim Q.10 ($\Box$)

By Claim Q.10, we assume without loss of generality that both vertices in Q_1 and in Q_2 have degree 1 in H', which implies that the vertices in Q_1 have degree 1 in H_1^* and all vertices in the edge h_2^* (which contains Q_2) have degree 2 in H_1^*.

Claim Q.11: $Q' \subseteq V(Y_1^*)$.

Proof of Claim Q.11: By Claim Q.9 we note that $Q_i \subseteq V(Y_1^*)$ for $i = 2, 3, 4$. For the sake of contradiction, suppose to the contrary that $Q_1 \not\subseteq V(Y_1^*)$. As Y_1^* is an H_{14}-component and all vertices in h_2^* have degree 2 in H_1^* and therefore also in Y_1^*, we note that $Y_1^* - h_2^*$ is a component containing fourteen vertices. This implies that the component containing $\{x_{23}, x_{24}, x_{34}\}$ in H_2^* has more than fourteen vertices as it contains all vertices of $V(Y_1^*)$ as well as Q_1. This contradiction to Claim Q.9 completes the proof of Claim Q.11. ($\Box$)

By Claim Q.11 we note that Y_1^* contains two degree-1 vertices (namely, the two vertices in Q_1) and an edge consisting of only degree-2 vertices (namely, the edge h_2^*). However no H_{14} component has these properties, a contradiction. This completes the proof of Claim Q. ($\Box$)

By Claim M and Claim P, if x is an arbitrary vertex of H of degree 3, then $H - x$ contains an H_4-component that is intersected by all three edges incident with x, and $\mathrm{def}(H - x) = 8$. By Claim Q, no edge in H contains two degree-3 vertices.

Claim R: Every degree-3 vertex in H has at most one neighbor of degree 1.

Proof of Claim R: Suppose, to the contrary, that there exists a vertex x of degree 3 in H with at least two degree-1 neighbors, say y and z. Let $H' = H - \{x, y, z\}$, and note that $n(H') = n(H) - 3$ and $m(H') = m(H) - 3$. Let e be the edge in the H_4-component of $H - x$. By Claim M and Claim P, the edge e is intersected by all three edges incident with x, and contains a vertex (of degree 1 in H) not adjacent to x. Further by Claim M, $\mathrm{def}(H - x) = 8$. We note that $\mathrm{def}(H') = \mathrm{def}(H - x)$ since H' is obtained from $H - x$ by deleting the two isolated vertices y and z in $H - x$. Every transversal in H' can be extended to a transversal in H by adding to it the vertex x, and so $\tau(H) \le \tau(H') + 1$. Applying the inductive hypothesis to H', we have that $45\tau(H) \le 45(\tau(H') + 1) \le 6n(H') + 13m(H') + \mathrm{def}(H') + 45 = 6(n(H) - 3) + 13(m(H) - 3) + 8 + 45 = 6n(H) + 13m(H) - 4 < 6n(H) + 13m(H)$, a contradiction. ($\Box$)

By Claim R, every degree-3 vertex in H has at most one neighbor of degree 1. We now define the operation of *duplicating* a degree-3 vertex x as follows.

Let e_1, e_2 and e_3 be the three edges incident with x. By Claim Q and Claim R, every neighbor of x has degree 2, except possibly for one vertex which has degree 1. Renaming edges if necessary, we may assume that the edge e_1 contains no vertex of degree 1, and therefore every vertex in e_1 different from x has degree 2. We now delete the edge e_1 from

H, and add a new vertex x' and a new edge $e_1' = (V(e_1) \setminus \{x\}) \cup \{x'\}$ to H. We note that in the resulting hypergraph the vertex x now has degree 2 (and is incident with the edges e_2 and e_3) and the new vertex x' has degree 1 with all its three neighbors of degree 2. We call x' the *vertex duplicated copy* of x.

Let H' be obtained from H by duplicating every degree-3 vertex as described above. By construction, H' is a linear 4-uniform connected hypergraph with minimum degree $\delta(H') \ge 1$ and maximum degree $\Delta(H') \le 2$. For $i \in [2]$, let $n_i(H')$ be the number of vertices of degree i in H'. Then, $n(H') = n_1(H') + n_2(H')$ and $4m(H') = 2n_2(H') + n_1(H')$. We proceed further with the following properties of the hypergraph H'.

Claim S: The following properties hold in the hypergraph H'.
(a) $\tau(H) = \tau(H')$.
(b) $\mathrm{def}(H') = 0$.

Proof of Claim S: In order to show that $\tau(H) = \tau(H')$ we show that the operation of duplicating a degree-3 vertex x leaves the transversal number unchanged. Let x', e_1, e_2 and e_3 be defined as in the description of duplication. Let e_x be the edge in the H_4-component in $H - x$ and let H_x be the hypergraph obtained from H by duplicating x. Let x_1^* be the vertex (e_1e_x) in $V(e_x) \cap V(e_1)$.

Let T_x be a transversal in H_x. As $d_{H'}(x') = 1$, we may assume that $x' \notin T_x$. Now we note that T_x is also a transversal in H and therefore $\tau(H) \le \tau(H_x)$. Conversely assume that T is a transversal in H. If $x \notin T$, let $T' = T$. If $x \in T$, then since T is a transversal in H, there exists a vertex $y \in V(e_x) \cap T$. In this case, we let $T' = (T \setminus \{y\}) \cup \{x_1^*\}$. In both cases, $|T'| = |T|$ and T' is a transversal in H_x, implying that $\tau(H_x) \le |T'| = |T| = \tau(H)$. Consequently, $\tau(H) = \tau(H_x)$, which implies that $\tau(H) = \tau(H')$ and (a) holds.

In order to prove part (b) we, for the sake of contradiction, suppose that $\mathrm{def}(H') > 0$. Let Y be a special H'-set satisfying $\mathrm{def}_{H'}(Y) = \mathrm{def}(H') > 0$. By Claim N, no edge in H contains two vertices of degree 1. We note that this is still the case after every duplication and therefore no edge in H' contains two vertices of degree 1. This implies that every H_4-component in Y_4 is intersected by at least three edges of $E^*(Y)$.

If $Y \setminus Y_4 \neq \emptyset$, let $R \in Y \setminus Y_4$ be arbitrary. Since $\Delta(H') \le 2$, we note that $R \in \{H_{10}, H_{11}, H_{14,5}, H_{14,6}\}$. Let x be a vertex of degree 3 in H and let x' be the duplicated copy of x in H'. Furthermore let e_1' be the edge containing x' in H' and let e_x be the edge in the H_4-component in $H - x$. Finally let e_2 and e_3 be the edges containing x in H'. Since e_x and e_1' both contain a degree-1 vertex and intersect each other we note that they both cannot belong to R. If $e_1' \in E(R)$, then it would therefore contain two degree-1 vertices in R (namely x' and the vertex (e_xe_1') in $V(e_x) \cap V(e_1')$), a contradiction. Therefore, $e_1' \notin E(R)$. Analogously $e_x \notin E(R)$. The edges e_2 and e_3 cannot both belong to R as they intersect and would both contain degree-1 vertices in R (namely the vertices (e_2e_x) and (e_3e_x) in $V(e_2) \cap V(e_x)$ and $V(e_3) \cap V(e_x)$, respectively). However if $e_2 \in E(R)$, then e_2 would contain two degree-1 vertices in R (namely x and the vertex (e_2e_x) in $V(e_2) \cap V(e_x)$), a contradiction. Therefore, $e_2 \notin R$ and analogously $e_3 \notin R$. This implies that R was also a special hypergraph in H. By Claim J, R is not isomorphic to H_{10}, and so $R \in \{H_{11}, H_{14,5}, H_{14,6}\}$. Furthermore if

R is intersected by k edges in H', then it is also intersected by k edges in H, and so by Claim I(a) we note that at least three edges intersect R in H and therefore also in H'. Therefore, all components in Y are intersected by at least three edges from $|E^*(Y)|$. This implies that

$$4|E^*(Y)| \ge \sum_{e \in E^*(Y)} |V(e) \cap V(Y)| \ge 3|Y|,$$

and so $13|E^*(Y)| \ge \frac{39}{4}|Y| > 8|Y|$. As observed earlier, $Y_{10} = \emptyset$. Hence, $\mathrm{def}(Y) \le 8|Y| - 13|E^*(Y)| < 0$, a contradiction. Therefore $\mathrm{def}(H') = 0$. This completes the proof of Claim S. $(\Box)$

We now consider the multigraph G whose vertices are the edges of H' and whose edges correspond to the $n_2(H')$ vertices of degree 2 in H': if a vertex of H' is contained in the edges e and f of H', then the corresponding edge of the multigraph G joins vertices e and f of G. By the linearity of H', the multigraph G is in fact a graph, called the *dual* of H'. We shall need the following properties about the dual G of the hypergraph H'.

Claim T: The following properties hold in the dual, G, of the hypergraph H'.
(a) G is connected, $n(G) = m(H')$ and $m(G) = n_2(H')$.
(b) $\Delta(G) \le 4$, and so $m(G) \le 2n(G)$ and $8n(G) + 6m(G) \le 20n(G)$.
(c) $\tau(H') = m(H') - \alpha'(G)$.

Proof of Claim T: Since H' is connected, so too is G by construction. Further by construction, $n(G) = m(H')$ and $m(G) = n_2(H')$. Since H' is 4-uniform and $\Delta(H') \le 2$, we see that $\Delta(G) \le 4$, implying that $m(G) \le 2n(G)$ and $8n(G) + 6m(G) \le 20n(G)$. This establishes Part (a) and Part (b). Part (c) is well-known (see, for example, [12, 26]), so we omit the details of this property. $(\Box)$

Suppose that x is a vertex of degree 3 in H, and let x' be the vertex duplicated from x when constructing H. Adopting our earlier notation, let e_1, e_2 and e_3 be the three edges incident with x. Further, let e be the edge in the H_4-component of $H - x$, and let y be the vertex of degree 1 in e. Let y_i be the vertex common to e and e_i for $i \in [3]$, and so $y_i = (ee_i)$ and $V(e) = \{y, y_1, y_2, y_3\}$. We note that in the graph G, which is the dual of H', the vertex e has degree 3 and is adjacent to the vertices e_1, e_2 and e_3. Further, we note that in the graph G, the vertex e_1 has degree 3, while the vertices e_2 and e_3 are adjacent and have degree at most 4. Further, the edge y_i in G is the edge ee_i, while the edge x in G is the edge e_2e_3. The vertex e_1 is adjacent in G to neither e_2 nor e_3. This set of four vertices $\{e, e_1, e_2, e_3\}$ in the graph G we call a *quadruple* in G. We illustrate this quadruple in G in Figure 11. We denote the set of (vertex-disjoint) quadruples in G by Q.

We shall need the following additional property about the dual G of the hypergraph H'.

Claim U: If G is the dual of the hypergraph H', then

$$\begin{array}{rl} & 45\tau(H') \le 6n(H') + 13m(H') - 6|Q| \\ \Leftrightarrow & 45\alpha'(G) \ge 8n(G) + 6m(G) + 6|Q|. \end{array}$$

Proof of Claim U: Recall that $n(H') = n_1(H') + n_2(H')$ and $4m(H') = 2n_2(H') + n_1(H')$.

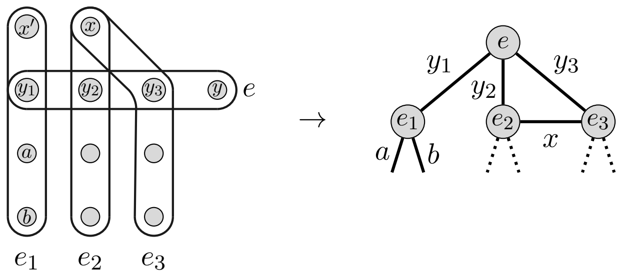


Figure 11: The transformation creating a quadruple.

The following holds by Claim T:

$$
\begin{array}{rrcl}
 & 45\tau(H') & \leq & 6n(H') + 13m(H') - 6|Q| \\
\Leftrightarrow & 45(m(H') - \alpha'(G)) & \leq & 6n(H') + 13m(H') - 6|Q| \\
\Leftrightarrow & 45\alpha'(G) & \geq & 32m(H') - 6n(H') + 6|Q| \\
\Leftrightarrow & 45\alpha'(G) & \geq & (16n_2(H') + 8n_1(H')) - (6n_2(H') + 6n_1(H')) + 6|Q| \\
\Leftrightarrow & 45\alpha'(G) & \geq & 10n_2(H') + 2n_1(H') + 6|Q| \\
\Leftrightarrow & 45\alpha'(G) & \geq & 8m(H') + 6n_2(H') + 6|Q| \\
\Leftrightarrow & 45\alpha'(G) & \geq & 8n(G) + 6m(G) + 6|Q|.
\end{array}
$$

This completes the proof of Claim U. $(\Box)$

Let S be a set of vertices in G such that $(n(G) + |S| - \mathrm{oc}(G - S))/2$ is minimum. By the Tutte-Berge Formula,

$$\alpha'(G) = \frac{1}{2}\left(n(G) + |S| - \mathrm{oc}(G - S)\right). \tag{9}$$

We now consider two cases, depending on whether $S = \emptyset$ or $S \neq \emptyset$.

Claim V: If $S \neq \emptyset$, then $45\tau(H') \leq 6n(H') + 13m(H') - 6|Q|$.

Proof of Claim V: Suppose that $S \neq \emptyset$. For $i \geq 1$, let $n_i(G - S)$ denote the number of components on $G - S$ of order i. Let $n_5^1(G - S)$ be the number of components of $G - S$ isomorphic to $K_5 - e$ and let $n_5^2(G - S)$ denote all remaining components of $G - S$ on five vertices (with at most eight edges), and so $n_5(G-S) = n_5^1(G-S) + n_5^2(G-S)$. For notational convenience, let $n = n(G)$, $m = m(G)$, $n_5^1 = n_5^1(G-S)$, $n_5^2 = n_5^2(G-S)$, and $n_i = n_i(G-S)$ for $i \geq 1$. Let $\mathbb{Z}^+$ denote the set of all positive integers, and let $\mathbb{Z}^+_{\text{even}}$ and $\mathbb{Z}^+_{\text{odd}}$ denote the set of all even and odd integers, respectively, in $\mathbb{Z}^+$. Further for a fixed $j \in \mathbb{Z}^+$, let $\mathbb{Z}_{\geq j} = \{i \in \mathbb{Z} \mid i \geq j\}$, $\mathbb{Z}^j_{\text{even}} = \{i \in \mathbb{Z}_{\geq j} \mid i \text{ even}\}$, and $\mathbb{Z}^j_{\text{odd}} = \{i \in \mathbb{Z}_{\geq j} \mid i \text{ odd}\}$. We note that

$$n = |S| + \sum_{i \in \mathbb{Z}^+} i \cdot n_i. \tag{10}$$

By Equation (9) and Equation (10), and since

$$\mathrm{oc}(G - S) = \sum_{i \in \mathbb{Z}^+_{\text{odd}}} n_i,$$

we have the equation

$$45\alpha'(G) = 45|S| + \frac{45}{2}\left((\sum_{i\in\mathbb{Z}^3_{\text{odd}}} (i-1)\cdot n_i) + \sum_{i\in\mathbb{Z}^2_{\text{even}}} i\cdot n_i \right). \tag{11}$$

Claim V.1: $m \le 4|S| + n_2 + 3n_3 + 6n_4 + 9n_5^1 + 8n_5^2 + \sum_{i\in\mathbb{Z}^6}(2i-1)n_i - |Q|$.

Proof of Claim V.1: Since G is connected and $\Delta(G) \le 4$, we note that if F is a component of $G-S$ of order i, then $m(F) \le 2i-1$. Further, every component of $G-S$ of order 5 is either isomorphic to $K_5 - e$ or contains at most eight edges, while every component of $G-S$ of order 2, 3 and 4 contains at most 1, 3 and 6 edges, respectively. The above observations imply that

$$m \le 4|S| + n_2 + 3n_3 + 6n_4 + 9n_5^1 + 8n_5^2 + \sum_{i\in\mathbb{Z}^6}(2i-1)n_i.$$

We show next that each quadruple in the graph G decreases the count on the right hand side expression of the above inequality by at least 1. Adopting our earlier notation, consider a quadruple $\{e, e_1, e_2, e_3\}$. Recall that the vertices e and e_1 both have degree 3 in G, and there is no vertex in G that is adjacent to both e and e_1. Further, recall that the vertices e_2 and e_3 are adjacent in G. If e or e_1 or if both e_2 and e_3 belong to the set S, then the quadruple decreases the count $4|S|$ by at least 1. Hence, we may assume that e, e_1 and e_2 all belong to a component, C say, of $G-S$. In particular, we note that C has order at least 3. Abusing notation, we say that the component C contains the quadruple $\{e, e_1, e_2, e_3\}$, although possibly the vertex e_3 may belong to S. Since no vertex in G is adjacent to both e and e_1, we note that if the component C has order 3, 4 or 5, then it contains at most 2, 4 and 7 edges, respectively. Further, since we define a component to contain a quadruple if it contains at least three of the four vertices in the quadruple, we note in this case when the component C has order at most 5 that it contains exactly one quadruple. Further, this quadruple decreases the count $3n_3 + 6n_4 + 9n_5^1 + 8n_5^2$ by at least 1.

It remains for us to consider a component F of $G-S$ of order $i \ge 6$ that contains q quadruples, and to show that these q quadruples decrease the count $2i-1$ by at least q. We note that each quadruple contains a pair of adjacent vertices of degree 3 in G. Further, at least one vertex v in F is joined to at least one vertex of S in G, implying that $d_F(v) < d_G(v)$. These observations imply that $2m(F) = \sum_{v\in V(F)} d_F(v) \le 4n(F) - 2q - 1 = 4i - 2q - 1$, and therefore that $m(F) \le 2i - q - 1$. Hence, these q quadruples contained in F combined decrease the count $2i-1$ by at least q. This completes the proof of Claim V.1. ($\Box$)

By Claim V.1 and by Equation (10), we have

$$\begin{aligned} 8n + 6m + 6|Q| \;\le\;& 32|S| + 8n_1 + 22n_2 + 42n_3 + 68n_4 \\ & + 88n_5^2 + 94n_5^1 + \sum_{i\in\mathbb{Z}^6}(20i-6)n_i. \end{aligned} \tag{12}$$

Let

$$\Sigma_{\rm even} = \sum_{i \in \mathbb{Z}^6_{\rm even}} \left(\frac{5}{2} i + 6 \right) \cdot n_i \qquad \text{and} \qquad \Sigma_{\rm odd} = \sum_{i \in \mathbb{Z}^7_{\rm odd}} \frac{1}{2} \left(5i - 33 \right) \cdot n_i.$$

We note that every (odd) component in G isomorphic to K_1 corresponds to a subhypergraph H_4 in H', while every (odd) component in G isomorphic to $K_5 - e$ corresponds to a subhypergraph H_{11} in H'. Hence the odd components of G isomorphic to K_1 or isomorphic to $K_5 - e$ correspond to a special H'-set, X say, where $|X| = |X_4| + |X_{11}|$, $|X_4| = n_1$ and $|X_{11}| = n_5^1$. Further, the set S of vertices in G correspond to the set $E^*(X)$ of edges in H', and so $|E^*(X)| \le |S|$. Thus,

$$\mathrm{def}_{H'}(X) = 8|X_4| + 4|X_{11}| - 13|E^*(X)| \ge 8n_1 + 4n_5^1 - 13|S|.$$

By Claim S(b), $\mathrm{def}_{H'}(X) \le \mathrm{def}(H') = 0$, and therefore we have that

$$13|S| \ge 8n_1 + 4n_5^1. \tag{13}$$

By Equation (11), and by Inequalities (12) and (13), and noting that $\Sigma_{\rm even} \ge 0$ and $\Sigma_{\rm odd} \ge 0$, the following now holds.

$$\begin{array}{rcl}
45\alpha'(G) & \stackrel{(11)}{=} & 32|S| + 8n_1 + 22n_2 + 42n_3 + 68n_4 + 88n_5^2 + 94n_5^1 + \displaystyle\sum_{i \in \mathbb{Z}^6} (20i - 6)n_i \\
& & +13|S| - 8n_1 + 23n_2 + 3n_3 + 22n_4 + 2n_5^2 - 4n_5^1 + \Sigma_{\rm even} + \Sigma_{\rm odd} \\
& \stackrel{(12)}{\ge} & (8n + 6m + 6|Q|) + (13|S| - 8n_1 - 4n_5^1) \\
& \stackrel{(13)}{\ge} & 8n + 6m + 6|Q|.
\end{array}$$

Claim V now follows from Claim U. (□)

Claim W: If $S = \emptyset$, then $45\tau(H) \le 6n(H) + 13m(H) - 6|Q|$.

Proof of Claim W: Suppose that $S = \emptyset$. Then, $\alpha'(G) = (n(G) - \mathrm{oc}(G))/2$. Since G is connected by Claim T, we have the following.

$$\alpha'(G) = \begin{cases} \frac{1}{2} n(G) & \text{if } n(G) \text{ is even} \\ \frac{1}{2}(n(G) - 1) & \text{if } n(G) \text{ is odd.} \end{cases}$$

By Claim T(b), $\Delta(G) \le 4$. As every quadruple in G contains two vertices of degree 3,

$$2m(G) = \sum_{v \in G} d_G(v) \le 4n(G) - 2|Q|,$$

implying that $12n(G) \ge 6m(G) + 6|Q|$. If $n(G)$ is even, then $\alpha'(G) = n(G)/2$, and so

$$45\alpha'(G) = \frac{45}{2} n(G) > 8n(G) + 12n(G) \ge 8n(G) + 6m(G) + 6|Q|.$$

This completes the case when $n(G)$ is even by Claim U. Suppose next that $n(G)$ is odd. In this case $45\alpha'(G) = 45(n(G)-1)/2$, and so

$$90\alpha'(G) = 45n(G) - 45 = 21n(G) + 24n(G) - 45 \geq 21n(G) + 12m(G) + 12|Q| - 45.$$

If $5n(G) \geq 45$, then $90\alpha'(G) \geq 16n(G) + 12m(G) + 12|Q|$, which completes the proof by Claim U. We may therefore assume that $5n(G) < 45$, implying that $n(G) \in \{1, 3, 5, 7\}$. Since every quadruple contains four vertices, we must therefore have $|Q| \leq 1$.

We first consider the case when $|Q| = 1$. In this case $n(G) \geq 6$, as the quadruple contains four vertices and one vertex (called e_1 in the definition of a quadruple) has two neighbours outside the quadruple. Therefore, $n(G) = 7$. Since two vertices in the quadruple have degree at most 3, $2m(G) = \sum_{v \in V(G)} d(v) \leq 4n(G) - 2 = 26$, and so $m(G) \leq 13$. If $m(G) \leq 12$, then $45\alpha'(G) = 45 \cdot 3 > 8 \cdot 7 + 6 \cdot 12 + 6 \geq 8n(G) + 6m(G) + 6|Q|$, and the desired result follows from Claim U. Therefore, we may assume that $m(G) = 13$, for otherwise the case is complete. In this case, all vertices in G have degree 4 except for two vertices in the quadruple which have degree 3. Let $\{e, e_1, e_2, e_3\}$ be the vertices in the quadruple in G, such that $d(e) = d(e_1) = 3$ and e, e_2 and e_3 form a 3-cycle in G. Define u_1 and u_2 such that $N(e_1) = \{e, u_1, u_2\}$ and define w, such that $V(G) = \{e, e_1, e_2, e_3, u_1, u_2, w\}$. As $d(w) = 4$ and w is not adjacent to e or e_1 we have $N(w) = \{e_2, e_3, u_1, u_2\}$. Therefore, e_2 is adjacent to e, e_3 and w. Its fourth neighbour is either u_1 or u_2. Renaming u_1 and u_2 if necessary, we may assume that $N(e_2) = \{e, e_3, w, u_2\}$. This implies that u_1 must be adjacent to u_2 and e_3 and G is the graph shown in Figure 12.

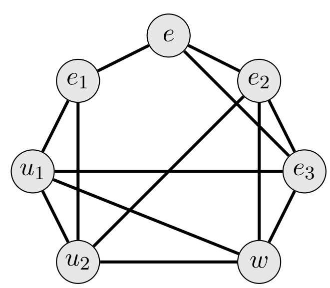


Figure 12: The graph G if $|Q| = 1$.

If we draw the corresponding hypergraph whose dual is the graph G, we note that it is obtained by duplicating the degree-3 vertex in $H_{14,3}$. However, H is not equal to $H_{14,3}$ by Claim C, implying that G cannot be the graph in Figure 12, a contradiction. This completes the case when $|Q| = 1$. Therefore, we may assume that $|Q| = 0$.

If $n(G) = 1$, then $H = H_4$, contradicting Claim C. Hence, $n(G) \in \{3, 5, 7\}$. Suppose that $n(G) = 3$. Then, $\alpha'(G) = 1$ and $m(G) \leq 3$. In this case, $8n(G) + 6m(G) \leq 8 \cdot 3 + 6 \cdot 3 = 42 < 45 = 45\alpha'(G)$, which by Claim U completes the proof. Hence we may assume that $n(G) = 5$ or $n(G) = 7$.

Suppose that $n(G) = 5$. Then, $\alpha'(G) = 2$ and by Claim T(b), $m(G) \leq 10$. If $m(G) = 10$, then $G = K_5$. In this case, H is a 4-uniform 2-regular linear intersecting hypergraph. However, H_{10} is the unique such hypergraph as shown, for example, in [12, 26]. Thus if

$m(G) = 10$, then $H = H_{10}$, contradicting Claim C. Hence, $m(G) \le 9$. If $m(G) = 9$, then $G = K_5 - e$, where e denotes an arbitrary edge in K_5. In this case, $H = H_{11}$, contradicting Claim C. Hence, $m(G) \le 8$. Thus, $8n(G) + 6m(G) \le 8 \cdot 5 + 6 \cdot 8 = 88 < 90 = 45\alpha'(G)$, which by Claim U completes the proof in this case.

Finally suppose that $n(G) = 7$. Then, $\alpha'(G) = 3$ and by Claim T(b), $m(G) \le 14$. Suppose that $m(G) = 14$. Then, G is a 4-regular graph of order 7. Equivalently, the complement, $\overline{G}$, of G is a 2-regular graph of order 7. If $\overline{G} = C_3 \cup C_4$, then $H = H_{14,2}$. If $\overline{G} = C_7$, then $H = H_{14,4}$. Both cases contradict Claim C. Hence, $m(G) \le 13$. Thus, $8n(G) + 6m(G) \le 8 \cdot 7 + 6 \cdot 13 = 134 < 135 = 45\alpha'(G)$, which by Claim U completes the proof. ($\Box$)

Recall that $n(H') = n(H) + |Q|$ and $m(H') = m(H)$. By Claim S(a), Claim V and Claim W, $45\tau(H) = 45\tau(H') \le 6n(H') + 13m(H') - 6|Q| = 6n(H) + 13m(H)$, a contradiction. This completes the proof of Theorem 21. $\Box$

7.5 Proof of Theorem 7

We are finally in a position to present a proof of our main result, namely Theorem 7. Recall its statement.

Theorem 7. *If H is a 4-uniform, linear hypergraph on n vertices with m edges, then $\tau(H) \le \frac{1}{5}(n + m)$.*

Proof of Theorem 7. Let H be a 4-uniform, linear hypergraph on n vertices with m edges. We show that $\tau(H) \le (n + m)/5$. We proceed by induction on n. If $n = 4$, then H consists of a single edge, and $\tau(H) = 1 = (n+m)/5$. Let $n \ge 5$ and suppose that the result holds for all 4-uniform, linear hypergraphs on fewer than n vertices. Let H be a 4-uniform, linear hypergraph on n vertices with m edges.

Suppose that $\Delta(H) \ge 4$. Let v be a vertex of maximum degree in H, and consider the 4-uniform, linear hypergraph $H' = H - v$ on n' vertices with m' edges. We note that $n' = n - 1$ and $m' = m - \Delta(H) \le m - 4$. Every transversal in H' can be extended to a transversal in H by adding to it the vertex v. Hence, applying the inductive hypothesis to H', we have that $\tau(H) \le \tau(H') + 1 \le (n' + m')/5 + 1 \le (n + m - 5)/5 + 1 = (n + m)/5$. Hence, we may assume that $\Delta(H) \le 3$, for otherwise the desired result follows. With this assumption, we note that $4m \le 3n$. Applying Theorem 21 to the hypergraph H, we have

$$45\tau(H) \le 6n(H) + 13m(H) + \mathrm{def}(H).$$

If $\mathrm{def}(H) = 0$, then $45\tau(H) \le 6n(H)+13m(H) = (9n+9m)+(4m-3n) \le 9(n+m)$, and so $\tau(H) \le (n + m)/5$. Hence, we may assume that $\mathrm{def}(H) > 0$, for otherwise the desired result follows. Among all special non-empty H-sets, let X be chosen so that $|E^*(X)| - |X|$ is minimum. We note that since $\mathrm{def}(H) > 0$, $|E^*(X)| - |X| < 0$. As in Section 7.3, we associate with the set X a bipartite graph, G_X, with partite sets X and $E^*(X)$, where an edge joins $e \in E^*(X)$ and $H' \in X$ in G_X if and only if the edge e intersects the subhypergraph H' of X in H. Suppose that there is no matching in G_X that matches $E^*(X)$ to a subset of X. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*(X)$ such that $|N_{G_X}(S)| < |S|$. We now consider the special H-set, $X' = X \setminus N_{G_X}(S)$, and note that $|X'| = |X| - |N_{G_X}(S)| > |E^*(X)| - |S| \geq 0$ and $|E^*(X')| = |E^*(X)| - |S|$. Thus, X' is a special non-empty H-set satisfying

$$\begin{array}{rcl} |E^*(X')| - |X'| & = & (|E^*(X)| - |S|) - (|X| - |N_{G_X}(S)|) \\ & = & (|E^*(X)| - |X|) + (|N_{G_X}(S)| - |S|) \\ & < & |E^*(X)| - |X|, \end{array}$$

contradicting our choice of the special H-set X. Hence, there exists a matching in G_X that matches $E^*(X)$ to a subset of X. By Observation 1(g), there exists a minimum X-transversal, T_X, that intersects every edge in $E^*(X)$. By Observation 1, every special hypergraph F satisfies $\tau(F) \leq (n(F) + m(F))/5$. Hence, letting

$$n(X) = \sum_{F \in X} n(F) \quad \text{and} \quad m(X) = \sum_{F \in X} m(F),$$

we note that

$$|T_X| = \sum_{F \in X} \tau(F) \leq \sum_{F \in X} \frac{n(F) + m(F)}{5} = \frac{n(X) + m(X)}{5}.$$

We now consider the 4-uniform, linear hypergraph $H' = H - V(X)$ on n' vertices with m' edges. We note that $n' = n - n(X)$ and $m' = m - m(X) - |E^*(X)| \leq m - m(X)$. Every transversal in H' can be extended to a transversal in H by adding to it the set T_X. Hence, applying the inductive hypothesis to H', we have that

$$\begin{array}{rcl} \tau(H) & \leq & \tau(H') + |T_X| \\ & \leq & \frac{1}{5}(n' + m') + |T_X| \\ & \leq & \frac{1}{5}(n + m) - \frac{1}{5}(n(X) + m(X)) + |T_X| \\ & \leq & \frac{1}{5}(n + m). \end{array}$$

This completes the proof of Theorem 7. $\square$

8 Proof of Theorem 15

In this section, we present a proof of Theorem 15. We first prove the following lemma. Let $c(H)$ denote the number of components of a hypergraph H. Recall that if X is a special H-set, we write $E^*(X)$ to denote the set $E^*_H(X)$ if the hypergraph H is clear from context.

Lemma 22 *If H is a 4-uniform, linear hypergraph and X is a special H-set, then* $3|E^*_H(X)| \geq |X| - c(H)$.

Proof. We proceed by induction on $|E^*_H(X)| = k \ge 0$. If $k = 0$, then $c(H) \ge |X|$, and so $3|E^*(X)| = 0 \ge |X| - c(H)$. This establishes the base case. Suppose $k \ge 1$ and the result holds for special H-sets, X, such that $|E^*_H(X)| < k$. Let X be a special H-set satisfying $|E^*_H(X)| = k$. Let $e \in E^*_H(X)$ and consider the hypergraph $H' = H - e$. We note that H' is a 4-uniform, linear hypergraph, and that $c(H') \le c(H) + 3$. Further, the set X is a special H'-set satisfying $|E^*_{H'}(X)| = k - 1$. Applying the inductive hypothesis to the hypergraph $H' \in \mathcal{H}_4$ and to the special H'-set, X, we have $3(|E^*_H(X)| - 1) = 3|E^*_{H'}(X)| \ge |X| - c(H') \ge |X| - (c(H) + 3)$, implying that $3|E^*_H(X)| \ge |X| - c(H)$. $\Box$

We are now in a position to present a proof of Theorem 15. Recall its statement, where $\mathcal{F}$ is the family defined in Section 6.3.

Theorem 15. *Let $H \ne H_{10}$ be a 4-uniform, connected, linear hypergraph with maximum degree $\Delta(H) \le 2$ on n vertices with m edges. Then, $\tau(H) \le \frac{3}{16}(n + m) + \frac{1}{16}$, with equality if and only if $H \in \{H_{14,5}, H_{14,6}\}$ or $H \in \mathcal{F}$.*

Proof of Theorem 15. Let $H \ne H_{10}$ be a 4-uniform, connected, linear hypergraph with maximum degree $\Delta(H) = 2$ on n vertices with m edges. Suppose firstly that H is a special hypergraph. By assumption, $H \ne H_{10}$. Since $\Delta(H) \le 2$, we note that $H \in \{H_4, H_{11}, H_{14,5}, H_{14,6}\}$. If $H = H_{11}$, then by Observation 1(c), $\tau(H) = \frac{3}{16}(n + m)$. If $H \in \{H_4, H_{14,5}, H_{14,6}\}$, then by Observation 1(a) and 1(d), $\tau(H) = \frac{3}{16}(n+m) + \frac{1}{16}$. Hence, we may assume that H is not a special hypergraph, for otherwise the desired result holds, noting that $H_4 \in \mathcal{F}$.

Since $\Delta(H) \le 2$, we observe that $m \le \frac{1}{2}n$. By Theorem 21, $45\tau(H) \le 6n + 13m + \mathrm{def}(H)$. Suppose that $\mathrm{def}(H) = 0$. Then, since $0 \le \frac{1}{2}n - m$, we have

$$
\begin{array}{rcl}
45\tau(H) & \le & 6n + 13m \\
 & \le & 6n + 13m + \frac{14}{3}(\frac{1}{2}n - m) \\
 & = & (6 + \frac{14}{6})n + (13 - \frac{14}{3})m \\
 & = & \frac{25}{3}(n + m),
\end{array}
$$

or, equivalently, $\tau(H) \le \frac{5}{27}(n+m) < \frac{3}{16}(n+m)$. Hence, we may assume that $\mathrm{def}(H) > 0$, for otherwise the desired result follows. Let X be a special H-set such that $\mathrm{def}(H) = \mathrm{def}_H(X)$. If H_{10} belongs to X, then, since $\Delta(H) \le 2$ and H is connected, $H = H_{10}$, a contradiction. If $H_{14,5}$ or $H_{14,6}$ belong to X, then, analogously, $H \in \{H_{14,5}, H_{14,6}\}$, contradicting our assumption that H is not a special hypergraph. Thus, if $F \in X$, then $F \in \{H_4, H_{11}\}$, noting that $\Delta(H) \le 2$. Recall that if F is a hypergraph, we denote by $n_1(F)$ the number of vertices of degree 1 in H. Let

$$
n_1(X) = \sum_{F \in X} n_1(F),
$$

and note that $n_1(H) \ge n_1(X) - 4|E^*(X)|$, since every edge in $E^*(X)$ contains at most four vertices whose degree is 1 in some subhypergraph $F \in X$. Since $\Delta(H) \le 2$ and H is 4-uniform, we note that $4m = 2n - n_1(H)$, or, equivalently, $n_1(H) = 2n - 4m$. Let $\beta = \frac{73}{64}$. If $F = H_4$, then $n_1(F) = 4$ and $\mathrm{def}(F) = 8 = (8 - 4\beta) + 4\beta = (8 - 4\beta) + n_1(F) \cdot \beta$. If

$F = H_{11}$, then $n_1(F) = 2$ and $\text{def}(F) = 4 < 5.71875 = 8 - 2\beta = (8-4\beta) + n_1(F)\cdot\beta$. Hence, if $F \in X$, then $\text{def}(F) \le (8-4\beta) + n_1(F)\cdot\beta$, with strict inequality if $F = H_{11}$. Therefore,

$$\begin{aligned}
\text{def}(H) &= 8|X_4| + 4|X_{11}| - 13|E^*(X)| \\
&= \left(\sum_{F\in X} \text{def}(F)\right) - 13|E^*(X)| \\
&\le (8-4\beta)|X| + n_1(X)\cdot\beta - 13|E^*(X)|,
\end{aligned}$$

with strict inequality if $X \ne X_4$. By Lemma 22, $|E^*(X)| \ge \frac{1}{3}(|X|-1)$. We also note that $n \ge 4|X_4| + 11|X_{11}| \ge 4|X|$, and so $|X| \le \frac{n}{4}$. Thus by our previous observations,

$$\begin{aligned}
45\tau(H) &\le 6n + 13m + \text{def}(H) \\
&\le 6n + 13m + (8-4\beta)|X| + n_1(X)\cdot\beta - 13|E^*(X)| \\
&\le 6n + 13m + (8-4\beta)|X| + (n_1(H) + 4|E^*(X)|)\cdot\beta - 13|E^*(X)| \\
&= 6n + 13m + (8-4\beta)|X| + n_1(H)\cdot\beta - |E^*(X)|(13-4\beta) \\
&\le 6n + 13m + (8-4\beta)|X| + n_1(H)\cdot\beta - \tfrac{1}{3}(|X|-1)(13-4\beta) \\
&= 6n + 13m + (8-4\beta-\tfrac{1}{3}(13-4\beta))|X| + (2n-4m)\cdot\beta + \tfrac{1}{3}(13-4\beta) \\
&\le 6n + 13m + (8-4\beta-\tfrac{1}{3}(13-4\beta))\tfrac{n}{4} + (2n-4m)\cdot\beta + \tfrac{1}{3}(13-4\beta) \\
&= (\tfrac{83}{12} + \tfrac{4}{3}\beta)n + (13-4\beta)m + \tfrac{1}{3}(13-4\beta) \\
&= (13-4\beta)(n+m) + \tfrac{1}{3}(13-4\beta) \\
&= \tfrac{135}{16}(n+m) + \tfrac{135}{48},
\end{aligned}$$

or, equivalently, $\tau(H) \le \frac{3}{16}(n+m) + \frac{1}{16}$. This establishes the desired upper bound.

Recall that H is a 4-uniform, connected, linear hypergraph with maximum degree at most 2. Suppose that $\tau(H) = \frac{3}{16}(n+m) + \frac{1}{16}$. Then we must have equality throughout the above inequality chain. This implies that $X = X_4$, $V(H) = V(X)$, $n = 4|X|$, $E(H) = E(X) \cup E^*(X)$, and $|E^*(X)| = \frac{1}{3}(|X|-1)$. We show by induction on $n \ge 4$ that these conditions imply that $H \in \mathcal{F}$. When $n = 4$, $|X| = 1$ and $|E^*(X)| = 0$, and so $H = H_4 \in \mathcal{F}$. This establishes the base case. Suppose that $n > 4$. Thus, $|X| \ge 2$ and $n = 4|X| \ge 8$. We now consider the bipartite graph, G_X, with partite sets X and $E^*(X)$, where an edge joins $e \in E^*(X)$ and $H' \in X$ in G_X if and only if the edge e intersects the subhypergraph H' of X in H. Since H is 4-uniform and linear, each vertex in $E^*(X)$ has degree 4 in G_X. Let $n_1 = n_1(G)$, and so n_1 is the number of vertices of degree 1 in G. Counting the edges in G, we note that $\frac{4}{3}(|X|-1) = 4|E^*(X)| = m(G) \ge n_1 + 2(|X| - n_1)$, implying that $n_1 \ge \frac{1}{3}(2|X|+4)$. By the Pigeonhole Principle, there is a vertex of $E^*(X)$ adjacent in G to at least

$$\frac{n_1}{|E^*(X)|} = \frac{\left(\frac{2|X|+4}{3}\right)}{\left(\frac{|X|-1}{3}\right)} = \frac{2|X|+4}{|X|-1} = 2 + \frac{6}{|X|-1}$$

vertices of degree 1 (that belong to X). Thus, since $|X| \ge 2$ here, some vertex $e \in E^*(X)$ in G_X is adjacent to three vertex of degree 1, say x_1, x_2 and x_3. Let x_4 be the remaining neighbor of e in G_X. We now consider the hypergraph H' obtained from H by deleting the 12 vertices from the three special H_4-subhypergraphs, say F_1, F_2, and F_3, corresponding to x_1,

x_2 and x_3, respectively, and deleting the hyperedge corresponding to e. Since H is connected and linear, so too is H'. Let $X' = X \setminus \{F_1, F_2, F_3\}$, and so $|X'| = |X| - 3$. We note that $|E^*_{H'}(X')| = |E^*_H(X)| - 1 = \frac{1}{3}(|X| - 1) - 1 = \frac{1}{3}(|X'| - 1)$. Further, $X' = X_4$, $V(H') = V(X')$, $n' = 4|X'|$, and $E(H') = E(X') \cup E^*_{H'}(X')$. Applying the inductive hypothesis to H', we deduce that $H' \in \mathcal{F}$. The original hypergraph H can now be reconstructed from H' by adding back the three deleted edges and 12 deleted vertices in $F_1 \cup F_2 \cup F_3$, and adding back the deleted edge e that contains the vertex $x_4 \in V(H')$ and contains one vertex from each edge in F_1, F_2 and F_3. Thus, $H \in \mathcal{F}$. This completes the proof of Theorem 15. $\Box$

9 Closing Conjecture

In this paper, we have shown that Conjecture 2 fails for large k but holds for small k. More precisely, we prove (see Theorem 8) that $5 \le k_{\min} \le 166$, where recall that in Problem 1 we define $k_{\min}$ as the smallest value of k for which Conjecture 2 fails. We believe that Conjecture 2 holds when $k = 5$, and state this new conjecture formally as follows.

Conjecture 4 *$k_{\min} \ge 6$.*

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