On the spectrum of hypergraphs

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Abstract

Hypergraphs are well represented by hypermatrices (tensors) and are extensively studied by the eigenvalues of these hypermatrices. Due to higher order nonlinear equations, their eigenvalues are not easy to compute and it makes the study of spectral theory of hypergraphs difficult. Here, we introduce adjacency, Laplacian and normalized Laplacian matrices of uniform hyperhraphs. We show that different structural properties of hypergrpahs, like, diameter, vertex strong chromatic number, Cheeger constant, etc., can also be well studied using spectral properties of these matrices. Random walk on a hypergraph can be explored by using the spectrum of transition probability operator defined on the hypergraph. We also introduce Ricci curvature(s) on hypergraphs and show that if the Laplace operator, Δ, on a hypergraph satisfies a curvature-dimension type inequality $CD(\mathbf{m}, \mathbf{K})$ with $\mathbf{m} > 1$ and $\mathbf{K} > 0$ then any non-zero eigenvalue of $-\Delta$ can be bounded below by $\frac{\mathbf{mK}}{\mathbf{m}-1}$. Eigenvalues of a normalized Laplacian operator defined on a connected hypergraph can be bounded by the Ollivier's Ricci curvature of the hypergraph.

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1 Introduction

In spectral graph theory, eigenvalues of an operator or a matrix, defined on a graph, are investigated and different properties of the graph structure are explored from these eigenvalues. Adjacency matrix, Laplacian matrix, normalized Laplacian matrix are the popular matrices to study in spectral graph theory [6, 11, 14]. Depending on graph structure, various bounds on eigenvalues have been estimated. Different relations of graph spectrum with graph diameter, coloring and connectivity have been established. Eigenvalues also play an important role to characterize graph connectivity by edge boundary, vertex boundary, isoperimetric number, Cheeger constant, etc. Isoperimetric problems deal with optimal relations between size of a cut and the size of the separated parts. Similarly, Cheeger constant shows how difficult it is to cut the Riemannian manifold into two large pieces [9]. The concept of Cheeger constant in spectral geometry has been incorporated in very similar way in spectral graph theory. The Cheeger constant of a graph can be bounded above and below by the smallest nontrivial eigenvalue of Laplacian matrix and normalized Laplacian matrix, respectively, of the graph [11, 25]. Ricci curvature on a graph [4, 20, 23] has been introduced which is analogous to the notion of Ricci curvature in Riemannian geometry [2, 28]. Many results have been proved on manifolds with Ricci curvature bounded below. Lower Ricci curvature bounds have been derived in the context of finite connected graphs. Random walk on graphs is also studied by defining transition probability operator on the same [16]. The eigenvalues of the transition probability operator can be estimated from the spectrum of normalized graph Laplacian [1].

Unlike in a graph, an edge of a hypergraph can be formed with more than two vertices. Thus the edge set of a hypergraph is the subset of the power set of the vertex set of that hypergraph [37]. Different aspects of a hypergraph like, Helly property, fractional transversal number, connectivity, chromatic number have been studied [5, 36]. A hypergraph is used to be represented by an incidence graph which is a bipartite graph with vertex classes, the vertex set and the edge set of the hypergraph and it has been exploited to study Eulerian property, existence of different cycles, vertex and edge coloring in hypergraphs.

Hypergraphs can also be represented by tensors, i.e., by hypermatrices. A recent trend has been developed to explore spectral hypergraph theory using different connectivity tensors. An m-uniform hypergraph on n vertices, where each edge contains the same, m, number of vertices can easily be represented by a tensor (or hypermatrix), $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$, $1 \leq i_1, \dots, i_m \leq n$, of order m and dimension n. In 2005 [29], Liqun Qi introduced the concept of different eigenvalues of a real supersymmetric tensor. Let $u \in \mathbb{R}^n$. If we write u^m as an m order and n dimension hypermatrix with $(i_1, i_2, \dots, i_m)$-th entry $u(i_1)u(i_2)\dots u(i_m)$ then $\mathcal{A}u^{m-1}$, where the multiplication is taken as tensor contraction over all indices, is an n-tuple whose i-th component is $\sum_{i_2, i_3, \dots, i_m=1}^{n} a_{i i_2 i_3 \dots i_m} u(i_2)u(i_3)\dots u(i_m)$. Let $\mathcal{A}$ be a nonzero hypermatrix. A pair $(\lambda, u) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenpair (i.e., λ is an eigenvalue and u is an eigenvector) if it satisfies the equation $\mathcal{A}u^{m-1} = \lambda u^{[m-1]}$, where $u^{[m]}$ is a vector with i-th entry $(u(i))^m$. Afterwards, many researchers started to analyze different eigenvalues of several connectivity tensors (or hypermatrices), namely, adjacency tensor, Laplacian tensor, normalized Laplacian tensors, etc. Various properties of eigenvalues of a tensor have been studied in [7, 8, 21, 27, 34, 35, 38, 39]. Using characteristic polynomial, the spectrum of adjacency matrix of a graph is extended for uniform hypergraphs in [12]. Different properties of eigenvalues of Laplacian and signless Laplacian tensors of a uniform hypergraph have been studied in [17, 18, 19, 30, 31]. We also refer to [32] for detailed reading on spectral analysis of hypergraphs using different tensors.

A tensor of order m and dimension n, which can represent an m-uniform hypergraph on n vertices, possesses $n(m-1)^{n-1}$ number of eigenvalues. Thus the computational complexity is very high to compute the eigenvalues of a tensor. Moreover, tensors are nonlinear operators and hence they are not easy to handle. In this article, we introduce linear operators (connectivity matrices) on hypergraphs which are easy to study. A rich number of tools are already available for linear operators and matrices. We show that spectrum of these matrices (or operators) can reveal many structural properties of hypergraphs. We have studied hypergraph connectivity by their eigenvalues. Spectral radii of these operators have been bounded by the degrees of a hypergraph. We have also bounded diameter of a hypergraph by the eigenvalues of its connectivity matrices. Different properties of a regular hypergraph are characterized by the spectrum. Strong (vertex) chromatic number of a hypergraph is bounded by the eigenvalues of the hypergraph. We have also defined Cheeger constant on a hypergraph and showed that it can be bounded above and below by the smallest nontrivial eigenvalues of Laplacian matrix and normalized Laplacian matrix, respectively, of a connected hypergraph. We have shown that the study of random walk on a hypergraph can be performed by analyzing the spectrum of the transition probability operator defined on that hypergraph. Ricci curvature on hypergraphs has been introduced. We have showed that if the Laplace operator, Δ, on a hypergraph satisfies a curvature-dimension type inequality $CD(\mathbf{m}, \mathbf{K})$ with $\mathbf{m} > 1$ and $\mathbf{K} > 0$ then any non-zero eigenvalue of $-\Delta$ can be bounded below by $\frac{\mathbf{mK}}{\mathbf{m-1}}$. Spectrum of normalized Laplacian operator on a connected hypergraph has also been bounded by the Olliviers Ricci curvature of the hypergraph. Before we start, we recall some basics from linear algebra and hypergraph theory.

2 Preliminary

2.1 Linear algebra

A matrix is called nonnegative matrix if all of its entries are nonnegative real number.

Definition 2.1. A matrix $A \in M_n$ is reducible if there is a permutation matrix P such that

$$P^T AP = \begin{bmatrix} B & C \\ \mathbf{0}_{n-r,r} & D \end{bmatrix}, 1 \leq r \leq n-1,$$

where $\mathbf{0}_{n-r,r}$ is the $n-r \times r$ null matrix. A square matrix A is irreducible if it is not reducible.
Equivalently, a square matrix $A = [(A)_{ij}]$ is irreducible if and only if the underlying (directed) graph with the vertices $\{1, 2, \ldots, n\}$ and edges (i, j), whenever $(A)_{ij} \neq 0$, is strongly connected.
In other words, a nonnegative $n \times n$ matrix A is irreducible if and only if $(I + A)^{n-1} > 0$, where $A = [(A)_{ij}] > 0$ means $(A)_{ij} > 0 \ \forall i, j$.

Definition 2.2. If $A \in M_n$, then the spectral radius of A is $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$.
It is well known that, (Gelfand formula) for $A \in M_n$ the spectral radius can be estimated as $\rho(A) = \lim_{k\to\infty} ||A^k||^{\frac{1}{k}}$, where $|| \cdot ||$ be a matrix norm on M_n.

Theorem 2.1. *(Geršgorin[15]) The eigenvalues of an $n \times n$ complex matrix $A = [a_{ij}]$ lie in the region*

$$G_A = \bigcup_{i=1}^{n} \Big\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \Big\}.$$

Theorem 2.2 (Perron-Frobenius theorem (weak form))**.** *For any nonnegative square matrix A,*

1. *the spectral radius $\rho(A)$, is an eigenvalue of A;*
2. *there exists a nonnegative vector $X \neq 0$, such that, $AX = \rho(A)X$.*

Theorem 2.3 (Perron-Frobenius theorem)**.** *For any irreducible nonnegative square matrix A,*

1. *the spectral radius $\rho(A) > 0$ is an eigenvalue of A with algebraic and geometric multiplicity one;*
2. *there exists a vector $X > 0$ (i.e., all components of X are positive), such that, $AX = \rho(A)X$;*
3. *if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(A)$;*
4. *$|\lambda| \leq \rho(A)$, where λ is any eigenvalue of A.*

Definition 2.3. Let A be an $n \times n$ matrix. The Rayleigh Quotient of a vector $X \in \mathbb{R}^n$ with respect to A is defined as the fraction

$$\mathcal{R}_A(X) = \frac{X^t AX}{X^t X}.$$

2.2 Hypergraph and hypermatrices

Definition 2.4. An m-uniform hypergraph $\mathcal{G}$ is a pair $\mathcal{G} = (V, E)$ where V is a set of elements called vertices, and E is a set of non-empty subsets, of order m, of V called edges.

Definition 2.5. Let $\mathcal{G} = (V, E)$ be a hypergraph with the vertex set $V = \{1, \dots, n\}$. Two vertices $i, j \in V$ are called adjacent if they belong to an edge together, i.e., $i, j \in e$ for some $e \in E$ and it is denoted by $i \sim j$.

Example 2.1. *Let $\mathcal{G} = (V, E)$, where $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \big\{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}\big\}$. Here, $\mathcal{G}$ is a 4-uniform hypergraph of 6 vertices and 2 edges. Here, vertices 3 and 4 are adjacent, i.e., $3 \sim 4$, but, 3 is not adjacent to 5.*

Let $\mathcal{G}(V, E)$ be an m-uniform hypergraph with n vertices and let K_n^m be the complete m-uniform hypergraph with n vertices. Further let $\bar{\mathcal{G}}(V, \bar{E})$ be the (m-uniform) complement of $\mathcal{G}$ which is also an m-uniform hypergraph such that an edge $e \in \bar{E}$ if and only if $e \notin E$. Thus the edge set of K_n^m is $E \cup \bar{E}$.

Definition 2.6. An $m(> 2)$-uniform hypergraph $\mathcal{G} = (V, E)$ is called bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that for each edge $e \in E$, $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$. An m-uniform complete bipartite hypergraph is denoted by K_{n_1,n_2}^m, where $|V_1| = n_1$ and $|V_2| = n_2$.

Definition 2.7. The Cartesian product, $\mathcal{G}_1 \Box \mathcal{G}_2$, of two hypergraphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ is defined by the vertex set $V(\mathcal{G}_1 \Box \mathcal{G}_2) = V_1 \times V_2$ and the edge set $E(\mathcal{G}_1 \Box \mathcal{G}_2) = \big\{\{a\} \times e : a \in V_1, e \in E_2\big\} \bigcup \big\{e \times \{x\} : e \in E_1, x \in V_2\big\}$.

Thus, the vertices $(a, x), (b, y) \in V(\mathcal{G}_1 \Box \mathcal{G}_2)$ are adjacent,$(a, x) \sim (b, y)$, if and only if, either $a = b$ and $x \sim y$ in $\mathcal{G}_2$, or $a \sim b$ in $\mathcal{G}_1$ and $x = y$. Clearly, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are two m-uniform hypergraphs with n_1 and n_2 vertices, respectively, then $\mathcal{G}_1 \Box \mathcal{G}_2$ is also an m-uniform hypergraph with $n_1 n_2$ vertices.

Definition 2.8. Let $\mathcal{G}(V, E)$ be a hypergraph. Then, for a set $S \subset V$, the edge boundary $\partial S = \partial_{\mathcal{G}} S$ is the set of edges in $\mathcal{G}$ with vertices in both S and $V \setminus S$, i.e., $\partial S = \{e \in E : i, j \in e, i \in S \text{ and } j \in V \setminus S\}$.

Similarly we define the vertex boundary δS for S to be the set of all vertices in $V \setminus S$ adjacent to some vertex in S, i.e., $\delta S = \{i \in V \setminus S : i, j \in e \in E, j \in S\}$.

Definition 2.9. The Cheeger constant (isoperimetric number) $h(\mathcal{G})$ of a hypergraph $\mathcal{G}(V, E)$ is defined as

$$h(\mathcal{G}) := \inf_{\phi \neq S \subset V} \left\{ \frac{|\partial S|}{\min(\mu(S), \mu(V \setminus S))} \right\},$$

where μ is a measure on subsets of vertices.

Note: Depending on the choice of measure μ we use different tools. For example, if we consider equal weights 1 for all vertices in subset S then $\mu(S)$ becomes the number of vertices in S, i.e., $|S|$ and combinatorial Laplacian is better tool to use here. On the other hand, if we choose weight of a vertex equal to its degree, then $\mu(S) = \sum_{i \in S} d_i$ and normalized Laplacian will be a better choice in this case.

Sometimes, for a weighted graph, we take $\mu'(\partial S) = \sum_{e \equiv (i,j) \in \partial S} w_{ij}$ instead of $|\partial S|$ in the numerator of $h(\mathcal{G})$, where $\mu'(\partial S)$ is a measure on the set of edges, ∂S, and w_{ij} is weight of an edge (i, j).

Definition 2.10. The adjacency relation in an m-uniform hypergraph $\mathcal{G} = (V, E)$ with n vertices can be represented by an m order and n dimensional adjacency hypermatrix (or tensor)

$$\mathcal{A}_{\mathcal{G}} = [a_{i_1 i_2 \dots i_m}],\ 1 \leq i_1, i_2, \dots, i_m \leq n.$$

Here,

$$a_{i_1 i_2 \dots i_m} = \frac{1}{(m-1)!} \begin{cases} 1 & \text{if } \{i_1 i_2 \dots i_m\} \in E(\mathcal{G}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that, in this article we always consider finite hypergraph $\mathcal{G}(V, E)$, i.e., $|V| < \infty$.

3 Adjacency matrix for hypergraphs

Now we construct different hypermatrices of order $m' \leq m$ and dimension n to represent an m-uniform hypergraph $\mathcal{G} = (V, E)$ with a vertex set $V = \{1, 2, \ldots, n\}$. Let $\mathcal{A}_{\mathcal{G}} = [a_{i_1 i_2 \ldots i_m}]$ be the adjacency hypermatrix related to the hypergraph $\mathcal{G}$. We define an adjacency hypermatrix, $\mathcal{A}_{\mathcal{G}}^{m'}$, by contracting $\mathcal{A}_{\mathcal{G}}$, of order $m' \leq m$ with the entries $(\mathcal{A}_{\mathcal{G}}^{m'})_{i_1 \ldots i_{m'}}$, for $\mathcal{G}$ as

$$(\mathcal{A}_{\mathcal{G}}^{m'})_{i_1 \ldots i_{m'}} = \sum_{i_{m'+1}, \ldots, i_m = 1}^{n} a_{i_1 i_2 \ldots i_m}.$$

Clearly, when $m' = m$, $\mathcal{A}_{\mathcal{G}}^{m'}$ becomes the adjacency hypermatrix $\mathcal{A}_{\mathcal{G}}$ of $\mathcal{G}$. Thus the adjacency matrix $A_{\mathcal{G}} = [(A_{\mathcal{G}})_{ij}]$, which is the adjacency hypermatrix of order 2, of an m-uniform hypergraph $\mathcal{G} = (V, E)$ is defined as

$$(A_{\mathcal{G}})_{ij} = d_{ij} \frac{1}{m-1},$$

where d_{ij} is the *codegree* of the vertices i and j. The codegree d_{ij} of vertices i and j is the number of edges (in $\mathcal{G}$) that contain the vertices i and j both, i.e., $d_{ij} = |\{e \in E : i, j \in e\}|$.

Example 3.1. *Let $\mathcal{G}$ be a hypergraph in Example 2.1. The order m of $\mathcal{G}$ is 3. The codegrees between the vertices of G are: $d_{12} = 2, d_{13} = d_{14} = d_{15} = d_{16} = d_{23} = d_{24} = d_{25} = d_{26} = d_{34} = d_{56} = 1$. The codegrees between other pair of vertices are zero. Hence the adjacency matrix of $\mathcal{G}$ is $A_{\mathcal{G}} = [(A_{\mathcal{G}})_{ij}]$, where $1 \leq i, j \leq 6$, and $(A_{\mathcal{G}})_{12} = (A_{\mathcal{G}})_{21} = \frac{2}{3}, (A_{\mathcal{G}})_{13} = (A_{\mathcal{G}})_{31} = \frac{1}{3}, (A_{\mathcal{G}})_{14} = (A_{\mathcal{G}})_{41} = \frac{1}{3}, (A_{\mathcal{G}})_{15} = (A_{\mathcal{G}})_{51} = \frac{1}{3}, (A_{\mathcal{G}})_{16} = (A_{\mathcal{G}})_{61} = \frac{1}{3}, (A_{\mathcal{G}})_{23} = (A_{\mathcal{G}})_{32} = \frac{1}{3}, (A_{\mathcal{G}})_{24} = (A_{\mathcal{G}})_{42} = \frac{1}{3}, (A_{\mathcal{G}})_{25} = (A_{\mathcal{G}})_{52} = \frac{1}{3}, (A_{\mathcal{G}})_{26} = (A_{\mathcal{G}})_{62} = \frac{1}{3}, (A_{\mathcal{G}})_{34} = (A_{\mathcal{G}})_{43} = \frac{1}{3}, (A_{\mathcal{G}})_{56} = (A_{\mathcal{G}})_{65} = \frac{1}{3}$ and the other elements of $A_{\mathcal{G}}$ are zero.*

Definition 3.1. Let $\mathcal{G} = (V, E)$ be a hypergraph with the vertex set $V = \{1, \ldots, n\}$. Then the degree, d_i, of a vertex $i \in V$ is the number of edges that contain i and it is given by

$$d_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} = \sum_{j=1}^{n} (A_{\mathcal{G}})_{ij}.$$

3.1 Operator form of adjacency matrix

Let $\mathcal{G}(V, E)$ be an m-uniform hypergraph on n vertices. Now, let us consider a real-valued function f on $\mathcal{G}$, i.e., on the vertices of $\mathcal{G}$, $f : V \to \mathbb{R}$. The set of such function forms a vector space (or a real inner product space) which is isomorphic to $\mathbb{R}^n$. For such two functions f_1 and f_2 on $\mathcal{G}$ we take their inner product

$$\langle f_1, f_2 \rangle = \sum_{i \in V} f_1(i) f_2(i).$$

This inner product space is also isomorphic to $\mathbb{R}^n$. Let us choose a basis $\mathcal{B} = \{g_1, g_2, \ldots, g_n\}$ such that $g_i(j) = \delta_{ij}$. Now we find the adjacency operator T such that $[T]_{\mathcal{B}} = A_{\mathcal{G}}$. Here, we also denote T by $A_{\mathcal{G}}$. Now, our adjacency operator $A_{\mathcal{G}}$ (which is a linear operator) is defined as

$$(A_{\mathcal{G}} f)(i) = \frac{1}{m-1} \sum_{j, i \sim j} d_{ij} f(j).$$

It is easy to verify that

$$\langle A_{\mathcal{G}} f_1, f_2 \rangle = \langle f_1, A_{\mathcal{G}} f_2 \rangle,$$

for all $f_1, f_2 \in \mathbb{R}^n$, i.e., the operator $A_{\mathcal{G}}$ is symmetric w.r.t. $\langle .,. \rangle$. So the eigenvalues of $A_{\mathcal{G}}$ are real. Now onwards we shall use the operator and the matrix from of $A_{\mathcal{G}}$ interchangeably.

Clearly, for a connected hypergraph $\mathcal{G}$ the adjacency matrix $A_{\mathcal{G}}$, which is real and non-negative, possesses a Perron eigenvalue with positive real eigenvector. Moreover, for an undirected hypergraph $A_{\mathcal{G}}$ is symmetric. The hypergraph $\mathcal{G}$, the corresponding weighted graph $G[A_{\mathcal{G}}]$ (constructed from the adjacency matrix $A_{\mathcal{G}}$ of $\mathcal{G}$) and the graph $G_0[A_{\mathcal{G}}]$ have the similar property regarding graph connectivity and coloring. Here $G_0[A_{\mathcal{G}}]$ is the underlying unweighted graph of $G[A_{\mathcal{G}}]$.

Lemma 3.1. *$\mathcal{G}$ is connected $\iff$ $G[A_{\mathcal{G}}]$ is connected.*

Proof. $\mathcal{G}$ is connected $\iff$ $\mathcal{A}_{\mathcal{G}}$ is irreducible $\iff$ $A_{\mathcal{G}}$ is irreducible $\iff$ $G[A_{\mathcal{G}}]$ is connected. □

Theorem 3.1. *A hypergraph $\mathcal{G}$ is connected if and only if the highest eigenvalue of $A_{\mathcal{G}}$ is simple and possesses a positive eigenvector.*

Proof. The proof follows from the Corollary 1.3.8 (in [14]) which states that a graph is connected if and only if its index is a simple eigenvalue with a positive eigenvector.

□

Theorem 3.2. *Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two m-uniform hypergraphs on n_1 and n_2 vertices, respectively. If λ and μ are eigenvalues of $A_{\mathcal{G}_1}$ and $A_{\mathcal{G}_2}$, respectively, then $\lambda+\mu$ is an eigenvalue of $A_{\mathcal{G}_1 \square \mathcal{G}_2}$.*

Proof. Let $a, b \in V_1$ and $x, y \in V_2$ be any vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then clearly

$$\begin{aligned}\left(A_{\mathcal{G}_1 \square \mathcal{G}_2}\right)_{(a,x),(b,y)} &= \frac{d_{(a,x)(b,y)}}{m-1} \\ &= \frac{d_{ab} + d_{xy}}{m-1}.\end{aligned}$$

Let α and β be the eigenvectors corresponding to the eigenvalues λ and μ, respectively. Let $\gamma \in \mathbb{C}^{n_1 n_2}$ be a vector with the entries $\gamma(a,x) = \alpha(a)\beta(x)$, where $(a,x) \in [n_1] \times [n_2]$. Now we show that γ is an eigenvector of $A_{\mathcal{G}_1 \square \mathcal{G}_2}$ corresponding to the eigenvalue $\lambda + \mu$.

$$\begin{aligned}\sum_{(b,y),(a,x)\sim(b,y)} \frac{d_{(a,x)(b,y)}}{m-1}\gamma(b,y) &= \sum_{(b,y),(a,x)\sim(b,y)} \frac{d_{ab}+d_{xy}}{m-1}\alpha(b)\beta(y) \\ &= \sum_{(b,x),(a,x)\sim(b,x)} \frac{d_{ab}}{m-1}\alpha(b)\beta(x) + \sum_{(a,y),(a,x)\sim(a,y)} \frac{d_{xy}}{m-1}\alpha(a)\beta(y) \\ &= \beta(x)\sum_{b,a\sim b}\frac{d_{ab}}{m-1}\alpha(b) + \alpha(a)\sum_{y,x\sim y}\frac{d_{xy}}{m-1}\beta(y) \\ &= \beta(x)\lambda\alpha(a) + \alpha(a)\mu\beta(x) \\ &= (\lambda+\mu)\gamma(a,x).\end{aligned}$$

Thus the proof follows. □

3.2 Bounds on eigenvalues of adjacency matrix

Lemma 3.2. *For any real symmetric matrix M*

$$\lambda_{max} = max_{||X||=1}\langle X, A_{\mathcal{G}}X\rangle = max_{X\neq 0}\{\mathcal{R}_M(X)\},$$

and

$$\lambda_{min} = min_{||X||=1}\langle X, A_{\mathcal{G}}X\rangle = min_{X\neq 0}\{\mathcal{R}_M(X)\}.$$

If the hypergraph $\mathcal{G}$ has at least one edge, then $\langle X, A_{\mathcal{G}}X\rangle > 0$ if $X = (1, 1, 0, \dots)^t$ and $\langle X, A_{\mathcal{G}}X\rangle < 0$ when $X = (1, -1, 0, \dots)^t$. Thus $\lambda_{min} < 0 < \lambda_{max}$, where λ_{min} and λ_{max} are smallest and largest eigenvalues of $(A_{\mathcal{G}})$, respectively. From Theorem 2.1, it follows that for any eigenvalue λ of $A_{\mathcal{G}}$ we have $|\lambda| \leq d_{max}$, where d_{max} is the maximum degree of $\mathcal{G}$. We also see that $\lambda_{max} \geq \frac{1}{\sqrt{n}}\mathbf{1}_n^t A_{\mathcal{G}} \frac{1}{\sqrt{n}}\mathbf{1}_n = (\sum d_i)/n$, where $\mathbf{1}_n \in \mathbb{R}^n$ is a vector whose all entries are unity. So, if $\mathcal{G}$ is connected and k-regular on n vertices then $A_{\mathcal{G}}$ contains a Perron eigenvalue k with an eigenvector $\mathbf{1}_n$.

Lemma 3.3 (Cor 2.5,[13])**.** *Let G be a weighted graph which is simple, connected, in which the edge weights are positive numbers, and ρ_1 is the spectral radius of the (weighted) adjacency matrix of G. Then*

$$\rho_1 \leq \max_{i\sim j}\{\sqrt{w_i w_j}\},$$

where w_i is the sum of the weights of the edges that are incident to vertex i. Moreover, equality holds if and only if G is a regular graph or G is a bipartite semi-regular graph.

Now from the above lemma we have the following theorem.

Theorem 3.3. *For a connected hypergraph $\mathcal{G}(V, E)$,*

$$\rho(A_{\mathcal{G}}) \leq \max_{\substack{i,j,\\ i\sim j}}\{\sqrt{d_i d_j}\},$$

where d_i is the degree of vertex i in $\mathcal{G}$. The equality holds if and only if $\mathcal{G}$ is a regular hypergraph.

A *graph-sum* of two hypergraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is a hypergraph with the adjacency hypermatrix which is the sum of adjacency hypermatrices of $\mathcal{G}_1$ and $\mathcal{G}_2$. The graph-sum is defined on two hypergraphs with same uniformity and same number of vertices.

Lemma 3.4. *Let A and B be two nonnegative square matrices. Then*

$$\lambda_{max}(A + B) \leq \lambda_{max}(A) + \lambda_{max}(B).$$

Proposition 3.1. *Let $\mathcal{G}$ be the graph-sum of the hypergraphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Then,*

$$\lambda_{max}(A_{\mathcal{G}}) \leq \lambda_{max}(A_{\mathcal{G}_1}) + \lambda_{max}(A_{\mathcal{G}_2}).$$

Proof. $\mathcal{A}_{\mathcal{G}} = \mathcal{A}_{\mathcal{G}_1} + \mathcal{A}_{\mathcal{G}_2} \Rightarrow A_{\mathcal{G}} = A_{\mathcal{G}_1} + A_{\mathcal{G}_2}$. Since $A_{\mathcal{G}_1}$ and $A_{\mathcal{G}_2}$ are nonnegative symmetric matrices, hence the proof follows from the above lemma. □

3.3 Diameter of a hypergraph and eigenvalues of adjacency matrix

Definition 3.2. A path $v_0 - v_1$ of length l between two vertices $v_0, v_1 \in V$ in a hypergraph $\mathcal{G}(V, E)$ is an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{l-1} e_l v_l$ of distinct vertices $v_0, v_1, v_2, \dots, v_l$ and distinct edges $e_1, e_2, \dots, e_l$, such that, $v_{i-1}, v_i \in e_i$ for $i = 1, \dots, l$.

Definition 3.3. The distance, $d(i, j)$, between two vertices i, j in a hypergraph $\mathcal{G}$ is the minimum length of a $i - j$ path. The diameter, $diam(\mathcal{G})$, of a hypergraph $\mathcal{G}(V, E)$ is the maximum distance between any pair of vertices in $\mathcal{G}$, i.e.,

$$diam(\mathcal{G}) = \max\{d(i, j) : i, j \in V\}.$$

Theorem 3.4. *The diameter of a uniform hypergraph $\mathcal{G}$ is less than the number of distinct eigenvalues of $A_{\mathcal{G}}$.*

Proof. Let the number of distinct eigenvalues of $A_\mathcal{G}$ be r and $k \leq diam(\mathcal{G})$. Let $i, j \in V(\mathcal{G})$ be such that $d(i,j) = k$. So, $(A^k_\mathcal{G})_{ij} \neq 0$, but, $(A^l_\mathcal{G})_{ij} = 0$ for all $l < k$. Thus, $A^k_\mathcal{G}$ is not a linear combination of the smaller powers of $A_\mathcal{G}$.

On the other hand, since the number of distinct eigenvalues of $A_\mathcal{G}$ is r, $A_\mathcal{G}$ satisfies a polynomial of degree r, i.e., some nonzero linear combination of $A^0_\mathcal{G}, A^1_\mathcal{G}, \ldots, A^r_\mathcal{G}$ is zero. Thus, r must be strictly greater than $diam(\mathcal{G})$. Hence the proof follows. □

Theorem 3.5. *Let $m > 2$ and let $\mathcal{G}(V,E)$ be an m-uniform connected hypergraph with n vertices. Let θ be the second largest eigenvalue (in absolute value) of $A_\mathcal{G}$. Then*

$$diam(\mathcal{G}) \leq \left\lfloor 1 + \frac{\log((1-\alpha^2)/\alpha^2)}{\log(\lambda_{max}/\theta)} \right\rfloor,$$

where λ_{max} is the largest eigenvalue of $A_\mathcal{G}$ with the unit eigenvector $X_1 = ((X_1)_1, (X_1)_2, \ldots, (X_1)_n)^t$ and $\alpha = \min_i\{(X_1)_i\}$.

Proof. $A_\mathcal{G}$ is real symmetric and thus have orthonormal eigenvectors X_l with $A_\mathcal{G} X_l = \lambda_l X_l$, where $\lambda_1 = \lambda_{max}$. Let us choose $i, j \in V$ such that $d(i,j) = diam(\mathcal{G})$ and $r \geq diam(\mathcal{G})$ be a positive integer. We try to find the minimum value of r such that $(A^r_\mathcal{G})_{ij} > 0$. Using spectral decomposition of $A_\mathcal{G}$, $(A^r_\mathcal{G})_{ij}$ can be express as

$$\begin{aligned}
(A^r_\mathcal{G})_{ij} &= \sum_{l=1}^{n} \lambda_l^r (X_l X_l^t)_{ij} \\
&\geq \lambda^r_{max}(X_1)_i (X_1)_j - \left| \sum_{l=2}^{n} \lambda_l^r (X_l)_i (X_l)_j \right| \\
&\geq \alpha^2 \lambda \neq^r_{max} -\theta^r \left(\sum_{l=2}^{n} |(X_l)_i|^2 \right)^{1/2} \left(\sum_{l=2}^{n} |(X_l)_j|^2 \right)^{1/2} \\
&\geq \alpha^2 \lambda^r_{max} - \theta^r (1-\alpha^2).
\end{aligned}$$

Now, $(A^r_\mathcal{G})_{ij} > 0$ if $(\lambda_{max}/\theta)^r > (1-\alpha^2)/\alpha^2$, which implies that

$$r > \frac{\log((1-\alpha^2)/\alpha^2)}{\log(\lambda_{max}/\theta)}.$$

This proves that

$$diam(\mathcal{G}) \leq \left\lfloor 1 + \frac{\log((1-\alpha^2)/\alpha^2)}{\log(\lambda_{max}/\theta)} \right\rfloor.$$

□

Corollary 3.1. *Let $\mathcal{G}(V,E)$ be an uniform k-regular connected hypergraph with n vertices. Let θ be the second largest eigenvalue (in absolute value) of $A_\mathcal{G}$. Then*

$$diam(\mathcal{G}) \leq \left\lfloor 1 + \frac{\log(n-1)}{\log(k/\theta)} \right\rfloor.$$

Remark: When $m = 2$, the above bounds are more sharp [10]. Also note that, in Theorem 3.5, $\theta \neq \lambda_{max}$ since the underlying graph is not bipartite.

3.4 Subhypergraphs and eigenvalues of adjacency matrices

Definition 3.4. A hypergraph $\mathcal{G}'(V', E')$ is said to be a subhypergraph of a hypergraph $\mathcal{G}(V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

Let $\mathcal{G}(V, E)$ be a hypergraph and, for any $i \in V$, $E(i) = \{e : i \in e \in E\}$.

Definition 3.5. Strong deletion of a vertex $i \in V$ from $\mathcal{G}$ is the removal of all the edges $e \in E(i)$ and i from V, whereas, weak deletion of a vertex $i \in V$ from $\mathcal{G}$ is the removal of i from V and from each hyperedge $e \in E(i)$.

Example 3.2. *Let $\mathcal{G}$ be a 4-uniform hypergraph in Example 2.1. Now, if we weakly delete the vertex 5 from $\mathcal{G}$, the remaining hypergraph possesses the vertex set $\{1, 2, 3, 4, 6\}$ and the edge set $\big\{\{1, 2, 3, 4\}, \{1, 2, 6\}\big\}$, whereas, if we strongly delete the vertex 5 from $\mathcal{G}$, the remaining hypergraph possesses the vertex set $\{1, 2, 3, 4, 6\}$ and the edge set $\big\{\{1, 2, 3, 4\}\big\}$.*

Definition 3.6. An induced subhypergraph $\mathcal{H}(V', E')$ of a hypergraph $\mathcal{G}(V, E)$ is the subhypergraph obtained by strongly deleting all the vertices $v \in V \setminus V'$ from $\mathcal{G}$.

Clearly, an induced subhypergraph $\mathcal{H}$ of an m-uniform hypergraph $\mathcal{G}$ is also m-uniform.

Proposition 3.2. *If $\mathcal{H}$ is an induced subhypergraph of a uniform hypergraph $\mathcal{G}$, then*

$$\lambda_{max}(A_{\mathcal{H}}) \leq \lambda_{max}(A_{\mathcal{G}}) \text{ and } \lambda_{min}(A_{\mathcal{H}}) \geq \lambda_{min}(A_{\mathcal{G}}).$$

Proof. Let $|V(\mathcal{G})| = n$ and $|V(\mathcal{H})| = n_{\mathcal{H}}(\leq n)$. By permuting the vertices of $\mathcal{G}$, we can get an $(n_{\mathcal{H}} \times n_{\mathcal{H}})$ upper left principal submatrix $A'_{\mathcal{H}}$ of $A_{\mathcal{G}}$, such that $(A'_{\mathcal{H}})_{ij} \geq (A_{\mathcal{H}})_{ij}$ for all $i, j \in [n_{\mathcal{H}}]$. Let $X_{\mathcal{H}}$ be the unit eigenvector of $A_{\mathcal{H}}$, such that $A_{\mathcal{H}} X_{\mathcal{H}} = \lambda_{max}(A_{\mathcal{H}}) X_{\mathcal{H}}$. Let X be the unit vector in $\mathbb{R}^n$ obtained by appending zeros to $X_{\mathcal{H}}$. So,

$$\lambda_{max}(A_{\mathcal{H}}) = X^t_{\mathcal{H}} A_{\mathcal{H}} X_{\mathcal{H}} \leq X^t_{\mathcal{H}} A'_{\mathcal{H}} X_{\mathcal{H}} \leq X^t A_{\mathcal{G}} X \leq \lambda_{max}(A_{\mathcal{G}}).$$

Similarly we can have $\lambda_{min}(A_{\mathcal{H}}) \geq \lambda_{min}(A_{\mathcal{G}})$. □

Theorem 3.6. *Let $\mathcal{G}(V, E)$ be an m-uniform hypergraph and let $\mathcal{G}'(V', E')$ be any m-uniform subgraph of $\mathcal{G}$. Then $\lambda_{max}(A_{\mathcal{G}'}) \leq \lambda_{max}(A_{\mathcal{G}})$.*

Proof. It is known that $A_{\mathcal{G}}$ and $A_{\mathcal{G}'}$ are non-negative real symmetric matrices and, for any two vertices i and j, $d_{ij}(\mathcal{G}) \geq d_{ij}(\mathcal{G}')$, where $d_{ij}(\mathcal{G})$ is the codegree of i, j in $\mathcal{G}$. Let $\lambda_{max}(A_{\mathcal{G}'})$ be the maximum eigenvalue of $A_{\mathcal{G}'}$. Let $|V| = n$, $|V'| = n'(\leq n)$. Then

$$\begin{aligned}
\lambda_{max}(A_{\mathcal{G}'}) &= \max_{||X||=1} \left(\sum_{i,j=1}^{n'} (A_{\mathcal{G}'})_{ij} (X)_i (X)_j \right) \\
&= \max_{||X||=1} \left(\sum_{i,j=1}^{n} (A_{\mathcal{G}'})_{ij} (X)_i (X)_j \right) \\
&\leq \left(\sum_{i,j=1}^{n} (A_{\mathcal{G}})_{ij} (X)_i (X)_j \right) \\
&\leq \lambda_{max}(A_{\mathcal{G}}).
\end{aligned}$$

The second equality follows because $(A_{\mathcal{G}'})_{ij} = 0$ when i or $j > n'$ and $(X)_i = 0$ if $i > n'$. The first inequality holds because, each component of X is nonnegative (by weak Perron-Frobenious theorem), the number of edges in $\mathcal{G}$ is greater than or equal to the number of edges in $\mathcal{G}'$ and $(A_{\mathcal{G}})_{ij} \geq (A_{\mathcal{G}'})_{ij}$. □

3.5 Regular hypergraphs and eigenvalues of adjacency matrices

Lemma 3.5. *Let $\mathcal{G}$ be an m-uniform hypergraph with n vertices. Then $A_{\mathcal{G}} + A_{\bar{\mathcal{G}}} = A_{K_n^m} = \theta(J_n - I_n)$, where $\theta = \frac{\binom{n-2}{m-2}}{m-1}$, J_n is the $(n \times n)$ matrix with all the entries are 1 and I_n is the $(n \times n)$ identity matrix.*

Lemma 3.6. *Let $\mathcal{G}$ be an m-uniform hypergraph with n vertices and let d_i be the degree of the vertex i in $\mathcal{G}$. Further, let $\bar{d}_i$ be the degree of i in $\bar{\mathcal{G}}$. Then $d_i + \bar{d}_i = \binom{n-1}{m-1}$.*

Theorem 3.7. *Let $\mathcal{G}$ be an m-uniform k-regular hypergraph with n vertices and let $\bar{\mathcal{G}}(V, \bar{E})$ be the complement of $\mathcal{G}$. Then we have the following*

1. *$A_{\bar{\mathcal{G}}}$ contains a Perron eigenvalue $\binom{n-1}{m-1} - k$ with an eigenvector $\mathbf{1}_n$.*

2. *If X is a non-Perron eigenvector of $A_{\mathcal{G}}$ for an eigenvalue λ, then its associated non-Perron eigenvalue of $A_{\bar{\mathcal{G}}}$ is $-\theta - \lambda$.*

Proof. 1. $\mathcal{G}$ is k-regular $\Rightarrow d_i = k \ \forall i$ in $\mathcal{G}$. Hence $d_i + \bar{d}_i = \binom{n-1}{m-1} \Rightarrow \bar{\mathcal{G}}$ is $\left(\binom{n-1}{m-1} - k\right)$-regular. Since $A_{\bar{\mathcal{G}}}$ is a symmetric non-negative matrix, the proof follows.

2. Let $X = (x_i)$ be a non-Perron eigenvector of $A_{\mathcal{G}}$ in an orthogonal basis of eigenvectors, such that $A_{\mathcal{G}}X = \lambda X$. Hence, $\sum_i x_i = 0$. Now,

$$\begin{aligned} A_{\bar{\mathcal{G}}}X &= \theta(J_n - I_n)X - A_{\mathcal{G}}X \text{ (using Lemma 3.5)}, \\ &= (-\theta - \lambda)X. \end{aligned}$$

□

Corollary 3.2. *If $\mathcal{G}$ is an m-uniform k-regular hypergraph with n vertices, then the minimum eigenvalue λ_n of $A_{\mathcal{G}}$ satisfies the inequality $\lambda_n \geq k - \theta - \binom{n-1}{m-1}$.*

Proof. Let us order the eigenvalues of $A_{\mathcal{G}}$ as $k = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues of $A_{\bar{\mathcal{G}}}$ can be ordered as $\binom{n-1}{m-1} - k \geq -\theta - \lambda_n \geq \cdots \geq -\theta - \lambda_2$, which implies $\lambda_n \geq k - \theta - \binom{n-1}{m-1}$. □

Corollary 3.3. *Let $\mathcal{G}$ be an m-uniform k-regular hypergraph. Then $A_{\mathcal{G}}$ and $A_{\bar{\mathcal{G}}}$ have the same eigenvectors.*

3.6 Hypergraph coloring and adjacency eigenvalues

Definition 3.7. A strong vertex coloring of a hypergraph $\mathcal{G}$ is a coloring where any two adjacent vertices get different colors. The strong (vertex) chromatic number $\gamma(\mathcal{G})$ of a hypergraph $\mathcal{G}$ is the minimum number of colors needed to have a strong vertex coloring of $\mathcal{G}$.

Lemma 3.7. *If $\chi(G[A_{\mathcal{G}}])$ and $\chi(G_0[A_{\mathcal{G}}])$ are the (vertex) chromatic numbers of the weighted graph $G[A_{\mathcal{G}}]$ and the unweighted graph $G_0[A_{\mathcal{G}}]$, respectively, then*

$$\gamma(\mathcal{G}) = \chi(G[A_{\mathcal{G}}]) = \chi(G_0[A_{\mathcal{G}}]).$$

Lemma 3.8. *Let $A, B \in M_n$ be two nonnegative matrices such that $(A)_{ij} \geq (B)_{ij}$. Then $\lambda_{max}(A) \geq \lambda_{max}(B)$.*

Proof. By Perron-Frobenius theorem (Weak Form), the eigenvectors corresponding to the eigenvalues $\lambda_{max}(A)$ and $\lambda_{max}(B)$ are nonnegative. Let $X = ((X)_1, (X)_2, \ldots, (X)_n)^t$ be a unit eigenvector corresponding to the eigenvalue $\lambda_{max}(B)$. Then $\lambda_{max}(B) = \sum_{i,j=1}^n (B)_{ij}(X)_i(X)_j \leq \sum_{i,j=1}^n (A)_{ij}(X)_i(X)_j \leq \lambda_{max}(A)$. □

Lemma 3.9 ([40])**.** *Let G be a simple unweighted graph. Then*

$$\chi(G) \leq 1 + \lambda_{max}(G),$$

where $\lambda_{max}(G)$ is the maximum eigenvalue of the adjacency matrix of G.

Theorem 3.8. *Let $\gamma(\mathcal{G})$ be an m-uniform hypergraph. Then*

$$\gamma(\mathcal{G}) \leq 1 + (m-1)\lambda_{max}(A_{\mathcal{G}}).$$

Proof. Using Lemmas 3.7 and 3.9 we have

$$\gamma(\mathcal{G}) = \chi(G_0[A_{\mathcal{G}}]) \leq 1 + \lambda_{max}(G_0[A_{\mathcal{G}}]).$$

Now, using Lemma 3.8 we have

$$\lambda_{max}(G_0[A_{\mathcal{G}}]) \leq (m-1)\lambda_{max}(G[A_{\mathcal{G}}]) = (m-1)\lambda_{max}(A_{\mathcal{G}}).$$

The proof follows from the above two inequalities. □

Corollary 3.4. $\gamma(\mathcal{G}) \leq 1 + \lambda_{max}(G_0[A_{\mathcal{G}}])$.

Lemma 3.10 (Lemma 3.22, [3])**.** *Let B be a symmetric matrix partitioned as*

$$\mathbf{B} = \begin{bmatrix} 0 & B_{12} & \dots & B_{1k} \\ B_{21} & 0 & \dots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \dots & 0 \end{bmatrix}.$$

Then $\lambda_{max}(B) + (k-1)\lambda_{min}(B) \leq 0$.

Theorem 3.9. *Let $\mathcal{G}$ be a uniform hypergraph with at least one edge. Then*

$$\gamma(\mathcal{G}) \geq 1 - \frac{\lambda_{max}(A_{\mathcal{G}})}{\lambda_{min}(A_{\mathcal{G}})}.$$

Proof. Let $k = \gamma(\mathcal{G})$. Now $A_{\mathcal{G}}$ can be partitioned as

$$\mathbf{A}_{\mathcal{G}} = \begin{bmatrix} 0 & A_{\mathcal{G}_{12}} & \dots & A_{\mathcal{G}_{1k}} \\ A_{\mathcal{G}_{21}} & 0 & \dots & A_{\mathcal{G}_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\mathcal{G}_{k1}} & A_{\mathcal{G}_{k2}} & \dots & 0 \end{bmatrix}.$$

Using the above lemma we have

$$\lambda_{max}(A_{\mathcal{G}}) + (k-1)\lambda_{min}(A_{\mathcal{G}}) \leq 0.$$

Since $\mathcal{G}$ has at least one edge, $\lambda_{min}(A_{\mathcal{G}}) < 0$. Hence the proof follows. □

Corollary 3.5. $\gamma(\mathcal{G}) \geq 1 - \frac{\lambda_{max}(G_0[A_{\mathcal{G}}])}{\lambda_{min}(G_0[A_{\mathcal{G}}])}$.

From Theorems 3.8 and 3.9 we have the following corollary.

Corollary 3.6. *Let $\mathcal{G}$ be an m-uniform hypergraph with at least one edge. Then*

$$|\lambda_{min}(A_{\mathcal{G}})| \geq 1/(m-1).$$

4 Combinatorial Laplacian matrix and operator of a hypergraph

Now we define our (combinatorial) Laplacian operator $L_{\mathcal{G}}$ for an m-uniform hypergraph $\mathcal{G}(V,E)$ on n vertices. We take the same usual inner product $\langle f_1, f_2\rangle = \sum_{i\in V} f_1(i)f_2(i)$ for the n dimensional Hilbert space $L^2(\mathcal{G})$ constructed with all real-valued functions f on $\mathcal{G}$, i.e., $f : V \to \mathbb{R}$. Now our Laplacian operator

$$L_{\mathcal{G}} : L^2(\mathcal{G}) \to L^2(\mathcal{G})$$

defined as

$$(L_{\mathcal{G}}f)(i) := \frac{1}{m-1}\sum_{j,i\sim j} d_{ij}(f(i)-f(j)) = d_i f(i) - \frac{1}{m-1}\sum_{j,i\sim j} d_{ij} f(j).$$

It is easy to verify that $L_{\mathcal{G}}$ is symmetric (self-adjoint) w.r.t. the usual inner product $\langle ., .\rangle$, i.e.,

$$\langle L_{\mathcal{G}}f_1, f_2\rangle = \langle f_1, L_{\mathcal{G}}f_2\rangle,$$

for all $f_1, f_2 \in \mathbb{R}^n$. So the eigenvalues of $L_{\mathcal{G}}$ are real. Since

$$\langle L_{\mathcal{G}}f, f\rangle = \frac{1}{m-1}\sum_{i\sim j} d_{ij}(f(i)-f(j))^2 \geq 0,$$

for all $f \in \mathbb{R}^n$, $L_{\mathcal{G}}$ is nonnegative, i.e., the eigenvalues of $L_{\mathcal{G}}$ are nonnegative. The Rayleigh Quotient $\mathcal{R}_{L_{\mathcal{G}}}(f)$ of a function $f : V \to \mathbb{R}$ is defined as

$$\mathcal{R}_{L_{\mathcal{G}}}(f) = \frac{\langle L_{\mathcal{G}}f, f\rangle}{\langle f, f\rangle} = \frac{\frac{1}{m-1}\sum_{i\sim j} d_{ij}(f(i)-f(j))^2}{\sum_{i\in V} f(i)^2}.$$

For standard basis we get the matrix from Laplacian operator $L_{\mathcal{G}}$ as

$$(L_{\mathcal{G}})_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -\frac{d_{ij}}{m-1} & \text{if } i \sim j, \\ 0 & \text{elsewhere.} \end{cases}$$

So, $L_{\mathcal{G}} = D_{\mathcal{G}} - A_{\mathcal{G}}$, where $D_{\mathcal{G}}$ is the diagonal matrix where the entries are the degrees d_i of the vertices i of $\mathcal{G}$. Any $\lambda(L_{\mathcal{G}}) \in \mathbb{R}$ becomes an eigenvalue of $L_{\mathcal{G}}$ if, for a nonzero $u \in \mathbb{R}^n$, it satisfies the equation

$$L_{\mathcal{G}}u = \lambda(L_{\mathcal{G}})u. \tag{1}$$

Let us order the eigenvalues of $L_{\mathcal{G}}$ as $\lambda_1(L_{\mathcal{G}}) \leq \lambda_2(L_{\mathcal{G}}) \leq \cdots \leq \lambda_n(L_{\mathcal{G}})$. Now find an orthonormal basis of $L^2(\mathcal{G})$ consisting of eigenfunctions of $L_{\mathcal{G}}$, $u_k, k = 1, \ldots, n$ as follows. First we find u_1 from the expression

$$\lambda_1(L_{\mathcal{G}}) = \inf_{u\in L^2(\mathcal{G})}\{\langle L_{\mathcal{G}}u, u\rangle : ||u|| = 1\}.$$

Now iteratively define Hilbert space of all real-valued functions on $\mathcal{G}$ with the scalar product $\langle ., .\rangle$,

$$H_k := \{v \in L^2(\mathcal{G}) : \langle v, u_l\rangle = 0 \text{ for } l \leq k\}.$$

Then we start with the function u_1 (eigenfunction for the eigenvalue $\lambda_1(L_{\mathcal{G}})$) and find all the eigenvalues of $L_{\mathcal{G}}$ as

$$\lambda_k(L_{\mathcal{G}}) = \inf_{u\in H_k-\{0\}}\left\{\frac{\langle L_{\mathcal{G}}u, u\rangle}{\langle u, u\rangle}\right\}.$$

Thus

$$\lambda_2(L_{\mathcal{G}}) = \inf_{0\neq u\perp u_1}\left\{\frac{\langle L_{\mathcal{G}}u, u\rangle}{\langle u, u\rangle}\right\}.$$

We can also find $\lambda_n(L_{\mathcal{G}})$ as

$$\lambda_n(L_{\mathcal{G}}) = \sup_{u\in L^2(\mathcal{G})}\{\langle L_{\mathcal{G}}u, u\rangle : ||u|| = 1\}.$$

We see that Theorem 2.1 provides rough bounds on any eigenvalue λ of $L_{\mathcal{G}}$ as $|\lambda| \leq 2d_{max}$. Now Lemma 3.5 directly gives the following lemma.

Lemma 4.1. *Let $\mathcal{G}$ be an m-uniform hypergraph with n vertices. Then $L_{\mathcal{G}} + L_{\bar{\mathcal{G}}} = L_{K_n^m} = \phi_m(n)I_n - \theta J_n$, where $\phi_m(n) = \frac{n}{n-1}\binom{n-1}{m-1}$, $\theta = \frac{\binom{n-2}{m-2}}{m-1}$, J_n is the $(n\times n)$ matrix with all the entries 1 and I_n is the $(n\times n)$ identity matrix.*

Now we have the following proposition.

Proposition 4.1. *Let $\mathcal{G}$ be an m-uniform hypergraph with n vertices and let $\bar{\mathcal{G}}(V, \bar{E})$ be the complement of $\mathcal{G}$. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L_{\mathcal{G}}$ and the corresponding eigenvectors be $\mathbf{1}_n = X_1, X_2, \ldots, X_n$, respectively. Then the eigenvalues of $L_{\bar{\mathcal{G}}}$ are $0 = \lambda_1, \phi_m(n) - \lambda_2, \ldots, \phi_m(n) - \lambda_n$ with the same set of corresponding eigenvectors $X_1, X_2, \ldots, X_n$, respectively.*

Proof. Since $\langle X_1, X_i\rangle = 0$ for all $i = 2, \ldots, n$, the proof directly follows from the above lemma. □

Corollary 4.1.

1. *The eigenvalues of $L_{K_n^m}$ are 0 and $\phi_m(n)$ with the multiplicity[1] 1 and $n-1$, respectively.*
2. *The eigenvalues of $L_{K^m_{n_1,n_2}}$ are $0, \phi_m(n_1+n_2), \phi_m(n_1+n_2) - \phi_m(n_1)$ and $\phi_m(n_1+n_2) - \phi_m(n_2)$ with the multiplicity $1, 1, n_1 - 1$ and $n_2 - 1$, respectively.*
3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two m-uniform hypergraphs with n_1 and n_2, respectively, number of vertices. Let $0, \lambda_2, \ldots, \lambda_{n_1}$ and $0, \mu_2, \ldots, \mu_{n_2}$ be the eigenvalues of $L_{\mathcal{G}_1}$ and $L_{\mathcal{G}_2}$, respectively. Then the eigenvalues of $L_{\mathcal{G}_1+\mathcal{G}_2}$ are $0, \phi_m(n_1+n_2)$, $\phi_m(n_1+n_2) - \phi_m(n_1) + \lambda_2, \ldots, \phi_m(n_1+n_2) - \phi_m(n_1) + \lambda_{n_1}$, $\phi_m(n_1+n_2) - \phi_m(n_2) + \mu_2, \ldots, \phi_m(n_1+n_2) - \phi_m(n_2) + \mu_{n_2}$, where $\mathcal{G}_1 + \mathcal{G}_2 = \overline{\bar{\mathcal{G}}_1 \cup \bar{\mathcal{G}}_2}$.*

Theorem 4.1. *Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two m-uniform hypergraphs on n_1 and n_2 vertices, respectively. If λ and μ are eigenvalue of $L_{\mathcal{G}_1}$ and $L_{\mathcal{G}_2}$, respectively, then $\lambda+\mu$ is an eigenvalues of $L_{\mathcal{G}_1\Box\mathcal{G}_2}$.*

Proof. The proof is similar to that of Theorem 3.2. Here,

$$\begin{aligned}
L_{\mathcal{G}_1\Box\mathcal{G}_2}\gamma(a,x) &= \sum_{(b,y),(a,x)\sim(b,y)} \frac{d_{(a,x)(b,y)}}{m-1}(\gamma(a,x) - \gamma(b,y)) \\
&= \sum_{(b,x),(a,x)\sim(b,x)} \frac{d_{ab}}{m-1}(\gamma(a,x) - \gamma(b,x)) + \sum_{(a,y),(a,x)\sim(a,y)} \frac{d_{xy}}{m-1}(\gamma(a,x) - \gamma(a,y)) \\
&= \beta(x)\sum_{b,a\sim b}\frac{d_{ab}}{m-1}(\alpha(a) - \alpha(b)) + \alpha(a)\sum_{y,x\sim y}\frac{d_{xy}}{m-1}(\beta(x) - \beta(y)) \\
&= \beta(x)\lambda\alpha(a) + \alpha(a)\mu\beta(x) \\
&= (\lambda+\mu)\gamma(a,x).
\end{aligned}$$

Thus the proof follows. □

[1] Here the multiplicity means algebraic multiplicity.

4.1 Hypergraph connectivity and eigenvalues of a (combinatorial) Laplacian matrix

Now it is easy to verify that $\mathcal{G}$ is connected *iff* $\lambda_2(L_\mathcal{G}) \neq 0$ and then any constant function $u \in \mathbb{R}^n$ becomes the eigenfunction with the eigenvalue $\lambda_1(L_\mathcal{G}) = 0$. If $\mathcal{G}(V, E)$ has k connected components, then the (algebraic) multiplicity of the eigenvalue 0 of $L_\mathcal{G}$ is exactly k. So, we call $\lambda_2(L_\mathcal{G})$ algebraic weak connectivity of hypergraph $\mathcal{G}$. The following theorems show more relation of $\lambda_2(L_\mathcal{G})$ with the different aspects of connectivity of hypergraph $\mathcal{G}$.

Definition 4.1. A set of vertices in a hypergraph $\mathcal{G}$ is a weak vertex cut of $\mathcal{G}$ if weakly deletion of the vertices from that set increases the number of connected components in $\mathcal{G}$. The weak connectivity number $\kappa_W(\mathcal{G})$ is the minimum size of a weak vertex cut in $\mathcal{G}$.

Theorem 4.2. *Let $\mathcal{G}$ be an m-uniform connected hypergraph with $n(\geq 3)$ vertices, such that, $\mathcal{G}$ contains at least one pair of nonadjacent vertices and $d_{max} \leq m$. Then $\lambda_2(L_\mathcal{G}) \leq \kappa_W(\mathcal{G})$.*

Proof. Let W be a weak vertex cut of $\mathcal{G}$ such that $|W| = \kappa_W(\mathcal{G})$. Let us partition the vertex set of $\mathcal{G}$ as $V_1 \cup W \cup V_2$ such that no vertex in V_1 is adjacent to a vertex in V_2. Let $|V_1| = n_1$ and $|V_2| = n_2$.

Since $\mathcal{G}$ is connected, u_1 is constant. Let us construct a real-valued function u, orthogonal to u_1, as

$$u(i) = \begin{cases} n_2 & \text{if } i \in V_1, \\ 0 & \text{if } i \in W, \\ -n_1 & \text{if } i \in V_2. \end{cases}$$

Since any vertex $j \in W$ is adjacent to a vertex in V_1 and also to a vertex in V_2, and $d_{max} \leq m$, then $d_{ij} \leq m - 1$ for all $j \in W$ and $i \notin W$. Now, for any vertex $i \notin W$, we define $k_i = \sum_{j \in W, j \sim i} \frac{d_{ij}}{m-1} \leq \kappa_W(\mathcal{G})$.

Thus, for all $i \in V_1$,

$$\begin{aligned} (L_\mathcal{G} u)(i) &= d_i u(i) - \sum_{j \in V_1, j \sim i} \frac{d_{ij}}{m-1} u(j) \\ &= n_2 d_i - n_2(d_i - k_i) = n_2 k_i. \end{aligned}$$

Similarly, for all $i \in V_2$, we have $(L_\mathcal{G} u)(i) = -n_1 k_i$.

Hence,

$$\lambda_2(L_\mathcal{G}) ||u||^2 \leq \langle L_\mathcal{G} u, u \rangle \leq n_1 n_2^2 \kappa_W(\mathcal{G}) + n_1^2 n_2 \kappa_W(\mathcal{G}) = \kappa_W(\mathcal{G}) ||u||^2.$$

□

Theorem 4.3. *Let $\mathcal{G}(V, E)$ be an m-uniform hypergraph on n vertices. Then, for a nonempty $S \subset V$, we have*

$$|\partial S| \geq \frac{m-1}{\lfloor m^2/4 \rfloor} \frac{\lambda_2(L_\mathcal{G}) |S| |V \setminus S|}{n}.$$

Proof. Let us construct a real-valued function u, orthogonal to u_1, as

$$u(i) = \begin{cases} n - n_S & \text{if } i \in S, \\ -n_S & \text{if } i \in V \setminus S, \end{cases}$$

where $n_S = |S|$. Now we have

$$\begin{aligned} \langle L_\mathcal{G} u, u \rangle &= \sum_{e \in E} \Big(\sum_{i,j \in e} \frac{1}{m-1} (u(i) - u(j))^2 \Big) \qquad (2) \\ &\leq |\partial S| \frac{1}{m-1} \lfloor m^2/4 \rfloor n^2. \end{aligned}$$

The inequality holds because if $i \in S$ and $j \in V \setminus S$, then the number of terms in the parentheses in (2) is maximum when there are equal number of vertices in e from S and $V \setminus S$, respectively, and is equal to $\lfloor m^2/4 \rfloor$. Thus

$$\lambda_2(L_{\mathcal{G}})||u||^2 = \lambda_2(L_{\mathcal{G}})nn_S(n-n_S) \leq \langle L_{\mathcal{G}}u, u\rangle \leq |\partial S|\frac{1}{m-1}\lfloor m^2/4 \rfloor n^2.$$

Hence the proof follows. □

Now we bound the Cheeger constant

$$h(\mathcal{G}) := \min_{\phi \neq S \subset V} \left\{ \frac{|\partial S|}{\min(|S|, |V \setminus S|)} \right\}$$

of a hypergraph $\mathcal{G}(V, E)$ from below and above through $\lambda_2(L_{\mathcal{G}})$.

Theorem 4.4. *Let $\mathcal{G}(V, E)$ be an $m(\geq 3)$-uniform connected hypergraph with n vertices. Then*

$$h(\mathcal{G}) \geq 2\lambda_2(L_{\mathcal{G}})(1 - \frac{1}{m}).$$

Proof. Let S be a nonempty subset of V, such that, $h(\mathcal{G}) = |\partial S|/|S|$ and $|S| \leq |V|$. Let us define a real-valued function u as

$$u(i) = \begin{cases} \frac{1}{\sqrt{|S|}} & \text{if } i \in S, \\ 0 & \text{elsewhere.} \end{cases}$$

Let us define $t_S(e) = |\{v : v \in e \cap S, e \in E\}|$ and $t(S) = \frac{\sum_{e \in \partial S} t_S(e)}{|\partial S|}$. Now we have

$$\begin{aligned} \frac{\langle L_{\mathcal{G}}u, u\rangle}{||u||^2} &= \sum_{e \in E} \sum_{\substack{i \sim j, \\ \{i,j\} \in e}} \frac{1}{m-1}(u(i) - u(j))^2 \\ &= \sum_{e \in \partial S} \sum_{i \in e \cap S} \frac{1}{m-1} u(i)^2 \\ &= \frac{1}{m-1} \sum_{e \in \partial S} \frac{t_S(e)}{|S|}. \end{aligned}$$

Thus we have

$$\lambda_2(L_{\mathcal{G}}) \leq \frac{1}{m-1} \frac{|\partial S| t(S)}{|S|}$$

and similarly

$$\lambda_2(L_{\mathcal{G}}) \leq \frac{1}{m-1} \frac{|\partial \bar{S}| t(\bar{S})}{|\bar{S}|},$$

where $\bar{S} = V \setminus S$. Now from the above two inequalities we have

$$\lambda_2(L_{\mathcal{G}}) \leq \frac{m}{2(m-1)} h(\mathcal{G}).$$

Thus the proof follows. □

Theorem 4.5. *Let $\mathcal{G}(V,E)$ be an $m(\geq 3)$-uniform connected hypergraph on n vertices. If d_{max} is the maximum degree of $\mathcal{G}$ and $\lambda_2 = \lambda_2(L_{\mathcal{G}})$ then*

$$h(\mathcal{G}) < (m-1)\sqrt{(2d_{max}-\lambda_2)\lambda_2}.$$

Proof. Let u_2 be the eigenfunction with the eigenvalue λ_2, such that, $||u_2|| = 1$. Let $\phi \neq S \subset V$, such that $h(\mathcal{G}) = \frac{|\partial S|}{|S|}$ and $|S| \leq |V|/2$. Let $u : V \to \mathbb{R}$ be a function defined by

$$u(i) = |u_2(i)| \text{ for all } i \in V.$$

Thus,

$$\lambda_2 \quad > \quad \frac{1}{m-1}\sum_{i\sim j} d_{ij}(u(i)-u(j))^2 =: M \text{ (say)}. \tag{3}$$

The rest of the proof is similar to the proof for graphs [25]. Equation (3), by using Cauchay-Schwarz inequality, implies that

$$M \quad \geq \quad \frac{1}{m-1}\frac{\Big(\sum_{i\sim j} d_{ij}|u^2(i)-u^2(j)|\Big)^2}{\sum_{i\sim j} d_{ij}(u(i)+u(j))^2}. \tag{4}$$

Now proceed in a similar way as in the proof given in [25]. Let $t_0 < t_1 < \cdots < t_h$ be all different values of $u(i)$, $i \in V$. For $k = 0, 1, \ldots, h$, let us define $V_k := \{i \in V : u(i) \geq t_k\}$, and we denote $\delta_k(e) = \min\{|V_k \cap e|, |(V\setminus V_k)\cap e|\}$ for each edge $e \in \partial V_k$ and $\delta(V_k) = \min_{e\in\partial V_k}\{\delta_k(e)\}$. Let $\delta(\mathcal{G}) = \min_{k\in[h]}\{\delta(V_k)\}$, where $[h] = \{1, 2, \ldots, h\}$. Now

$$\begin{aligned}
\sum_{i\sim j} d_{ij}|u^2(i)-u^2(j)| &= \sum_{\substack{i\sim j\\ u(i)\geq u(j)}} d_{ij}\big(u^2(i)-u^2(j)\big)\\
&= \sum_{k=1}^{h}\sum_{\substack{i\sim j\\ u(i)=t_k\\ u(j)=t_l<t_k}} d_{ij}\big(t_k^2-t_{k-1}^2+t_{k-1}^2-\cdots-t_{l+1}^2+t_{l+1}^2-t_l^2\big)\\
&= \sum_{k=1}^{h}\sum_{\substack{i\sim j\\ i\in V_k}}\sum_{j\notin V_k} d_{ij}(t_k^2-t_{k-1}^2)\\
&\geq \sum_{k=1}^{h}\delta(V_k)|\partial V_k|(t_k^2-t_{k-1}^2)\\
&\geq \delta(\mathcal{G})h(\mathcal{G})\sum_{k=1}^{h}|V_k|(t_k^2-t_{k-1}^2)\\
&= \delta(\mathcal{G})h(\mathcal{G})\sum_{k=1}^{h}t_k^2(|V_k|-|V_{k+1}|), \text{ (here } V_{h+1}=\phi)\\
&= \delta(\mathcal{G})h(\mathcal{G})\sum_{k=1}^{n}u(k)^2.
\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}\sum_{i\sim j} d_{ij}(u(i)+u(j))^2 &= 2\sum_{i\sim j} d_{ij}\big(u(i)^2+u(j)^2\big) - \sum_{i\sim j} d_{ij}\big(u(i)-u(j)\big)^2\\ &\le 2d_{max}(m-1)\sum_{i=1}^n u(i)^2 - \sum_{i=1}^n u(i)^2 \sum_{i\sim j} d_{ij}\big(u(i)-u(j)\big)^2\\ &= 2d_{max}(m-1)\sum_{i=1}^n u(i)^2 - (m-1)M\sum_{i=1}^n u(i)^2, \text{ using Equation (3)}\\ &= (2d_{max}-M)(m-1)\sum_{i=1}^n u(i)^2. \qquad (6)\end{aligned}$$

Now, from Equations (4), (5) and (6), we get

$$\begin{aligned} M &> \frac{\frac{1}{m-1}\delta(\mathcal{G})^2 h(\mathcal{G})^2\big(\sum_{k=1}^n u(k)^2\big)^2}{(2d_{max}-M)(m-1)\sum_{i=1}^n u(i)^2}\\ &> \frac{1}{(m-1)^2}\frac{\delta(\mathcal{G})^2 h(\mathcal{G})^2}{(2d_{max}-M)}\\ &\ge \frac{1}{(m-1)^2}\frac{h(\mathcal{G})^2}{(2d_{max}-M)}, \text{ since } \delta(\mathcal{G})^2 \ge 1\end{aligned}$$

Hence,

$$\lambda_2 > \frac{h(\mathcal{G})^2}{(m-1)^2}\frac{1}{(2d_{max}-M)} \Rightarrow h(\mathcal{G}) < (m-1)\sqrt{(2d_{max}-\lambda_2)\lambda_2}.$$

□

4.2 Diameter and eigenvalues of Laplacian matrix of a hypergraph

Lemma 4.2 (Theorem 4.2, [26])**.** $\lambda_2(L_{G_0[A_\mathcal{G}]}) \ge 4/\big(n\cdot diam(G_0[A_\mathcal{G}])\big)$.

Lemma 4.3. *$diam(\mathcal{G}) = diam(G_0[A_\mathcal{G}])$.*

Theorem 4.6. *For an m-uniform connected hypergraph $\mathcal{G}(V,E)$ on n vertices,*

$$diam(\mathcal{G}) \ge \frac{4}{n(m-1)\lambda_2(L_\mathcal{G})}.$$

Proof. Let us consider the eigenfunction u_2 with the eigenvalue $\lambda_2(L_\mathcal{G})$. Then we have

$$\begin{aligned}\lambda_2(L_\mathcal{G}) &= \frac{1}{m-1}\frac{\sum_{i\sim j} d_{ij}(u_2(i)-u_2(j))^2}{||u_2||^2}\\ &\ge \frac{1}{m-1}\lambda_2(L_{G_0[A_\mathcal{G}]}).\end{aligned}$$

Now the proof follows from the above two lemmas. □

Definition 4.2. Let $\mathcal{G}(V,E)$ be a hypergraph and let $V_1, V_2 \subset V$. Then the distance $d(V_1,V_2)$ between V_1 and V_2 is defined as

$$d(V_1,V_2) = \min\{d(i,j) : i\in V_1, j\in V_2\}.$$

Lemma 4.4. *Let M denote an $n \times n$ matrix with rows and columns indexed by the vertices of a graph G and let $M(i,j) = 0$ if i, j are not adjacent. Now, if for some integer t and some polynomial $p_t(x)$ of degree t, $(p_t(M))_{ij} \neq 0$ for any i and j, then $diam(G) \leq t$.*

Theorem 4.7. *Let $\mathcal{G}(V, E)$ be an m-uniform connected hypergraph on n vertices with at least one pair of nonadjacent vertices. Then, for $V_1, V_2 \subset V$ such that $V_2 \neq V_1 \neq V \setminus V_2$, we have*

$$d(V_1, V_2) \leq \left\lceil \frac{\log \sqrt{\frac{(n-|V_1|)(n-|V_2|)}{|V_1||V_2|}}}{\log \left(\frac{\lambda_n(L_\mathcal{G}) + \lambda_2(L_\mathcal{G})}{\lambda_n(L_\mathcal{G}) - \lambda_2(L_\mathcal{G})} \right)} \right\rceil.$$

Proof. For $V_1 \subset V$, let us construct a function u_{V_1} as

$$u_{V_1}(i) = \begin{cases} 1/\sqrt{|V_1|} & \text{if } i \in V_1, \\ 0 & \text{elsewhere.} \end{cases}$$

Let f_i be orthonormal eigenfunctions of $L_\mathcal{G}$, such that $L_\mathcal{G} f_i = \lambda_i(L_\mathcal{G}) f_i$, for $i = 1, \ldots, n$. Then we have

$$u_{V_1} = \sum_{i=1}^{n} a_i f_i.$$

Let us take $f_1 = \frac{1}{\sqrt{n}} \mathbf{1}_n$. Then

$$a_1 = \langle u_{V_1}, f_1 \rangle = \sqrt{\frac{|V_1|}{n}}.$$

Similarly, for $V_2 \subset V$, we construct a function

$$u_{V_2} = \sum_{i=1}^{n} b_i f_i,$$

where

$$b_1 = \sqrt{\frac{|V_2|}{n}}.$$

Now, choose a polynomial $p_t(x) = \left(1 - \frac{2x}{\lambda_2(L_\mathcal{G}) + \lambda_n(L_\mathcal{G})}\right)^t$. Clearly

$$|p_t(\lambda_i(L_\mathcal{G}))| \leq (1 - \lambda)^t,$$

for all $i = 2, 3, \ldots, n$, where $\lambda = \frac{2\lambda_2(L_\mathcal{G})}{\lambda_2(L_\mathcal{G}) + \lambda_n(L_\mathcal{G})}$. If $\langle u_{V_2}, p_t(L_\mathcal{G}) u_{V_1} \rangle > 0$ for some t, then there is a path of length at most t between a vertex in V_1 and a vertex in V_2. Thus,

$$\begin{aligned} \langle u_{V_2}, p_t(L_\mathcal{G}) u_{V_1} \rangle &= p_t(0) a_1 b_1 + \sum_{i=2}^{n} p_t(\lambda_i(L_\mathcal{G})) a_i b_i \\ &\geq \frac{\sqrt{|V_1||V_2|}}{n} - \left| \sum_{i=2}^{n} p_t(\lambda_i(L_\mathcal{G})) a_i b_i \right| \\ &\geq \frac{\sqrt{|V_1||V_2|}}{n} - (1-\lambda)^t \sqrt{\sum_{i=2}^{n} a_i^2 \sum_{i=2}^{n} b_i^2} \qquad (7) \\ &= \frac{\sqrt{|V_1||V_2|}}{n} - (1-\lambda)^t \frac{\sqrt{(n-|V_1|)(n-|V_2|)}}{n}. \qquad (8) \end{aligned}$$

The last inequality follows from Cauchy-Schwarz inequality, whereas the last equality holds because

$$\sum_{i=2}^{n} a_i^2 = ||u_{V_1}||^2 - (|\langle u_{V_1}, f_1\rangle|)^2 = \frac{n-|V_1|}{n},$$

and

$$\sum_{i=2}^{n} b_i^2 = \frac{n-|V_2|}{n}.$$

The inequality in (7) is strict. This is because the equality in Cauchy-Schwarz inequality holds if and only if $a_i = cb_i$, for all i, for some constant c. However, it is possible only when $V_1 = V_2$ or $V_1 = V \setminus V_2$, which is not the case here. So, we get

$$\langle u_{V_2}, p_t(L_{\mathcal{G}})u_{V_1}\rangle > \frac{\sqrt{|V_1||V_2|}}{n} - (1-\lambda)^t \frac{\sqrt{(n-|V_1|)(n-|V_2|)}}{n}.$$

Now, if we choose

$$t \geq \frac{\log\sqrt{\frac{(n-|V_1|)(n-|V_2|)}{|V_1||V_2|}}}{\log\left(\frac{\lambda_n(L_{\mathcal{G}})+\lambda_2(L_{\mathcal{G}})}{\lambda_n(L_{\mathcal{G}})-\lambda_2(L_{\mathcal{G}})}\right)},$$

$\langle u_{V_2}, p_t(L_{\mathcal{G}})u_{V_1}\rangle$ becomes strictly positive. Thus the proof follows. $\square$

Corollary 4.2. *For an m-uniform connected hypergraph $\mathcal{G}(V,E)$ on n vertices and with at least one pair of nonadjacent vertices,*

$$diam(\mathcal{G}) \leq \left\lceil \frac{\log(n-1)}{\log\left(\frac{\lambda_n(L_{\mathcal{G}})+\lambda_2(L_{\mathcal{G}})}{\lambda_n(L_{\mathcal{G}})-\lambda_2(L_{\mathcal{G}})}\right)} \right\rceil.$$

Corollary 4.3. *Let $\mathcal{G}(V,E)$ be an m-uniform connected hypergraph on n vertices and with at least one pair of nonadjacent vertices. Then, for any $S \subset V$, we have*

$$\frac{|\delta S|}{|S|} \geq (n-|S|)\frac{1-(1-\lambda)^2}{(1-\lambda)^2(n-|S|)+|S|},$$

where $\lambda = \frac{2\lambda_2(L_{\mathcal{G}})}{\lambda_n(L_{\mathcal{G}})+\lambda_2(L_{\mathcal{G}})}$ and $\delta S = \{j \in V \setminus S : d(i,j) = 1,$ for some $i \in S\}$ is the vertex boundary of S. Moreover, if $|S| \leq n/2$ then we have

$$\frac{|\delta S|}{|S|} \geq \frac{2\lambda_n(L_{\mathcal{G}})\lambda_2(L_{\mathcal{G}})}{\lambda_n(L_{\mathcal{G}})^2+\lambda_2(L_{\mathcal{G}})^2}.$$

Proof. Take $V_1 = S$, $V_2 = V \setminus S - \delta S$ and $t = 1$. Then, using Equation (8) of Theorem 4.7, we have

$$\begin{aligned} 0 &= \langle u_{V_2}, p_{t=1}(L_{\mathcal{G}})u_{V_1}\rangle \\ &> \frac{\sqrt{|V_1||V_2|}}{n} - (1-\lambda)\frac{\sqrt{(n-|V_1|)(n-|V_2|)}}{n}. \end{aligned}$$

Since $V \setminus V_2 = S \cup \delta S$, this implies that

$$(1-\lambda)^2(n-|S|)(|S|+|\delta S|) > |S|(n-|S|-|\delta S|).$$

Thus the first part of the result follows.
Now, when $|S| \leq n - |S|$, from the above inequality we get

$$\begin{aligned}\frac{|\delta S|}{|S|} &\geq \frac{1-(1-\lambda)^2}{(1-\lambda)^2 + |S|/(n-|S|)} \\ &\geq \frac{1-(1-\lambda)^2}{(1-\lambda)^2+1} \\ &= \frac{2\lambda_n(L_{\mathcal{G}})\lambda_2(L_{\mathcal{G}})}{\lambda_n(L_{\mathcal{G}})^2 + \lambda_2(L_{\mathcal{G}})^2}.\end{aligned}$$

□

4.3 Bounds on $\lambda_2(L_{\mathcal{G}})$ and $\lambda_n(L_{\mathcal{G}})$

Let $e \equiv \{i_1, i_2, \ldots, i_m\} \in E$ be any edge in $\mathcal{G}$. Then, for e and $u \in \mathbb{R}^n$, we define a homogeneous polynomial of degree 2 in n variables by

$$L_{\mathcal{G}}(e)u^2 := \sum_{j=1}^{m} u(i_j)^2 - \frac{1}{m-1} \sum_{\substack{j=1,\\ i_j \in e}}^{m} \sum_{\substack{k=1,\\ i_j \neq i_k \in e}}^{m} u(i_j)u(i_k).$$

Thus,

$$\langle L_{\mathcal{G}}u, u\rangle = \sum_{e\in E} L_{\mathcal{G}}(e)u^2.$$

Theorem 4.8. *Let $\mathcal{G}(V,E)$ be an m-uniform connected hypergraph on $n(>2)$ vertices. Then*

$$\lambda_2(L_{\mathcal{G}}) \leq \min\left\{\frac{d_{i_1} + d_{i_2} + \cdots + d_{i_m} - m}{m} : \{i_1, i_2, \ldots, i_m\} \in E\right\}.$$

Proof. Let $e \equiv \{i_1, i_2, \ldots, i_m\} \in E$ be any edge in $\mathcal{G}$. Now, let us construct a function $u \in \mathbb{R}^n$, as

$$u(i) = \begin{cases} m^{-1/2} & \text{if } i \in e, \\ 0 & \text{elsewhere.}\end{cases}$$

Now,

$$\begin{aligned}\lambda_2(L_{\mathcal{G}}) &\leq \sum_{e\in E} L_{\mathcal{G}}(e)u^2 \\ &= \frac{1}{m}(d_{i_1} + d_{i_2} + \cdots + d_{i_m}) - \frac{1}{m-1}\cdot 2\binom{m}{2}\cdot\frac{1}{m}.\end{aligned}$$

Thus the proof follows. □

Corollary 4.4. *Let $\mathcal{G}(V,E)$ be an m-uniform connected hypergraph with $n(>2)$ vertices. Then*

$$\lambda_n(L_{\mathcal{G}}) \geq \min\left\{\frac{d_{i_1} + d_{i_2} + \cdots + d_{i_m} - m}{m} : \{i_1, i_2, \ldots, i_m\} \in E\right\}.$$

Theorem 4.9. *Let $\mathcal{G}(V,E)$ be an $m(>2)$-uniform connected hypergraph with n vertices. Then*

$$\lambda_n(L_{\mathcal{G}}) \leq \max\left\{\frac{2d_i(m-1)-1+\sqrt{4(m-1)^2d_im_iD_{max}^2-2d_i(m-1)+1}}{2(m-1)} : i \in V\right\},$$

where $m_i = (\sum_{j,j\sim i} d_j)/d_i(m-1)$ is the average 2-degree of the vertex i and $D_{max} = \max\{d_{xy} : x,y \in V\}$.

Proof. Let u_n be an eigenfunction of $L_{\mathcal{G}}$ with the eigenvalue $\lambda_n(L_{\mathcal{G}})$ $(= \lambda_n(\mathcal{G})$, say). Then we have

$$\sum_{i,j,i\sim j} d_{ij}(u_n(i)-u_n(j))^2 = (m-1)\lambda_n(\mathcal{G})\sum_{i=1}^{n} u_n(i)^2. \tag{9}$$

From the eigenvalue equation of $L_{\mathcal{G}}$ (i.e., from (1)) for the vertex i we have

$$(m-1)(d_i-\lambda_n(L_{\mathcal{G}}))u_n(i) \quad \leq \quad \sum_{j,j\sim i} D_{max}u_n(j).$$

Using Lagrange identity and summing both sides over i we get

$$\sum_{i=1}^{n}(m-1)^2(d_i-\lambda_n(L_{\mathcal{G}}))^2u_n(i)^2 \leq (m-1)\sum_{i=1}^{n} d_iD_{max}^2\sum_{j,j\sim i}u_n(j)^2 - \sum_{i=1}^{n}\sum_{\substack{1\leq j<k\leq n\\ j\sim i,k\sim i}} D_{max}^2(u_n(j)-u_n(k))^2 \tag{10}$$

Now, since

$$\sum_{i=1}^{n} d_i\sum_{j,j\sim i}u_n(j)^2 = (m-1)\sum_{i=1}^{n} d_im_iu_n(i)^2,$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{\substack{1\leq j<k\leq n\\ j\sim i,k\sim i}} D_{max}^2(u_n(j)-u_n(k))^2 \quad &\geq \quad \sum_{i,j,i\sim j} d_{ij}(u_n(i)-u_n(j))^2 \\ &= \quad (m-1)\lambda_n(\mathcal{G})\sum_{i=1}^{n} u_n(i)^2 \text{ (using (9))},\end{aligned}$$

Equation (10) becomes

$$\sum_{i=1}^{n}\left[(m-1)^2(d_i-\lambda_n(\mathcal{G}))^2-(m-1)^2d_im_iD_{max}^2+(m-1)\lambda_n(\mathcal{G})\right]u_n(i)^2 \leq 0. \tag{11}$$

Hence there exists a vertex i for which we have

$$(m-1)(d_i-\lambda_n(\mathcal{G}))^2-(m-1)d_im_iD_{max}^2+\lambda_n(\mathcal{G}) \leq 0.$$

This implies that

$$\lambda_n(L_{\mathcal{G}}) \leq \max\left\{\frac{2d_i(m-1)-1+\sqrt{4(m-1)^2d_im_iD_{max}^2-2d_i(m-1)+1}}{2(m-1)} : i \in V\right\}.$$

□

Corollary 4.5. *Let $\mathcal{G}(V,E)$ be an $m(>2)$-uniform connected hypergraph on n vertices. Then*

$$\lambda_n(L_{\mathcal{G}}) \leq \frac{2d_{max}(m-1)-1+\sqrt{4(m-1)^2d_{max}^2|E|^2-2d_{min}(m-1)+1}}{2(m-1)},$$

where d_{max} and d_{min} are the maximum and the minimum degrees, respectively, of $\mathcal{G}$.

Proof. Since $d_i m_i = (\sum_{j,j\sim i} d_j)/(m-1) \leq d_{max}^2$ and $D_{max} \leq |E|$ the result follows from the above theorem. $\square$

Lemma 4.5 (Theorem 5, [33])**.** *Let G be a simple connected weighted graph on n vertices, where each edge (i,j) is associated with a positive weighte w_{ij}. Then*

$$\lambda_n(L_G) \leq \frac{1}{2}\max_{i\sim j}\Big\{w_i + w_j + \sum_{k\sim i, k\nsim j} w_{ik} + \sum_{k\sim j, k\nsim i} w_{jk} + \sum_{k\sim i, k\sim j} |w_{ik}-w_{jk}|\Big\},$$

where $w_i = \sum_{j=1}^{n} w_{ij}$.

Now the above lemma provides us another upper bound for $\lambda_n(L_G)$ as follows.

Theorem 4.10. *Let $\mathcal{G}(V,E)$ be an m-uniform connected hypergraph with n vertices. Then*

$$\lambda_n(L_{\mathcal{G}}) \leq \frac{1}{2}\max_{i\sim j}\Big\{d_i + d_j + \frac{1}{m-1}\Big(\sum_{k\sim i, k\nsim j} d_{ik} + \sum_{k\sim j, k\nsim i} d_{jk} + \sum_{k\sim i, k\sim j} |d_{ik}-d_{jk}|\Big)\Big\}.$$

5 Normalized Laplacian matrix and operator of a hypergraph

Now we define normalized Laplacian operator and matrix $\Delta_{\mathcal{G}}$ for an m-uniform hypergraph $\mathcal{G}(V,E)$ on n vertices. Let μ be a natural measure on V given by $\mu(i) = d_i$. We consider the inner product $\langle f_1, f_2\rangle_\mu := \sum_{i\in V}\mu(i)f_1(i)f_2(i)$, for the n dimensional Hilbert space $l^2(V,\mu)$, given by

$$l^2(V,\mu) = \{f : V \to \mathbb{R}\}.$$

Now our normalized Laplacian operator

$$\Delta_{\mathcal{G}} : l^2(V,\mu) \to l^2(V,\mu)$$

is defined as

$$(\Delta_{\mathcal{G}} f)(i) := \frac{1}{m-1}\frac{1}{d_i}\sum_{j, i\sim j} d_{ij}(f(i)-f(j)) = f(i) - \frac{1}{m-1}\frac{1}{d_i}\sum_{j,i\sim j} d_{ij} f(j). \tag{12}$$

It is easy to verify that the eigenvalues of $\Delta_{\mathcal{G}}$ are real and nonnegative, since

$$\langle f_1, \Delta_{\mathcal{G}} f_2\rangle_\mu = \langle \Delta_{\mathcal{G}} f_1, f_2\rangle_\mu,$$

for all $f_1, f_2 \in \mathbb{R}^n$ and

$$\langle \Delta_{\mathcal{G}} f, f\rangle_\mu = \frac{1}{m-1}\sum_{i\sim j} d_{ij}(f(i)-f(j))^2 \geq 0,$$

for all $f \in \mathbb{R}^n$. The Rayleigh Quotient $\mathcal{R}_{\Delta_{\mathcal{G}}}(f)$ of a function $f : V \to \mathbb{R}$ is defined as

$$\mathcal{R}_{\Delta_{\mathcal{G}}}(f) = \frac{\langle \Delta_{\mathcal{G}} f, f\rangle_\mu}{\langle f, f\rangle_\mu} = \frac{\frac{1}{m-1}\sum_{i\sim j} d_{ij}(f(i) - f(j))^2}{\sum_{i\in V} d_i f(i)^2}. \tag{13}$$

For standard basis we get the matrix form of normalized Laplacian operator $\Delta_{\mathcal{G}}$ as

$$(\Delta_{\mathcal{G}})_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{d_{ij}}{d_i(m-1)} & \text{if } i \sim j, \\ 0 & \text{elsewhere.} \end{cases} \tag{14}$$

So, $\Delta_{\mathcal{G}} = I - R_{\mathcal{G}}$, where $R_{\mathcal{G}} = A_{\mathcal{G}} D_{\mathcal{G}}^{-1}$ is normalized adjacency matrix, which is a row-stochastic matrix. $R_{\mathcal{G}}$ can be considered as a probability transition matrix of a random walk on $\mathcal{G}$.

Now we order the eigenvalues of $\Delta_{\mathcal{G}}$ as $\lambda_1(\Delta_{\mathcal{G}}) \le \lambda_2(\Delta_{\mathcal{G}}) \le \cdots \le \lambda_n(\Delta_{\mathcal{G}})$ and find an orthonormal basis of $l^2(V, \mu)$ consisting of eigenfunctions of $\Delta_{\mathcal{G}}$, $u_k, k = 1, \ldots, n$, as we did it for Laplacian operator. The expression

$$\lambda_1(\Delta_{\mathcal{G}}) = \inf_{u \in l^2(V,\mu)-\{0\}} \{\langle \Delta_{\mathcal{G}} u, u\rangle_\mu : ||u|| = 1\}$$

provides u_1 and $\lambda_1(\Delta_{\mathcal{G}})$. The rest of the eigenvalues are iteratively estimated from the expression

$$\lambda_k(\Delta_{\mathcal{G}}) = \inf_{u \in \mathcal{H}_k-\{0\}} \left\{\frac{\langle \Delta_{\mathcal{G}} u, u\rangle_\mu}{\langle u, u\rangle_\mu}\right\},$$

where

$$\mathcal{H}_k := \{v \in l^2(V, \mu) : \langle v, u_l\rangle_\mu = 0 \text{ for } l \le k\}.$$

$\lambda_2(\Delta_{\mathcal{G}})$ can also be expressed as

$$\lambda_2(\Delta_{\mathcal{G}}) = \inf_{u \perp u_1} \left\{\frac{\langle \Delta_{\mathcal{G}} u, u\rangle_\mu}{\langle u, u\rangle_\mu}\right\}.$$

We may also define normalized Laplacian operator (and matrix) on an m-uniform hypergraph $\mathcal{G}(V, E)$ on n vertices as follows. Here, we consider the usual inner product $\langle f_1, f_2\rangle := \sum_{i\in V} f_1(i) f_2(i)$, for the n dimensional Hilbert space $L^2(V)$ constructed with all real-valued functions $f : V \to \mathbb{R}$ and the other normalized Laplacian operator

$$\mathcal{L}_{\mathcal{G}} : L^2(V) \to L^2(V)$$

and is defined as

$$(\mathcal{L}_{\mathcal{G}} f)(i) := \frac{1}{m-1} \frac{1}{\sqrt{d_i d_j}} \sum_{j, i\sim j} d_{ij}(f(i) - f(j)) = f(i) - \frac{1}{m-1} \frac{1}{\sqrt{d_i d_j}} \sum_{j, i\sim j} d_{ij} f(j). \tag{15}$$

For standard basis we get the matrix form of the above normalized Laplacian operator $\mathcal{L}_{\mathcal{G}}$ as

$$(\mathcal{L}_{\mathcal{G}})_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{d_{ij}}{(m-1)} \frac{1}{\sqrt{d_i d_j}} & \text{if } i \sim j, \\ 0 & \text{elsewhere.} \end{cases} \tag{16}$$

Two normalized Laplacian operators in (12) and (15) are equivalent. Hence, the matrices in (14) and (16) are similar and thus have same spectrum. In this article we use the normalized Laplacian operator defined in (12) and its matrix form[2] in (16). It is easy to verify that the eigenvalues of $\Delta_{\mathcal{G}}$ for an $m(>2)$-unform hypergraph lie in $[0,2)$ and the number of connected components in $\mathcal{G}$ is equal to the (algebraic) multiplicity of eigenvalue 0. When $\mathcal{G}$ is connected, u_1 is constant. Many theorems for normalized Laplacian matrix can be constructed similar to the theorems for Laplacian matrix (operator). We see that $\lambda_2(\Delta_{\mathcal{G}})$ can also bound the Cheeger constant

$$h(\mathcal{G}) := \inf_{\phi \neq S \subset V} \left\{ \frac{|\partial S|}{\min(vol(S), vol(\bar{S}))} \right\}$$

of a hypergraph $\mathcal{G}(V,E)$ from below and above. Here $vol(S) = \sum_{i \in S} d_i$.

Theorem 5.1. *Let $\mathcal{G}(V,E)$ be an $m(\geq 3)$-uniform connected hypergraph on n vertices. Then*

$$2\lambda_2(\Delta_{\mathcal{G}})(1 - \frac{1}{m}) \leq h(\mathcal{G}) < (m-1)\sqrt{(2 - \lambda_2(\Delta_{\mathcal{G}}))\lambda_2(\Delta_{\mathcal{G}})}.$$

Proof. The proof is similar to the proofs of Theorem 4.4 and Theorem 4.5. □

Lemma 5.1. *The diameter of a uniform hypergraph $\mathcal{G}$ is less than the number of distinct eigenvalues of $R_{\mathcal{G}}$.*

Proof. The proof is similar to the proof of Theorem 3.4. □

Proposition 5.1. *For a uniform hypergraph $\mathcal{G}$, $diam(\mathcal{G})$ is less than the number of distinct eigenvalues of $\Delta_{\mathcal{G}}$.*

Proof. Since $\Delta_{\mathcal{G}} = I - R_{\mathcal{G}}$, the proof follows from the above lemma. □

Theorem 5.2. *For an m-uniform hypergraph $\mathcal{G}$ on n vertices,*

$$diam(\mathcal{G}) \geq \frac{4}{n(m-1)d_{max}\lambda_2(\Delta_{\mathcal{G}})}.$$

Proof. Since $\lambda_2(\Delta_{\mathcal{G}}) \geq \frac{1}{(m-1)d_{max}}\lambda_2(L_{G_0[A_{\mathcal{G}}]})$, the proof is similar to the proof of Theorem 4.6. □

Theorem 5.3. *Let $\mathcal{G}(V,E)$ be an m-uniform connected hypergraph with n vertices and at least one pair of nonadjacent vertices. Then, for $V_1, V_2 \subset V$ such that $V_2 \neq V_1 \neq V \setminus V_2$, we have*

$$d(V_1,V_2) \leq \left\lceil \frac{\log \sqrt{\frac{vol(\bar{V_1})vol(\bar{V_2})}{vol(V_1)vol(V_2)}}}{\log\left(\frac{\lambda_n(\Delta_{\mathcal{G}})+\lambda_2(\Delta_{\mathcal{G}})}{\lambda_n(\Delta_{\mathcal{G}})-\lambda_2(\Delta_{\mathcal{G}})}\right)} \right\rceil.$$

Proof. The proof is similar to the proof of Theorem 4.7. □

[2] Note that, two non-isomorphic hypergraphs of order $m > 2$ may have the same normalized Laplacian matrix $\Delta_{\mathcal{G}}$ (or the normalized adjacency matrix $R_{\mathcal{G}}$). It happens when all the 2-element subsets of the vertex set of the hypergraph are subsets of a fixed number of edges. For example, existence of a (combinatorial) simple incomplete 2-design on the vertex set of a hypergraph where each edge is considered as a block. A particular example is Fano plane, which is a finite projective plane of order 2 with 7 points, represents a 3-uniform hypergraph on 7 vertices with the vertex set $\{1,2,3,4,5,6,7\}$ and the edge set $\{\{1,2,3\},\{1,4,7\},\{1,5,6\},\{2,4,6\},\{2,5,7\},\{3,4,5\},\{3,6,7\}\}$, where each pair of vertices belongs to exactly one edge. Fano plane is a regular balanced incomplete block $(7,3,1)$-design. Thus, the normalized Laplacian (adjacency) matrices for Fano plane and K_7^3, respectively, are the same.

Corollary 5.1. *For an m-uniform connected hypergraph $\mathcal{G}(V, E)$ with n vertices and at least one pair of nonadjacent vertices,*

$$diam(\mathcal{G}) \leq \left\lceil \frac{\log(\frac{(n-1)d_{max}}{d_{min}})}{\log\left(\frac{\lambda_n(\Delta_{\mathcal{G}})+\lambda_2(\Delta_{\mathcal{G}})}{\lambda_n(\Delta_{\mathcal{G}})-\lambda_2(\Delta_{\mathcal{G}})}\right)} \right\rceil.$$

Corollary 5.2. *For an m-uniform connected regular hypergraph $\mathcal{G}(V, E)$ with n vertices and at least one pair of nonadjacent vertices,*

$$diam(\mathcal{G}) \leq \left\lceil \frac{\log(n-1)}{\log\left(\frac{\lambda_n(\Delta_{\mathcal{G}})+\lambda_2(\Delta_{\mathcal{G}})}{\lambda_n(\Delta_{\mathcal{G}})-\lambda_2(\Delta_{\mathcal{G}})}\right)} \right\rceil.$$

Lemma 5.2. *Let $\mathcal{G}$ be a hypergraph containing at least one pair of nonadjacent vertices. Then $\lambda_2(\Delta_{\mathcal{G}}) \leq 1$.*

Proof. Let j, k be two nonadjacent vertices in $\mathcal{G}$. Now, we choose a function $u \in \mathbb{R}^n$, orthogonal to u_1, as

$$u(i) = \begin{cases} d_k & \text{if } i = j, \\ d_j & \text{if } i = k, \\ 0 & \text{elsewhere.} \end{cases}$$

Then we have

$$\lambda_2(\Delta_{\mathcal{G}}) \leq \frac{\langle \Delta_{\mathcal{G}} u, u \rangle_\mu}{\langle u, u \rangle_\mu} = 1.$$

□

Corollary 5.3. *Let $\mathcal{G}$ be a hypergraph on n vertices and let $\mathcal{G}$ contains at least one pair of nonadjacent vertices. Then $\lambda_n(\Delta_{\mathcal{G}}) \geq 1$.*

Theorem 5.4. *Let $\mathcal{G}(V, E)$ be an m-uniform connected hypergraph with n vertices. Then*

$$\lambda_n(\Delta_{\mathcal{G}}) \leq \max\left\{ \frac{2(m-1)d_i - 1 + \sqrt{1 - 4(m-1)d_i + 4(m-1)^2 d_i D_{max}^2 m_i}}{2(m-1)d_i} : i \in V \right\},$$

where $m_i = (\sum_{j, j\sim i} d_j)/d_i(m-1)$ is the average 2-degree of the vertex i and $D_{max} = \max\{d_{xy} : x, y \in V\}$.

Proof. Let u_n be an eigenfunction of $\Delta_{\mathcal{G}}$ with the eigenvalue $\lambda_n(\Delta_{\mathcal{G}})$. Then, from the eigenvalue equation for $\lambda_n(\Delta_{\mathcal{G}})$ and u_n, we have

$$\sum_{i,j,i\sim j} d_{ij}(u_n(i) - u_n(j))^2 = (m-1)\lambda_n(\Delta_{\mathcal{G}}) \sum_{i=1}^{n} d_i u_n(i)^2. \tag{17}$$

The rest of the proof is similar to the proof of Theorem 4.9. Now the expression in (11) becomes

$$\sum_{i=1}^{n} \left[(m-1)(1 - \lambda_n(\Delta_{\mathcal{G}})^2)d_i - (m-1)m_i D_{max}^2 + \lambda_n(\Delta_{\mathcal{G}})\right] u_n(i)^2 \leq 0.$$

Using similar arguments as in Theorem 4.9 we have

$$(m-1)d_i\lambda_n(\Delta_{\mathcal{G}})^2 + (1 - 2(m-1)d_i)\lambda_n(\Delta_{\mathcal{G}}) + (m-1)(d_i - m_i D_{max}^2) \leq 0,$$

which proves the theorem. □

When $m = 2$, $\mathcal{G}$ becomes a *triangulation* and the above upper bound coincides with the result proved in [22] for a triangulation.

Corollary 5.4. *For an m-uniform connected hypergraph $\mathcal{G}(V, E)$ with the maximum and the minimum degrees d_{max} and d_{min}, respectively, we have*

$$\lambda_n(\Delta_{\mathcal{G}}) \leq \frac{2(m-1)d_{max} - 1 + \sqrt{1 - 4(m-1)d_{min} + 4(m-1)^2 d_{max}^2 |E|^2}}{2(m-1)d_{min}}.$$

5.1 Random walk on hypergraphs

A random walk on an m-uniform hypergraph $\mathcal{G}(V, E)$ can be considered as a sequence of vertices $v_0, v_1, \ldots, v_t$ and it can be determined by the transition probabilities $P(u, v) = Prob(x_{i+1} = v | x_i = u)$ which is independent of i. Thus, a simple random walk on an m-uniform hypergraph $\mathcal{G}(V, E)$ is a *Markov chain*, where a *Markov kernel* on V is a function

$$P(\cdot, \cdot) : V \times V \to [0, +\infty),$$

such that $\sum_{y \in V} P(x, y) = 1 \ \forall x \in V$. Here $P(x, y)$ is called reversible if there exists a positive function $\mu(\cdot)$ on the state space V, such that $P(x, y)\mu(x) = P(y, x)\mu(y)$. A random walk is reversible if its underlying Markov kernel is reversible. It is easy to see that a random walk on a connected $m(> 2)$-uniform hypergraph is *ergodic*, i.e., P is (i) *irreducible*: i.e., for all $x, y \in V$, $P^t(x, y) > 0$ for some $t \in \mathbb{N}$ and (ii) *aperiodic*: i.e., g.c.d $\{t : P^t(x, y)\} = 1$. We can consider the transition probabilities $P_{\mathcal{G}}(x, y)$ for a connected $m(> 2)$-uniform hypergraph $\mathcal{G}(V, E)$ as

$$P_{\mathcal{G}}(x, y) = \frac{d_{xy}/(m-1)}{d_x}.$$

Now, let us consider $P_{\mathcal{G}} : l^2(V, \mu) \to l^2(V, \mu)$ as a transition probability operator for the random walk on $\mathcal{G}$. Thus $\Delta_{\mathcal{G}} = \mathbf{I} - P_{\mathcal{G}}$, where $\mathbf{I}$ is the identical operator in $l^2(V, \mu)$. Hence, for an eigenvalue $\lambda(\Delta_{\mathcal{G}})$ of $\Delta_{\mathcal{G}}$, we always get an eigenvalue $(1 - \lambda(\Delta_{\mathcal{G}}))$ of $P_{\mathcal{G}}$. Let $\alpha_k = 1 - \lambda_k(\Delta_{\mathcal{G}})$ be the eigenvalues of $P_{\mathcal{G}}$ of an $m(> 2)$ hypergraph $\mathcal{G}$ on n vertices, for $k = 0, \ldots, n$. Then, $-1 < \alpha_n \leq \alpha_{n-1} \leq \alpha_2 \leq \alpha_1 = 1$. Hence $||P_{\mathcal{G}}|| \leq 1$, since the Spec $P_{\mathcal{G}} \subset (-1, 1]$. Let us consider the powers $P_{\mathcal{G}}^t$ of $P_{\mathcal{G}}$ for $t \in \mathbb{Z}^+$ as composition of operators. Below we recall the theorem for convergence of random walk on graphs [16] in the context of connected $m(> 2)$-uniform hypergraphs $\mathcal{G}(V, E)$ on n vertices.

Theorem 5.5. *For any function $f \in l^2(V, \mu)$, take*

$$\overline{f} = \frac{1}{vol(V)} \sum_{i \in V} d_i f(i).$$

Then, for any positive integer t, we have

$$||P_{\mathcal{G}}^t f - \overline{f}|| \leq \rho^t ||f||,$$

where $\rho = \max_{k \neq 1} |1 - \lambda_k(\Delta_{\mathcal{G}})| = \max(|1 - \lambda_2(\Delta_{\mathcal{G}})|, |1 - \lambda_n(\Delta_{\mathcal{G}})|)$ is the spectral radius of $P_{\mathcal{G}}$ and $||f|| = \sqrt{\langle f, f \rangle_\mu}$.
Consequently,

$$||P_{\mathcal{G}}^t f - \overline{f}|| \to 0$$

as $t \to \infty$, i.e., $P_{\mathcal{G}}^t f$ converges to a constant $\overline{f}$ as $t \to \infty$.

Thus, after $t \geq \frac{1}{1-\rho}\log(1/\epsilon)$ steps $||P_{\mathcal{G}}^t f - \overline{f}||$ becomes less than $\epsilon||f||$. We define the equilibrium transition probability operator $\overline{P_{\mathcal{G}}} : l^2(V,\mu) \to l^2(V,\mu)$ as

$$\overline{P_{\mathcal{G}}}u(i) = \frac{1}{vol(V)} \sum_{j \in V} d_j u(j).$$

Thus, $\overline{P_{\mathcal{G}}} f = \overline{f}$, for all functions $f \in l^2(V,\mu)$. Using the above theorem we find that $P_{\mathcal{G}}^t$ converges to $\overline{P_{\mathcal{G}}}$ as $t \to \infty$.

We also refer our readers to [24] where a set of Laplacians for hypergraphs have been defined to study high-order random walks on hypergraphs.

5.2 Ricci curvature on hypergraphs

Here we discuss two aspects of Ricci curvature on hypergraphs. Let us recall our transition probability operator

$$P_{\mathcal{G}}(x,y) = \frac{d_{xy}/(m-1)}{d_x}$$

for an $m(\geq 2)$-uniform hypergraph $\mathcal{G}(V,E)$. Clearly $P_{\mathcal{G}}(x,y)$ is reversible. Let us define the Laplace operator

$$\Delta := -\Delta_{\mathcal{G}}$$

which is also acting on $l^2(V,\mu)$. Thus $\Delta = P_{\mathcal{G}} - \mathbf{I}$ and for any $f \in l^2(V,\mu)$ we have $(\Delta f)(i) = \frac{1}{m-1}\frac{1}{d_i}\sum_{j,i\sim j} d_{ij}(f(j)-f(i))$. Now we discuss two aspects of Ricci curvature in the sense of Bakry and Emery [2] and Ollivier [28]. For graphs, readers may also see [4, 20, 23].

Ricci curvature on hypergraphs in the sense of Bakry and Emery

Let us define a bilinear operator

$$\Gamma : l^2(V,\mu) \times l^2(V,\mu) \to l^2(V,\mu)$$

as

$$\Gamma(f_1,f_2)(i) := \frac{1}{2}\{\Delta(f_1(i)f_2(i)) - f_1(i)\Delta f_2(i) - f_2(i)\Delta f_1(i)\}.$$

Then the Ricci curvature operator, Γ_2, is defined as

$$\Gamma_2(f_1,f_2)(i) := \frac{1}{2}\{\Delta\Gamma(f_1,f_2)(i) - \Gamma(f_1,\Delta f_2)(i) - \Gamma(f_2,\Delta f_1)(i)\}.$$

Now, for our hypergraph $\mathcal{G}$ we have

$$\Gamma(f,f)(i) = \frac{1}{2}\frac{1}{d_i}\sum_{j,j\sim i}\frac{d_{ij}}{(m-1)}(f(i)-f(j))^2.$$

Then, from the proof of Theorem 1.2 in [23], we can express our Ricci curvature operator on a hypergraph $\mathcal{G}$ as

$$\begin{aligned}\Gamma_2(f,f)(i) =& \frac{1}{4}\frac{1}{d_i}\sum_{j,j\sim i}\frac{d_{ij}}{d_j(m-1)}\sum_{k,k\sim j}\frac{d_{jk}}{(m-1)}(f(i)-2f(j)+f(k))^2 \\ &- \frac{1}{2}\frac{1}{d_i}\sum_{j,j\sim i}\frac{d_{ij}}{(m-1)}(f(i)-f(j))^2 + \frac{1}{2}\left(\frac{1}{d_i}\sum_{j,j\sim i}\frac{d_{ij}}{(m-1)}(f(i)-f(j))\right)^2.\end{aligned}$$

We have omitted the variable i in the above equation. For simplicity, we do the same for the following equations which hold for all $i \in V$.

Let $\mathbf{m}$ and $\mathbf{K}$ be the *dimension* and the lower bound of the Ricci curvature, respectively, of Laplacian operator Δ. Then we say that Δ satisfies *curvature-dimension type inequality* $CD(\mathbf{m}, \mathbf{K})$ for some $\mathbf{m} > 1$ if

$$\Gamma_2(f,f) \geq \frac{1}{\mathbf{m}}(\Delta f)^2 + \mathbf{K}\Gamma(f,f) \text{ for all } f \in l^2(V,\mu).$$

If $\Gamma_2 \geq \mathbf{K}\Gamma$, then Δ satisfies $CD(\infty, \mathbf{K})$. Any connected m-uniform (finite) hypergraph $\mathcal{G}(V,E)$ satisfies $CD(2, \frac{1}{d_*} - 1)$, where $d_* = \sup_{i \in V} \sup_{j \sim i} \frac{d_i(m-1)}{d_{ij}}$.

Now, Theorem 2.1 in [4] can be stated in the context of hypergraphs as follows.

Theorem 5.6. *If Δ satisfies a curvature-dimension type inequality $CD(\mathbf{m}, \mathbf{K})$ with $\mathbf{m} > 1$ and $\mathbf{K} > 0$ then*

$$\lambda_2(\Delta_{\mathcal{G}}) \geq \frac{\mathbf{m}\mathbf{K}}{\mathbf{m} - 1}.$$

Ricci curvature on hypergraphs in the sense of Ollivier

The Ollivier's Ricci curvature (also known as Ricci-Wasserstein curvature) is introduced on a separable and complete metric space (X, d), where each point $x \in X$ has a probability measure $p_x(\cdot)$. Let us denote the structure by (X, d, p). Let $\mathcal{C}(\mu, \nu)$ be the set of probability measures on $X \times X$ projecting to μ and ν. Now $\xi \in \mathcal{C}(\mu, \nu)$ satisfies

$$\xi(A \times X) = \mu(A), \xi(X \times B) = \nu(B), \forall A, B \subset X.$$

Then the *transportation distance (or Wasserstein distance)* between two probability measures μ, ν on a metric space (X, d) is defined as

$$\mathcal{T}_1(\mu, \nu) := \inf_{\xi \in \mathcal{C}(\mu,\nu)} \int_{X \times X} d(x,y) d\xi(x,y).$$

Now on (X, d, p), the Ricci curvature of (X, d, p) for distinct $x, y \in X$ is defined as

$$\kappa(x,y) := 1 - \frac{\mathcal{T}_1(p_x, p_y)}{d(x,y)}.$$

For a connected m-uniform hypergraph $\mathcal{G}(V,E)$ we take $d(x,y) = 1$ for two distinct adjacent vertices x, y and we consider the probability measure

$$p_x(y) = \begin{cases} \frac{1}{d_x}\frac{d_{xy}}{m-1}, & \text{if } y \sim x, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in V$. Now Theorem 3.1 in [4] also holds for a connected hypergraph as follows

Theorem 5.7. *Let $\mathcal{G}$ be a connected hypergraph on n vertices. Then $\kappa \leq \lambda_2(\Delta_{\mathcal{G}}) \leq \lambda_n(\Delta_{\mathcal{G}}) \leq 2 - \kappa$, where the Ollivier's Ricci curvature of $\mathcal{G}$ is at least κ.*

As in [20], we also introduce a scalar curvature (suggested in Problem Q in [28]) for a vertex x in $\mathcal{G}$ as

$$\kappa(x) := \frac{1}{d_x} \sum_{y, x \sim y} \kappa(x,y).$$

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