Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier.

– Gilbert Strang

Linear vector spaces

A *vector space* is simply a collection of things that obeys certain abstract (but mostly familiar) algebraic properties. We will start by detailing these properties.

- A vector space $\mathcal{S}$ is composed of a set of elements, called *vectors*, and members of a field[1] $\mathbb{F}$ called *scalars*.
- The space also has rules for adding vectors and multiplying them by scalars
  - *vector addition*, which we will write as '+' combines two vectors to get a third
  - *scalar multiplication*, combines a scalar and a vector to get another vector
- The '+' operation must obey the following four rules for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$:
  1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ (commutative)
  2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ (associative)
  3. There is a unique *zero vector* $\mathbf{0}$ such that

$$\boldsymbol{x} + \mathbf{0} = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \mathcal{S}$$

[1] A field is simply a set of numbers for which multiplication and addition are defined, and distribute/associate in the same manner as the reals.

4. For each vector $\boldsymbol{x} \in \mathcal{S}$, there is a unique vector (called $-\boldsymbol{x}$) such that
$$\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$$

- Scalar multiplication must obey the following four rules for all $a, b \in \mathbb{F}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$:
  1. $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$
     $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$ (distributive)
  2. $(ab)\boldsymbol{x} = a(b\boldsymbol{x})$ (associative)
  3. For the multiplicative identity of $\mathbb{F}$, which we write as 1, we have
  $$1\boldsymbol{x} = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \mathcal{S}$$
  4. For the additive identity of $\mathbb{F}$, which we write as 0, we have
  $$0\boldsymbol{x} = \mathbf{0}$$
  (that's the scalar zero on the left, and the vector zero on the right).
- $\mathcal{S}$ is closed under scalar multiplication and vector addition:
$$\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} \;\Rightarrow\; a\boldsymbol{x} + b\boldsymbol{y} \in \mathcal{S}, \quad \forall a, b \in \mathbb{F}.$$

This last point is really the most salient piece of algebraic structure. In light of it, we will often use the more descriptive terminology **linear vector space**.

Examples of vector spaces

1. $\mathbb{R}^N$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad \text{where the } x_i \text{ are real}$$

   and we use the standard rules for vector addition and scalar multiplication

2. $\mathbb{C}^N$, same as above, except the x_i are complex

3. Bounded, continuous functions $f(t)$ on the interval $[a, b]$ that are real valued.
   Vector addition = adding functions pointwise,
   scalar multiplication = multiplying by $a \in \mathbb{R}$ pointwise,
   it should be easy to see that adding two bounded, continuous functions gives you another bounded and continuous function.

4. $GF(2)^N$
   Here, the scalar field is $\{0, 1\}$, and so vectors are lists of N bits.
   Addition for the field is modulo 2, so

$$\begin{aligned} &0 + 0 = 0 \\ &0 + 1 = 1 + 0 = 1 \\ &1 + 1 = 0 \end{aligned}$$

   For example,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

   This space is super useful in information/coding theory

Here is an example of something which is not a vector space:

5. Bounded, continuous functions $f(t)$ on $[a, b]$ such that

$$|f(t)| \leq 2.$$

Why is this not a linear vector space?

Linear subspaces

A (non-empty) subset $\mathcal{T}$ of $\mathcal{S}$ is call a **linear subspace** of $\mathcal{S}$ if

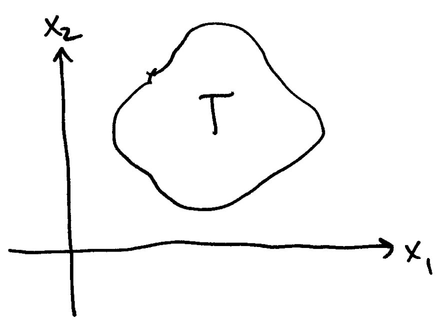
$$\forall\ a, b \in \mathbb{F}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{T} \ \Rightarrow\ a\boldsymbol{x} + b\boldsymbol{y} \in \mathcal{T}$$

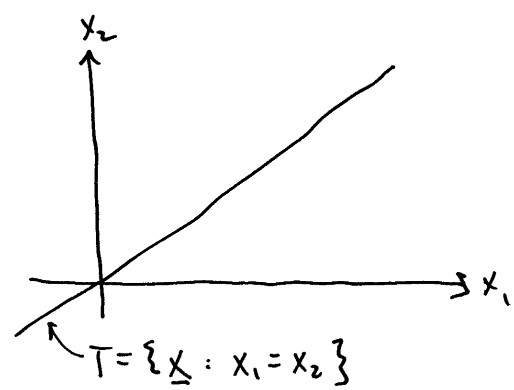
Note that is has to be true that

$$\mathbf{0} \in \mathcal{T}.$$

It is easy to show that $\mathcal{T}$ can be treated as a linear vector space by itself.

Easy examples: Are these subspaces of $\mathcal{S} = \mathbb{R}^2$?





Which of these are subspaces?

1. $\mathcal{S} = \mathbb{R}^5$
   $\mathcal{T} = \{\boldsymbol{x} \;:\; x_4 = 0,\; x_5 = 0\}$

2. $\mathcal{S} = \mathbb{R}^5$
   $\mathcal{T} = \{\boldsymbol{x} \;:\; x_4 = 1,\; x_5 = 1\}$

3. $\mathcal{S} = \mathcal{C}([0,1])$ (bounded, continuous functions on $[0,1]$)
   $\mathcal{T} = \{\text{polynomials of degree } p\}$

4. $\mathcal{S} =$ continuous functions on the real line
   $\mathcal{T} = \{f(t) \;:\; f \text{ is bandlimited to } \Omega\}$

5. $\mathcal{S} = \mathbb{R}^N$
   $\mathcal{T} = \{\boldsymbol{x} \;:\; \boldsymbol{x} \text{ has no more than 5 non-zero components}\}$

6. $\mathcal{S} = \mathbb{R}^N$
   $\mathcal{T} = \{\boldsymbol{x} \;:\; \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = 3\}$, where $\boldsymbol{c} \in \mathbb{R}^N$ is a fixed vector
   (Recall the standard dot product $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \sum_{n=1}^{N} c[n]x[n]$)

7. $\mathcal{S} = \mathcal{C}([0,1])$
   $\mathcal{T} = \{f(t) \;:\; f(t) = a\cos(2\pi t) + b\sin(2\pi t) \text{ for some } a, b \in \mathbb{R}\}$

Linear combinations and spans

Let $\mathcal{M} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$ be a set of vectors in a linear space $\mathcal{S}$.

Definition: A **linear combination** of vectors in $\mathcal{M}$ is a sum of the form

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_N\boldsymbol{v}_N$$

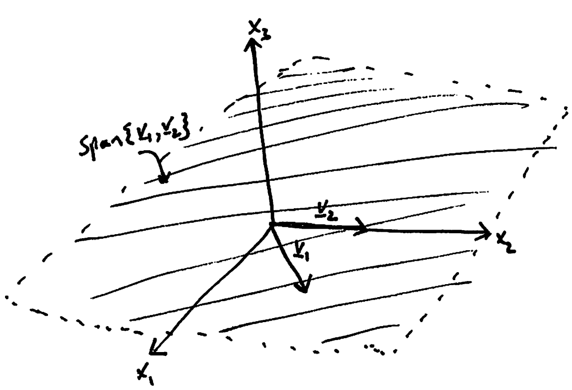
for some $a_1, \ldots, a_N \in \mathbb{F}$.

Definition: The **span** of $\mathcal{M}$ is the set of all linear combinations of $\mathcal{M}$. We write this as

$$\operatorname{span}(\mathcal{M}) \;=\; \operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\})$$

Example:

$$\mathcal{S} = \mathbb{R}^3, \qquad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



$$\operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\}) = (x_1, x_2) \text{ plane}$$

i.e. for any x_1, x_2 we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$

Question: What is the span of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad ?$$

What about if

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad ?$$

Example:

$$\begin{aligned} \mathcal{S} &= \{f(t) \ : \ f(t) \text{ is periodic with period } 2\pi\} \\ \mathcal{M} &= \{e^{jkt}\}_{k=-B}^{B} \end{aligned}$$

Then span$(\mathcal{M})$ = periodic, bandlimited (to B) functions, i.e.

$$f(t) = \sum_{k=-B}^{B} c_k \, e^{jkt}$$

for some $c_{-B}, c_{-B+1}, \ldots, c_0, c_1, \ldots, c_B \in \mathbb{C}$.

Linear dependence

A set of vectors $\{\boldsymbol{v}_j\}_{j=1}^N$ is said to be **linearly dependent** if there exists scalars $a_1, \ldots, a_N$, not all $= 0$, such that

$$\sum_{n=1}^{N} a_n \, \boldsymbol{v}_n = \mathbf{0}$$

Likewise, if $\sum_n a_n \boldsymbol{v}_n = \mathbf{0}$ only when all the $a_j = 0$, then $\{\boldsymbol{v}_n\}_{n=1}^N$ is said to be **linearly independent**.

Example 1:

$$\mathcal{S} = \mathbb{R}^3, \quad \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Find a_1, a_2, a_3 such that

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = \mathbf{0}$$

Note that any two of the vectors above are linearly independent:

$$\operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}) \;=\; \operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\}) \;=\; \operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_3\}) \;=\; \operatorname{span}(\{\boldsymbol{v}_2, \boldsymbol{v}_3\})$$

Example 2:

$$\begin{aligned} \mathcal{S} &= \mathcal{C}([0,1]) \\ \boldsymbol{v}_1 &= \cos(2\pi t) \\ \boldsymbol{v}_2 &= \sin(2\pi t) \\ \boldsymbol{v}_3 &= 2\cos(2\pi t + \pi/3) \end{aligned}$$

Find a_1, a_2, a_3 such that

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = \mathbf{0}$$

Repeat for

$$\boldsymbol{v}_3 = A\cos(2\pi t + \phi) \qquad \text{for some } A > 0, \quad \phi \in [0, 2\pi).$$

Suppose that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\}$ are linearly dependent. Then

$$\sum_n a_n\,\boldsymbol{v}_n \;=\; \mathbf{0} \quad \Rightarrow \quad \boldsymbol{v}_k = \frac{1}{a_k}\sum_{n\neq k} a_n\boldsymbol{v}_n \quad \text{for every } a_k \neq 0.$$

Thus there is at least one vector we can remove from the set without changing its span. This process can be repeated until we are left with a set that is linearly independent.

Bases

Definition: A **basis** of a linear vector space $\mathcal{S}$ is a (countable) set of vectors $\mathcal{B}$ such that[2]

1. $\text{span}(\mathcal{B}) = \mathcal{S}$
2. $\mathcal{B}$ is linearly independent

The second condition ensures that all bases of $\mathcal{S}$ will have the same (possibly infinite) number of elements.

The **dimension** of $\mathcal{S}$ is the number of elements required in a basis for $\mathcal{S}$. (Again, this could very easily be ∞.)

Examples:

1. $\mathbb{R}^N$ with

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

   This is the **standard basis** for $\mathbb{R}^N$.

2. $\mathbb{R}^N$ with any set of N linearly independent vectors.

[2] In infinite dimensions, we really need to be more careful with this definition than what is being said here. In that setting, there are multiple definitions of a basis, the most useful of which require the notion of an inner product, which we will get to soon. We will return to this technical issue then.

3. $\mathcal{S} = \{\text{polynomials of degree at most } p\}$.
   A basis for $\mathcal{S}$ is $\mathcal{B} = \{1, t, t^2, \ldots, t^p\}$.
   The dimension of $\mathcal{S}$ is $p+1$.

4. $\mathcal{S} = \{f(t) \; : \; f(t) \text{ is periodic with period } 2\pi\}$
   A basis for $\mathcal{S}$ is $\mathcal{B} = \{e^{jkt}\}_{k=-\infty}^{\infty}$ (Fourier Series)
   $\mathcal{S}$ is infinite dimensional.

5. $\mathcal{S} = GF(2)^3$ (length 3 bit vectors with mod 2 arithmetic).
   A basis for $\mathcal{S}$ is

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   How would you write

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{\qquad}\,\boldsymbol{v}_1 + \underline{\qquad}\,\boldsymbol{v}_2 + \underline{\qquad}\,\boldsymbol{v}_3 \qquad ?$$