Independence of continuous random variables

Just as in the discrete case, we say that two continuous random variables X and Y are **independent** if

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

That is, we can factor their joint pdf as the marginal pdf for X times the marginal pdf for Y.

Recall that it is always true that $f_{X,Y}(x, y) = f_{X|Y}(x|y)f_Y(y)$, and so

$$X, Y \text{ independent } \Leftrightarrow f_{X|Y}(x|y) = f_X(x),$$

that is, the knowledge that $Y = y$ does not at all affect how we model X — learning Y tells us nothing about X. By symmetry, learning X also tells us nothing about Y:

$$X, Y \text{ independent } \Leftrightarrow f_{Y|X}(y|x) = f_Y(y)$$

The definition extends to more than two random variables in the obvious way:
X, Y, and Z are independent if

$$f_{X,Y,Z}(x, y, z) = f_X(x)f_Y(y)f_Z(z),$$

$X_1, \ldots, X_n$ are independent if

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n).$$

If X and Y are independent random variables, then any pair of events of the form $\{X \in A\}$ and $\{Y \in B\}$ are also independent, as

$$\begin{aligned}
\mathrm{P}\left(X \in A,\ Y \in B\right) &= \int_{x \in A} \int_{y \in B} f_{X,Y}(x,y)\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_{x \in A} \int_{y \in B} f_X(x)\, f_Y(y)\ \mathrm{d}y\ \mathrm{d}x \\
&= \left(\int_{x \in A} f_X(x)\ \mathrm{d}x\right) \left(\int_{y \in B} f_Y(y)\ \mathrm{d}y\right) \\
&= \mathrm{P}\left(X \in A\right) \cdot \mathrm{P}\left(Y \in B\right).
\end{aligned}$$

Thus we can also factor the joint cdf:

$$X, Y \text{ independent } \Leftrightarrow F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y),$$

since

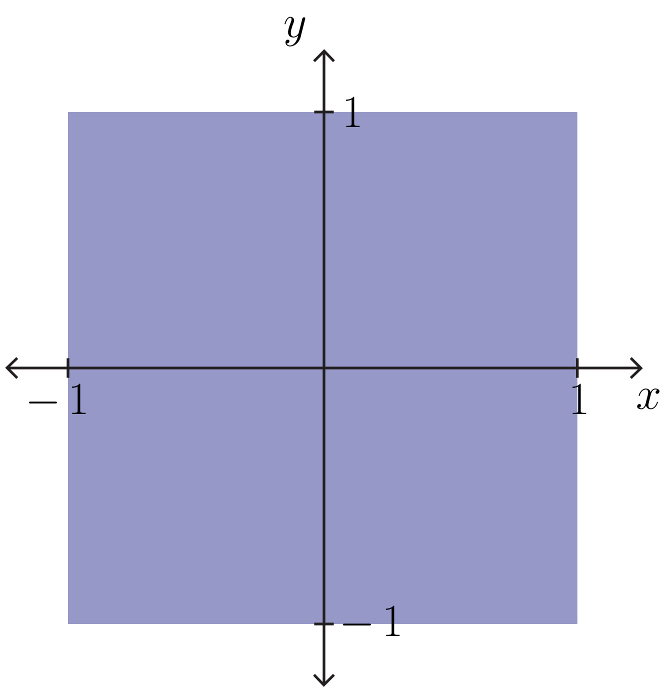
$$F_{X,Y}(x,y) = \mathrm{P}\left(X \le x,\ Y \le y\right) = \mathrm{P}\left(X \le x\right) \cdot \mathrm{P}\left(Y \le y\right)$$

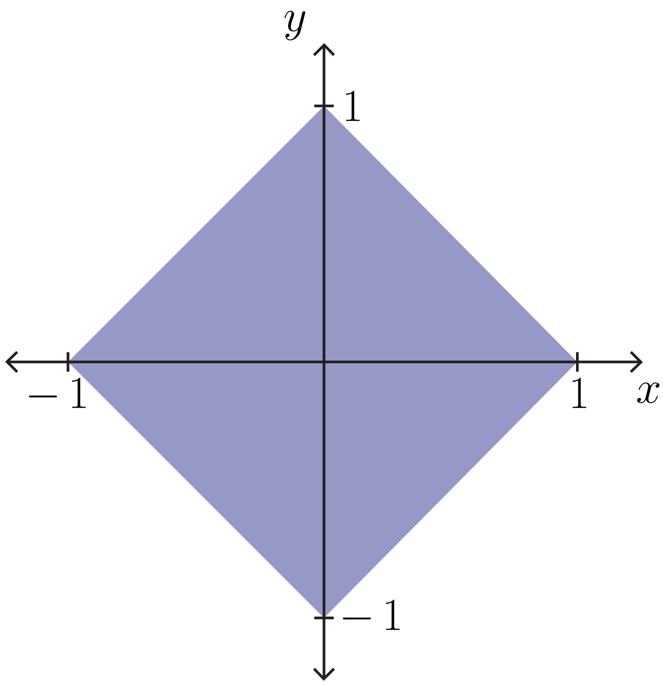
Exercise:
Let X, Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area}(S)} & (x,y) \in S \\ 0 & (x,y) \notin S \end{cases}$$

for each of the shapes S below.

In which of these cases are X and Y independent?





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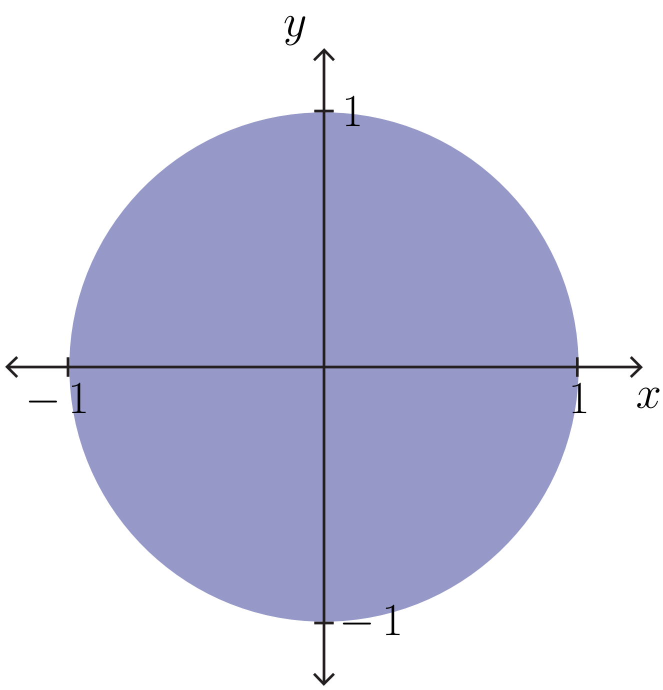

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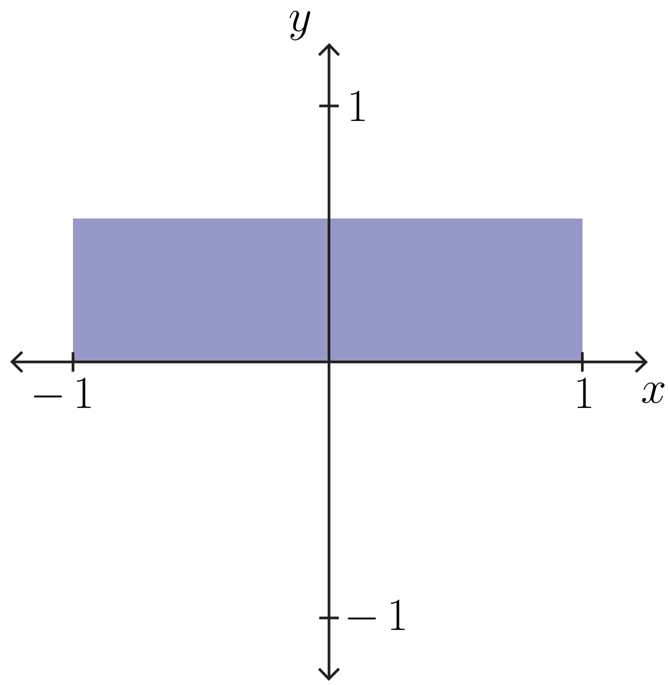
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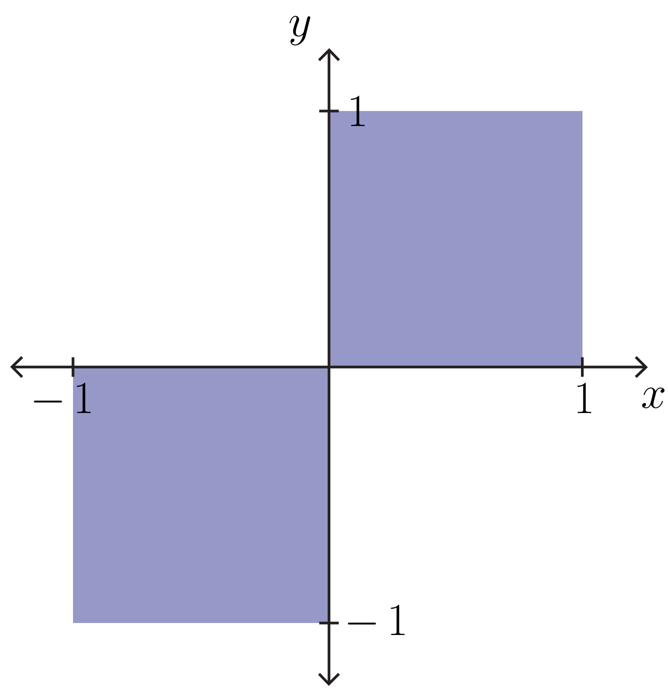
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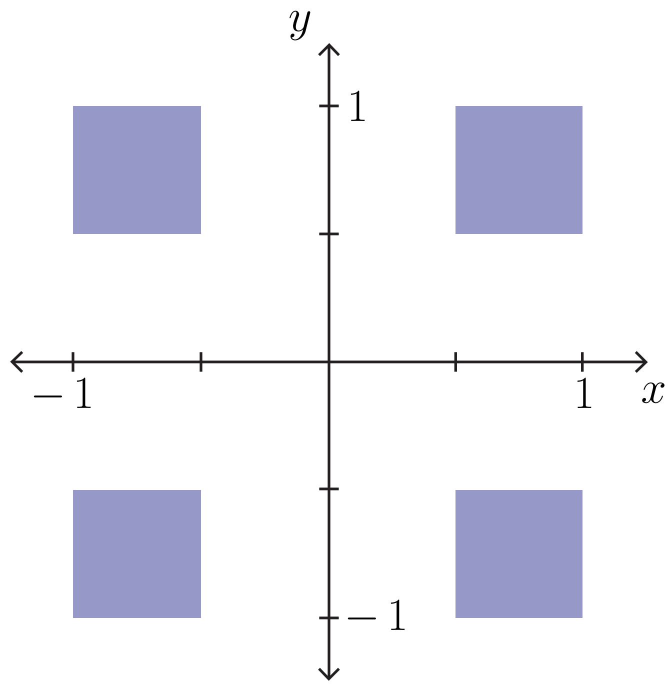

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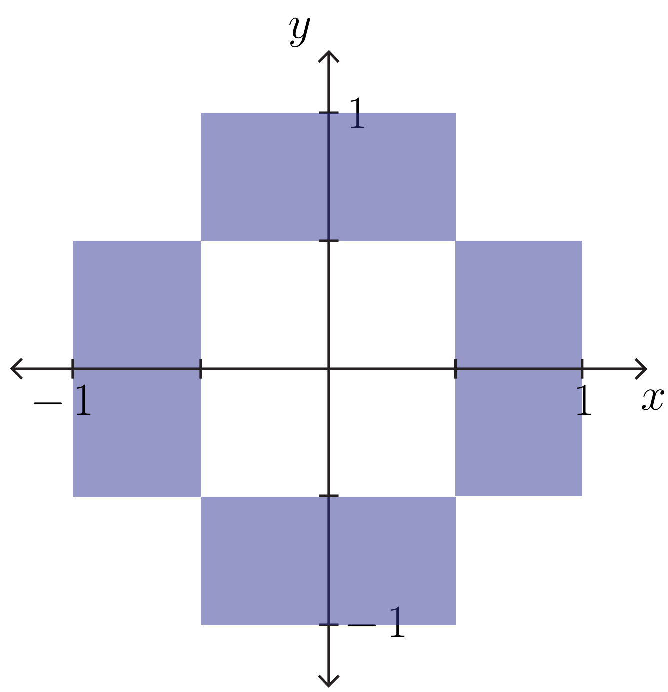

y
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−1
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x
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Exercise:

Which of these joint pdfs describe independent random variables?

(a)

$$f_{X,Y}(x,y) = \begin{cases} \lambda_1\lambda_2\,\mathrm{e}^{-\lambda_1 x-\lambda_2 y}, & x,y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi}\,\mathrm{e}^{-(y^2+1)x}, & x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\,\mathrm{e}^{-(x^2+y^2)/2}$$

(d)

$$f_{X,Y}(x,y) = \begin{cases} \mathrm{e}^{-(x+y)^2/2}, & x,y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(e)

$$f_{X,Y}(x,y) = \begin{cases} \mathrm{e}^{-x}, & 0 \le x < \infty,\ 1 \le y \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(f)

$$f_{X,Y}(x,y) = \begin{cases} 2\mathrm{e}^{-(x+y)}, & x \ge y \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$