A SHORT NONALGORITHMIC PROOF OF THE CONTAINERS THEOREM FOR HYPERGRAPHS

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Abstract. Recently the breakthrough method of hypergraph containers, developed independently by Balogh, Morris, and Samotij [1] as well as Saxton and Thomason [12], has been used to study sparse random analogues of a variety of classical problems from combinatorics and number theory. The previously known proofs of the containers theorem use the so-called *scythe algorithm*—an iterative procedure that runs through the vertices of the hypergraph. (Saxton and Thomason [13] have also proposed an alternative, randomized construction in the case of simple hypergraphs.) Here we present the first known deterministic proof of the containers theorem that is not algorithmic, i.e., it requires no induction on the vertex set. This proof is less than 4 pages long while being entirely self-contained. Although our proof is completely elementary, it was inspired by considering hypergraphs in the setting of nonstandard analysis, where there is a notion of dimension capturing the logarithmic rate of growth of finite sets.

1. Introduction

Hypergraph containers theorems. An important and extremely active line of research in recent years, especially in combinatorics and number theory, is extending classical results to the so-called "sparse random setting." One breakthrough tool for obtaining such results is the method of hypergraph containers developed independently by Balogh, Morris, and Samotij [1] as well as by Saxton and Thomason [12].

The hypergraph containers theorem gives a tool for analyzing the structure of all the independent subsets in a hypergraph by "capturing" each independent set in one of a small number of "containers." Let H be a k-uniform hypergraph with n vertices and $n^{1+(k-1)\delta}$ edges. In general, H can have close to 2^n independent sets, for instance, when all the edges of H span only a small portion of its vertex set. To avoid this, one considers *homogeneous* hypergraphs, i.e., those in which the degree of a vertex cannot significantly exceed the average value $n^{(k-1)\delta}$, and similar upper bounds hold for the codegrees of the sets of $\ell < k$ vertices; for details, see Definition 4. According to the containers theorem, if H is sufficiently homogeneous, then each independent set contains a *fingerprint*, which is a subset

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of size roughly $n^{1-\delta}$. Furthermore, each fingerprint F determines a *container* $C(F)$ of size less than $(1-\alpha)n$, where α is a positive constant, with the property that if F is a fingerprint of an independent set I, then $I \subseteq C(F)$. Each container can host at most $2^{(1-\alpha)n}$ independent sets, and the number of containers is bounded by the number of fingerprints, which is at most $2^{o(n)}$, so the total number of independent sets must be much smaller than 2^n. Note that in this calculation we still used the trivial upper bound $2^{(1-\alpha)n}$ on the number of independent sets inside a given container; in practice, the above approach is usually iterated, leading to particularly strong results.

The containers method has been used to prove (or reprove) sparse versions of theorems originally established for dense hypergraphs; see [1, 12] and the survey [2]. For example, Szemerédi's theorem [15] in number theory states that for every $k \in \mathbb{N}$, the largest subset of $[n] := \{1, \ldots, n\}$ which does not contain a k-term arithmetic progression (*k-AP-free*) is very small, having $o(n)$ elements. This classical result implies that the number of k-AP-free subsets of $[n]$ is also small, namely at most $2^{o(n)}$. Considering the hypergraph with vertex set $[n]$ whose edges are the k-term arithmetic progressions, the containers method leads to a new combinatorial proof [1, 12] of a stronger statement known as the random sparse version of Szemerédi's theorem, which was originally obtained by Schacht [14] and, independently, by Conlon and Gowers [3].

The statements and proofs of the core version of the containers theorem originally appeared in [1, Proposition 3.1] and [12, Theorem 3.4]. We state it here as Theorem 8. Our main result is a new proof of this theorem, whose advantages are described below.

Our proof. All previously known proofs of the containers theorem are based on the so-called *scythe algorithm*—in other words, they use induction on the number of vertices of the hypergraph. (In [13], a different, *randomized* approach was developed for *simple* hypergraphs, i.e., those in which every pair of vertices lies in at most one edge.) In contrast to that, our proof is not algorithmic and provides a deterministic way of building containers in a single step (or, rather, k steps, since it still involves induction on k, the uniformity of the hypergraph). It is also conceptually transparent and rather short—under 4 pages.

Our proof was inspired by an attempt to reprove the containers theorem in the setting of nonstandard analysis, i.e., for ultraproducts of finite hypergraphs. Of course, the theorem for ultraproducts follows from that for finite hypergraphs via the transfer principle, but the present authors were hoping to find a direct proof that would take advantage of the notion of dimension available in the ultraproduct that captures the logarithmic rate of growth. However, it turned out that our approach in the nonstandard setting translated into an even more concise proof for finite hypergraphs, to which we devote the current paper (abandoning ultraproducts altogether).

Organization. The rest of this paper is organized as follows. Section 2 establishes standard hypergraph notation and terminology. Section 3 begins with our definitions of a *homogeneous hypergraph* and a *print/container pair*, and ends with the statement of the containers theorem in these terms, namely Theorem 8. In Section 4, we sketch the idea behind our proof inspired by nonstandard analysis. Finally, our proof of Theorem 8 is presented in Section 5.

2. Basic notation and terminology

The set $\mathbb{N}$ of natural numbers includes 0 and we denote $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. For a set X and $k \in \mathbb{N}^+$, we call a k-element subset of X a *k-edge* and denote the set of all k-edges by $[X]^k$. We refer to a subset $H \subseteq [X]^k$ as a *k-uniform hypergraph* (on X).

A set $I \subseteq X$ is said to be *H-independent* if $H \cap [I]^k = \emptyset$ and we denote

$$\mathcal{I}_X(H) := \text{the set of all } H\text{-independent subsets of } X.$$

Remark 1. Note that $[X]^1$ is the set of all singletons of X, so for any 1-hypergraph $H \subseteq [X]^1$, H-independent subsets of X are precisely those that are disjoint from $\bigcup H$.

Notation 2. Let X be a finite set, $k \in \mathbb{N}^+$, $H \subseteq [X]^k$, and $\ell \in \{1, \ldots, k-1\}$.

- For $U \subseteq [X]^\ell$, $V \subseteq [X]^{k-\ell}$, we denote

$$[U, V]_H := \{e \in H : e = u \cup v \text{ for some } u \in U \text{ and } v \in V\},$$

$$H_U := \left\{v \in [X]^{k-\ell} : u \cup v \in H \text{ for some } u \in U\right\},$$

  We refer to H_U as the *fiber of H over U* and if $U = \{u\}$, we write H_u instead of H_U. (Another common term for H_u is the *link graph of u*.)
- For each $u \in [X]^\ell$, we denote $\deg_H(u) := |H_u|$.
- We put $\Delta_\ell(H) := \max_{u \in [X]^\ell} \deg_H(u)$.

Notation 3. For sets A, B and a relation $R \subseteq A \times B$, we denote

$$\mathrm{dom}(R) := \{a \in A : \exists b \in B \text{ with } aRb\}$$

$$\mathrm{im}(R) := \{b \in B : \exists a \in A \text{ with } aRb\}$$

and refer to these sets, respectively, as the *domain* and the *image* of R.

3. Statement of the containers theorem

Throughout, let X denote a finite nonempty set and $k \in \mathbb{N}^+$.

Definition 4. Let $H \subseteq [X]^k$ and $\delta \in [0, 1]$.

(4.a) We define the *logarithmic degree* of H as

$$\delta_X(H) := \max_{1 \le \ell < k} \frac{1}{k - \ell} \cdot \log_{|X|} \Delta_\ell(H).$$

In other words, $\delta_X(H)$ is the least $\delta \in [0, 1]$ such that $\Delta_\ell(H) \le |X|^{(k-\ell)\delta}$ for all $\ell \in \{1, \ldots, k-1\}$.

(4.b) We say that H is *δ-bounded* if $\delta_X(H) \le \delta$.

(4.c) We let $|H|_\delta$ denote the maximum size of a δ-bounded subhypergraph of H, i.e.

$$|H|_\delta := \max\{|H'| : H' \subseteq H \text{ and } H' \text{ is } \delta\text{-bounded}\}.$$

(4.d) For $\varepsilon > 0$, we say that H is *(δ, ε)-homogeneous* if it is δ-bounded and $\log_{|X|} |H| \ge 1 + (k-1)\delta - \varepsilon$.

As the name suggests, (δ, ε)-homogeneity implies that H is "close to evenly distributed," in the sense that for most $x \in X$, $\log_{|X|} |H_x|$ is close to $(k-1)\delta_X(H)$.

Definition 5. Let $\pi \in [0, 1]$.

- A π-*fingerprint* (in X) is a subset $F \subseteq X$ with $\log_{|X|}|F| \le \pi$.
- A (π,k)-*print* (in X) is a sequence $\boldsymbol{F} := (F_i)_{0\le i<\ell}$ of π-fingerprints (in X), where $\ell \le k-1$. We put $\bigcup \boldsymbol{F} := \bigcup_{i<\ell} F_i$ and denote the set of all (π,k)-prints by $\mathcal{F}^k_\pi(X)$.

Remark 6. In the definition of a (π,k)-print $\boldsymbol{F}$, it is possible that $\ell = 0$ and $\boldsymbol{F} = \emptyset$.

For $\sigma \in [0,1]$, we denote $\mathscr{P}^\sigma(X) := \left\{C \subseteq X : \log_{|X|}|X \setminus C| \ge 1-\sigma\right\}$.

Definition 7. Let $k \ge 1$, $H \subseteq [X]^k$, $\pi,\sigma \in [0,1]$. For relations $\searrow \subseteq \mathcal{I}_X(H) \times \mathcal{F}^k_\pi(X)$ and $\nearrow \subseteq \mathcal{F}^k_\pi(X) \times \mathscr{P}(X)$, the pair $(\searrow, \nearrow)$ is called a (π,σ)-*print/container pair* for H if

(7.i) $\mathrm{dom}(\searrow) = \mathcal{I}_X(H)$;

(7.ii) $\mathrm{dom}(\nearrow) \supseteq \mathrm{im}(\searrow)$;

(7.iii) for each $I \in \mathcal{I}_X(H)$, $\boldsymbol{F} \in \mathcal{F}^k_\pi(X)$, and $C \in \mathscr{P}(H)$, if $I \searrow \boldsymbol{F} \nearrow C$, then

$$\bigcup \boldsymbol{F} \subseteq I \subseteq \bigcup \boldsymbol{F} \cup C;$$

(7.iv) $\mathrm{im}(\nearrow) \subseteq \mathscr{P}^\sigma(X)$ — we refer to the sets in $\mathrm{im}(\nearrow)$ as *containers*.

Our main result is a new proof of the following version of the containers theorem:

Theorem 8. *For any $k \in \mathbb{N}^+$, $\pi \in [0,1]$, and $\varepsilon > 0$, putting $\delta := 1-\pi$ and $\sigma := 3^{k-1}\varepsilon$, the following holds: For any finite nonempty set X with*

$$\varepsilon \ge 2k\log_{|X|} 2 \qquad \text{and} \qquad \pi \ge (k-1)\log_{|X|} 2,$$

any (δ,ε)-homogeneous hypergraph $H \subseteq [X]^k$ admits a (π,σ)-print/container pair.

Remark 9. We point out that in most applications of the above theorem, π and δ are constants independent of $|X|$, while ε and σ are parameters of order $O(\log_{|X|} 2)$. In particular, saying that for a container C, we have $\log_{|X|}|X\setminus C| \ge 1-\sigma$, usually means that $|C| \le (1-\alpha)|X|$ for some positive constant α.

4. Idea of proof

Heuristically, we would like to talk about the "dimension" rather than the actual cardinality of the sets appearing in the proof. If the set X has, say, dimension 1, then the sets whose cardinality has the same "order of magnitude" as $|X|$, maybe $|X|/2$ or $|X|/17$, should also have dimension 1. On the other hand, a set with size $\sqrt{|X|}$ should have dimension 1/2, while a set with size $|X|^k$ or $\binom{|X|}{k}$ should have dimension k. When $|X|$ is a fixed finite number, this is not well defined. Hence, it makes sense to take a sequence of sets X_n with $|X_n| \to \infty$, and consider the rates of growth of various sets that appear in the proof.

This informal idea can be made rigorous by passing to the ultraproduct (as in [5, 7]) and working with the *fine pseudofinite dimension* [8, 9, 10, 6], which captures this property: the dimension of a set is essentially its "rate of growth relative to $|X_n|$," and the dimension is valued in ${}^*\mathbb{R}^+/\mathcal{N}$, where ${}^*\mathbb{R}$ is the ultrapower of the real numbers and $\mathcal{N}$ is the convex subgroup consisting of the "negligible" values, namely, those bounded by $\log_{|X|} n$ for some $n \in \mathbb{N}^+$. (Taking the quotient by the negligible values corresponds to identifying the dimension of Y and Z if $|Y| = c|Z|$ for some fixed real number c.)

We will now informally outline the proof based on the assumption that a well-behaved notion of dimension exists. We take $\mathfrak{d}_X(H)$ to be the dimension of the value $\delta_X(H)$. Analogous to the definitions above but for a dimension $\mathfrak{d}$, we say H is $\mathfrak{d}$-bounded if, for every $u \in [X]^\ell$,

$$\dim(H_u) \leq (k-\ell)\mathfrak{d},$$

and define $|H|_\mathfrak{d} := \max\{|H'| : H' \subseteq H \text{ and } H' \text{ is } \mathfrak{d}\text{-bounded}\}$.

Then we may attempt to prove our theorem by induction on k. Given $H \subseteq [X]^k$ and an independent set I, we take a maximal fingerprint $F \subseteq I$ so that H_F is *homogeneously expanding*, i.e.

$$\dim(|H_F|_{\mathfrak{d}_X(H)}) \geq \dim(F) + (k-1)\mathfrak{d}_X(H).$$

If $\dim(F) = 1 - \mathfrak{d}_X(H)$ then $\dim(|H_F|_{\mathfrak{d}_X(H)}) = 1 + (k-2)\mathfrak{d}_X(H)$, so H_F contains a subhypergraph G with $\mathfrak{d}_X(G) = \mathfrak{d}_X(H)$ and $\dim(G) = 1 + (k-2)\mathfrak{d}_X(H)$. Then by the inductive hypothesis, there is a print/container pair $\left(\overset{\star}{\searrow}, \overset{\star}{\nearrow}\right)$ for G; using this pair we proceed to define a print/container pair for H as follows. Put $I \searrow (F, F_1, \ldots, F_{\ell-1})$ whenever $I \overset{\star}{\searrow} (F_1, \ldots, F_{\ell-1})$, and note that I, being G-independent, must admit such $(F_1, \ldots, F_{\ell-1})$. As for the container relation, we put $(F_0, F_1, \ldots, F_{\ell-1}) \nearrow C$ whenever $(F_1, \ldots, F_{\ell-1}) \overset{\star}{\nearrow} C$. Verifying that this works is straightforward.

Let us now suppose that $\dim(F) < 1 - \mathfrak{d}_X(H)$. Then the maximality of F guarantees that for any $x \in I \setminus F$,

$$\dim(H_x \setminus H_F) < (k-1)\mathfrak{d}_X(H).$$

Thus, we set $I \searrow (F)$ and $(F) \nearrow C$, where

$$C := \{x \in X : \dim(H_x \setminus H_F) < (k-1)\mathfrak{d}_X(H)\}.$$

We certainly have $F \subseteq I \subseteq C \cup F$ and all that remains to check is that C has codimension 1. We observe that

$$H \subseteq [X, H_F]_H \cup [C, X^{k-1} \setminus H_F]_H \cup [X \setminus C, X^{k-1}]_H,$$

and therefore,

$$\dim(H) \leq \max\left\{\dim[X, H_F]_H, \dim[C, X^{k-1} \setminus H_F]_H, \dim[X \setminus C, X^{k-1}]_H\right\}.$$

But

$$\begin{aligned}
\dim[X, H_F]_H &\leq \dim(H_F) + \mathfrak{d}_X(H) \\
&= \dim(F) + (k-1)\mathfrak{d}_X(H) + \mathfrak{d}_X(H) \\
&< 1 - \mathfrak{d}_X(H) + k\mathfrak{d}_X(H) \\
&= 1 + (k-1)\mathfrak{d}_X(H) = \dim(H)
\end{aligned}$$

and, using the Fubini property of dimension,

$$\dim[C, X^{k-1} \setminus H_F]_H \leq \dim(C) + \max_{x \in C} \dim(H_x \setminus H_F) < 1 + (k-1)\mathfrak{d}_X(H) < \dim(H).$$

Therefore,

$$\dim(H) = \dim[X \setminus C, X^{k-1}]_H \leq \dim(X \setminus C) + (k-1)\mathfrak{d}_X(H),$$

which forces $\dim(X \setminus C) = 1$.

For formal reasons, this argument does not quite go through in the rigorous setting of nonstandard analysis: the notion of dimension is "external" (not defined by a formula of first-order logic), and therefore such a maximal set F need not exist; in fact it *cannot* exist because adding one point to a set does not affect its dimension. To fix this, one has to replace the notion of dimension with *logarithmic size*. This is precisely the argument we give below, using bounds on the logarithmic sizes of sets as an approximation to the notion of dimension.

5. Proof

This section is devoted to our proof of Theorem 8, so we let $k, \pi, \varepsilon, \delta, \sigma, X$ and H be as in its hypothesis and we let log stand for $\log_{|X|}$. We adopt the convention that $\log 0 = -\infty$.

We define a (π, σ)-print/container pair by induction on k. For the base case $k = 1$, we let $I \searrow \boldsymbol{F}$ exactly when $\boldsymbol{F} = \emptyset$ and $\boldsymbol{F} \nearrow C$ exactly when $C = X \setminus (\bigcup H)$. The complement of C is $\bigcup H$ and $\log|\bigcup H| = \log|H| \geq 1 - \varepsilon = 1 - \sigma$. The rest of the conditions clearly hold as well.

Thus, we may assume that $k > 1$ and that the statement is true for all $1 \leq k' < k$.

5.I. Choice of constants.

We take

- $\delta' := \delta + \log 2$
- $\pi' := 1 - \delta'$
- $\tilde{\pi} := \pi - \varepsilon - k \log 2$
- $\tilde{\varepsilon} := \varepsilon + (k+1)\log 2$
- $\varepsilon' := 2\varepsilon + 2k \log 2$
- $\sigma' := 3^{k-2}\varepsilon'$.

Note that since $(k-1)\log 2 \leq \pi$, we have $(k-2)\log 2 \leq \pi'$. Also,

$$\varepsilon' = 2\varepsilon + 2k\log 2 \leq 2\varepsilon + 2k \cdot \frac{\varepsilon}{2k} \leq 2\varepsilon + \varepsilon = 3\varepsilon,$$

and hence $\sigma' \leq \sigma$.

Definition 10. Call a π-fingerprint F *expanding* if

$$\log|H_F|_{\delta'} \geq 1 + (k-2)\delta' - \varepsilon'.$$

Notice that a π-fingerprint F is expanding if and only if the fiber H_F contains a (δ', ε')-homogeneous subhypergraph. For each expanding π-fingerprint F, fix an arbitrary (δ', ε')-homogeneous subhypergraph $G_F \subseteq H_F$. By the induction hypothesis, G_F admits a (π', σ')-print/container pair; fix any such (π', σ')-print/container pair $\left(\overset{F}{\searrow}, \overset{F}{\nearrow} \right)$.

5.II. The print relation.

Given $I \in \mathcal{I}_X(H)$ and $\boldsymbol{F} = (F_0, F_1, \ldots, F_{\ell-1}) \in \mathcal{F}_\pi^k(X)$, we set $I \searrow \boldsymbol{F}$ to hold exactly when at least one of the following conditions holds:

Condition 11. We have $\ell \geq 1$, F_0 is expanding, $F_0 \subseteq I$, and $I \overset{F_0}{\searrow} (F_1, F_2, \ldots, F_{\ell-1})$.

Condition 12. We have $\ell = 1$, F_0 is not expanding, $\log|F_0| < \tilde{\pi}$, and F_0 is maximal among the π-fingerprints F that are contained in I and satisfy

$$\log|H_F|_{\delta'} \geq \log|F| + (k-1)\delta' - \tilde{\varepsilon}. \tag{13}$$

Remark 14. Condition 11 makes sense, since if $F_0 \subseteq I$ and $G \subseteq H_{F_0}$, then I is G-independent.

5.III. **Condition (7.i).** For a fixed $I \in \mathcal{I}_X(H)$, there are two cases.

Case 1: There is an expanding π-fingerprint $F \subseteq I$. Since I is G_F-independent, there is a print $\boldsymbol{F}' = (F_1, \ldots, F_{\ell-1}) \in \mathcal{F}_\pi^{k-1}(X)$ with $I \overset{F}{\searrow} \boldsymbol{F}'$. Therefore, taking $\boldsymbol{F} := (F, F_1, \ldots, F_{\ell-1})$, we see that Condition 11 holds, so $I \searrow \boldsymbol{F}$.

Case 2: There is no expanding π-fingerprint $F \subseteq I$.

Claim 15. There is a (possibly empty) set $F \subseteq I$ with $\log|F| < \tilde{\pi}$ that is maximal among the π-fingerprints contained in I and satisfying (13).

Proof of Claim. Because $F = \emptyset$ satisfies (13), there is a maximal π-fingerprint F contained in I satisfying (13). Then $\log|F| < \tilde{\pi}$, for otherwise we have

$$\log|H_F|_{\delta'} \geq \tilde{\pi} + (k-1)\delta' - \tilde{\varepsilon} = (1 - \delta' - \varepsilon') + (k-1)\delta' = 1 + (k-2)\delta' - \varepsilon',$$

which means that F is expanding, contradicting the assumption of our case. ⊠

The print $\boldsymbol{F} := (F)$, where F is given by Claim 15, satisfies Condition 12, so $I \searrow \boldsymbol{F}$.

5.IV. **The container relation.** Given $\boldsymbol{F} = (F_0, F_1, \ldots, F_{\ell-1}) \in \mathcal{F}_\pi^k(X)$ and $C \in \mathscr{P}(X)$, we set $\boldsymbol{F} \nearrow C$ to hold exactly when at least one of Conditions 16 and 18 below holds.

Condition 16. We have $\ell \geq 1$, F_0 is expanding, and $(F_1, F_2, \ldots, F_{\ell-1}) \overset{F_0}{\nearrow} C$.

To state Condition 18, we need a definition first.

Definition 17. For $k' \geq 1$, a hypergraph $H' \subseteq [X]^{k'}$, $1 \leq t < k'$, and $\delta \in [0,1]$, let $\nabla_t^\delta(H')$ denote the set of all $u \in [X]^t$ with $\log \deg_{H'}(u) \geq (k'-t)\delta$ in H'.

Condition 18. We have $\ell = 1$, F_0 is not expanding, and the following holds. Define

$$H^- := H \setminus \hat{H}, \quad \text{where} \quad \hat{H} := [H_{F_0}, X]_H \cup \bigcup_{t=1}^{k-2} [\nabla_t^\delta(H_{F_0}), [X]^{k-t}]_H. \tag{19}$$

Then we have

$$C = \{x \in X : \log \deg_{H^-}(x) < (k-1)\delta' - \tilde{\varepsilon}\}. \tag{20}$$

5.V. **Condition (7.ii).** Let $\boldsymbol{F} = (F_0, F_1, \ldots, F_{\ell-1}) \in \operatorname{im}(\searrow)$. It follows from Conditions 11 and 12 that $\ell \geq 1$.

Case 1: F_0 is expanding. Then Condition 11 holds. This means that $(F_1, F_2, \ldots, F_{\ell-1}) \in \operatorname{im}(\overset{F_0}{\searrow}) \subseteq \operatorname{dom}(\overset{F_0}{\nearrow})$. Hence, for some $C \in \mathscr{P}(X)$ we have $(F_1, F_2, \ldots, F_{\ell-1}) \overset{F_0}{\nearrow} C$, which yields $\boldsymbol{F} \nearrow C$ by Condition 16.

Case 2: F_0 is not expanding. Then Condition 12 holds. This means that $\ell = 1$ and there is a (unique) set C satisfying Condition 18, so $\boldsymbol{F} \nearrow C$.

5.VI. **Condition (7.iii).** We fix $I \in \mathcal{I}_X(H)$, $\boldsymbol{F} = (F_0, F_1, \ldots, F_{\ell-1}) \in \mathcal{F}_\pi^k(X)$, and $C \in \mathscr{P}(X)$ with $I \searrow \boldsymbol{F} \nearrow C$. It follows that $\ell \geq 1$.

Case 1: F_0 is expanding. Set $\boldsymbol{F}' := (F_1, F_2, \ldots, F_{\ell-1})$. By the case assumption, Conditions 11 and 16 hold, so $F_0 \subseteq I$ and $I \overset{F_0}{\searrow} \boldsymbol{F}' \overset{F_0}{\nearrow} C$. Therefore, (7.iii) applied to $\left(\overset{F_0}{\searrow}, \overset{F_0}{\nearrow}\right)$ yields $\bigcup \boldsymbol{F}' \subseteq I \subseteq \bigcup \boldsymbol{F}' \cup C$.

Case 2: F_0 is not expanding. Then Conditions 12 and 18 hold. In particular, $\ell = 1$. For brevity, let $F := F_0$. By Condition 12, $F \subseteq I$, so it remains to show that each $x \in I \setminus F$ belongs to C. Letting H^- be as in (19), we suppose towards a contradiction that $x \notin C$. By

Condition 12, F satisfies (13), so let $G \subseteq H_F$ be a δ'-bounded hypergraph with $\log|G| \geq \log|F| + (k-1)\delta' - \tilde{\varepsilon}$.

Claim 21. $G' := G \cup H_x^-$ is δ'-bounded.

Proof of Claim. We fix $\ell \in \{1, \ldots, k-2\}$ and $u \in [X]^\ell$ and show that $\log \deg_{G'}(u) \leq (k-1-\ell)\delta'$.

If $u \in \nabla_\ell^\delta(H_F)$ or $x \in u$, then $G'_u = G_u$, so $\deg_{G'}(u) = \deg_G(u) \leq |X|^{(k-1-\ell)\delta'}$.

Otherwise, $\deg_{G'}(u) \leq \deg_G(u) + \deg_{H_x^-}(u)$. Since $u \notin \nabla_\ell^\delta(H_F)$, $\deg_G(u) \leq |X|^{(k-1-\ell)\delta}$. Also, because $x \notin u$,

$$\deg_{H_x^-}(u) = \deg_{H^-}\left(\{x\} \cup u\right) \leq \deg_H\left(\{x\} \cup u\right) \leq |X|^{(k-1-\ell)\delta},$$

so $\deg_{G'}(u) \leq 2 \cdot |X|^{(k-1-\ell)\delta} = |X|^{(k-1-\ell)\delta'}$. ⊠

Furthermore, H^- and $[H_F, X]_H$ are disjoint, in particular, H_x^- and $H_F \supseteq G$ are disjoint, so

$$\begin{aligned} |G'| &= |G| + |H_x^-| \\ \left[\text{Because } x \notin C\right] \quad &\geq |F| \cdot |X|^{(k-1)\delta' - \tilde{\varepsilon}} + |X|^{(k-1)\delta' - \tilde{\varepsilon}} \\ &= (|F| + 1) \cdot |X|^{(k-1)\delta' - \tilde{\varepsilon}}. \end{aligned}$$

Therefore,

$$\log\left|H_{F \cup \{x\}}\right|_{\delta'} \geq \log|F \cup \{x\}| + (k-1)\delta' - \tilde{\varepsilon},$$

i.e., $F \cup \{x\}$ satisfies (13). Since $|X|^{\tilde{\pi}} + 1 \leq |X|^\pi$, the set $F \cup \{x\}$ is a π-fingerprint contained in I. This contradicts the properties of F given by Condition 12.

5.VII. **Condition (7.iv).** For a given $C \in \operatorname{im}(\nearrow)$, fix any $\boldsymbol{F} = (F_0, F_1, \ldots, F_{\ell-1}) \in \mathcal{F}_\pi^k(X)$ with $\boldsymbol{F} \nearrow C$. It follows that $\ell \geq 1$.

Case 1: F_0 is expanding. Then Condition 16 holds, so $(F_1, \ldots, F_{\ell-1}) \overset{F_0}{\nearrow} C$, and thus $|X \setminus C| \geq 1 - \sigma' \geq 1 - \sigma$.

Case 2: F_0 is not expanding. Then Condition 18 holds, so $\ell = 1$ and C is defined as in (20). For brevity, let $F := F_0$.

Claim 22. $\log\left|[H_F, X]_H\right| \leq \log|F| + k\delta$.

Proof of Claim. $\log\left|[H_F, X]_H\right| \leq \log|H_F| + \delta \leq \log|F| + (k-1)\delta + \delta = \log|F| + k\delta$. ⊠

Claim 23. For each $\ell \in \{1, \ldots, k-2\}$, $\log\left|[\nabla_\ell^\delta(H_F), [X]^{k-\ell}]_H\right| \leq \log\binom{k-1}{\ell} + \log|F| + k\delta$.

Proof of Claim. Because each edge $e \in H_F$ is counted in the degrees of at most $\binom{k-1}{\ell}$-many points in $[X]^\ell$, we have that

$$\log|\nabla_\ell^\delta(H_F)| + (k-1-\ell)\delta \leq \log \sum_{u \in [X]^\ell} \deg_{H_F}(u) \leq \log\binom{k-1}{\ell} + \log|H_F|.$$

But $\log|H_F| \leq \log|F| + (k-1)\delta$, so

$$\log|\nabla_\ell^\delta(H_F)| \leq \log\binom{k-1}{\ell} + \log|F| + (k-1)\delta - (k-1-\ell)\delta = \log\binom{k-1}{\ell} + \log|F| + \ell\delta.$$

Thus,

$$\log\left|[\nabla^{\delta}_{\ell}(H_F), [X]^{k-\ell}]_H\right| \leq \log\left|\nabla^{\delta}_{\ell}(H_F)\right| + (k-\ell)\delta \leq \log\binom{k-1}{\ell} + \log|F| + k\delta. \qquad \boxtimes$$

Let H^- and $\hat{H}$ be defined as in (19).

Claim 24. $\log|H^-| \geq 1 + (k-1)\delta - \varepsilon - \log 2$.

Proof of Claim. It follows from the last two claims that

$$\begin{aligned}
|\hat{H}| &\leq |F| \cdot |X|^{k\delta} + \sum_{\ell=1}^{k-2}\binom{k-1}{\ell} \cdot |F| \cdot |X|^{k\delta} \\
&= \sum_{\ell=1}^{k-1}\binom{k-1}{\ell} \cdot |F| \cdot |X|^{k\delta} \\
&< 2^{k-1} \cdot |F| \cdot |X|^{k\delta}, \\
&< 2^{k-1} \cdot |X|^{\tilde{\pi}} \cdot |X|^{k\delta} \\
&= 2^{k-1} \cdot |X|^{\pi-\varepsilon-k\log 2} \cdot |X|^{k\delta} \\
&= 2^{k-1} \cdot 2^{-k} \cdot |X|^{1+(k-1)\delta-\varepsilon} \leq \frac{1}{2} \cdot |H|,
\end{aligned}$$

so $|H^-| = |H| - |\hat{H}| \geq \frac{1}{2} \cdot |H|$. $\boxtimes$

On the other hand,

$$\begin{aligned}
\log\left|[C, [X]^{k-1}]_{H^-}\right| &< |C| + (k-1)\delta' - \tilde{\varepsilon} \\
&= |C| + (k-1)\delta + (k-1)\log 2 - \varepsilon - (k+1)\log 2 \\
&\leq 1 + (k-1)\delta - \varepsilon - 2\log 2,
\end{aligned}$$

and

$$\log\left|[X \setminus C, [X]^{k-1}]_{H^-}\right| \leq \log\left|[X \setminus C, [X]^{k-1}]_H\right| \leq \log|X \setminus C| + (k-1)\delta,$$

so,

$$\begin{aligned}
\frac{1}{4} \cdot |X|^{1+(k-1)\delta-\varepsilon} + |X \setminus C| \cdot |X|^{(k-1)\delta} &\geq \left|[C, [X]^{k-1}]_{H^-}\right| + \left|[X \setminus C, [X]^{k-1}]_{H^-}\right| \\
&\geq |H^-| \geq \frac{1}{2} \cdot |X|^{1+(k-1)\delta-\varepsilon}.
\end{aligned}$$

Therefore, $|X \setminus C| \geq \frac{1}{4} \cdot |X|^{1-\varepsilon}$, so

$$\begin{aligned}
\log|X \setminus C| &\geq 1 - \varepsilon - 2\log 2 \\
&\geq 1 - \varepsilon - 2 \cdot \frac{\varepsilon}{2k} \\
\left[\text{Because } k \geq 2\right] \quad &> 1 - 2\varepsilon > 1 - \sigma.
\end{aligned}$$

The proof of Theorem 8 is now complete. $\square$

References

[1] J. Balogh, R. Morris, and W. Samotij, Independent sets in hypergraphs, *J. Am. Math. Soc.*, 28 (2015), no. 3, 669–709.

[2] D. Conlon, Combinatorial theorems relative to a random set, *Proceedings of the International Congress of Mathematicians*, Volume 184 (2016), Issue 2, 367–454.

[3] D. Conlon and T. Gowers, Combinatorial theorems in sparse random sets, *Annals of Mathematics*, Volume 184 (2016), Issue 2, 367–454.

[4] P. Erdős, P. Frankl, and V. Rödl, The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, *Graphs and Combin. 2* (2) (1986) 113–121.

[5] M. Di Nasso, I. Goldbring, and M. Lupini, Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory, draft, https://arxiv.org/abs/1709.04076

[6] D. García, A note on pseudofinite dimensions and forking, note, https://arxiv.org/abs/1402.5212

[7] R. Goldblatt, Lectures on the hyperreals, Springer-Verlag, New York, 1998

[8] E. Hrushovski, Stable group theory and approximate subgroups, *J. Amer. Math. Soc*, 25 (2012), 189–243.

[9] E. Hrushovski, On pseudo-finite dimensions, *Notre Dame J. Form. Log*, 54 (2013), 463–495

[10] E. Hrushovski and F. Wagner, Counting and dimensions, in *Model theory with applications to algebra and analysis*, Vol 2 (2008), 161–176.

[11] T. Łuczak, On triangle-free random graphs, *Random Structures Algorithms*, 16 (2000), 260–276.

[12] D. Saxton and A. Thomason, Hypergraph containers, *Inventiones Math.*, 201 (2015), 1–68.

[13] D. Saxton and A. Thomason, Simple containers for simple hypergraphs, *Combin. Probab. Comput.*, 25 (2016), no. 3, 448–459.

[14] M. Schacht, Extremal results for random discrete structures, *Annals of Mathematics*, Volume 184 (2016), Issue 2, 333–365.

[15] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.*, 27 (1975), 199–245.