G53FUZ
Fuzzy Sets and Systems

Mamdani Inference and Defuzzification

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Definition of Logical Implication

- Logical implication can be defined in terms of other primitives
  - $(p \Rightarrow q) \equiv ((\neg p) \vee q)$
- Or as a truth table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$
F	F	T	F	F	T
T	F	F	F	T	F
F	T	T	F	T	T
T	T	F	T	T	T

Logical Inference

- Modus ponens
  - $p \Rightarrow q, p; q$
  - $((p \Rightarrow q) \wedge p) \perp q$
  - IF p THEN q; p is TRUE; hence q is TRUE
  - IF p THEN q; p is FALSE; hence q is ???
    - we will return to this question later
- Modus tollens
  - $((p \Rightarrow q) \wedge \neg q) \perp \neg p$
  - IF p THEN q; q is FALSE; hence p is FALSE

Example

- IF raining THEN cloudy
  - modus ponens
    - it is raining: it must be cloudy
  - modus tollens
    - it is not cloudy: it is not raining
  - incorrect inference
    - it is not raining: it is not cloudy
  - correct inference
    - $((F \Rightarrow F)$: $T)$ AND $((F \Rightarrow T)$: $T)$
    - it is not raining: it may or may not be cloudy

If-Then Rules

- Inference is performed by utilising a set of rules connecting premises to conclusions
  - premise (if part) is called the *antecedent(s)*
  - conclusion (then part) is called the *consequent(s)*
- These rules are similar to the production rules of expert systems
- Inference is simplified by putting aside formal considerations of logical implication

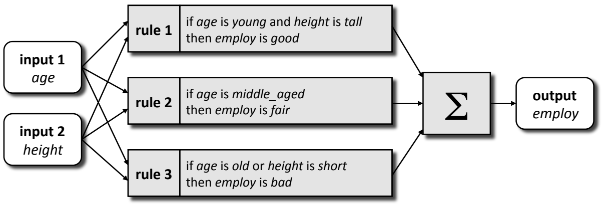
If-Then Rules

- Essential operation
  - each of the antecedent(s) is evaluated to a number in [0, 1] and combined into a single number
    - the truth of the rule premise
  - each of the consequent(s) is considered to be true to the same degree as the premise
- IF p THEN q
  - p is **true**, hence q is **true**
  - p is **half true**, hence q is **half true**
  - p is **not true**, hence q is **not true**!

Does This Make Sense?

- IF *p* THEN *q*; *p* is FALSE, hence *q* is FALSE
  - IF the moon is made of cheese THEN I'm a fool!
  - the moon is NOT made of cheese
  - hence I'm not a fool?
    - NO: logically, you have no evidence to support the conclusion one way or another
- We need to specify alternative antecedents
  - IF moon is NOT made of cheese THEN I'm NOT a fool

Outline



1. Fuzzify inputs
2. Combine inputs
3. Perform implication
4. Aggregate output
5. (Defuzzify)

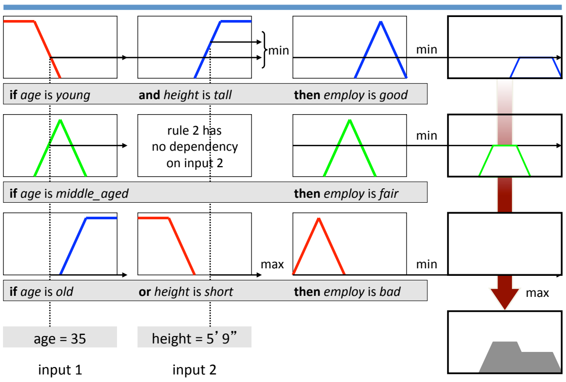
Background

- Mamdani introduced the first successful form of fuzzy inferencing in 1975
  - E.H. Mamdani and S. Assilian
    - "An experiment in linguistic synthesis with a fuzzy logic controller"; *International Journal of Man-Machine Studies;* Vol. 7, No. 1, pp. 1-13, 1975
- The fuzzy system was developed to control kiln temperature in a cement factory
  - it is based on pragmatic considerations rather than any theoretical correctness

Methodology

- Comprises a set of rules of the form
  - IF x is A [AND/OR y is B ...] THEN z is C
  - IF crisp_input matches fuzzy_input_term AND/OR ... THEN add fuzzy_output_term to fuzzy_output
- For each rule
  - for each antecedent
    - evaluate m.f. (μ) of the crisp input value at the fuzzy term
  - combine all μ using appropriate fuzzy operator
  - fire the consequence at strength of resultant truth
    - add the output term to a (fuzzy) output set
- Interpret the output set in some way

Mamdani Example



Example: Variables

- Age
  - *young* = 1/0 + 1/10 + .75/20 +.5/30 + .25/40
  - *middle_aged* = 0/30 + .5/40 + 1/50 +.5/60 + 0/70
  - *old* = .25/60 + .5/70 + .75/80 + 1/90 + 1/100
- Height
  - *short* = 1/1.4 + .75/1.5 + .5/1.6 + .25/1.7 + 0/1.8
  - *tall* = .25/1.6 + .5/1.7 + .75/1.8 + 1/1.9 + 1/2.0
- Employ
  - *bad* = 0/0 + .5/1 + 1/2 + .5/3 + 0/4
  - *fair* = 0/3 + .5/4 + 1/5 + .5/6 + 0/7
  - *good* = 0/6 + .5/7 + 1/8 + .5/9 + 0/10

Example: Rules

- Three rules
  - IF *Age* is *young* AND *Height* is *tall* THEN *Employ* is *good*
  - IF *Age* is *middle_aged* THEN *Employ* is *fair*
  - IF *Age* is *old* OR *Height* is *short* THEN *Employ* is *bad*
- Inputs
  - Age = 40 (years)
  - Height = 1.8 (metres)

Rule 1

- Antecedent 1
  - *Age* is *young*: $\mu_{young}(40) = 0.25$
- Antecedent 2
  - *Height* is *tall*: $\mu_{tall}(1.8) = 0.75$
- Rule strength = $Ante_1$ AND $Ante_2$
  - min(0.25, 0.75) = 0.25
- Consequent
  - *Employ* is *good*
    - min(0.25, 0/6 + .5/7 + 1/8 + .5/9 + 0/10)
    - **0/6 + 0.25/7 + 0.25/8 + 0.25/9 + 0/10**

Rule 2

- Antecedent 1
  - *Age* is *middle_aged*: $\mu_{middle\_aged}(40) = 0.5$
- Antecedent 2
  - *BLANK*
- Rule strength = $Ante_1$
  - 0.5
- Consequent
  - *Employ* is *fair*
    - min(0.5, 0/3 + .5/4 + 1/5 + .5/6 + 0/7)
    - **0/3 + 0.5/4 + 0.5/5 + 0.5/6 + 0/7**

Rule 3

- Antecedent 1
  - *Age* is *old*: $\mu_{old}(40) = 0$
- Antecedent 2
  - *Height* is *short*: $\mu_{short}(1.8) = 0$
- Rule strength = $Ante_1$ OR $Ante_2$
  - max(0, 0) = 0
- Consequent
  - *Employ* is *bad*
    - min(0.0, 0/0 + .5/1 + 1/2 + .5/3 + 0/4)
    - **0/0 + 0/1 + 0/2 + 0/3 + 0/4**

Rule Combination

- The three rule results
  - R_1: 0/6 + 0.25/7 + 0.25/8 + 0.25/9 + 0/10
  - R_2: 0/3 + 0.5/4 + 0.5/5 + 0.5/6 + 0/7
  - R_3: 0/0 + 0/1 + 0/2 + 0/3 + 0/4
- Rule combination
  - max(R_1, R_2, R_3)
  - max(0/6 + 0.25/7 + 0.25/8 + 0.25/9 + 0/10, 0/3 + 0.5/4 + 0.5/5 + 0.5/6 + 0/7, 0/0 + 0/1 + 0/2 + 0/3 + 0/4)
  - max(0)/0 + max(0)/1 + max(0)/2 + max(0,0)/3 + max(.5,0)/4 + max(.5)/5 + max(.5)/6 + max(.25,0)/7 + max(.25)/8 + max(.25)/9 + max(0)/10
  - **0/0 + 0/1 + 0/2 + 0/3 + .5/4 + .5/5 + .5/6 + .25/7 + .25/8 + .25/9 + 0/10**

Operators

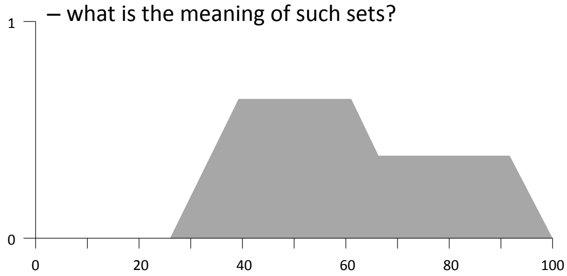
- Mamdani inference features union and intersection operators, both in two places
  - intersection
    - combining antecedants joined by AND
    - implication operator to derive each consequent
  - union
    - combining antecedants joined by OR
    - operator used to combine all consequents overall
- Operator families should be used consistently
  - in practice, often AND-OR pair is varied independently of implication/combination

Defuzzification

- In general, the result of Mamdani inference is a complex output fuzzy set
  - what does this mean?
- Often, for example in Mamdani's case, a single (crisp) number is required for output
  - the fuzzy output set is converted to a number
  - this process is termed *defuzzification*
- Mamdani chose to use a method whereby the centre of the area under the output set is used
  - this is called the *centroid* or *centre-of-gravity*

The Problem

- The result of Mamdani inference is one or more arbitrary output fuzzy set(s)
  - what is the meaning of such sets?



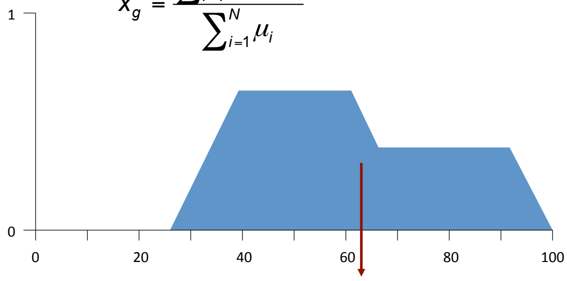
Defuzzification

- There are two principal forms of defuzzification
  - numeric defuzzification
  - linguistic defuzzification
- Numeric defuzzification
  - often, a single (crisp) number is required as output
    - e.g. fuzzy control
  - there are many different options
    - COG (centroid), mean-of-maxima, centre-of-area
- Linguistic defuzzification
  - a linguistic term representing the output set is found
  - some form of similarity or distance metric used

Centre of Gravity

- The imaginary balance point of the shape

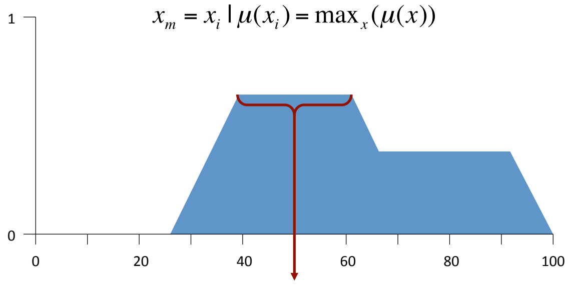
$$x_g = \frac{\sum_{i=1}^{N}(\mu_i \cdot x_i)}{\sum_{i=1}^{N}\mu_i}$$



Mean of Maxima

- The mean of the *x*'s which attain the maximal membership grade

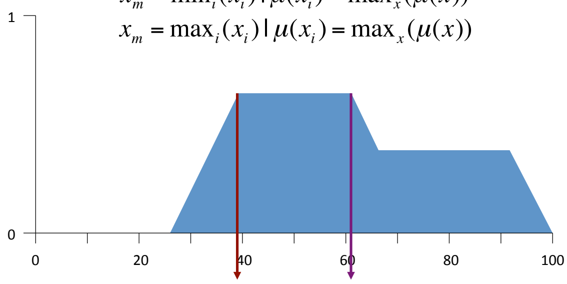
$$x_m = \overline{x_i \mid \mu(x_i) = \max_x(\mu(x))}$$



Smallest/Largest of Maxima

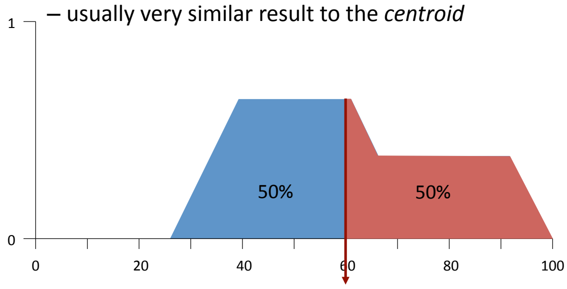
- The smallest or largest of *x*'s with the maximal grade

$$x_m = \min_i(x_i) \mid \mu(x_i) = \max_x(\mu(x))$$
$$x_m = \max_i(x_i) \mid \mu(x_i) = \max_x(\mu(x))$$



Bisector

- The value of x which splits the total area into two equal subareas
  - usually very similar result to the *centroid*



Problems

- Information is lost
  - this is inevitable when reducing to a single number

Other Metrics

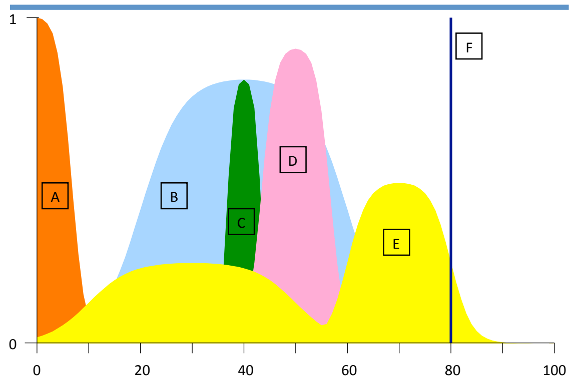
- Membership grade at defuzzification point (μ_g)
  - provides an indication of confidence in the result
- Maximum membership grade (μ_h, height)
  - provides a direct measure of strength of rules fired
- Normalised area

$$A = \frac{\sum_{i=1}^{N} \mu_i}{N}$$

- Fuzzy entropy

$$S = \frac{\sum_{i=1}^{N} \left(-\mu_i \ln(\mu_i) - (1-\mu_i)\ln(1-\mu_i)\right)}{N}$$

Metrics Illustrated



Metric Values

set	x_g	μ_g	μ_h	A	S
A	3	0.95	1.00	0.07	0.06
B	40	0.80	0.80	0.32	0.50
C	40	0.80	0.80	0.05	0.08
D	50	0.90	0.90	0.12	0.15
E	50	0.16	0.50	0.19	0.60
F (singleton)	80	1.00	1.00	0.00	0.00
unknown 1.0/x	50	1.00	1.00	1.00	0.00
indeterm. 0.5/x	50	0.50	0.50	0.50	1.00
undefined 0/x	50	0.00	0.00	0.00	0.00

Linguistic Approximation

- A similarity measure is used to compute the distance between
  - the actual output set
  - the set of all terms of the linguistic variable
    - collection of primitive terms, connectives and hedges
- Search to find the best term while limiting the complexity to produce comprehensible output
  - e.g. *medium or high* may be prefered to *not extremely low or fairly medium or fairly high*
- Special level sets may also be included in search

Similarity Measures

- Euclidean distance

$$\delta^2 = \sum_{i=1}^{N} (\mu_i - \eta_i)^2$$

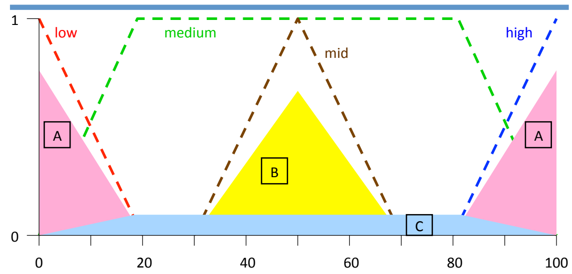
  - where η_i is membership grade of linguistic term
  - minimum will determine the best match

- Degree of overlap

$$\gamma = \frac{A \cap B}{A \cup B}$$

  - maximum will determine the best match

Examples

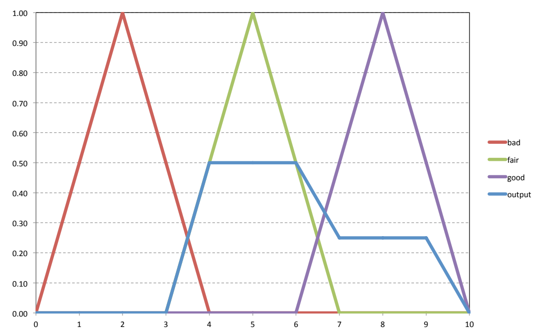


- *A* is best approximated by *low or high*
- *B* is best approximated by *mid*
- *C* is best approximated by *undefined* (0/*x*)

Example - Sets

- Recall, our output set
  - .5/4 + .5/5 + .5/6 + .25/7 + .25/8 + .25/9
- Recall, the output sets for *Employ*
  - *bad* = 0/0 + .5/1 + 1/2 + .5/3 + 0/4
  - *fair* = 0/3 + .5/4 + 1/5 + .5/6 + 0/7
  - *good* = 0/6 + .5/7 + 1/8 + .5/9 + 0/10
- And, three level sets
  - *undefined*= 0/0 + 0/1 + 0/2 + … + 0/8 + 0/9 + 0/10
  - *indeterminate* = .5/0 + .5/1 + .5/2 + … + .5/8 + .5/9 + .5/10
  - *unknown* = 1/0 + 1/1 + 1/2 + … + 1/8 + 1/9 + 1/10

Example - Plots



Example - Similarities

x	*output*	*bad*	*dist(bad)*	*fair*	*dist(fair)*	*good*	*dist(good)*	*undefined*	*d(undefined)*	*indet.*	*d(indet.)*	*unknown*	*d(unknown)*
0	0.00	0.00	0.00	0.00	**0.00**	0.00	0.00	0.00	0.00	0.50	0.25	1.00	1.00
1	0.00	0.50	0.25	0.00	**0.00**	0.00	0.00	0.00	0.00	0.50	0.25	1.00	1.00
2	0.00	1.00	1.00	0.00	**0.00**	0.00	0.00	0.00	0.00	0.50	0.25	1.00	1.00
3	0.00	0.50	0.25	0.00	**0.00**	0.00	0.00	0.00	0.00	0.50	0.25	1.00	1.00
4	0.50	0.00	0.25	0.50	**0.00**	0.00	0.25	0.00	0.25	0.50	0.00	1.00	0.25
5	0.50	0.00	0.25	1.00	**0.25**	0.00	0.25	0.00	0.25	0.50	0.00	1.00	0.25
6	0.50	0.00	0.25	0.50	**0.00**	0.00	0.25	0.00	0.25	0.50	0.00	1.00	0.25
7	0.25	0.00	0.06	0.00	**0.06**	0.50	0.06	0.00	0.06	0.50	0.06	1.00	0.56
8	0.25	0.00	0.06	0.00	**0.06**	1.00	0.56	0.00	0.06	0.50	0.06	1.00	0.56
9	0.25	0.00	0.06	0.00	**0.06**	0.50	0.06	0.00	0.06	0.50	0.06	1.00	0.56
10	0.00	0.00	0.00	0.00	**0.00**	0.00	0.00	0.00	0.00	0.50	0.25	1.00	1.00
		Σ	2.44		**0.44**		1.44		0.94		1.44		7.44

- The best linguistic match is '*fair*'

Summary

- Lecture summary
  - Mamdani inference uses a heuristic approximation of inference, inspired by production rules
  - with reasonable choices of variables, terms and rules, it produces reasonable results
  - defuzzification is required as the output is fuzzy
    - there are alternative numeric and linguistic methods
    - no defuzzification technique is 'correct'
- Next lecture
  - Sugeno inference