G53FUZ
Fuzzy Sets and Systems

**Fuzzy Concepts**

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## Characteristic Functions

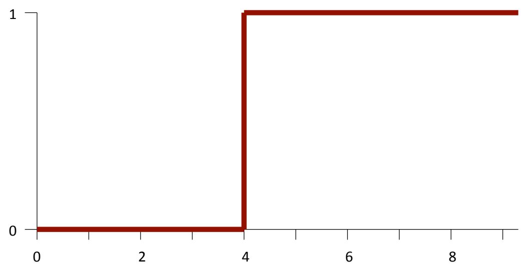
- Elements of the universal set $X$ are defined to be either members or not of a set $A$ by a *characteristic function*
  - for a given set $A$, this function assigns a value $\mu_A(x)$ to every $x \in X$, such that

$$\mu_A(x) = \begin{cases} 1 & \text{if and only if (iff) } x \in A \\ 0 & \text{if and only if (iff) } x \notin A \end{cases}$$

- Thus, the function maps elements of the universal set to the set containing 0 and 1
  - this can be denoted by $\mu_A(x): X \to \{ 0, 1 \}$

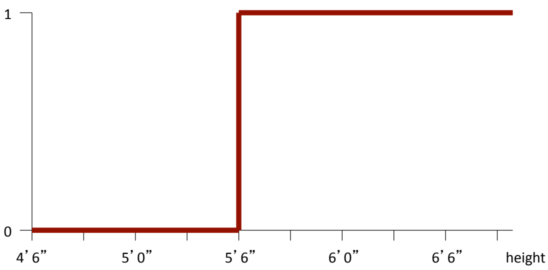
## Diagramatically

- The set of real numbers greater than 4, $\mu_A(x) = 1$ iff $x \geq 4$, may be illustrated as



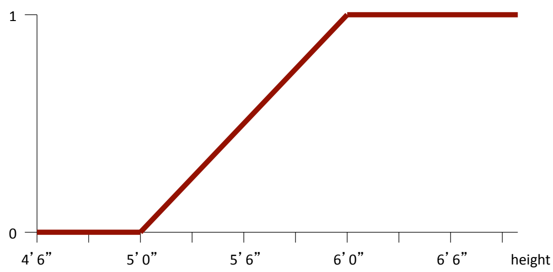

## Tall People

- But what about the set of tall people? Say, $\mu_{tall}(x) = 1$ iff $x \geq 66$ (inches, i.e. 5’ 6”)???




## Fuzzy Tall People

- Let’s modify the sharp (crisp) cut-off for the set of tall people into a smooth transition, e.g.




## Formal Definition

- The Boolean characteristic function of conventional sets is replaced by a *membership function* that returns a real value in [0, 1]
  - this can be denoted by $\mu_A(x): X \to [\,0, 1\,]$
- So, for the tall example previously

$$\mu_{tall}(x) = \begin{cases} 1 & x \geq 72'' \\ \dfrac{x-60}{12} & 60'' < x < 72'' \\ 0 & x \leq 60'' \end{cases}$$

- Note that membership values can also be listed
  - *tall* = { 0/*Danny DeVito*, 0.9/*Jon G*, 1/*Michael Jordan* }

## Notation

- Fuzzy sets (as with crisp sets) can be either discrete or continuous
  - fuzzy set notation can (initially) be confusing
- Discrete sets

  $A = \mu_1/x_1 + \mu_2/x_2 + \mu_3/x_3 + \ldots + \mu_n/x_n$

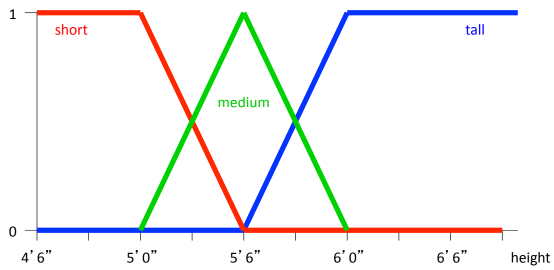
  or

  $A = \Sigma_{i=1,n}\, \mu_i / x_i$
- Continuous sets

  $A = \int_x \mu(x) / x$

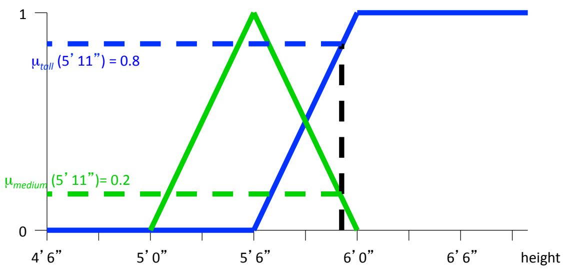
## Examples

- Examples of further fuzzy sets for height




## Multiple Memberships

- A given element or value can belong to multiple fuzzy sets with differing memberships




## Exercises

- Write down the fuzzy set (memberships/ elements) of your three closest neighbours for the fuzzy set *tall*
- What do membership values mean?
  - e.g. Jon is 0.99 in fuzzy set *tall* (people)
- On the universe of real numbers, draw a fuzzy set of the concept *about five*
- Draw a fuzzy set for the concept *middle-aged* (people)

## $\alpha$-Cuts

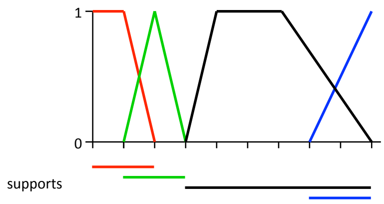
- An important concept which establishes a relationship between crisp sets and fuzzy sets is the concept of an $\alpha$-*cut*
  - an $\alpha$-cut of a fuzzy set *A* is a crisp set $A_\alpha$ that contains all the elements of *A* with membership greater than or equal to the specified value of $\alpha$
  - this can be written as

    $A_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha \}$
- The *strong* $\alpha$-*cut*, $A_{\alpha+}$, can also be defined

  $A_{\alpha+} = \{ x \in X \mid \mu_A(x) > \alpha \}$

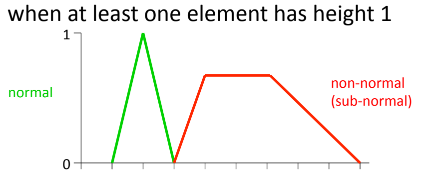
## Support

- The *support* of a fuzzy set, *A*, is the stong $\alpha$-cut of *A* for $\alpha = 0$ ($A_{0+}$)
  - i.e. the crisp set of elements where the membership is greater than zero



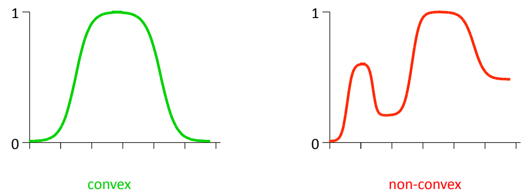
# Normality

- The *height* of a fuzzy set is the largest membership grade attained by any element of that set
- A fuzzy set is *normalised* if at least one of its elements attains the maximum possible grade
  - if membership grades are in [0,1], it is normalised when at least one element has height 1




# Convexity

- A fuzzy set is convex if and only if each of its $\alpha$-cuts is a convex set
  - iff $\mu_A(\lambda \mathbf{r} + (1 - \lambda)\mathbf{s}) > \min[\mu_A(\mathbf{r}), \mu_A(\mathbf{s})]$ $\forall\, \mathbf{r},\mathbf{s} \in \Re^n$ and $\forall\, \lambda \in [0, 1]$




# Exercises

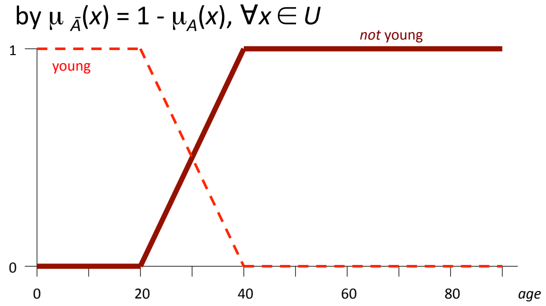
- Given the following fuzzy sets
  - $A$ = 0.0/1 + 0.1/2 + 0.7/3 + 0.6/4 + 0.7/5 + 0.4/6 + 0.1/7
  - $B$ = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.5/6 + 0.1/7
  - $C$ = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.9/6 + 1.0/7
- What are the alpha cuts
  - $A_{0.2}$, $B_{0.5}$, $C_{0.9}$
- What is the support of each?
- Which of the sets are normal? / convex?

# Fuzzy Sets and Probabilities

- Fuzzy memberships are *not* probabilities
  - there is no probability involved in a person's height
  - memberships are better interpreted as *compatibilities*
- Consider you are given two bottles of liquid
  - bottle A
    - the liquid is drinkable with probability 0.9
  - bottle B
    - the liquid is drinkable with fuzzy membership 0.9
- Which do you drink, and why?

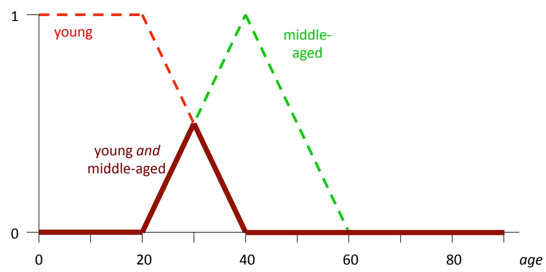
# Complement

- The fuzzy complement, $\bar{A}$, of fuzzy set $A$ is given by $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$, $\forall x \in U$



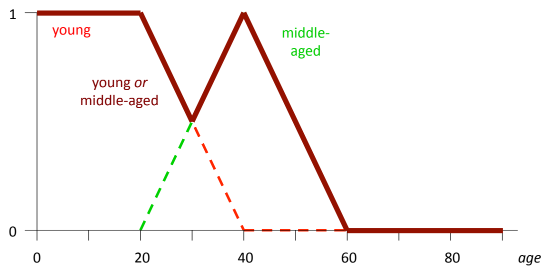

# Intersection

- The fuzzy intersection, $A \cap B$, of two fuzzy sets $A$ and $B$, is given by $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$



# Union

- The fuzzy union, $A \cup B$, of two fuzzy sets $A$ and $B$, is given by $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$




# Exercises

- Given the following two fuzzy sets
  - $A$ = 0.0/1 + 0.1/2 + 0.5/3 + 1.0/4 + 0.7/5 + 0.4/6 + 0.1/7
  - $B$ = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.9/6 + 1.0/7
  - write down the fuzzy sets
    - NOT $B$
    - $A$ AND $B$?
    - $A$ OR $B$?
    - $A$ AND $\bar{A}$

# Complement

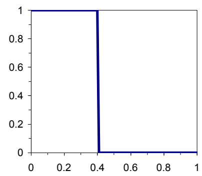
- A complement is a function which converts a fuzzy set, $A$, to another set, $\bar{A}$
  - $c$: [0,1] → [0,1]
- The function *must* satisfy the following axioms
  - **axiom *c*1**: $c(0) = 1$ and $c(1) = 0$
    - must behave like crisp sets (*boundary conditions*)
  - **axiom *c*2**: for all $a, b \in [0,1]$:
    if $a < b$, then $c(a) \geq c(b)$
    - $c$ is *monotonic non-increasing*
  - where $a = \mu_A(x)$ and $b = \mu_B(x)$

# Complement

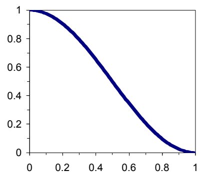
- Very often (in most cases), further requirements may be placed
  - **axiom *c*3**: $c$ should be a *continuous* function
  - **axiom *c*4**: $c$ should be *involutive*
    - that is $c(c(a)) = a$ for all $a \in [0,1]$
- Functions satisfying axiom 3 form a special sub-class of general fuzzy complements
  - all functions satisfying axiom 4 are necessarily continuous, and so form a further nested sub-class

# Example Complements

(a)



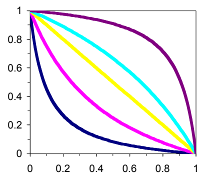

(b)



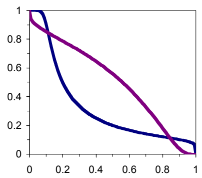

$$(a) \quad \overline{\mu(x)} = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{if } a > t \end{cases} \qquad (b) \quad \overline{\mu(x)} = \tfrac{1}{2}(1 + \cos(\pi a))$$

# Example Complements

(c)




(d)




$$(c) \quad \overline{\mu(x)} = \frac{1-a}{1+\lambda a} \qquad (d) \quad \overline{\mu(x)} = \left(\frac{1-a^w}{1+\lambda a^w}\right)^{1/w}$$

## Intersection

- Fuzzy unions are represented by an established class of functions called *triangular norms* or *t-norms*
- A t-norm is a function which takes two arguments in [0,1] and returns a value in [0,1]
  - $u$: [0,1] x [0,1] → [0,1]
- Thus
  - $\mu_{A \cup B}(x) = u(\mu_A(x), \mu_B(x))$

## Intersection Axioms

- The function *must* satisfy the following axioms
  - **axiom *i*1**: $i(1,1)=1$, $i(0,1)=i(1,0)=i(0,0)=0$
    - must behave like crisp sets (*boundary conditions*)
    - sometimes written like $1 \otimes a = a$
  - **axiom *i*2**: $i(a, b) = i(b, a)$ (*commutative*)
  - **axiom *i*3**: if $a \le a'$ and $b \le b'$, then $i(a,b) \le i(a',b')$ (*monotonic*)
  - **axiom *i*4**: $i(i(a,b), c) = i(a, i(b,c))$ (*associative*)

## Optional Intersection Axioms

- The function *may* satisfy the following axioms
  - **axiom *i*5**: $i$ is *continuous*
  - **axiom *i*6**: $i(a, a) = a$ (*idempotent*)
- The minimum min($a$, $b$) is the *only* t-norm which satisfies axioms $i$1 to $i$6

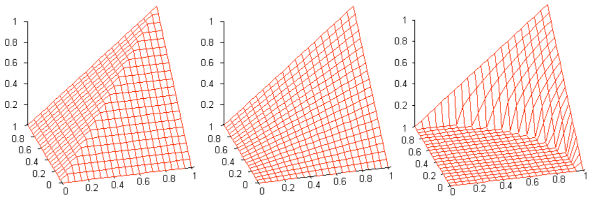
## Common T-Norms

- Schweizer and Sklar (1963)

  $a \otimes b = \max(0, a^p + b^p - 1)^{1/p}$
- This gives

| | | |
|---|---|---|
| $a \otimes_{-\infty} b$ | $= \min(a, b)$ | (*standard*) |
| $a \otimes_0 b$ | $= ab$ | (*product*) |
| $a \otimes_1 b$ | $= \max(0, a + b - 1)$ | (*bounded diff.*) |
| $a \otimes_\infty b$ | = ($a$ if $b$ = 1; $b$ if $a$ = 1; else 0) | (*drastic sum*) |

## Example Intersections



min product bounded diff

## Union

- Fuzzy unions are represented by an established class of functions called *triangular conorms* or *t-conorms*
- A t-conorm is a function which takes two arguments in [0,1] and returns a value in [0,1]
  - $u$: [0,1] x [0,1] → [0,1]
- Thus
  - $\mu_{A \cup B}(x) = u(\mu_A(x), \mu_B(x))$

## Union Axioms

- The function *must* satisfy the following axioms
  - **axiom *u*1**: $u(0,0)=0$, $u(0,1)=u(1,0)=u(1,1)=1$
    - must behave like crisp sets (*boundary conditions*)
    - sometimes written like $0 \oplus a = a$
  - **axiom *u*2**: $u(a, b) = u(b, a)$ (*commutative*)
  - **axiom *u*3**: if $a \le a'$ and $b \le b'$, then $u(a,b) \le u(a',b')$ (*monotonic*)
  - **axiom *u*4**: $u(u(a,b), c) = u(a, u(b,c))$ (*associative*)

## Optional Union Axioms

- The function *may* satisfy the following axioms
  - **axiom *u*5**: $u$ is *continuous*
  - **axiom *u*6**: $u(a, a) = a$ (*idempotent*)
- The maximum $\max(a, b)$ is the *only* t-conorm which satisfies axiom u6
  - as well as all the others!

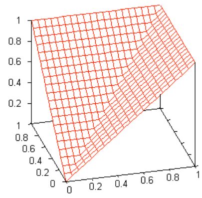
## Common T-Conorms

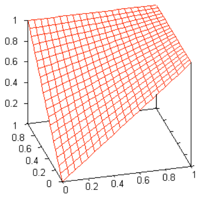
- Schweizer and Sklar (1963)

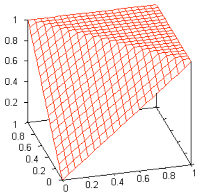
  $a \oplus b = 1 - \max(0, (1 - a)^{-p} + (1 - b)^{-p} - 1)^{1/p}$
- This gives

| | | | |
|---|---|---|---|
| $a \oplus_{-\infty} b$ | = | $\max(a, b)$ | (*standard*) |
| $a \oplus_0 b$ | = | $a + b - ab$ | (*probabilistic sum*) |
| $a \oplus_1 b$ | = | $\min(1, a + b)$ | (*bounded sum*) |
| $a \oplus_\infty b$ | = | ($a$ if $b = 0$; $b$ if $a = 0$; else 1) | (*drastic*) |

## Example Unions


max


probabilistic sum


bounded sum

## Exercises

- Given the following two fuzzy sets
  - $A$ = 0.0/1 + 0.1/2 + 0.5/3 + 1.0/4 + 0.7/5 + 0.4/6 + 0.1/7
  - $B$ = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.9/6 + 1.0/7
  - write down ($A$ AND $B$) and ($A$ OR $B$), using
    - min, max
    - product, probabilistic sum
    - bounded difference, bounded sum
- Calculate ANDs using min and product for
  - Jack is 0.9/tall and 0.9/old
  - Fred is 0.9/tall and 0.2/old
  - which do you think makes more sense, and why?

## Summary

- Lecture summary
  - fuzzy sets are extensions of conventional (crisp) sets that allow everyday notions to be represented
  - fuzzy memberships are not probabilities
    - but their precise meaning is open to interpretation
  - min and max are used for basic AND and OR
  - there are many alternative operator families
- Next lecture
  - linguistic variables