PSTAT 5A Practice Worksheet 4 - SOLUTIONS

Comprehensive Review: Discrete Random Variables and Distributions

Complete Solutions with Detailed Work

2025-07-23

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1 Section A: Basic Concepts and Identification - SOLUTIONS

1.1 Problem A1: Distribution Identification

Important

Instructions: For each scenario below, identify the appropriate probability distribution and specify its parameters. Justify your choice by identifying the key characteristics.

1.1.1 (a) Coin Flipping Until First Head

A fair coin is flipped until the first head appears. Let X = number of flips needed.

Solution:

Geometric Distribution with parameter p = 0.5

Key Characteristics:

- We count the number of trials until the first success
- Each flip is **independent** with constant probability of success
- Only two outcomes per trial (head or tail)
- We stop as soon as we get a success

Notation: $X \sim \text{Geometric}(p = 0.5)$

1.1.2 (b) Quality Control Inspection

A quality control inspector tests 20 randomly selected items from a production line where 5% are defective. Let X = number of defective items found.

Solution:

Binomial Distribution with parameters n = 20, p = 0.05

Key Characteristics:

- Fixed number of trials (n = 20)
- Each item has the **same probability** of being defective (p = 0.05)
- We count the **number of successes** (defective items)
- Each test is **independent**

Notation: $X \sim \text{Binomial}(n = 20, p = 0.05)$

1.1.3 (c) Website Visitor Count

A website receives visitors at an average rate of 3 per minute. Let X = number of visitors in a 2-minute period.

Solution:

Poisson Distribution with parameter $\lambda = 6$

Key Characteristics:

- Events occurring **over time** at a constant average rate
- Events are **independent** and **rare**
- Rate calculation: 3 visitors/minute \times 2 minutes = 6 expected visitors

Notation: $X \sim \text{Poisson}(\lambda = 6)$

1.1.4 (d) Single Free Throw

A basketball player shoots one free throw with an 80% success rate. Let X=1 if successful, 0 if unsuccessful.

Solution:

Bernoulli Distribution with parameter p = 0.8

Key Characteristics:

- Single trial with exactly two outcomes
- Success (make shot) vs. Failure (miss shot)
- Binary outcome: $X \in \{0, 1\}$

Notation: $X \sim \text{Bernoulli}(p = 0.8)$

1.1.5 (e) Driving Test Attempts

A student keeps taking a driving test until they pass. The probability of passing on any attempt is 0.7. Let X = number of attempts needed to pass.

Solution:

Geometric Distribution with parameter p = 0.7

Key Characteristics:

- We count **trials until first success** (passing the test)
- Each attempt is **independent** with constant probability
- Student continues until success occurs

Notation: $X \sim \text{Geometric}(p = 0.7)$

1.2 Summary Table

Decision Framework Visualization

Table 1: Distribution Identification Summary

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Figure 1: Decision Framework for Distribution Identification



Quick Reference Guide Ask these key questions to identify distributions: How many trials?

- One trial \rightarrow Bernoulli
- Fixed number \rightarrow Binomial (if counting successes)
- Until first success \rightarrow Geometric

What are we counting?

- Successes in fixed trials \rightarrow Binomial
- Trials until success \rightarrow Geometric
- Events over time/space \rightarrow Poisson

Time component?

- Events at constant rate over time \rightarrow Poisson
- No time component \rightarrow Binomial/Bernoulli/Geometric



Geometric vs. Binomial: Geometric counts trials until success;

- Binomial counts successes in fixed trials
- Poisson parameter: Remember to multiply rate by time period (e.g., 3/minute \times 2 minutes = λ = 6)

Independence assumption: All these distributions require independent trials/events

1.3 Problem A2: Probability Mass Function

Given distribution:

X	1	2	3	4	5
P(X=k)	0.1	0.3	0.4	a	0.1

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Figure 2: Probability Mass Function

1.3.1 (a) Find the value of a.

Solution. Since probabilities must sum to 1:

$$0.1 + 0.3 + 0.4 + a + 0.1 = 1$$

$$0.9 + a = 1$$

$$\boxed{a = 0.1}$$

1.3.2 (b) Calculate $P(X \le 3)$.

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Figure 3: PMF showing
$$P(X = 3) = 0.8$$

1.3.3 (c) Calculate P(X > 2).

Solution.
$$P(X > 2) = P(X = 3) + P(X = 4) + P(X = 5)$$

$$P(X > 2) = 0.4 + 0.1 + 0.1 = \boxed{0.6}$$

(Check: 0.8 + 0.2 = 1 and the full PMF sums to 1, so the results are consistent.)

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Figure 4: PMF showing
$$P(X > 2) = 0.6$$

Putting everything together:



Key Insights from Visualizations

Distribution Shape: The PMF shows X = 3 has the highest probability (0.4), making it the mode

Cumulative Probability: $P(X \le 3) = 0.8$ means 80% of outcomes are 3 or less

Complement Relationship: P(X > 2) = 0.6 and $P(X \le 2) = 0.4$ sum to 1

Symmetry: The distribution has some symmetry around the center, with equal probabili-

ties at the extremes (X = 1 and X = 5 both have P = 0.1)

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Figure 5: PMF showing both P(X = 3) and P(X > 2) regions

2 Section B: Expected Value and Variance - SOLUTIONS

2.1 Problem B1: Manual Calculations

Using the distribution from Problem A2:

X	1	2	3	4	5
P(X=k)	0.1	0.3	0.4	0.1	0.1

2.1.1 (a) Compute the expected value (E[X])

Solution. For a **discrete random variable**, the expected value is the *probability-weighted average* of all possible outcomes:

$$E[X] = \sum_{k=1}^{5} k \times P(X = k).$$

1. Set up the sum

$$E[X] = 1(0.1) + 2(0.3) + 3(0.4) + 4(0.1) + 5(0.1).$$

2. Multiply each outcome by its probability

$$= 0.1 + 0.6 + 1.2 + 0.4 + 0.5.$$

3. Add the terms

$$E[X]=2.8$$

Probability Mass Function Expected Value = 2.8

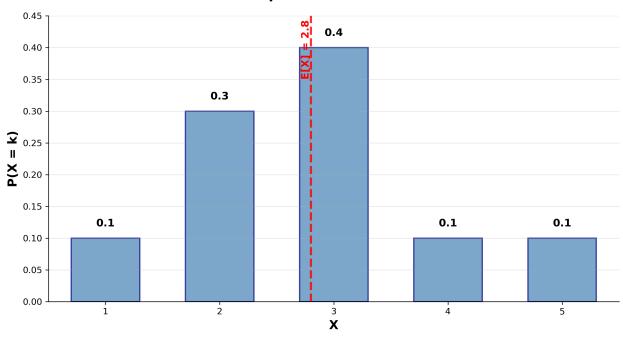


Figure 6: Probability Mass Function showing E[X] = 2.8

Tip

Visual Interpretation

Looking at the PMF plot:

- The highest probability (0.4) occurs at X=3
- The second highest (0.3) occurs at X=2

Together, these two values account for 70% of the probability mass

The expected value E[X] = 2.8 (red dashed line) falls between these two most likely out-

This visual confirms our intuition that the "center of gravity" should be close to, but slightly less than, 3

Note

Interpretation & quick check

Interpretation: If we were to observe this experiment many, many times, the long-run **average** value of X would settle down around **2.8**. Although 2.8 itself isn't an attainable outcome (only integers 1–5 are), it represents the center of gravity of the distribution.

Check: Notice most probability mass is on 2 and 3 (0.3 + 0.4 = 0.7). A mix that skews slightly toward the larger of those two values should indeed give an average a bit below 3, exactly what we see with 2.8.

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Figure 7: Multiple Simulation Runs Showing Convergence

2.1.2 (b) Compute the variance Var(X)

Solution. The variance measures how far the values of (X) tend to deviate from the mean. We use the shortcut formula

$$Var(X) = E[X^2] - (E[X])^2,$$

where E[X] = 2.8 was found in part (a).

1. Find $E[X^2]$ (the mean of the squared outcomes)

$$E[X^{2}] = \sum_{k=1}^{5} k^{2} P(X = k)$$

$$= 1^{2}(0.1) + 2^{2}(0.3) + 3^{2}(0.4) + 4^{2}(0.1) + 5^{2}(0.1)$$

$$= 1(0.1) + 4(0.3) + 9(0.4) + 16(0.1) + 25(0.1)$$

$$= 0.1 + 1.2 + 3.6 + 1.6 + 2.5$$

$$= 9.0$$

2. Apply the variance formula

$$Var(X) = 9.0 - (2.8)^2 = 9.0 - 7.84 = \boxed{1.16}$$

Note

Interpretation & quick check

Interpretation: A variance of 1.16 tells us that typical values of X deviate from the mean (2.8) by a little over one unit (figure 8).

Check: Most probability mass is on 2 and 3; the only "far" value is 5 (probability 0.1). So we expect a modest spread, larger than 0 but well below the maximum possible of $(5-2.8)^2 = 4.84$. The calculated 1.16 fits this intuition.

2.1.3 (c) Compute the standard deviation σ

Solution. The **standard deviation** is the square root of the variance:

$$\sigma = \sqrt{\operatorname{Var}(X)} = \sqrt{1.16} \approx \boxed{1.08}.$$

Note

Interpretation

A standard deviation (σ) of about 1.08 means typical observations of X lie roughly one unit above or below the mean value 2.8. This agrees with our earlier intuition that the distribution is fairly concentrated around 2–3, with only a small chance of the extreme value 5.

Let's visualise this!

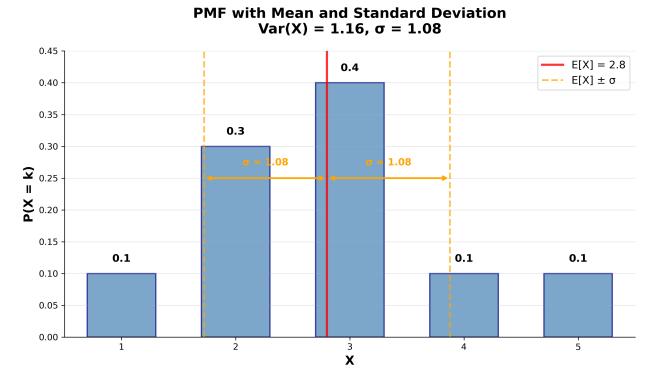


Figure 8: PMF with Variance Illustration

2.2 Problem B2: Bernoulli and Binomial Applications

Manufacturing Scenario: A manufacturing process has a 15% defect rate.

2.2.1 (a) Single Item Selection

If you select one item randomly, what is the expected value and variance of X = number of defective items?

Solution. This is a **Bernoulli distribution** with parameter p = 0.15

$$X \sim \text{Bernoulli}(p = 0.15)$$

Step 1: Expected Value

$$E[X] = p = \boxed{0.15}$$

Step 2: Variance

$$Var(X) = p(1-p) = 0.15 \times 0.85 = \boxed{0.1275}$$

Step 3: Standard Deviation

$$\sigma = \sqrt{\text{Var}(X)} = \sqrt{0.1275} = \boxed{0.357}$$

Note

Interpretation:

- $\bullet\,$ On average, 15% of items selected will be defective
- Since this is a single trial, X can only be 0 (not defective) or 1 (defective)
- The variance measures the uncertainty in this binary outcome

2.2.2 (b) Multiple Items Selection

If you select 25 items randomly, what is the expected number of defective items and the standard deviation?

Solution. This is a **Binomial distribution** with parameters n = 25, p = 0.15

$$X \sim \text{Binomial}(n = 25, p = 0.15)$$

Step 1: Expected Value

$$E[X] = np = 25 \times 0.15 = \boxed{3.75}$$

Step 2: Variance

$$Var(X) = np(1-p) = 25 \times 0.15 \times 0.85 = \boxed{3.1875}$$

Step 3: Standard Deviation

$$\sigma = \sqrt{\text{Var}(X)} = \sqrt{3.1875} = \boxed{1.785}$$

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Note

Interpretation:

- On average, we expect about 3.75 defective items out of 25
- The actual number will typically be within ± 1.785 items of this average
- Values between 2 and 6 defective items would be quite common

2.3 Visualizations

Let's visualize this to build more intuition

2.3.1 Bernoulli Distribution (Single Item)

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Figure 9: Bernoulli Distribution: P(X=k) for Single Item

2.3.1.1 Binomial Distribution (25 Items)

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Figure 10: Binomial Distribution: Number of Defective Items in 25 Trials

2.3.1.2 Comparison: Bernoulli vs Binomial Relationship

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Figure 11: Relationship Between Bernoulli and Binomial Distributions

Important

$\mathbf{Bernoulli} \to \mathbf{Binomial}$ Connection:

- A Binomial distribution is the sum of n independent Bernoulli trials
- If X_1, X_2, \dots, X_{25} are independent Bernoulli (0.15), then $X_1 + X_2 + \dots + X_{25} \sim \text{Binomial}(25, 0.15)$

Scaling Formulas:

- Expected Value: $E[Binomial] = n \times E[Bernoulli] = 25 \times 0.15 = 3.75$
- Variance: $Var(Binomial) = n \times Var(Bernoulli) = 25 \times 0.1275 = 3.1875$

Tip

- Single inspection: 15% chance of finding a defect
- Batch inspection (25 items): Expect 3-4 defective items typically Acceptable range: 2-6 defective items would be within 1 standard deviation
- Red flag: Finding 7+ defective items might indicate process issues (beyond 2 σ)

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3 Optional: Conceptual Understanding - SOLUTIONS

! Important

Objective: Deepen understanding of key differences between probability distributions and their applications.

3.0.1 (a) Binomial vs. Geometric Distributions

Explain the key difference between a Binomial distribution and a Geometric distribution in terms of what they count.

Solution:

Key Difference:	What We Count		
Distribution	What We Count	Fixed Parameter	Variable
Binomial	Number of successes	Number of trials (n)	Number of
Geometric	Number of trials	Until first success	successes Number of trials

- Binomial Distribution: Counts the number of successes in a fixed number of trials
 - Example: "How many heads in 10 coin flips?"
 - We know we'll flip exactly 10 times, but don't know how many heads
- Geometric Distribution: Counts the number of trials needed to get the first success
 - Example: "How many coin flips until the first head?"
 - We know we'll get exactly 1 head, but don't know how many flips it takes

3.0.2 Visual Comparison: Binomial vs Geometric - Fundamental Difference

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Figure 12: Binomial vs Geometric: What They Count



Binomial: "How many successes in a fixed box of trials?"

Fixed trials, variable successes

Geometric: "How many attempts until first success?"

Fixed successes (1), variable trials

3.0.3 (b) Poisson vs. Binomial: When to Use Each

When would you use a Poisson distribution instead of a Binomial distribution?

Solution. Use Poisson when:

- Events occur over time or space at a constant rate
- The number of possible events is very large but the probability of each is very small
- We don't have a fixed number of trials Examples: arrivals, defects per unit area, accidents per day

Decision Framework: Poisson vs. Binomial				
Use Binomial	Use Poisson			
Fixed number (n)	No fixed limit			
Not the focus	Events over time/space			
Moderate p	Very small p			
Not applicable	Constant rate (λ)			
Coin flips, surveys	Phone calls, defects			
	Use Binomial Fixed number (n) Not the focus Moderate p Not applicable			

Comparative Examples: Poisson vs Binomial - Choosing the Right Distribution

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Figure 13: Poisson vs Binomial: When to Use Each



Warning

Common Mistake

- Don't use Poisson just because events are "rare." The key criteria are:
- No fixed number of trials
- Events over time/space
- Constant rate (λ)

A rare event in a fixed number of trials is still Binomial!

3.1.1 (c) Variance Maximization in Binomial Distribution

If $X \sim \text{Binomial}(n, p)$, under what conditions would the variance be maximized?

Solution. For a Binomial distribution: Var(X) = np(1-p)

For **fixed** n, variance is maximized when p(1-p) is maximized.

Approach:

Taking the derivative with respect to p:

$$\tfrac{d}{dp}\big[p(1-p)\big] = \tfrac{d}{dp}\big[p-p^2\big] = 1-2p$$

Setting equal to zero:

$$1 - 2p = 0 \implies p = 0.5$$

Second derivative = -2 < 0, confirming this is a **maximum**.

The variance is maximized when p = 0.5 (fair coin scenario).

3.1.2 Visualization of Variance vs. Probability

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Figure 14: Binomial Variance Maximization: Effect of p

3.1.3 Intuitive Understanding of Variance Maximization

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Figure 15: Why p=0.5 Maximizes Variance: Mathematical Intuition

Important

Key Insights

Maximum: p = 0.5 maximizes p(1-p) for any fixed n

Intuitive Explanation: Maximum uncertainty occurs when success and failure are equally

likely

Practical Meaning: A fair coin (50-50) has the highest variability in outcomes Extremes: When p approaches 0 or 1, outcomes become predictable (low variance)