PSTAT 5A Practice Worksheet 3 - SOLUTIONS

Comprehensive Review: Probability, Counting, and Conditional Probability

Solution Key

2025-07-23

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1 Section A: Probability - SOLUTIONS

Estimated time: 8 minutes

Problem A1: Probability Distributions - SOLUTION

For a valid probability distribution, two conditions must be met:

1. All probabilities must be non-negative (0)
2. The sum of all probabilities must equal 1

Analysis:

(a) Invalid

- Sum $= 0.3 + 0.3 + 0.3 + 0.2 + 0.1 = 1.2 > 1$ The probabilities sum to more than 1, violating the second condition.

(b) Valid

- Sum $= 0 + 0 + 1 + 0 + 0 = 1$ All probabilities are non-negative and sum to 1. This represents a class where everyone receives a C.

(c) Invalid

- Sum $= 0.3 + 0.3 + 0.3 + 0 + 0 = 0.9 < 1$ The probabilities sum to less than 1, violating the second condition.

(d) Invalid

- Contains F = -0.1 < 0 Although the sum would equal 1.0, the probability for grade F is negative, violating the first condition.

(e) Valid

- Sum = 0.2 + 0.4 + 0.2 + 0.1 + 0.1 = 1.0 All probabilities are non-negative and sum to 1.

(f) Invalid

- Contains B = -0.1 < 0 Although the sum equals 1.0, the probability for grade B is negative, violating the first condition.

2 Section B: Permutations and Combinations - SOLUTIONS

Estimated time: 15 minutes

Problem B1: Permutations and Combinations - SOLUTION

Part (a): How many 6-character passwords can be formed using 3 specific letters and 3 specific digits if repetitions are not allowed and letters must come before digits?

Solution: Since letters must come before digits, we have a fixed structure: LLL DDD

- Step 1: Arrange 3 letters in the first 3 positions
  - This is a permutation: P(3,3) = 3! = 6 ways
- Step 2: Arrange 3 digits in the last 3 positions
  - This is a permutation: P(3,3) = 3! = 6 ways
- Step 3: Apply multiplication principle
  - Total passwords = 6 × 6 = **36 passwords**

Part (b): If the team wants to select 4 people from 12 employees to form a security committee where order doesn't matter, how many ways can this be done?

Solution: Since order doesn't matter, this is a combination problem.

$$C(12,4) = \binom{12}{4} = \frac{12!}{4!(12-4)!} = \frac{12!}{4! \cdot 8!}$$

$$= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = \frac{11880}{24} = \mathbf{495\ ways}$$

3 Section C: Conditional Probability - SOLUTIONS

Estimated time: 15 minutes

Problem B1: Conditional Probability and Medical Testing - SOLUTION

Given Information:

- P(has variant) = 0.03
- P(test positive | has variant) = 0.95 (sensitivity)
- P(test negative | no variant) = 0.92 (specificity)

- Therefore: P(test positive | no variant) = 1 - 0.92 = 0.08

Part (a): What is the probability that a randomly selected person tests positive?

Solution:

Using the Law of Total Probability:

$$P(\text{test positive}) = P(\text{test positive} \mid \text{has variant}) \times P(\text{has variant}) + P(\text{test positive} \mid \text{no variant}) \times P(\text{no variant})$$

$$P(\text{test positive}) = 0.95 \times 0.03 + 0.08 \times 0.97$$
$$= 0.0285 + 0.0776 = \mathbf{0.1061}$$

Part (b): If someone tests positive, what is the probability they actually have the variant?

Solution: Using Bayes' Theorem:

$$P(\text{has variant} \mid \text{test positive}) = \frac{P(\text{test positive} \mid \text{has variant}) \times P(\text{has variant})}{P(\text{test positive})}$$

$$= \frac{0.95 \times 0.03}{0.1061} = \frac{0.0285}{0.1061} = \mathbf{0.2686}$$

Part (c): If someone tests negative, what is the probability they actually don't have the variant?

Solution: First, find P(test negative):

$$P(\text{test negative}) = 1 - P(\text{test positive}) = 1 - 0.1061 = 0.8939$$

Using Bayes' Theorem:

$$P(\text{no variant} \mid \text{test negative}) = \frac{P(\text{test negative} \mid \text{no variant}) \times P(\text{no variant})}{P(\text{test negative})}$$

$$= \frac{0.92 \times 0.97}{0.8939} = \frac{0.8924}{0.8939} = \mathbf{0.9983}$$

Part (d) [Challenge]: Two consecutive positive tests - what is the probability they actually have the variant?

Solution: Assuming independence between tests:

$$P(\text{two positive} \mid \text{has variant}) = 0.95^2 = 0.9025$$
$$P(\text{two positive} \mid \text{no variant}) = 0.08^2 = 0.0064$$

$$P(\text{two positive}) = 0.9025 \times 0.03 + 0.0064 \times 0.97 = 0.027075 + 0.006208 = 0.033283$$

$$P(\text{has variant} \mid \text{two positive}) = \frac{0.027075}{0.033283} = \mathbf{0.8134}$$

Problem C1: Advanced Counting with Restrictions - SOLUTION

Part (a): How many valid meal combinations are possible?

Solution: We need to consider cases based on the restrictions.

Case 1: Seafood appetizer is chosen

- 1 appetizer option (seafood)
- 7 main course options (cannot choose vegetarian)
- 5 dessert options
- Combinations: $1 \times 7 \times 5 = 35$

Case 2: Non-seafood appetizer + chocolate dessert

- 5 appetizer options (non-seafood)
- 3 main course options (only beef or chicken allowed with chocolate)
- 1 dessert option (chocolate)
- Combinations: $5 \times 3 \times 1 = 15$

Case 3: Non-seafood appetizer + non-chocolate dessert - 5 appetizer options (non-seafood)

- 8 main course options (no restrictions)
- 4 dessert options (non-chocolate)
- Combinations: $5 \times 8 \times 4 = 160$

Total valid combinations: $35 + 15 + 160 =$ **210 combinations**

Part (b): If customers choose randomly among valid combinations, what is the probability someone chooses the chocolate dessert?

Solution: Combinations with chocolate dessert: 15 (from Case 2 above) Total valid combinations: 210

$$P(\text{chocolate dessert}) = \frac{15}{210} = \frac{1}{14} = \mathbf{0.0714}$$

4 Section D: Review - SOLUTIONS

Estimated time: 12 minutes

Problem B3: Daily Expenses - SOLUTION

Given:

- Coffee: Mean = \$1.40, SD = \$0.30
- Muffin: Mean = \$2.50, SD = \$0.15

- Prices are independent

Part (a): What is the mean and standard deviation of the amount she spends on breakfast daily?

Solution: For the sum of independent random variables:

Mean of daily expenses:

$$E[\text{Daily}] = E[\text{Coffee}] + E[\text{Muffin}] = \$1.40 + \$2.50 = \mathbf{\$3.90}$$

Variance of daily expenses:

$$\text{Var}[\text{Daily}] = \text{Var}[\text{Coffee}] + \text{Var}[\text{Muffin}] = (0.30)^2 + (0.15)^2 = 0.09 + 0.0225 = 0.1125$$

Standard deviation of daily expenses:

$$SD[\text{Daily}] = \sqrt{0.1125} = \mathbf{\$0.3354}$$

Part (b): What is the mean and standard deviation of the amount she spends on breakfast weekly (7 days)?

Solution: For the sum of 7 independent daily expenses:

Mean of weekly expenses:

$$E[\text{Weekly}] = 7 \times E[\text{Daily}] = 7 \times \$3.90 = \mathbf{\$27.30}$$

Variance of weekly expenses:

$$\text{Var}[\text{Weekly}] = 7 \times \text{Var}[\text{Daily}] = 7 \times 0.1125 = 0.7875$$

Standard deviation of weekly expenses:

$$SD[\text{Weekly}] = \sqrt{0.7875} = \mathbf{\$0.8874}$$