PSTAT 5A Practice Worksheet 3 - SOLUTIONS

Comprehensive Review: Probability, Counting, and Conditional Probability

Narjes Mathlouthi

2025-07-29

Table of contents

Key Formulas Reference:

Basic Probability:

- **Conditional Probability:** $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- **Law of Total Probability:** $P(A) = \sum P(A|B_i) \cdot P(B_i)$
- **Addition Rule:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Multiplication Rule:** $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$

Counting:

- **Multiplication Rule:** If a procedure consists of k steps, with n_1 ways for step 1, n_2 for step 2, ..., n_k for step k, then total ways: $n_1 \times n_2 \times \cdots \times n_k$
- **Factorial:** $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
- **Permutations:** $P(n, r) = \frac{n!}{(n-r)!}$
- **Combinations:** $C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

1 Section A: Probability - SOLUTIONS

Estimated time: 8 minutes

Problem A1: Probability Distributions - SOLUTION

For a valid probability distribution, two conditions must be met:

1. All probabilities must be non-negative (0)
2. The sum of all probabilities must equal 1

Analysis:

(a) Invalid

$$\text{Sum} = 0.3 + 0.3 + 0.3 + 0.2 + 0.1 = 1.2 > 1$$

The probabilities sum to more than 1, violating condition 2.

(b) Valid

$$\text{Sum} = 0 + 0 + 1 + 0 + 0 = 1$$

All probabilities are non-negative and sum to 1. This represents a class where everyone receives a C.

(c) Invalid

$$\text{Sum} = 0.3 + 0.3 + 0.3 + 0 + 0 = 0.9 < 1$$

The probabilities sum to less than 1, violating condition 2.

(d) Invalid Contains $P(F) = -0.1 < 0$ Although the sum would equal 1.0, the probability for grade F is negative, violating condition 1.

(e) Valid

$$\text{Sum} = 0.2 + 0.4 + 0.2 + 0.1 + 0.1 = 1.0$$

All probabilities are non-negative and sum to 1.

(f) Invalid Contains $P(B) = -0.1 < 0$ Although the sum equals 1.0, the probability for grade B is negative, violating condition 1.

2 Section B: Permutations and Combinations - SOLUTIONS

Estimated time: 15 minutes

Problem B1: Permutations and Combinations - SOLUTION

Part (a): How many 6-character passwords can be formed using 3 specific letters and 3 specific digits if repetitions are not allowed and letters must come before digits?

Solution: Since letters must come before digits, we have a fixed structure: **LLL DDD**

Step 1: Arrange 3 letters in the first 3 positions

- This is a permutation: $P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 3! = 6$ ways

Step 2: Arrange 3 digits in the last 3 positions

- This is a permutation: $P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 3! = 6$ ways

Step 3: Apply multiplication principle

$$\text{Total passwords} = 6 \times 6 = \boxed{36 \text{ passwords}}$$

Part (b): If the team wants to select 4 people from 12 employees to form a security committee where order doesn't matter, how many ways can this be done?

Solution: Since order doesn't matter, this is a **combination** problem.

$$C(12, 4) = \binom{12}{4} = \frac{12!}{4!(12-4)!} = \frac{12!}{4! \cdot 8!}$$

$$= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = \frac{11,880}{24} = \boxed{495 \text{ ways}}$$

3 Section C: Conditional Probability - SOLUTIONS

Estimated time: 12 minutes

Problem C1: Drawing Cards (Without Replacement) - SOLUTION

Given Information:

- Standard 52-card deck
- Drawing two cards without replacement
- $A = \{\text{"first card is a heart"}\}$
- $B = \{\text{"second card is an ace"}\}$

Solution:

1. P(A)

There are 13 hearts in a 52-card deck.

$$P(A) = \frac{13}{52} = \boxed{\frac{1}{4} = 0.2500}$$

2. P(A and B)

We need both events to occur: first card is a heart AND second card is an ace.

Case 1: First card is the Ace of Hearts - $P(\text{1st card is Ace of Hearts}) = \frac{1}{52}$

- $P(\text{2nd card is an ace} \mid \text{1st card is Ace of Hearts}) = \frac{3}{51}$ (3 aces left)
- $P(\text{Case 1}) = \frac{1}{52} \times \frac{3}{51} = \frac{3}{2652}$

Case 2: First card is a non-ace heart - $P(\text{1st card is non-ace heart}) = \frac{12}{52}$ (12 non-ace hearts)

- $P(\text{2nd card is an ace} \mid \text{1st card is non-ace heart}) = \frac{4}{51}$ (4 aces left)
- $P(\text{Case 2}) = \frac{12}{52} \times \frac{4}{51} = \frac{48}{2652}$

$$P(A \text{ and } B) = \frac{3}{2652} + \frac{48}{2652} = \frac{51}{2652} = \boxed{\frac{1}{52} = 0.0192}$$

3. P(B|A)

Using the definition of conditional probability:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{1}{52}}{\frac{1}{4}} = \frac{1}{52} \times \frac{4}{1} = \boxed{\frac{4}{52} = \frac{1}{13} = 0.0769}$$

Alternative approach: Given that the first card is a heart:

- If it's the Ace of Hearts: 3 aces remain out of 51 cards
- If it's a non-ace heart: 4 aces remain out of 51 cards
- $P(B|A) = \frac{1}{13} \times \frac{3}{51} + \frac{12}{13} \times \frac{4}{51} = \frac{3+48}{13 \times 51} = \frac{51}{663} = \frac{4}{52} = \frac{1}{13}$

4. P(B)

Using the Law of Total Probability. Let A^c = "first card is not a heart"

$$P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)$$

We know:

- $P(A) = \frac{1}{4}$ and $P(A^c) = \frac{3}{4}$
- $P(B|A) = \frac{1}{13}$ (from part 3)
- $P(B|A^c) = \frac{4}{51}$ (if first card isn't a heart, all 4 aces remain)

$$P(B) = \frac{1}{13} \times \frac{1}{4} + \frac{4}{51} \times \frac{3}{4} = \frac{1}{52} + \frac{12}{204} = \frac{1}{52} + \frac{3}{51}$$

$$= \frac{51 + 156}{52 \times 51} = \frac{207}{2652} = \boxed{\frac{4}{51} = 0.0784}$$

5. Comparison of P(B|A) vs P(B)

$$P(B|A) = \frac{1}{13} = 0.0769$$
$$P(B) = \frac{4}{51} = 0.0784$$

Analysis: $P(B|A) < P(B)$

Explanation: The probability of getting an ace on the second draw is slightly **lower** when we know the first card is a heart compared to when we don't know anything about the first card. This happens because:

- When the first card is a heart, there's a $\frac{1}{13}$ chance it's the Ace of Hearts, removing one ace from the deck
- This makes it slightly less likely to draw an ace on the second draw
- This demonstrates **dependence** - the events are not independent because drawing without replacement creates dependence between successive draws

4 Section D: Advanced Counting with Restrictions - SOLUTIONS

Estimated time: 15 minutes

Problem D1: Advanced Counting with Restrictions - SOLUTION

Given:

- 6 appetizer options (including 1 seafood)
- 8 main course options (including 1 vegetarian, and 3 that are beef or chicken)
- 5 dessert options (including 1 chocolate)

Restrictions:

1. Seafood appetizer $\rightarrow$ cannot choose vegetarian main course
2. Chocolate dessert $\rightarrow$ must choose beef or chicken main course (3 specific options)

Part (a): How many valid meal combinations are possible?

Solution using cases:

Case 1: Seafood appetizer is chosen

- 1 appetizer choice (seafood)
- 7 main course choices (8 total minus 1 vegetarian)
- 5 dessert choices (no restrictions)
- Total: $1 \times 7 \times 5 = 35$ combinations

Case 2: Non-seafood appetizer + chocolate dessert

- 5 appetizer choices (6 total minus 1 seafood)
- 3 main course choices (only beef or chicken allowed with chocolate)
- 1 dessert choice (chocolate)
- Total: $5 \times 3 \times 1 = 15$ combinations

Case 3: Non-seafood appetizer + non-chocolate dessert

- 5 appetizer choices (6 total minus 1 seafood)
- 8 main course choices (no restrictions since no seafood appetizer)
- 4 dessert choices (5 total minus 1 chocolate)
- Total: $5 \times 8 \times 4 = 160$ combinations

Total valid combinations:

$$35 + 15 + 160 = \boxed{210 \text{ combinations}}$$

Verification using complementary counting:

- Total unrestricted combinations: $6 \times 8 \times 5 = 240$

- Invalid combinations to subtract:
  - Seafood + vegetarian + any dessert: $1 \times 1 \times 5 = 5$
  - Non-seafood + chocolate + non-beef/chicken: $5 \times 5 \times 1 = 25$
- Valid combinations: $240 - 5 - 25 = 210$

Part (b): If customers choose randomly among valid combinations, what is the probability someone chooses the chocolate dessert?

Solution: From our case analysis, combinations with chocolate dessert come only from Case 2:

- Combinations with chocolate dessert: 15
- Total valid combinations: 210

$$P(\text{chocolate dessert}) = \frac{15}{210} = \frac{1}{14} = \boxed{0.0714}$$

Alternative verification:

We can also calculate this directly:

- Non-seafood appetizers: 5 choices
- With chocolate dessert, must choose from 3 main courses
- Valid chocolate combinations: $5 \times 3 = 15$
- Probability: $\frac{15}{210} = \frac{1}{14} = 0.0714$

5 Summary of Key Concepts

Probability Distributions

- Valid distributions require: all probabilities ≥ 0 and sum $= 1$
- Check both conditions systematically

Permutations vs Combinations

- Permutations: Order matters, use $P(n, r) = \frac{n!}{(n-r)!}$
- Combinations: Order doesn't matter, use $C(n, r) = \frac{n!}{r!(n-r)!}$
- Multiplication principle: Combine independent choices

Conditional Probability

- Without replacement: Creates dependence between events
- Use definition: $P(B|A) = \frac{P(A \cap B)}{P(A)}$
- Law of Total Probability: For calculating marginal probabilities

Advanced Counting

- Case analysis: Break complex problems into manageable parts

- Handle restrictions: Consider what's allowed vs. not allowed
- Verification: Use complementary counting or direct calculation