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## No Integer Solutions for $x^3 = y^2 + k$

Suppose  $k = 1, 2 \pmod{4}$ , that  $k$  is squarefree, and  $k$  is not of the form  $3t^2 \pm 1$  for some  $t \in \mathbb{Z}$ .

Also assume  $3 \nmid \text{cl}(\mathbb{Q}(\sqrt{-k}))$ .

Then  $x^3 = y^2 + k$  has no integer solution.

### $x$ is Odd

We start by brute-forcing all possible values mod 4 for  $x, y$ .

```
sage: for x in range(4):
.....:     for y in range(4):
.....:         if (x^3 - (y^2 + 1)) % 4 == 0:
.....:             print(x, y)
.....:
1 0
1 2
sage: for x in range(4):
.....:     for y in range(4):
.....:         if (x^3 - (y^2 + 2)) % 4 == 0:
.....:             print(x, y)
.....:
3 1
3 3
```

So in both cases  $x$  is odd.

### $(x, y)$ are Coprime

Let  $p|(x, y)$  then  $p|x^3 - y^2$  so  $p|k$ .

We also see  $p^3|x^3 \Rightarrow p^2|x^3$  but  $p^2 \nmid k$  since  $k$  is squarefree, so  $p^2 \nmid y^2 + k$ .

Hence  $(x, y)$  are coprime.

### $y + \sqrt{-k}$ and $y - \sqrt{-k}$ are in the Same Ideal

$$x^3 = (y + \sqrt{-k})(y - \sqrt{-k})$$

Suppose there is a prime ideal  $\mathfrak{p}$  such that  $(y \pm \sqrt{-k}) \in \mathfrak{p}$  which means they are both coprime. This means  $x^3 \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ , also by summing the ideals we see also  $2y \in \mathfrak{p}$ . Since  $x$  is odd, 2 is not in  $\mathfrak{p}$  otherwise it would be the whole ring. But  $\mathfrak{p}$  is prime  $\Rightarrow y \in \mathfrak{p}$ . But both  $x, y$  are coprime so this cannot be true.

### Both Ideals are Principal

Next we see both ideals are principal.

$$\langle y + \sqrt{-k} \rangle = \mathfrak{a}^3, \quad \langle y - \sqrt{-k} \rangle = \mathfrak{b}^3$$

We see  $[\mathfrak{a}^3] = [1]$  in the class group since it is principal. Therefore  $[\mathfrak{a}]^3 = [1]$  means that  $3 | \text{ord}([\mathfrak{a}])$ , but by lagrange's theorem  $\text{ord}([\mathfrak{a}] | \text{cl}(\mathbb{Q}(\sqrt{-k})))$  which means also  $3 | \text{cl}(\mathbb{Q}(\sqrt{-k}))$ . But we stated this is not true in the beginning so we conclude  $\mathfrak{a}$  and likewise  $\mathfrak{b}$  are both principal.

## Result

Lastly we see our result.

$y + \sqrt{-k} = u\alpha^3$  for some unit  $u$ . Note  $k \equiv 1, 2 \pmod{4}$  means  $-k \equiv 3, 2 \pmod{4}$ . For all  $-k$ , the units are  $\{\pm 1\}$  except  $-k = -1$  which includes  $\{\pm i\}$ . But  $k = 1$  is of the form  $3t^2 + 1$  so we ignore that value.

In all cases, these units have integer cube roots so  $y + \sqrt{-k} = \alpha^3$  for some  $\alpha = a + b\sqrt{-k}$ . Then

$$y + \sqrt{-k} = (a + b\sqrt{-k})^3$$

```
sage: var("a b k")
(a, b, k)
sage: ( (a + b*sqrt(-k))^3 ).expand()
b^3*(-k)^(3/2) - 3*a*b^2*k + 3*a^2*b*sqrt(-k) + a^3
```

By comparing coefficients, we see

$$\begin{aligned}\sqrt{-k} &= b^3\sqrt{-k}^3 + 3a^2b\sqrt{-k} \\ &= (b^3\sqrt{-k}^2 + 3a^2b)\sqrt{-k} \\ &= (-kb^3 + 3a^2b)\sqrt{-k} \\ \Rightarrow 1 &= b(3a^2 - kb^2)\end{aligned}$$

So  $b = \pm 1$  and so  $3a^2 - kb^2 = 3a^2 - k = \pm 1$ , which means

$$k = 3a^2 \mp 1$$

which has no solutions as stated at the beginning.