Contents

Dimension

Example:

$$V : y^2 = x^3 - x, \bar{\mathbb{Q}}(V) = \bar{\mathbb{Q}}(x, \sqrt{x^3 - x})$$

$$\bar{\mathbb{Q}} \subseteq \bar{\mathbb{Q}}(x) \subseteq \bar{\mathbb{Q}}(x, \sqrt{x^3 - x})$$

x is transcendetal of deg $= 1$ over $\bar{\mathbb{Q}}$, but $\sqrt{x^3 - x}$ is an algebraic extension of $\bar{\mathbb{Q}}(x)$.

So the dimension of $V = 1$.

Another example: Let V be the y axis, then $I = \langle x \rangle \subseteq \mathbb{Q}[x, y]$

$$\bar{\mathbb{Q}}(V) \cong \bar{\mathbb{Q}}(y)/\bar{\mathbb{Q}}$$

Which has a transcendence degree of 1.

Questions

1.3

First an example:

$$V : y^2 = x^3 + x$$

We will align the curve f at the origin $P = (0, 0)$.

$$\begin{aligned} D_p(f) &= f_y(P) + f_x(P) \\ &= x \end{aligned}$$

The point will be singular when $D_p(f) = 0$ by I.1.5

We see here the curve above is smooth at P.

$$f \in M_p, f \notin M_p^2 \implies D_p(f) \neq 0$$

Which is equivalent to saying $f_{x_i}(P) \neq 0$ for some $i \iff \operatorname{rank}(f_{x_i}(P))_i = 1$.

By definition T is an affine hyperplane, and if P is smooth then $\dim T = \dim V$. Otherwise $T = \mathbb{A}^n$.

$$D_p : M_p \to (K^n)^*$$

$$D_p(f) = \sum f_{x_i}(P) x_i$$

$\ker D_p = M_p^2$, and $D_p(x_i) = x_i$ is a basis of $(K^n)^*$, so D_p is surjective.

$$M_p/M_p^2 \cong (K^n)^*$$

$$\dim(V) = n - 1$$

$$M_p/M_p^2 \to (K^n)^* \to \bar{K}$$

Likewise for all $t \in T$, $D_p(g) \neq 0, D_p(g)(t) = 0 \implies g \in \langle f \rangle$.

A smooth point has a well defined hyperplane with reduced dimension $n-1$, which is the dimension of V. When f contains linear terms, this allows us to reduce the dimension by 1, so creating a smooth point.

$$x \equiv y^2 - x^3 \equiv 0 \pmod{M_p^2}$$

1.6

The morphism is regular at all P. The only zero value is at $\infty = [0 : 1 : 0]$.

$$x^2 = \frac{z}{x}(y^2 - z^2)$$

$$\begin{aligned}
[x^2 : xy : z^2] &= [\frac{z}{x}(y^2 - z^2) : xy : z^2] \\
&= [z(y^2 - z^2) : x^2y : xz^2] \\
&= [z(y^2 - z^2) : \frac{z}{x}(y^2 - z^2)y : xz^2] \\
&= [xz(y^2 - z^2) : z(y^3 - yz^2) : x^2z^2] \\
&= [x(y^2 - z^2) : y^3 - yz^2 : x^2z]
\end{aligned}$$

$$\phi(\infty) = \infty$$

As expected.