Abstract Algebra by Pinter, Chapter 28

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Chapter 28 on Vector Spaces

Contents

A. Examples of Vector Spaces

Q1

$$\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n)$$
$$\mathbf{a} + \mathbf{b} = (a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$
$$k\mathbf{a} = k(a_1, \dots, a_n) = (ka_1, \dots, ka_n)$$

$$\begin{aligned} k(\mathbf{a} + \mathbf{b}) &= k[(a_1, \dots, a_n) + (b_1, \dots, b_n)] \\ &= k(a_1 + b_1, \dots, a_n + b_n) \\ &= (ka_1 + kb_1, \dots, ka_n + kb_n) \\ &= (ka_1, \dots, ka_n) + (kb_1, \dots, kb_n) \\ &= k\mathbf{a} + k\mathbf{b} \end{aligned}$$

$$\begin{aligned} (k + l)\mathbf{a} &= ((k + l)a_1, \dots, (k + l)a_n) \\ &= (ka_1 + la_1, \dots, ka_n + la_n) \\ &= (ka_1, \dots, ka_n) + (la_1, \dots, la_n) \\ &= k\mathbf{a} + l\mathbf{b} \end{aligned}$$

$$\begin{aligned} k(l\mathbf{a}) &= k(la_1, \dots, la_n) = (kla_1, \dots, kla_n) \\ &= (kl)\mathbf{a} \end{aligned}$$

$$1\mathbf{a} = \mathbf{a}$$

Q2

$$[f + g](x) = f(x) + g(x)$$
$$[af](x) = af(x)$$

All the vector space rules are obeyed.

Q3

$\mathcal{P}l$ is trivially easy to show it obeys the vector space rules.

Q4

Same for $\mathcal{M}_2(\mathbb{R})$.

B. Exmples of Subspaces

Q1

$U = \{(a, b, c) : 2a - 3b + c = 0\}$ and let $\mathbf{u} = (a_1, b_1, c_1), \mathbf{v} = (a_2, b_2, c_2) \in U$, then $\mathbf{u} + \mathbf{v} \implies 2a_1 - 3b_1 + c_1 = 2a_2 - 3b_2 + c_2 = 0 \implies 2(a_1 + b_1) - 3(b_1 + b_2) + (c_1 + c_2) = 0 \implies (\mathbf{u} + \mathbf{v}) \in U$. Also $k\mathbf{v} = (ka, kb, kc)$ and $2ka - 3kb + kc = 0 \implies k\mathbf{v} \in U$.

Q2

Let $\mathbf{u}, \mathbf{v} \in U$, then $\mathbf{u} + \mathbf{v}$ satisfies the conditions, and hence is also in U. Thus U is a closed subspace.

Q3

For any two functions in $\mathcal{F}(\mathbb{R})$, then $f(1) = 0, g(1) = 0 \implies (f + g)(1) = 0$.

Q4

Two functions which are constant on the interval $[0, 1]$ when summed will still be constant, hence it is a closed subspace.

Q5

$f(x) = f(-x), g(x) = g(-x) \implies (f + g)(x) = (f + g)(-x)$. Likewise for odd functions.

Q6

$f(x) = a_0 x + \cdots + a_n x^n, g(x) = b_0 + \cdots + b_n x^n, f(x) + g(x) = (a_0 + b_0) + \cdots + (a_n + b_n)x^n$.

C. Examples of Linear Independence and Bases

Q1

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = k\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + l\begin{pmatrix}0\\1\\1\\1\end{pmatrix} + m\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$0 = k \cdot 0 + l \cdot 0 + m \cdot 1 = m \cdot 1$$
$$\implies m = 0$$
$$0 = k \cdot 0 + l \cdot 1 + m \cdot 1 = l \cdot 1$$
$$\implies l = 0$$
$$0 = k \cdot 1 + l \cdot 1 + m \cdot 1 = k \cdot 1$$
$$\implies k = 0$$

$$\begin{aligned} 1 &= k \cdot 1 + l \cdot 1 + m \cdot 1 \\ &= 0 \cdot 1 + 0 \cdot 1 + 0 \cdot 1 \\ &= 0 \neq 1 \end{aligned}$$

Contradiction.

Q2

$a \neq kb$, they are linearly independent. With $c = (0, 1, 0, 0)$ and $d = (0, 0, 1, 0)$ and the vectors, then any element of $\mathbb{R}^4$ can be represented.

Q3

$$\begin{aligned}(1,0,0) &= (2,1,1)-(1,1,1)\\(0,1,0) &= (1,2,1)-(1,1,1)\\(0,0,1) &= (1,1,2)-(1,1,1)\end{aligned}$$

Every vector of $\mathbb{R}$ is a linear combination of these vectors

$$\{(1,1,1),(2,1,1),(1,2,1),(1,1,2)\}$$

Since $(1,1,1)=\frac{1}{3}[(2,1,1)+(1,2,1)+(1,1,2)]$, so $\{(2,1,1),(1,2,1),(1,1,2)\}$ is a basis of $\mathbb{R}^3$.

Q4

Any $a(x)$ is a linear combo of elements from $\{1,x,\dots,x^n\}$. Another basis is $\{k,\dots,kx^n\}$.

Q5

a.

There are three variables so the third can be calculated from the first two.

Let $x=1, y=1$, then $3-2+z=0$ or $z=-1$, so one value of S_1 is $(1,1,-1)$. Now let $x=0, y=1$, then $z=2$ or $(0,1,2)$. Both $(1,1,-1)$ and $(0,1,2)$ are linearly independent. That is for any k

$$k_1(1,1,-1)+k_2(0,1,2)\neq 0$$

$$\forall \mathbf{v}=(x,y,z)\in S_1, \exists k_1,k_2\in\mathbb{R}:\mathbf{v}=k_1(1,1,-1)+k_2(0,1,2)$$
$$\iff \begin{cases} x=k_1\\ y=k_1+k_2\\ z=-k_1+2k_2\end{cases}$$

For each choice of k_1,k_2 above, the equations always have a unique solution.

b.

$$\begin{aligned}(x+y-z)+(2x-y+z)&=0\\ \implies x&=0\\ \implies y&=z\end{aligned}$$

Basis is therefore $(0,1,1)$.

Q6

According to this answer, it is simply any basis for $\mathbb{R}^3$ such as $(0,0,1),(0,1,0),(1,0,0)$.

Q7

$$\cos 2x=\cos^2 x-\sin^2 x$$

Thus dimension of U is 2.

Since U is a subspace of $\mathcal{F}(\mathbb{R})$ thus the basis is $(\cos^2 x,\sin^2 x)$.

Q8

Seems that the given vectors are all independent and cannot be reduced, hence they are also the basis.

D. Properties of Subspaces and Bases

Q1

U is a subspace of V, then U has a basis the size of $\dim U$. Since the basis consists of vectors from V, so the basis of U must have fewer or equal elements to the basis of V.

$$\dim U \leq \dim V$$

Q2

$\dim U = \dim V \implies$ they both have basis of matching length $\implies$ they are basis for the same vector space.

Q3

$$k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + \cdots + k_n\mathbf{a}_n = 0 : k_i \neq 0 \implies k_1\mathbf{a}_1 = -(k_2\mathbf{a}_2 + \cdots + k_n\mathbf{a}_n)$$

Q4

If $\mathbf{a} \neq \mathbf{0}$, then $k\mathbf{a} = 0 \implies k = 0$.

Q5

$$\{\mathbf{a}_1, \dots, \mathbf{a}_n\}, k_1\mathbf{a}_1 + \cdots + k_n\mathbf{a}_n \neq \mathbf{0} \implies k_1\mathbf{a}_1 + \cdots + k_i\mathbf{a}_i \neq \mathbf{0}$$

because otherwise if $k_{i+1} = \cdots = k_n = 0$, then not all k in $k_1\mathbf{a}_1 + \cdots + k_n\mathbf{a}_n$ are zero yet it equals $\mathbf{0}$. So any subset of an independent set is also independent.

A set of dependent vectors still remains dependent when contained in a larger set because

$$k_1\mathbf{a}_1 + \cdots + k_n\mathbf{a}_n + 0b_1 + \cdots + 0b_n = \mathbf{0}$$

Q6

$$k(\mathbf{a} + \mathbf{b}) + l(\mathbf{b} + \mathbf{c}) + m(\mathbf{a} + \mathbf{c}) = \mathbf{0}$$
$$k\mathbf{a} + k\mathbf{b} + l\mathbf{b} + l\mathbf{c} + m\mathbf{a} + m\mathbf{c} = \mathbf{0}$$
$$(k + m)\mathbf{a} + (k + l)\mathbf{b} + (l + m)\mathbf{c} = \mathbf{0}$$
$$\implies k + m = k + l = l + m = 0$$

So $\{\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{c}\}$ is linearly independent as well.

Q7

Both have the same number of elements so we just need to show that it is linearly independent to prove it's a basis of V.

$\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis and so is linearly independent. Thus multiply the elements by k, they remain linearly independent.

Q8

V is spanned by $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ so every vector in V including $\{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ is a linear combo of $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Argument also works both ways.

E. Properties of Linear Transformations

Q1

$$\mathbf{a}, \mathbf{b} \in U : h(\mathbf{a}) = h(\mathbf{b}) = \mathbf{0} \implies \mathbf{a}, \mathbf{b} \in \ker h$$
$$\implies h(\mathbf{a}) + h(\mathbf{b}) = \mathbf{0} = h(\mathbf{a} + \mathbf{b})$$
$$\implies \mathbf{a} + \mathbf{b} \in \ker h$$

so $\ker h$ is a subspace of U.

Q2

$$k_a h(\mathbf{a}) + k_b h(\mathbf{b}) = h(k_a\mathbf{a} + k_b\mathbf{b}) \in \operatorname{ran} h$$

Q3

$\ker h = \{\mathbf{0}\} \implies h(\mathbf{a}) = \mathbf{0}$ then $a = \mathbf{0} \implies h(\mathbf{a}) = h(\mathbf{b}) = \mathbf{0}$ then $\mathbf{a} = \mathbf{b}$ and so h is injective.

Likewise if h is injective then $h(\mathbf{a}) = h(\mathbf{0}) \implies \mathbf{a} = 0$, thus $\ker h = \{\mathbf{0}\}$.

Q4

$$\mathbf{a} \in \mathcal{N} \implies \mathbf{a} = k_1\mathbf{a}_1 + \cdots + k_r\mathbf{a}_r$$
$$h(\mathbf{a}) = \mathbf{0} = k_1 h(\mathbf{a}_1) + \cdots + k_r h(\mathbf{a}_r)$$

$$b \in U \implies \mathbf{b} = k_1\mathbf{a}_1 + \cdots + k_r\mathbf{a}_r + k_{r+1}\mathbf{a}_{r+1} + \cdots + k_n\mathbf{a}_n$$

$$\begin{aligned}\implies h(\mathbf{b}) &= (k_1 h(\mathbf{a}_1) + \cdots + k_r h(\mathbf{a}_r)) + k_{r+1} h(\mathbf{a}_{r+1}) + \cdots + k_n h(\mathbf{a}_n) \\ &= \mathbf{0} + k_{r+1} h(\mathbf{a}_{r+1}) + \cdots + k_n h(\mathbf{a}_n)\end{aligned}$$

Q5

If $\{h(\mathbf{a}_{r+1}), \dots, h(\mathbf{a}_n)\}$ is linearly independent, then $k_{r+1} h(\mathbf{a}_{r+1}) + \cdots + k_n h(\mathbf{a}_n) = \mathbf{0} \implies k_{r+1} = \cdots = k_n$.

If the vector is dependent, then there is a combination of the vectors that equals $\mathbf{0}$ and so they are part of the null space.

Q6

The vectors from $r+1$ to n are linearly independent, and span $\mathcal{R}$, so they are also a basis. Since they are a basis, the number of vectors is $n - r$ and this is also the dimension of $\mathcal{R} = \operatorname{ran} h$.

Q7

Null space of h is r and $ranh$ is $n - r$, so total is n, which is the domain of h.

Q8

If h is injective, then every element of U maps to a single element of V. Thus the codomain dimension is higher or equal to the domain's. They are equal so therefore h is surjective.

Likewise if h is surjective, then every element contains a preimage in the domain. The value $\mathbf{0} \in V$ has a single preimage so the nullspace is $\{\mathbf{0}\}$ and the range of h is $n - 1$. Thus the domain dimension is n, and so the function is injective since domain and codomain are equal.

F. Isomorphism of Vector Spaces

Q1

$$k_1 h(\mathbf{a}_1) + \cdots + k_r h(\mathbf{a}_r) = \mathbf{0} = h(k_1\mathbf{a}_1 + \cdots + k_r\mathbf{a}_r)$$

since h is injective, then the null space is $\{\mathbf{0}\}$.

$$k_1\mathbf{a}_1 + \cdots + k_r\mathbf{a}_r = \mathbf{0}$$

but $\{\mathbf{a}_1, \dots, \mathbf{a}_r\}$ is linearly dependent so

$$k_1 = \cdots = k_r = 0$$

so $\{h(\mathbf{a}_1, \dots, h(\mathbf{a}_r)\}$ is linearly independent.

Q2

Looking from google the dimension of a null space which is $\{\mathbf{0}\}$ is 0 since it has no basis.

From 28E7

$$\begin{aligned}\dim U &= \dim \mathcal{N} + \dim(\operatorname{ran} h)\\ &= 0 + (r - 0)\\ &= r\end{aligned}$$

since h is injective and $\dim(\operatorname{ran} h) = r$.

Likewise if the range of h is $r = \dim U$, then the kernel of h is a single element and the quotient group has the same structure as U.

Q3

Either h maps to $\{\mathbf{0}\}$ or h is isomorphic.

If h is injective (every image of h has a single preimage) or surjective (every element of V has a preimage for h), then because $\dim U = \dim V$, then h is an isomorphism.

Q4

$$V = \{k_1\mathbf{a}_1 + \dots + k_n\mathbf{a}_n : k_i \in F\}$$

where $\{a_1, \dots, a_n\}$ is the basis of V. Which is all the possible n-dimensional vectors over F.

$$V \cong F^n$$

G. Sums of Vector Spaces

Q1

$T + U$ and $T \cap U$ are closed with respect to addition and scalar multiplication.

Let $\mathbf{a} \in T \cap U$, $k \in F$, then

$$k\mathbf{a} \in T, k\mathbf{a} \in U$$

Q2

For every $\mathbf{c} \in V, \mathbf{c} = \mathbf{a} + \mathbf{b} : \mathbf{a} \in T, \mathbf{b} \in U \implies V = T + U$.

Since $\mathbf{c}$ is uniquely expressible in terms of $\mathbf{a}$ and $\mathbf{b}$ then this means $T \cap U = \{\mathbf{0}\}$.

This works both ways. If every element of V is expressed as $T + U$ and $T \cap U = \{\mathbf{0}\}$ then every element $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

Q3

T has a basis $T = (\mathbf{t}_1, \dots, \mathbf{t}_k)$ and since T is a subspace of V, this can be extended to $V = (\mathbf{t}_1, \dots, \mathbf{t}_k, \mathbf{u}_1, \dots, \mathbf{u}_{n-k})$. It is easily seen that $(\mathbf{u}_1, \dots, \mathbf{u}_{n-k})$ forms an independent basis and so

$$\begin{aligned}\mathbf{v} &= a_1\mathbf{t}_1 + \dots + a_k\mathbf{t}_k + b_1\mathbf{u}_1 + \dots + b_{n-k}\mathbf{u}_{n-k}\\ &= (a_1\mathbf{t}_1 + \dots + a_k\mathbf{t}_k) + (b_1\mathbf{u}_1 + \dots + b_{n-k}\mathbf{u}_{n-k})\end{aligned}$$

$$\implies \mathbf{v} = \mathbf{t}' + \mathbf{u}'$$

Q4

$$
\begin{gathered}
T = T \cap U + T \cap U^c \\
U = T \cap U + U \cap T^c \\
T + U = T \cap U + T \cap U^c + U \cap T^c
\end{gathered}
$$

$$
\begin{gathered}
\dim T = \dim(T \cap U) + \dim(T \cap U^c) \\
\dim U = \dim(T \cap U) + \dim(U \cap T^c)
\end{gathered}
$$

$$
\begin{aligned}
\dim(T + U) &= \dim(T \cap U) + (\dim T - \dim(T \cap U)) + (\dim U - \dim(T \cap U)) \\
&= \dim T + \dim U - \dim(T \cap U)
\end{aligned}
$$