

# Abstract Algebra by Pinter, Chapter 30

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Chapter 30 on Ruler and Compass

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## A. Constructible Numbers

### Q1

Measure the compass radius with from points  $(0, 0)$  to  $(b, 0)$  and put the center on  $(a, 0)$ , then draw out a circle. Where the circle intersects the line  $OI$  is at the point  $(a + b, 0)$ .

Likewise this time go backwards from  $(a, 0)$  and you will get  $(a - b, 0)$ .

### Q2

Because the lines of the triangles are parallel, they are similar, and the ratios between both sides are constant.

This triangle can be constructed using parallel line constructions (see on YouTube).

$$\frac{a}{1} = \frac{x}{b} \implies x = ab$$

Since we can construct  $x$ , so  $ab \in \mathbb{D}$ .

### Q3

Apply the same argument as above but swap the labels for  $x$  and  $a$  around, obtaining the ratio  $x = a/b$ .

### Q4

From Thales theorem, the angle where  $x$  intersects the circle is a right angled triangle. Call this point  $C$ . Call the point  $(1, 0)$  as  $O$ .

Angle  $ACO$  is the same as  $CBO$ , and so also  $CAO$  and  $BCO$ . Therefore the triangles are similar and the ratio of their sides is constant.

$$\frac{x}{a} = \frac{1}{x} \implies x^2 = a$$

Since  $x \in \mathbb{D}$  and  $x = \sqrt{a}$ , so  $\sqrt{a} \in \mathbb{D}$ .

### Q5

$\mathbb{D}$  contains all the integers and ratios between those integers which is the definition of  $\mathbb{Q}$ .

### Q6

Take a monic polynomial and

$$(x + \frac{a_1}{2})^2 - \frac{a_1^2}{4} + a_0 = 0$$

$$x = \sqrt{\frac{a_1^2}{4} - a_0} - \frac{a_1}{2} \in \mathbb{D}$$

## B. Constructible Points and Constructible Numbers

### Q1

Use parallel line construction (see youtube for howto) to construct parallel lines crossing  $a$  and  $b$ . Where the lines intersect is  $(a, b)$ .

See the answer for Q3 mistakenly which is actually for this question.

## **Q2**

$P$  is constructible from  $\{O, I\} \subseteq \mathbb{Q} \times \mathbb{Q}$ .

## **Q3**

Just with  $\{O, I\}$  we can construct the entire set of integers along the x axis.

We can also split segments in half using a compass and so move along the y axis.

From part A, we then have all the field operations, and since  $\mathbb{Q} \subseteq \mathbb{D}$ , so we have the entire space of  $\mathbb{Q} \times \mathbb{Q}$ .

## **Q4**

Since  $\mathbb{Q} \times \mathbb{Q}$  is constructible from  $\{O, I\}$ , so therefore any point constructible from  $\mathbb{Q} \times \mathbb{Q}$  would be an extension of  $\{O, I\}$ .

## **Q5**

$P$  is constructible and so are its coordinates.

## **C. Constructible Angles**

### **Q1**

$$\sin \alpha = \frac{AC}{AB}$$

If this wasn't constructible then neither would be the angle. Same for  $\cos \alpha$ .

### **Q2**

From pythagoras theorem, the sides are all related by operations that exist in the field, and so both operations must be constructible.

### **Q3**

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

Since  $\cos x$  being constructible implies  $\sin x$  is constructible, so the right hand sides above are constructible.

### **Q4**

$\cos(\alpha + \alpha)$  from above is constructible.

### **Q5**

From above, we have  $\cos \alpha = \cos(\frac{\alpha}{2} + \frac{\alpha}{2})$ , and likewise for  $n\alpha$ . All are constructible.

### **Q6**

For 30 degrees, use an equilateral triangle. All angles will be 60 degrees. Split it in half forming 2 right angled triangles. One angle will be 30 degrees, therefore it is constructible.

For 75 degrees, construct a half circle with a segment of 30 degrees. The remaining angle is 150 degrees. Split that in half.

22.5 degrees is half of 45 degrees which is formed by the unit right angled triangle.

## Q7

20 degrees is not constructible by theorem 3 on trisecting an angle. The polynomial has degree 3, so the extension field of the trisected element  $c = \cos 20$  would have degree  $[F(c) : F] = 2$  but all constructible extensions are a power of 2. So it is impossible.

Likewise 40 degrees because double 20 degrees is also not constructible. If it was then it would also mean 20 degrees is constructible.

Lastly 140 degrees is not constructible either using the half circle. If it was then it would mean 40 degrees would be constructible.

## D. Constructible Polygons

### Q1

Each vertex angle in an ngon is  $\pi - 2\pi/n$ . Since  $2\pi/n$  is constructible, so is  $\pi - 2\pi/n$ , and therefore the vertices for an ngon can be formed with line segments connecting them.

### Q2

A regular hexagon has 6 vertices each being 120 degrees. 120 degrees being a multiple of 30 degrees is constructible.

### Q3

A 9gon has vertices 140 degrees which is not constructible by 30C7.

## E. A Constructible Polygon

### Q1

From trigonmetry we know  $\cos^2 x + \sin^2 x = 1$  is always true.

$$r \cdot \frac{1}{r} = (\cos k + i \sin k)z = \cos^2 k + \sin^2 k = 1 \implies \frac{1}{r} = \cos k - i \sin k$$

### Q2

Note that  $\omega^5 = 1$  and since  $p(\omega) = 0$  then also  $(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$  because  $(x - 1) \neq 0$  and from the cancellation property.

Multiply both sides by  $\omega^{-2}$  and you get the result.

### Q3

```
sage: # x = cos k, y = sin k
sage: x, y = var("x y")
sage: r = x + I*y
sage: # s = 1 / r = cos k - i sin k
sage: s = x - I*y
sage: # Equation from Q2 above
sage: r^2 + r + 1 + s + s^2
(x + I*y)^2 + (x - I*y)^2 + 2*x + 1
sage: (r^2 + r + 1 + s + s^2).simplify_full()
2*x^2 - 2*y^2 + 2*x + 1
```

And we know that  $y^2 = 1 - x^2$ , so  $2x^2 - 2y^2 + 2x + 1 = 2x^2 - 2(1 - x^2) + 2x + 1 = 4x^2 + 2x - 1$ .

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 4 \cos^2 \frac{2\pi}{5} + 2 \frac{2\pi}{5} - 1 = 0$$

#### **Q4**

From 30A6, any quadratic root is constructible, so therefore  $\frac{2\pi}{5}$  being a quadratic root is constructible.

#### **Q5**

From 30D1, an n-gon is constructible iff the angle  $\frac{2\pi}{n}$  is constructible. Since  $\frac{2\pi}{5}$  is constructible, that means a pentagon is constructible using a compass and straight edge.

## **F. A Nonconstructible Polygon**

#### **Q1**

Multiply both sides by  $\omega^{-3}$

#### **Q2**

```
sage: (r^3 + r^2 + r + 1 + s + s^2 + s^3).simplify_full()
2*x^3 - 2*(3*x + 1)*y^2 + 2*x^2 + 2*x + 1
sage: # now y^2 = 1 - x^2
sage: # copy past above expr in but substitute
sage: (2*x^3 - 2*(3*x + 1)*(1 - x^2) + 2*x^2 + 2*x + 1).simplify_full()
8*x^3 + 4*x^2 - 4*x - 1
```

#### **Q3**

From chapter 22 page 263, the rational root of  $p(x)$  must be a fraction  $s/t$  where  $s \mid -1$  and  $t \mid 8$ .

Possible values for  $s = \{-1, 1\}$  and  $t = \{1, 2, 4, 8\}$ .

```
sage: p = lambda x: 8*x^3 + 4*x^2 - 4*x - 1
sage: p(1/1), p(1/2), p(1/4), p(1/8)
(7, -1, -13/8, -91/64)
sage: p(-1/1), p(-1/2), p(-1/4), p(-1/8)
(-1, 1, 1/8, -29/64)
```

Therefore no rational root and irreducible.

#### **Q4**

Since there is no rational root, we must rely on the polynomial being a power of 2 extension of the rational field. But it is a cubic polynomial and so its degree is 3 which makes it impossible to be constructible.

#### **Q5**

From the answer above because  $\cos 2\pi/7$  is not constructible, that means the angle isn't constructible by definition.

#### **Q6**

So therefore a 7 sided polygon cannot be constructed.

## **G. Further Properties of Constructible Numbers and Figures**

#### **Q1**

Constructible elements must form an extension field that's degree over the base field is a power of 2. You cannot express elements which are algebraic over a field using a non power of 2 degree polynomial.

#### **Q2**

See section A.

**Q3**

See section A.

**Q4**

Degree of polynomial is a power of 2.

**Q5**

$y = (-c - ax)/b$  is a degree 1 polynomial which is in  $\mathbb{D}$ .

**Q6**

The equation uses operations defined in part A and so is constructible.