Abstract Algebra by Pinter, Chapter 33

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Chapter 33 on Solving Equations By Radicals

Contents

A. Finding Radical Extensions

Q1

a

$$\begin{aligned} L_1 &= \mathbb{Q}(\alpha_0), & \alpha_0^2 &= 5 \in \mathbb{Q} \\ L_2 &= L_1(\alpha_1), & \alpha_1^5 &= 2 \in L_1 \\ L_3 &= L_2(\alpha_2), & \alpha_2^4 &= 3 \in L_2 \\ L_4 &= L_3(\alpha_3), & \alpha_3^3 &= 4 \in L_3 \end{aligned}$$

b

$$\begin{aligned} L_1 &= \mathbb{Q}(\alpha_0), & \alpha_0^9 &= 2 \in \mathbb{Q} \\ L_2 &= L_1(\alpha_1), & \alpha_1^2 &= 5 \in L_1 \\ L_3 &= L_2(\alpha_2), & \alpha_2^3 &= 1 - \sqrt{5} \in L_2 \\ L_4 &= L_3(\alpha_3), & \alpha_3^2 &= \frac{1 - \alpha_0}{\alpha_2} \in L_3 \end{aligned}$$

c

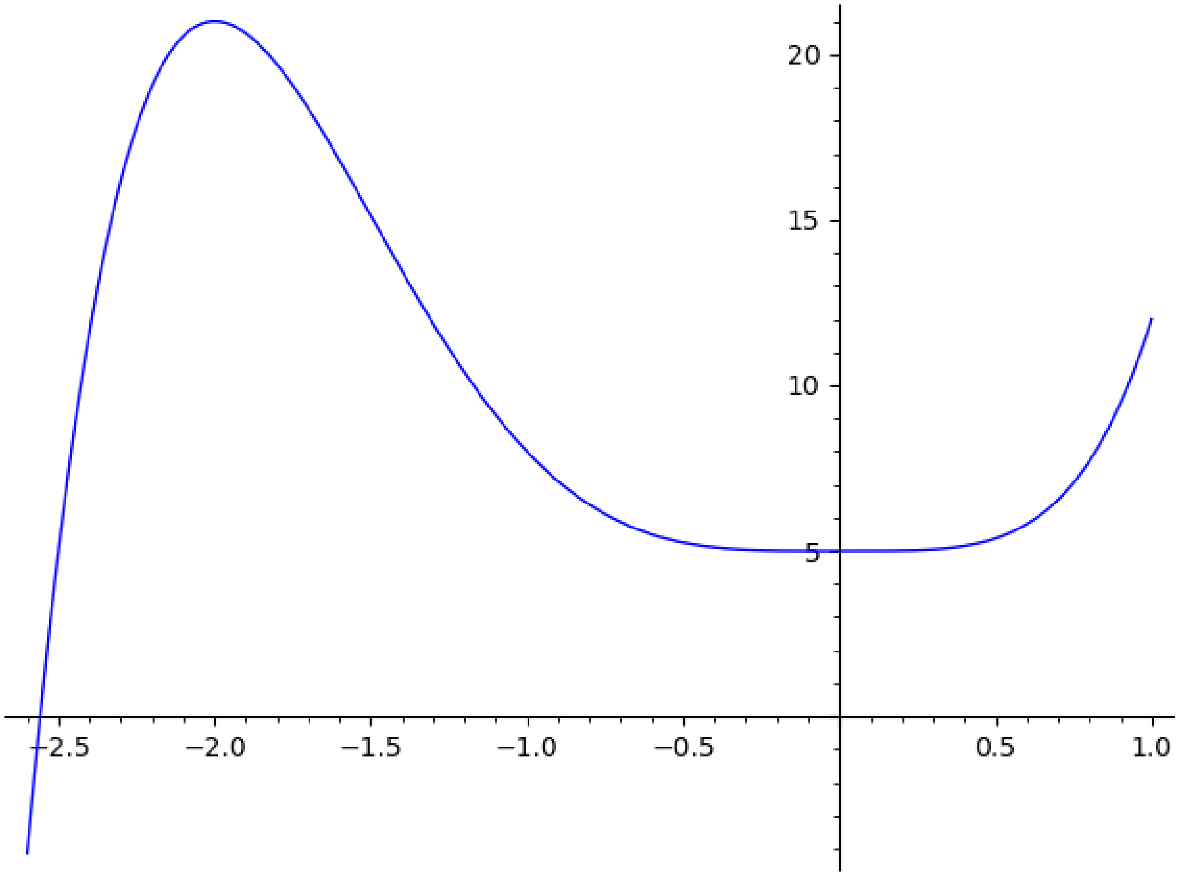
$$\begin{aligned} L_1 &= \mathbb{Q}(i), & i^2 &= 1 \in \mathbb{Q} \\ L_2 &= L_1(\alpha_1), & \alpha_1^2 &= 3 \in L_1 \\ L_3 &= L_2(\alpha_2), & \alpha_2^2 &= 11 \in L_2 \\ L_4 &= L_3(\alpha_3), & \alpha_3^5 &= \frac{(\alpha_1 - 2i)^3}{i - \alpha_2} \in L_3 \end{aligned}$$

Q2

a

The polynomial $2x^5 - 5x^4 + 5$ is irreducible by Eisenstein's criteria using divisor 5 given that $5 \mid a_i : i \neq 5$, $5 \nmid a_5$, and $5^2 \nmid a_0$.

```
sage: plot(2 * x ^ 5 + 5 * x ^ 4 + 5, (-2.6, 1))
```



Point of inflection at $(0, 5)$.

```
sage: diff(2 * x ^ 5 + 5 * x ^ 4 + 5)
10*x^4 + 20*x^3
sage: solve(_, x)
[x == -2, x == 0]
sage: a = 2 * x ^ 5 + 5 * x ^ 4 + 5
sage: a(x=-2), a(x=0)
(21, 5)
```

Minimum is $(0, 5)$ and maximum is $(-2, 21)$.

$a(x)$ thus only crosses the x-axis once and has one real root r_1, and four complex roots r_2, r_3, r_4, r_5 with r_2, r_3 and r_4, r_5 being complex conjugates of each other.

The permutation group of r_1, r_2, r_3, r_4, r_5 which is $\mathbf{G}$ is a subgroup of S_5.

Since $a(x)$ is irreducible in $\mathbb{Q}$, so $[\mathbb{Q}(r_1) : \mathbb{Q}] = 5$. So 5 is a factor of $[K : \mathbb{Q}]$. Thus $\mathbf{G}$ contains an element of order 5.

The automorphism $(r_1 r_2 r_3 r_4)$ has order 4, $(r_1 r_2 r_3)$ has order 3, $(r_1 r_2 r_3)(r_4 r_5)$ order 6, which only leaves the cycle of length 5 $(r_1 r_2 r_3 r_4 r_5)$ which has order 5. Thus $\mathbf{G}$ contains an automorphism which is a cycle permutation of the roots r_1, r_2, r_3, r_4, r_5 of length 5.

From 8H5, we saw that a transposition (12) and a cycle (12345) will generate S_5. The proof follows: $(12345)(12)(12345)^{-1} = (23), (12)(23)(12) = (13)$. Repeating the process we get $(12345)(13)(12345)^{-1} = (24), (12)(24)(12) = (14), (12345)(14)(12345)^{-1} = (25), (12)(25)(12) = (15)$. Finally the set $T_1 = \{(12), (13), \dots, (15)\}$ generates S_5.

Thus $\mathbf{G} = S_5$ and since S_5 is not solvable, there is no radical solution for $a(x)$.

b

```
sage: a = x ^ 5 - 4 * x ^ 2 + 2
sage: diff(a)
5*x^4 - 8*x
```

```
sage: solve(_, x)
[x == 1/5*5^(2/3)*(I*sqrt(3) - 1), x == 1/5*5^(2/3)*(-I*sqrt(3) - 1), x == 2/5*5^(2/3), x == 0]
sage: plot(a, (-1, 2))
```

The graph has a maximum at $x = 0$, and a minimum at $x = \frac{2}{5}\sqrt[3]{5}^2$. $a(x)$ crosses the x-axis 3 times and so has two imaginary roots.

By the complex conjugate root theorem, the complex roots of $a(x)$ are conjugate pairs. Therefore $a(x)$ has three real roots, and two imaginary roots. By the reasoning in the question above, there is a cycle of length 5 and a transposition between the imaginary roots.

Thus the group for $a(x)$ is S_5 which is unsolvable implying there is no radical solution.

c

```
sage: a = x ^ 5 - 4 * x ^ 4 + 2 * x + 2
sage: ad = diff(a)
sage: sols = solve(ad, x)
sage: for s in sols:
....:     print(s.rhs().n())
-0.259418669419159 - 0.411017935127584*I
-0.259418669419159 + 0.411017935127584*I
0.531186796300305
3.18765054253801
sage: plot(a, (-1, 4))
Launched png viewer for Graphics object consisting of 1 graphics primitive
sage: (sols[0].rhs() * sols[1].rhs()).n()
0.236233789039750 - 4.16333634234434e-17*I
sage: (sols[0].rhs() * sols[1].rhs()).n().imag_part() < 0.000000000001
True
sage: # rounding error, so ignore that part
sage: # 3 roots:
sage: 0.236233789039750, 0.531186796300305, 3.18765054253801  # for max and mins
(0.236233789039750, 0.531186796300305, 3.18765054253801)
```

We can see from the differentiated curve, that $a(x)$ is decreasing below $x = -1$ and increasing above $x = 4$. Thus it crosses the x-axis three times, and so has three real roots, and two imaginary roots.

By the argument before this implies the group for this curve is S_5 which is unsolvable.

Q3

$$a(x) = (x-2)^5 - (x-2)$$

Let $a(x) = 0$ then

$$(x-2)^4 = 1$$

The fourth roots of 1 are $\pm i, \pm 1$. The remaining root of $a(x)$ is $x = 2$. All these roots are real and solvable.

Q4

Substituting $y = x^2$, we get $a(x) = ay^4 + by^3 + cy^2 + dy + e$ which is easily solvable. Any solution is then solvable for x since $x = \pm\sqrt{y}$ which is itself a solvable equation.

Q5

There is no general solution for polynomials of degree 5, but there are polynomials of degree 5 which have a solvable group.

B. Solvable Groups

Q1

Every subgroup of an abelian group is a normal subgroup.

Let G be an abelian group with $x \in G$, and H a subgroup with $a \in H$. Since $xax^{-1} = axx^{-1} = a \in H$, H is a normal subgroup of G.

The set of commutators for an abelian group is $\{e\} \implies Hxyx^{-1}y^{-1} = Hxy(yx)^{-1} = H \implies Hxy = Hyx \implies G/H$ is abelian.

From these two derivable properties of an abelian group, we see that every abelian group is also a solvable group.

Q2

The intersection of two subgroups of G is a subgroup of G. For example $e \in J_0 = K \cap H_0$. For any $a \in J_0$, both K and H_0 contain $a^{-1} \implies a^{-1} \in J_0$, likewise for products $a, b \in J_0 \implies ab \in J_0$.

All the iterated groups J_i are subgroups of K, with $J_i \lhd J_{i+1}$. Observe that J_{i+1} is a subgroup of H_{i+1}. Let $x \in J_{i+1}, a \in J_i$ then $xax^{-1} \in K \cap H_i = J_i$. Thus J_i is a normal subgroup of J_{i+1}.

Thus the sequence $J_0, \dots, J_n$ is a normal series of K.

Q3

H_{i+1}/H_i is abelian $\implies H_i$ contains all the commutators $xyx^{-1}y^{-1} \in H_{i+1}$. Let $x, y \in J_{i+1}$, then $xyx^{-1}y^{-1} \in J_{i+1}$ and also K. Observe $xyx^{-1}y^{-1} \in H_i \cap K = J_i \implies J_{i+1}/J_i$ is abelian. Thus the series $\{e\} = J_0 \lhd J_1 \lhd \cdots \lhd J_n = K$ is a solvable series of K.

Q4

Combining the above two parts, we see that given a solvable group, any subgroup $K \subseteq G$ is also a solvable group.

Q5

S_3, the dihedral group of order 6 has six elements generated by $\langle a, b \rangle = \{e, \alpha = a, \beta = b, \delta = aba, \kappa = ab, \gamma = ba\}$. The subgroup $\{e, \beta = b, \delta = aba\}$

$$\alpha^{-1} = \alpha \quad \beta^{-1} = \delta \quad \gamma^{-1} = \gamma$$
$$\delta^{-1} = \beta \quad \kappa^{-1} = \kappa$$

$$\begin{aligned}
\alpha\beta\alpha^{-1} &= \alpha\beta\alpha = \alpha\kappa = \delta \\
\kappa\beta\kappa^{-1} &= \kappa\beta\kappa = \kappa\gamma = \delta \\
\gamma\beta\gamma^{-1} &= \gamma\beta\gamma = \gamma\alpha = \delta \\
\alpha\delta\alpha^{-1} &= \alpha\delta\alpha = \alpha\gamma = \beta \\
\kappa\delta\kappa^{-1} &= \kappa\delta\kappa = \kappa\alpha = \beta \\
\gamma\delta\gamma^{-1} &= \gamma\delta\gamma = \gamma\kappa = \beta
\end{aligned}$$

So $\{\epsilon, \beta, \delta\}$ absorbs products from S_3 and is a normal subgroup. Since S_3 has a solvable series, we conclude S_3 is a solvable group, and by part 4, that every subgroup is also solvable.

Q6

$$B = \{e, (12)(34), (13)(24), (14)(23)\}$$
$$A_4 = \{e, (12)(34), (13)(24), (14)(23), (13)(12), (12)(13), (14)(13), (13)(14), (14)(12), (12)(14), (24)(23), (23)(24)\}$$
$$[S_4 : A_4] = 2$$

Every index 2 subgroup is abelian.

B is the Klein subgroup of A_4 which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is an abelian normal subgroup. Therefore also A_4/B is abelian.

C. pth Roots of Elements in a Field

Q1

All roots of $x^p - a$ are of the form $d = \omega^k \sqrt[p]{a} : k \leq n - 1$. Since $\omega^{-k} \in F(\omega)$, $\sqrt[p]{a} \in F(\omega, d)$. See 31E5 and 31E6.

Q2

The question has a typo as explained here and should say "degree ≥ 2".

$x^p - a$ is reducible in $F[x]$ by the question, so $x^p - a = p(x)f(x) = (x - z_1)(x - z_2) \cdots (x - z_p)$.

Since $x^p - a$ reduces to factors $p(x)$ and $f(x)$, and $z_i \notin F$, we conclude that $\deg p(x) \geq 2$. Therefore $p(x) = (x - z_1)(x - z_2) \cdots (x - z_m)$ for some m and $b = z_1 z_2 \cdots z_m \in F$. Likewise for $f(x)$.

Q3

From above $p(x)$ splits into linear terms of $p(x) = (x - z_1) \cdots (x - z_m)$ with a constant term $b = z_1 \cdots z_m$.

Since the roots of $x^p - a$ are of the form $\omega^j \sqrt[p]{a}$ so $b = (\omega^j \sqrt[p]{a})^m = \omega^{jm} \sqrt[p]{a}^m$. But $d = \omega^i \sqrt[p]{a}$ or $\sqrt[p]{a} = \omega^{-i} d \implies b = \omega^{jm}(\omega^{-i} d)^m = \omega^k d^m$ for some k.

Q4

$$b^p = (\omega^k d^m)^p = (\omega^p)^k (d^p)^m = a^m$$

Q5

$$\deg p(x) = m \geq 2, \deg f(x) = p - m \geq 2 \implies m \neq p$$

p is prime $\implies m \nmid p \implies \exists s, t : sm + tp = 1$.

Q6

$$b^p = a^m \implies b^{sp} = (b^p)^s = (a^m)^s = a^{sm}$$

But $sm + tp = 1$

$$\begin{aligned} a^{sm} &= a^{1-tp} = a \cdot a^{-tp} \\ &\implies b^{sp} a^{tp} = a \\ &\implies (b^s a^t)^p = a \end{aligned}$$

Q7

$c = b^s a^t$ is a solution for the equation $x^p - a$, so when $x^p - a$ is reducible it has a root in the field F. Since $p(x) \in F[x]$ which means its constant term $b \in F$.

Otherwise we conclude that F is irreducible over F.

D. Another Way of Defining Solvable Groups

Q1

By the definition, a subgroup is always contained in a maximal subgroup (if it's not maximal itself).

Because every finite group is a finite set, every chain of proper subgroups of a finite group has a maximal element and thus every finite group has a maximal subgroup. The same applies to maximal normal subgroups.

Q2

$J \lhd H$ and $\operatorname{ran} f = H$, so there exists a set X of input values such that $f(X) = J$. Then $X = f^{-1}(J)$. For $a, b \in X, f(ab) = f(a)f(b) \in J$ which preserves group structure. Also $e_G \in J \implies e_H \in X$. And for the normal property let $g \in G$ then $f(g)f(a)(f(g))^{-1} = f(gag^{-1}) \in J \implies gag^{-1} \in X$.

Q3

Let $f : G \to G/K$ by $f(a) = Ka$. $\mathcal{F} \lhd G/K \implies f^{-1}(\mathcal{F}) \lhd G$. But $f^{-1}(\mathcal{F}) = \hat{\mathcal{F}}$.

Q4

This question is equivalent to proving the quotient group G/K of a maximal normal subgroup K is simple.

Let H be a normal subgroup in G/K. Then $\hat{H}$ is the union of cosets in H. Then $K \lhd \hat{H} \lhd G \implies \hat{H} = G$.

Q5

Let $|G| = n = p_1 \cdots p_k$. Then for each p_i there is an element $a \in G : \text{ord}(a) = p_i \implies \langle a \rangle \subseteq G$. Thus there are k subgroups in G of order p_i.

G has only the trivial subgroups $\{e\}$ and $G \implies |G| = p$ for some prime p with an element $a : \text{ord}(a) = p$. Therefore $G = \langle p \rangle$.

Q6

$$f : G/H \to G/K$$
$$f(Ha) = Ka$$

$$f(H) = K$$
$$f(Hab) = Kab = (Ka)(Kb) = f(a)f(b)$$

Q7

$$|G/H| = 1 \implies H = G$$

$$|G/H| = 2 \implies H = H_0 \lhd H_q = G$$

and H_{i+1}/H_i is cyclic of order 2.

Now let $|G/H| = n$ and assume the statement is true for groups $|G/H| < n$.

If there is no subgroup J between H and G such that $H \subseteq J \subseteq G$, then H is a maximal normal subgroup of G. By part 4 above G/H only contains trivial subgroups. By part 5, since the group is trivial it can only contain generators of the group which are prime order by Cauchy's theorem, and therefore G/H is a cyclic group of prime order.

Lastly we deal with the case that H is not a maximal normal subgroup where

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_{q-1} = J \lhd H_q = G$$

Let $|H| = k$, then since $|G/H| = n$, $|G| = nk$.

$$H_0 \subseteq H_{q-1} \implies |H_{q-1}| > k \implies |H_q/H_{q-1}| < n$$

Therefore by our inductive assumption, H_q/H_{q+1} is cyclic of prime order.

Likewise $H_1 \subseteq G \implies |H_1| < nk \implies |H_1/H_0| < n$. This can be generalized to

$$|H_i/H_0| < n \text{ for } i > 0$$

And $H_0 \subseteq H_i \implies |H_i| > k$
$$\implies |G/H_i| < n$$
$$\implies |H_{i+1}/H_i| < n \text{ (by combining both statements)}$$

Which by our inductive assumption means that H_{i+1}/H_i is a cyclic group of prime order.

E. If $\text{Gal}(K : F)$ Is Solvable, K is a Radical Extension of F

Q1

$$K = F_0$$
$$H_q = \mathbf{G}$$
$$H_0 = \text{Gal}(K : K) = \{e\}$$
$$H_i = \text{Gal}(K : F_i)$$
$$H_{i+1} = \text{Gal}(K : F_{i+1})$$
$$H_i \subseteq H_{i+1} \iff F_{i+1} \subseteq F_i$$

The lemma after theorem 2 in chapter 32 states that the number of elements in H_i is equal to $[K : F_i]$.

$$|H_i| = [K : F_i]$$

$$\begin{aligned} |H_{i+1}| &= [K : F_{i+1}] \\ &= [K : F_i][F_i : F_{i+1}] \end{aligned}$$

$$\frac{|H_{i+1}|}{|H_i|} = |H_{i+1}/H_i| = p$$

To show F_i forms an iterated normal extension, first we let $F_i = F_{i+1}(c)$. Assume F_{i+1} is a normal extension of F_0. $c^n = a \in F_{i+1}$.

$$H_{i+1} = \{h_1, \dots, h_r\}$$
$$b(x) = [x^n - h_1(a)] \cdots [x^n - h_r(a)]$$
$$\bar{h_i} : K[x] \to K[x]$$
$$h_i(\bar{b}(x)) = b(x) \implies b(x) \in F_{i+1}[x]$$

F_i is the splitting field of $b(x)$ over $F_{i+1} \implies F_i$ is a normal extension of F_{i+1}.

Q2

$$\omega \in F_{i+1} \implies \pi(\omega) = \omega$$
$$\pi(c) = \pi(b) + \omega b + \omega^2 \pi^{-1}(b) + \cdots + \omega^{p-1} \pi^{-(p-2)}(b)$$
$$[F_i : F_{i+1}] = p \implies \langle \pi \rangle = p$$
$$\omega c = \omega b + \omega^2 \pi^{-1}(b) + \cdots + \omega^{p-1} \pi^{-(p-2)}(b) + \omega^p \pi^{-(p-1)}(b)$$
$$\omega^p = 1, \pi^{-p} = \pi^0$$
$$\implies \omega c = \omega b + \omega^2 \pi^{-1}(b) + \cdots + \omega^{p-1} \pi^{-(p-1)}(b) + \pi(b)$$
$$\pi(c) = \omega c$$

Q3

$$\pi(c)\pi(c) = \omega^2 c^2$$
$$\implies \pi^k(c) = \omega^k c^k$$

$$\begin{aligned} \pi^p(c) &= \omega^p c^p \\ &= c^p \end{aligned}$$

But $\text{Gal}(F_i : F_{i+1}) = \langle \pi \rangle$ so $\pi^p = e$ so $\pi^p(c) = c$

$$c^p = c$$
$$\pi^k(c) = \pi^k(c^p) = c^p$$

since $\text{Gal}(F_i : F_{i+1}) = \langle \pi \rangle$, and $\pi^k(c^p) = c^p$, all automorphisms fix c^p, and so F_{i+1} is the fixfield of $\text{Gal}(F_i : F_{i+1})$.

Q4

There are p automorphisms $\pi^i \in \langle \pi \rangle$ which permute any root of $x^p - c^p$ to another unique root.

$x^p - c^p \in F_{i+1}[x]$, and we know at least one root $b \in F_i$ so all roots are in F_i.

Q5

$F_q = F$ and $F_0 = K$, such that $F_q \subseteq \cdots \subseteq F_0$ such that each F_i contains all roots of $x^p - c^p$ where $[F_i : F_{i+1}] = p$. Thus each extension is radical over the previous one.

We conclude that K is a radical extension of F.