Contents

$\dim \mathrm{col}(A) = \dim \mathrm{row}(A)$

Let $A = (\mathbf{c_1} \cdots \mathbf{c_n})$ with basis for column space $\{\mathbf{v_1}, ..., \mathbf{v_k}\}$, then

$$\mathbf{c_i} = \gamma_{1i}\mathbf{v_1} + \cdots + \gamma_{ki}\mathbf{v_k}$$

$$B := (\mathbf{v_1} \cdots \mathbf{v_n}) \in \mathbb{F}^{m \times k}$$

$$C := (\gamma_{ij}) = \begin{pmatrix} \gamma_{1i} \\ \dots \vdots \dots \\ \gamma_{ki} \end{pmatrix} \in \mathbb{F}^{k \times n}$$

$$\Rightarrow A = BC$$

so A is a linear combo of rows of $C \Rightarrow \dim \mathrm{row}(A) \leq \dim \mathrm{row}(C) = k = \dim \mathrm{col}(A)$.

Now applying the same argument to A^T we see that $\dim \mathrm{col}(A) \leq \dim \mathrm{row}(A) \Rightarrow \dim \mathrm{col}(A) = \dim \mathrm{row}(A)$. ∎

a and b are orthogonal $\Leftrightarrow |a+b|^2 = |a|^2 + |b|^2$

$$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 &= \langle \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b} \rangle = \langle \mathbf{a},\mathbf{a} \rangle + 2\langle \mathbf{a},\mathbf{b} \rangle + \langle \mathbf{b},\mathbf{b} \rangle \\ &= |\mathbf{a}|^2 + 2\langle \mathbf{a},\mathbf{b} \rangle + |\mathbf{b}|^2 \end{aligned}$$

but note that $\langle \mathbf{a},\mathbf{b} \rangle = 0$ since $\mathbf{a},\mathbf{b}$ are orthogonal.

Orthogonal Decomposition

There exists a unique λ such that $\mathbf{a} = \lambda\mathbf{b} + \mathbf{c}$ and $\langle \mathbf{b},\mathbf{c} \rangle = 0$.

Write $\mathbf{c} = \mathbf{a} - \lambda\mathbf{b}$, then

$$\langle \mathbf{b},\mathbf{c} \rangle = 0 \iff \langle \mathbf{b},\mathbf{a} \rangle - \lambda\langle \mathbf{b},\mathbf{b} \rangle = 0$$

$$\iff \lambda = \frac{\langle \mathbf{b},\mathbf{a} \rangle}{\langle \mathbf{b},\mathbf{b} \rangle}$$

Cachy-Schwarz Inequality

Since $\langle \lambda\mathbf{b}, \mathbf{c}\rangle = 0$, then $|\lambda\mathbf{b} + \mathbf{c}|^2 = \lambda^2|\mathbf{b}|^2 + |\mathbf{c}|^2$. So $|\mathbf{c}|^2 = |\mathbf{a}|^2 - \lambda^2|\mathbf{b}|^2$.

But $|\mathbf{c}|^2 \geq 0 \Rightarrow$

$$\begin{aligned}
|\mathbf{c}|^2 &= |\mathbf{a}|^2 - \lambda^2|\mathbf{b}|^2 \\
&= \langle \mathbf{a}, \mathbf{a}\rangle - \frac{\langle \mathbf{a}, \mathbf{b}\rangle^2}{\langle \mathbf{b}, \mathbf{b}\rangle^2}\langle \mathbf{b}, \mathbf{b}\rangle \\
&= |\mathbf{a}|^2 - \frac{\langle \mathbf{a}, \mathbf{b}\rangle^2}{|\mathbf{b}|^2} \geq 0
\end{aligned}$$

Multiplying both sides, rearranging and taking roots,

$$|\langle \mathbf{a}, \mathbf{b}\rangle| \leq |\mathbf{a}||\mathbf{b}|$$

From this we get that $-1 \leq \frac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}||\mathbf{b}|} \leq 1$, hence there exists a unique $\theta \in [0, \pi]$ such that

$$\langle \mathbf{a}, \mathbf{b}\rangle = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Fourier Coefficients

Let $\mathbf{u}_1, ..., \mathbf{u}_n$ be an orthonormal basis of V. Then if $w \in V$

$$\mathbf{w} = \sum_{i=1}^{n} \langle \mathbf{w}, \mathbf{u}_i\rangle \mathbf{u}_i$$

Proof. Let $\mathbf{w} = \sum_{i=1}^{n} x_i \mathbf{u}_i$. Then $\langle \mathbf{w}, \mathbf{u}_j\rangle = \sum_{i=1}^{n} x_i \langle u_i, u_j\rangle = x_j$ since the $\mathbf{u}_i$ are orthonormal.

Orthogonal Complement of a Subspace

$$U^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u}\rangle = 0 \ \ \forall \mathbf{u} \in U\}$$

Furthermore if $W = \operatorname{span} U$, then $W \cap U^{\perp} = \{\mathbf{0}\}$.

We can visualize $W^{\perp}$ in matrix terms. Let $A \in K^{m \times n} : \operatorname{col}(A) = \operatorname{span} U = W$ where $A = (\mathbf{u}_1 \cdots \mathbf{u}_n)$.

Now we are interested in $\mathbf{v} \in U^{\perp} : \langle \mathbf{v}, \mathbf{u}_i\rangle = 0$ for all i. This is equivalent to $A^T\mathbf{v} = (\langle \mathbf{u}_1, \mathbf{v}\rangle \cdots \langle \mathbf{u}_n, \mathbf{v}\rangle)^T = \mathbf{0}$.

$$\Rightarrow W^{\perp} = \mathcal{N}(A^T)$$

We also have $\dim \mathcal{N}(A^T) = \dim \mathcal{N}(A)$ and $\dim \operatorname{row}(A^T) = \dim \operatorname{col}(A)$ so

$$\dim W + \dim W^{\perp} = m$$

$V = W \oplus W^{\perp}$

Let $\mathbf{v} \in V$ and $\mathbf{u}_1, ..., \mathbf{u}_k$ be an orthonormal basis of W. Then we can put $\mathbf{y} = \mathbf{v} - \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{u}_i\rangle \mathbf{u}_i$. Since the $\mathbf{u}_i$ are orthonormal, we have $\langle \mathbf{y}, \mathbf{u}_i\rangle$ and so $\mathbf{y} \in W^{\perp}$.

Thus $V = W + W^{\perp}$ and since $W \cap W^{\perp} = \{\mathbf{0}\}$, we get $V = W \oplus W^{\perp}$. ■

Gram-Schmidt Method

This is an algorithm for producing orthonormal basis from a general basis. Let $\mathbf{v}_1, ..., \mathbf{v}_k$ be a basis.

1. Normalize $\mathbf{v}_1$ by putting $\mathbf{u}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$.
2. Remove the projection of $\mathbf{u}_1$ from $\mathbf{v}_2$ as follows:
   1. Set $\mathbf{v}_2' = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1\rangle \mathbf{u}_1$.
   2. Normalize by letting $\mathbf{u}_2 = \frac{\mathbf{v}_2'}{|\mathbf{v}_2'|}$.
3. Now repeat the process:
   1. $\mathbf{v}_3' = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{u}_1\rangle \mathbf{u}_1 - \langle \mathbf{v}_3, \mathbf{u}_2\rangle \mathbf{u}_2$.
   2. $\mathbf{u}_3 = \frac{\mathbf{v}_3'}{|\mathbf{v}_3'|}$.
4. And so on.

$d(\mathbf{a},\mathbf{c}) \leq d(\mathbf{a},\mathbf{b}) + d(\mathbf{b},\mathbf{c})$

Denote the sides of a triangle by $\mathbf{a} = \mathbf{b} + \mathbf{c}$.

$$\begin{aligned}
\langle \mathbf{a}, \mathbf{a} \rangle &= \langle \mathbf{b}, \mathbf{b} \rangle + 2\langle \mathbf{b}, \mathbf{c} \rangle + \langle \mathbf{c}, \mathbf{c} \rangle \\
&\leq \langle \mathbf{b}, \mathbf{b} \rangle + 2|\langle \mathbf{b}, \mathbf{c} \rangle| + \langle \mathbf{c}, \mathbf{c} \rangle \\
&\leq \langle \mathbf{b}, \mathbf{b} \rangle + 2|\mathbf{b}||\mathbf{c}| + \langle \mathbf{c}, \mathbf{c} \rangle \\
&= |\mathbf{b}| + 2|\mathbf{b}||\mathbf{c}| + |\mathbf{c}| \\
&= (|\mathbf{b}| + |\mathbf{c}|)^2
\end{aligned}$$

using the Cauchy-Schwarz inequality.

Least Squares Principle

Let $\mathbf{v} = \mathbf{w} + \mathbf{y}$, with $\mathbf{w} \in W$ and $\mathbf{y} \in W^{\perp}$. Then $d(\mathbf{v}, W) = |y|$.

Let $\mathbf{w}' = \mathbf{w} + \mathbf{m}$, then $\mathbf{v} - \mathbf{w}' = \mathbf{y} + \mathbf{m}$

$$d(\mathbf{v}, \mathbf{w}')^2 = |\mathbf{y} + \mathbf{m}|^2 = |\mathbf{y}|^2 + |\mathbf{m}|^2 \geq 0$$

since $\mathbf{y}$ and $\mathbf{m}$ are orthogonal.

But

$$d(\mathbf{v}, \mathbf{w})^2 = |\mathbf{y}|^2 \geq 0$$

so $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{w}')$ for all $\mathbf{w}' \in W$.

Vector Space Quotients

$$\dim V/W = \dim V - \dim W$$

Suppose $A \in \mathbb{F}^{m \times n}$, then $\mathbb{F}^n/\mathcal{N}(A)$ are the solution sets for $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ varies through $\mathbb{F}^m$.

$\mathcal{N}(A)$ corresponds to $A\mathbf{x} = \mathbf{0}$, and $\mathbf{p} + \mathcal{N}(A)$ corresponds to $A\mathbf{p} = \mathbf{b}$.

$\dim(V + W)/W = \dim V/(V \cap W)$

$$\begin{aligned}
\dim(V + W)/W &= (\dim V + \dim W - \dim(V \cap W)) - \dim W \\
&= \dim V - \dim(V \cap W) \\
&= \dim V/(V \cap W)
\end{aligned}$$