Discrete Mathematics
R204GA05401

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Objectives

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Objectives

- This course will introduce and illustrate in the elementary discrete mathematics for computer science and engineering students.
- To equip the students with standard concepts like formal logic notation, methods of proof, induction, sets, relations, graph theory, permutations and combinations, counting principles.

Course Outcomes

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Course Outcomes

1. Understand the logical connectives, normal forms, predicates and verify the validity of an argument by the rules of inference.
2. Explain functions and its properties such as homomorphism and isomorphism.
3. Explain the general Properties of Semigroups, Monoids, Groups, and Lattices.
4. Illustrate the concepts like partially ordered relation (POSET), compatibility relation and Equivalence relations.
5. Find Euler Trails and Circuits, Planar Graphs, Hamilton Paths and Cycles, Apply Chromatic number of a graph and spanning trees in a graph.
6. Apply the concepts of permutations, combinations, principle of inclusion and exclusion, binomial and multinomial theorems to solve the counting problems.

Unit I Mathematical Logic

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Unit I Mathematical Logic

Propositional Calculus: Statements and Notations, Connectives, Well Formed Formulas, Truth Tables, Tautologies, Equivalence of Formulas, Duality Law, Tautological Implications, Normal Forms, Theory of Inference for Statement Calculus, Consistency of Premises and Indirect Method of Proof.

Predicate Calculus: Predicative Logic, Statement Functions, Variables and Quantifiers, Free and Bound Variables, Inference Theory for Predicate Calculus.

Propositional Calculus

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Propositional Calculus

- Discrete mathematics is the branch of mathematics dealing with objects that can consider only distinct, separated values.

Statements and Notations:

- Basic Units of Object Language is called primary or primitive or Atomic statements.
- Object Language Contains a set of declarative sentence.
- **A Proposition or a statement or logical sentence or Assertion** is a declarative sentence which is either true or false.
- We can measure **effect** of assigning of particular truth value to declarative sentence rather than actual truth value.
- The type of logic is called two-value logic.

Propositional Calculus

- Exclamatory, Interrogative are not allowed to evaluate the logic.
- Declarative Sentences are two types of statements:
  1. Primitive statements. (atomic sentences) (primary statements)
  2. A collection of primitive statements using Connectives. (molecular or Compound statements)
- Capital letters A, B, C, … P, Q, are the mathematical symbolic notations to refer Primitive statements.
- Examples of Statements:
  1. Canada is a country.
  2. Moscow is a capital of Spain.
  3. This statement is false. (sematic Paradox)
  4. 1 + 101 = 110 (Context dependent)
  5. Close the door (Command)
  6. Toronto is an old City.
  7. Man will reach mars by 1980.

Propositional Calculus

Connectives:

- To Construct Complex sentence from simple sentences by using certain connecting words or expressions known **as sentential connectives.**
- Simple statements along with connectives define the algebra that satisfies a set of properties.
- These properties enable us to do some calculation by using statements as objects.
- A propositional variable denotes an arbitrary Proposition (statement) with an unspecified truth values.
- An assertion which contains at least one propositional variable is called **propositional form or statement formula or Assertion Representations.**

Propositional Calculus

Connectives:

- Logical Connectives, we use in this are 5
  1. And
  2. Or
  3. Negation
  4. Exclusive or
  5. Implication
  6. Equivalence

Symbol	Connective	Name
~	Not	Negation
∧	And	Conjunction
∨	Or	Disjunction
→	Implies or if...then	Implication or conditional
↔	If and only if	Equivalence or biconditional

Propositional Calculus

Connectives:

- Logical Connectives, we use in this are 5
  1. And
  2. Or
  3. Negation
  4. Exclusive or
  5. Implication
  6. Equivalence

A	B	A ∧ B
True	True	True
True	False	False
False	True	False
False	False	False

A	B	A ∨ B
True	True	True
True	False	True
False	True	True
False	False	False

A	¬ A
True	False
False	True

Propositional Calculus

Connectives:

- Logical Connectives, we use in this are 5
  1. And
  2. Or
  3. Negation
  4. Exclusive or
  5. Implication
  6. Equivalence

A	B	$A \rightarrow B$
True	True	True
True	False	False
False	True	True
False	False	True

A	B	$A \Leftrightarrow B$
True	True	True
True	False	False
False	True	False
False	False	True

Propositional Calculus

Connectives:

- Logical Connectives, we use in this are 5
  1. And
  2. Or
  3. Negation
  4. Exclusive or
  5. Implication
  6. Equivalence

A	B	A → B
True	True	True
True	False	False
False	True	True
False	False	True

A implies B means

1. A means premise, hypothesis, antecedent
2. B means Conclusion, Consequence

How do we Call this equation

- If A, then B
- A Only if B
- A is sufficient condition for B
- B is the necessary condition for A
- Q if P
- Q follows from P
- Q provided P
- Q is a logical Consequence of P
- Q whenever P
- $B \rightarrow \ A \ is \ called \ Converse$
- $\neg B \ \rightarrow \neg A \ is \ Called \ the \ Contrapositive$

Propositional Calculus

Connectives:

- Logical Connectives, we use in this are 5
  1. And
  2. Or
  3. Negation
  4. Exclusive or
  5. Implication
  6. Equivalence

A	B	A ⇔ B
True	True	True
True	False	False
False	True	False
False	False	True

A is equivalent to B

How do we Call this equation

- A if and only if B
- A is necessary and sufficient condition for B.

Propositional Calculus

Well Formed formulas:

- The notion of the statements are represented symbolically using symbols and its connectives.
- An expression which is a string consisting of variables, parentheses, and connective symbols is called **statement formula.**
- Not every string of these is a statement formula.
- A statement formula often **called well-formed formula(wff).** If the formula is generated by the following rules

1 A statement variable standing alone is a well-formed formula.

2 If A is a well-formed formula, then $\neg A$ is a well-formed formula.

3 If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \rightleftarrows B)$ are well-formed formulas.

4 A string of symbols containing the statement variables, connectives, and parentheses is a well-formed formula, iff it can be obtained by finitely many applications of the rules 1, 2, and 3.

Propositional Calculus

Well Formed formulas:

According to this definition, the following are well-formed formulas:

$$\neg(P \wedge Q) \qquad \neg(P \vee Q) \qquad (P \rightarrow (P \vee Q)) \qquad (P \rightarrow (Q \rightarrow R))$$

$$(((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R))$$

The following are not well-formed formulas.

1 $\neg P \wedge Q$. Obviously P and Q are well-formed formulas. A wff would be either $(\neg P \wedge Q)$ or $\neg(P \wedge Q)$.

2 $(P \rightarrow Q) \rightarrow (\wedge Q)$. This is not a wff because $\wedge Q$ is not.

3 $(P \rightarrow Q$. Note that $(P \rightarrow Q)$ is a wff.

4 $(P \wedge Q) \rightarrow Q)$. The reason for this not being a wff is that one of the parentheses in the beginning is missing. $((P \wedge Q) \rightarrow Q)$ is a wff, while $(P \wedge Q) \rightarrow Q$ is still not a wff.

Propositional Calculus

Truth Table:

- A truth table is a table or chart used to illustrate and determine the truth value of propositions and the validity of their resulting argument.
- Rules to construct the truth table.

1. For n variables construct n + k columns where are subset of all statement formulas (method 1) or **all symbols of a formula (method 2)**
2. For n variables we need to construct 2^n rows.
3. Represent Binary numbers from 0 to $2^n – 1$.
4. Optional Replace 0 with F, and 1 with T.
5. Apply Truth values for each formula or symbols of a formula in proper order.

Refer the website:

https://web.stanford.edu/class/cs103/tools/truth-table-tool/

Propositional Calculus

Truth Table:

Construct truth table of the following Formulas:

1. $((Q \wedge \neg P) \Rightarrow P)$
2. $((P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q))$
3. $(P \vee Q) \vee \sim P$
4. $P \wedge \neg P$

Pls. construct truth table for the above statement formula.

There are two methods to construct the truth table.

Table 1-2.4a

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Method 1

Table 1-2.4b

P	Q	P	$\vee$	$\neg$	Q
T	T	T	T	F	T
T	F	T	T	T	F
F	T	F	F	F	T
F	F	F	T	T	F
Step Number		1	3	2	1

Method 2

Propositional Calculus

Tautologies:

- **A tautology is a propositional form** whose truth value is true for all possible values its Propositional variables. It is also called a universally valid formula or a tautology or a logical truth.
- A simple fact about tautologies is that conjunction of two tautologies is also a tautology.
- e.g. $P \vee \neg P$
- **A contradiction or Absurdity** is a propositional form which is always false.
- e.g. $P \wedge \neg P$
- A propositional form which is neither a tautology nor a contradiction is called **contingency**

Propositional Calculus

Tautologies:

- A straight method to know whether a given formula is a tautology is to construct its truth table, which is shown in figure.

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Propositional Calculus

Tautologies:

- To identify the given formula is tautology, we use two methods

1. Constructing truth table.
2. Substituting method.

Substitution method:

- A **formula A is called substitution instance of another formula B** if A can be obtained from B by substituting formulas for some variables of B, with the condition that same formula is substituted each time it occurs.
- Note substitutions are made for the Atomic formula and never for molecular formulas.

$$B: P \to (J \wedge P)$$

Substitute $R \rightleftarrows S$ for P in B, and we get

$$A: (R \rightleftarrows S) \to (J \wedge (R \rightleftarrows S))$$

Then A is a substitution instance of B. Note that

$$(R \rightleftarrows S) \to (J \wedge P)$$

Propositional Calculus

Tautologies:

- The following substitution instances of P V¬ P are tautologies.

$$(R \to S) \vee \neg(R \to S)$$

$$((P \vee S) \wedge R) \vee \neg((P \vee S) \wedge R)$$

$$(((P \vee \neg Q) \to R) \rightleftarrows S) \vee \neg(((P \vee \neg Q) \to R) \rightleftarrows S)$$

Propositional Calculus

Equivalence of Formulas:

Let A and B are two statement formulas and let $P1, P2, \ldots, Pn$ denote all variables in both A and B. Consider an assignment of truth values to $P1, P2, \ldots, Pn$. If truth value of A is equal to the truth value of B for every one of the 2^n possible sets of truth values assigned to $P1, P2, \ldots, Pn$ then A and B are said to be equivalent.

1. $\neg\neg P$ is equivalent to P.
2. $P \vee P$ is equivalent to P.
3. $(P \wedge \neg P) \vee Q$ is equivalent to Q.
4. $P \vee \neg P$ is equivalent to $Q \vee \neg Q$.

Propositional Calculus

Equivalence of Formulas:

Note of equivalence:

1. It hold symmetric and transitive property.
2. We can identify independent variables.
3. $\Leftrightarrow$ is not a connective symbol but it is meta language symbol.
4. $\leftrightarrows$ is a connective symbol called bicondition.
5. Prove that two statement or statement formula A & B are equivalent or not .
6. We also know whether the statements or statement formula of A & B is a tautology or not.

Propositional Calculus

Equivalence of Formulas:

Table 4-1 Laws of the algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Propositional Calculus

Equivalence of Formulas:

Table 1-2.15 EQUIVALENT FORMULAS

$P \vee P \Leftrightarrow P$	$P \wedge P \Leftrightarrow P$	(Idempotent laws)	(1)
$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	(Associative laws)	(2)
$P \vee Q \Leftrightarrow Q \vee P$	$P \wedge Q \Leftrightarrow Q \wedge P$	(Commutative laws)	(3)
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	(Distributive laws)	(4)
$P \vee \mathbf{F} \Leftrightarrow P$	$P \wedge \mathbf{T} \Leftrightarrow P$		(5)
$P \vee \mathbf{T} \Leftrightarrow \mathbf{T}$	$P \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$		(6)
$P \vee \neg P \Leftrightarrow \mathbf{T}$	$P \wedge \neg P \Leftrightarrow \mathbf{F}$		(7)
$P \vee (P \wedge Q) \Leftrightarrow P$	$P \wedge (P \vee Q) \Leftrightarrow P$	(Absorption laws)	(8)
$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	(De Morgan's laws)	(9)

Propositional Calculus

Equivalence of Formulas:

- There are two methods to check the statement formula A and B are equivalent.
  - Constructing truth table.
  - Replacement process
- In replacement process, we replace any part of a statement which is itself a formula, be it atomic or molecular , by another formula.

Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.

$$
\begin{aligned}
&(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \\
&\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R) && (4) \\
&\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R) && (2) \\
&\Leftrightarrow ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R && (4) \\
&\Leftrightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R && (9), (3) \\
&\Leftrightarrow \mathbf{T} \wedge R && (7) \\
&\Leftrightarrow R && (5)
\end{aligned}
$$

Propositional Calculus

Duality Law:

We can represent any other connective like implication, bi-implication, nor, nand, xor etc**. using the connectives and, or and Not .**

- Two formulas A and A* , are said to be duals to each other , if one can be obtained from the other by replacing $\wedge$ *by* $\vee$ *and* $\vee$ *by* $\wedge$. These connectives $\wedge$ *and* $\vee$ **are also called duals of each other.**
- if the formula A contains the special variables T or F, then A*, its dual is obtained by replacing **T** by **F** and **F** by **T.**

EXAMPLE 1 Write the duals of (*a*) $(P \vee Q) \wedge R$; (*b*) $(P \wedge Q) \vee \mathbf{T}$; (*c*) $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$.

SOLUTION The duals are (*a*) $(P \wedge Q) \vee R$, (*b*) $(P \vee Q) \wedge \mathbf{F}$, and (*c*) $\neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg S))$. ////

Propositional Calculus

Duality Law:

Theorem 1-2.1 Let A and A^* be dual formulas and let $P_1, P_2, \ldots, P_n$ be all the atomic variables that occur in A and A^*. That is to say, we may write A as $A(P_1, P_2, \ldots, P_n)$ and A^* as $A^*(P_1, P_2, \ldots, P_n)$. Then through the use of De Morgan's laws

$$P \land Q \Leftrightarrow \neg(\neg P \lor \neg Q) \qquad P \lor Q \Leftrightarrow \neg(\neg P \land \neg Q)$$

we can show

$$\neg A(P_1, P_2, \ldots, P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, \ldots, \neg P_n) \tag{1}$$

Thus the negation of a formula is equivalent to its dual in which every variable is replaced by its negation. As a consequence of this fact, we also have

$$A(\neg P_1, \neg P_2, \ldots, \neg P_n) \Leftrightarrow \neg A^*(P_1, P_2, \ldots, P_n) \tag{2}$$

EXAMPLE 2 Verify equivalence (1) if $A(P, Q, R)$ is $\neg P \land \neg(Q \lor R)$.

SOLUTION Now $A^*(P, Q, R)$ is $\neg P \lor \neg(Q \land R)$, and $A^*(\neg P, \neg Q, \neg R)$ is $\neg\neg P \lor \neg(\neg Q \land \neg R) \Leftrightarrow P \lor (Q \lor R)$. On the other hand, $\neg A(P, Q, R)$ is $\neg(\neg P \land \neg(Q \lor R)) \Leftrightarrow P \lor (Q \lor R)$. ////

Propositional Calculus

Duality Law:

- We can also state that if any two formulas are equivalent, then their duals are equivalent to each other. In other words, if $A \Leftrightarrow B$, then $A^* \Leftrightarrow B^*$

EXAMPLE 2 Verify equivalence (1) if $A(P, Q, R)$ is $\neg P \wedge \neg(Q \vee R)$.

SOLUTION Now $A^*(P, Q, R)$ is $\neg P \vee \neg(Q \wedge R)$, and $A^*(\neg P, \neg Q, \neg R)$ is $\neg\neg P \vee \neg(\neg Q \wedge \neg R) \Leftrightarrow P \vee (Q \vee R)$. On the other hand, $\neg A(P, Q, R)$ is $\neg(\neg P \wedge \neg(Q \vee R)) \Leftrightarrow P \vee (Q \vee R)$. ////

Theorem 1-2.2 Let $P_1, P_2, \ldots, P_n$ be all the atomic variables appearing in the formulas A and B. Given that $A \Leftrightarrow B$ means "$A \rightleftarrows B$ is a tautology," then the following are also tautologies.

$$A(P_1, P_2, \ldots, P_n) \rightleftarrows B(P_1, P_2, \ldots, P_n)$$

$$A(\neg P_1, \neg P_2, \ldots, \neg P_n) \rightleftarrows B(\neg P_1, \neg P_2, \ldots, \neg P_n)$$

Using (2), we get

$$\neg A^*(P_1, P_2, \ldots, P_n) \rightleftarrows \neg B^*(P_1, P_2, \ldots, P_n)$$

Hence $A^* \Leftrightarrow B^*$.

Propositional Calculus

Tautology Implication:

- A statement A is said to tautology imply A statement B if and only if $A \to B$ is a tautology. We shall denote this idea by $A \Rightarrow B$ which is read as " A implies B“.
- From implication we get the following properties:
  - $A \to B$ converse $B \to A$
  - $A \to B$ inverse $\neg A \to \neg B$
  - $A \to B$ contrapositive $\neg B \to \neg A$

Propositional Calculus

Tautology Implication:

- A statement A is said to tautology imply A statement B if and only if $A \rightarrow B$ is a tautology. We shall denote this idea by $A \Rightarrow B$ which is read as " A implies B“.

Table 1-2.17 IMPLICATIONS

$P \wedge Q \Longrightarrow P$	(1)
$P \wedge Q \Longrightarrow Q$	(2)
$P \Longrightarrow P \vee Q$	(3)
$\neg P \Longrightarrow P \rightarrow Q$	(4)
$Q \Longrightarrow P \rightarrow Q$	(5)
$\neg(P \rightarrow Q) \Longrightarrow P$	(6)
$\neg(P \rightarrow Q) \Longrightarrow \neg Q$	(7)
$P \wedge (P \rightarrow Q) \Longrightarrow Q$	(8)
$\neg Q \wedge (P \rightarrow Q) \Longrightarrow \neg P$	(9)
$\neg P \wedge (P \vee Q) \Longrightarrow Q$	(10)
$(P \rightarrow Q) \wedge (Q \rightarrow R) \Longrightarrow P \rightarrow R$	(11)
$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Longrightarrow R$	(12)

Propositional Calculus

Tautology Implication:

- How to identify whether a given statement formula is a tautology implication.
  - Construction the truth table.
  - Assuming Antecedent is true and proving consequent is true.
  - Assuming Consequent is False and Proving Antecedent is False.
  - Refer Formulas 9, 10, 11, & 12.
- Important facts about implication and equivalence.

1. If a formula is equivalent to a tautology, then it must be a tautology.
2. If a formula is implied by tautology then it is a tautology.

Propositional Calculus

Tautology Implication:

Theorem 1-2.3 If $H_1, H_2, \ldots, H_m$ and P imply Q, then $H_1, H_2, \ldots, H_m$ imply $P \to Q$.

PROOF From our assumption we have

$$(H_1 \wedge H_2 \wedge \cdots \wedge H_m \wedge P) \Rightarrow Q$$

This assumption means $(H_1 \wedge H_2 \wedge \cdots \wedge H_m \wedge P) \to Q$ is a tautology. Using the equivalence (see Example 2, Sec. 1-2.9)

$$P_1 \to (P_2 \to P_3) \Leftrightarrow (P_1 \wedge P_2) \to P_3$$

we can say that

$$(H_1 \wedge H_2 \wedge \cdots \wedge H_m) \to (P \to Q)$$

is a tautology. Hence the theorem. ////

Propositional Calculus

Normal Forms:

- The problem of determining, in a finite number of steps, whether a given statement formula is a tautology or a contradiction or at least satisfiable is known as a **decision problem**.
- Construction a truth table for statement calculus is easy when compare to predicate calculus.
- This is not possible with aid of a computer, so another method known as **reduction to normal forms**

Propositional Calculus

Normal Forms: (Terminology)

- Let Assume the word “product” in place of Conjunction and “Sum” in place of disjunctive.
- A product of variables and their negations in a formula is called **elementary product.**
- A sum of variables and their negations in a formula is called **elementary sum.**
- Any part of the elementary sum or product which is itself an elementary sum or product is called a **factor of the original elementary sum or product.**

Propositional Calculus

Normal Forms: (Terminology)

- Basic Facts about elementary sums and products.

A necessary and sufficient condition for an elementary product to be identically false is that it contain at least one pair of factors in which one is the negation of the other.

A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of factors in which one is the negation of the other.

Propositional Calculus

Normal Forms: (Disjunctive Normal Form)

- A formula which is equivalent to a given formula and which consists a sum of elementary products is called a **disjunctive normal form.**

EXAMPLE 1 Obtain disjunctive normal forms of (*a*) $P \wedge (P \to Q)$; (*b*) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$.

SOLUTION

(*a*) $P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q) \Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

(*b*) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$$

Propositional Calculus

Normal Forms: (Conjunctive Normal Form)

- A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a **conjunctive normal form** of a given formula

Propositional Calculus

Normal Forms: (Conjunctive Normal Form)

EXAMPLE 1 Obtain a conjunctive normal form of each of the formulas given in Example 1 of Sec. 1-3.1.

SOLUTION

(a) $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$. Hence $P \wedge (\neg P \vee Q)$ is a required form.

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q) \Leftrightarrow (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))$

[using $R \rightleftarrows S \Leftrightarrow (R \rightarrow S) \wedge (S \rightarrow R)$]

$$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee (\neg P \wedge \neg Q))$$

$$\Leftrightarrow ((P \vee Q \vee P) \wedge (P \vee Q \vee Q)) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$$

$$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

////

Propositional Calculus

Normal Forms: (Conjunctive Normal Form)

EXAMPLE 2 Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

SOLUTION First we obtain a conjunctive normal form of the given formula.

$$
\begin{aligned}
Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) &\Leftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q) \\
&\Leftrightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q) \\
&\Leftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)
\end{aligned}
$$

Since each of the elementary sums is a tautology, the given formula is a tautology. ////

Propositional Calculus

Normal Forms: (Principle Disjunctive Normal Form)

- Let P and Q be two statement variables. Let us construct all possible formulas which consists of conjunctions of P or its negations and conjunctions of Q or its negations.

$$P \wedge Q \qquad P \wedge \neg Q \qquad \neg F \wedge Q \qquad \text{and} \qquad \neg P \wedge \neg Q$$

- These formulas are called minterms or Boolean Conjunctions of P and Q.

Propositional Calculus

Normal Forms: (Principle Disjunctive Normal Form)

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

- From a given formula, an equivalent formula consisting of disjunctions of minterms only is known as its **principle disjunctive normal form.** It is also called **sum-of-products canonical form**
- To obtain Unique normal form

Propositional Calculus

Normal Forms: (Principle Disjunctive Normal Form)

EXAMPLE 1 Obtain disjunctive normal forms of (a) $P \wedge (P \to Q)$; (b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$.

SOLUTION

(a) $P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q) \Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$$

[using $R \rightleftarrows S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)$]

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$$

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P)$$

$$\vee ((P \vee Q) \wedge \neg Q)$$

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P)$$

$$\vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$$

which is the required disjunctive normal form. ////

Propositional Calculus

Normal Forms: (Principle Disjunctive Normal Form)

EXAMPLE 2 Obtain the principal disjunctive normal forms of (a) $\neg P \vee Q$; (b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.

SOLUTION

(a) $\neg P \vee Q \Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$

$(A \wedge \mathbf{T} \Leftrightarrow A)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

(distributive laws)

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

(commutative law and $P \vee P \Leftrightarrow P$)

(See Example 1.)

(b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q))$

$\vee (Q \wedge R \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R)$

$\vee (\neg P \wedge \neg Q \wedge R)$ ////

Propositional Calculus

Normal Forms: (Principle Disjunctive Normal Form)

EXAMPLE 3 Show that the following are equivalent formulas.

(a) $P \vee (P \wedge Q) \Leftrightarrow P$

(b) $P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q$

SOLUTION We write the principal disjunctive normal form of each formula and compare these normal forms.

(a) $P \vee (P \wedge Q) \Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q) \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$

$$P \Leftrightarrow P \wedge (Q \vee \neg Q) \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$$

(b) $P \vee (\neg P \wedge Q) \Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (\neg P \wedge Q)$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

$$P \vee Q \Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \qquad ////$$

Propositional Calculus

Normal Forms: (Principle Disjunctive Normal Form)

EXAMPLE 4 Obtain the principal disjunctive normal form of

$$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$$

SOLUTION Using $P \to Q \Leftrightarrow \neg P \vee Q$ and De Morgan's law, we obtain

$$\begin{aligned}
&P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P)) \\
&\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P)) \\
&\Leftrightarrow \neg P \vee (\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P)) \\
&\Leftrightarrow \neg P \vee (Q \wedge P) \\
&\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge P) \\
&\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \qquad ////
\end{aligned}$$

Propositional Calculus

Normal Forms: (Principle Conjunctive Normal Form)

- For a given Formula, an equivalent formula consisting of conjunctions of maxterms only is known as its principle Conjunctive form. This normal form is also called product-of-sums canonical form.
- For a given number of variables, the maxterms consists of disjunction in which each variable or its negation, but not both, appears only once.
- Thus maxterms are the duals of minterms.

Propositional Calculus

Normal Forms: (Principle Conjunctive Normal Form)

EXAMPLE 1 Obtain the principal conjunctive normal form of the formula S given by $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$.

SOLUTION

$$
\begin{aligned}
&(\neg P \rightarrow R) \wedge (Q \rightleftarrows P) \\
&\Leftrightarrow (P \vee R) \wedge ((Q \rightarrow P) \wedge (P \rightarrow Q)) \\
&\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q) \\
&\Leftrightarrow (P \vee R \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P \vee (R \wedge \neg R)) \\
&\qquad \wedge (\neg P \vee Q \vee (R \wedge \neg R)) \\
&\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \\
&\qquad \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)
\end{aligned}
$$

Now the conjunctive normal form of $\neg S$ can easily be obtained by writing the conjunction of the remaining maxterms; thus, $\neg S$ has the principal conjunctive normal form

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$$

By considering $\neg\neg S$, we obtain

$$
\begin{aligned}
&\neg(P \vee Q \vee \neg R) \vee \neg(\neg P \vee \neg Q \vee R) \vee \neg(\neg P \vee \neg Q \vee \neg R) \\
&\qquad \Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)
\end{aligned}
$$

which is the principal disjunctive normal form of S. ////

Propositional Calculus

Normal Forms: (Principle Conjunctive Normal Form)

EXAMPLE 2 The truth table for a formula A is given in Table 1-3.3. Determine its disjunctive and conjunctive normal forms.

SOLUTION By choosing the minterms corresponding to each T value of A, we obtain

$$A \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

Similarly

$$A \Leftrightarrow (\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee \neg R)$$

Here the maxterms appearing in the normal form correspond to the F values of A. The maxterms are written down by including the variable if its truth value is F and its negation if the value is T. ////

Propositional Calculus

The Theory of inference for statement calculus:

- The main function of logic is to provide rules of inference, or principles of reasoning.
- The theory associated with such rules is known as inference theory.
- When a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called **a deduction, or a formal proof.**
- **In a formal proof,** every rule of inference that is used at any stage in the derivation is acknowledge.
- **In most literature of reasoning,** conclusions called theorems, and set of premises called axioms of the theory.
- The rules of inference are criteria for determining **the validity of argument.**
- Any conclusion which is arrived by following these rules is called a **valid conclusion** , and the argument is called **valid argument.**

Propositional Calculus

The Theory of inference for statement calculus:

1. Validity Using Truth Tables:

Let A and B be two statement formulas. We say that "*B logically follows from A*" or "B is a *valid conclusion* (*consequence*) of the premise A" iff $A \rightarrow B$ is a tautology, that is, $A \Rightarrow B$.

Just as the definition of implication was extended to include a set of formulas rather than a single formula, we say that from a set of premises $\{H_1, H_2, \ldots, H_m\}$ a conclusion C follows logically iff

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow C \tag{1}$$

Propositional Calculus

The Theory of inference for statement calculus:

1. Validity Using Truth Tables:

EXAMPLE 1 Determine whether the conclusion C follows logically from the premises H_1 and H_2.

(a) $H_1: P \to Q \quad H_2: P \quad C: Q$

(b) $H_1: P \to Q \quad H_2: \neg P \quad C: Q$

(c) $H_1: P \to Q \quad H_2: \neg(P \wedge Q) \quad C: \neg P$

(d) $H_1: \neg P \quad H_2: P \rightleftarrows Q \quad C: \neg(P \wedge Q)$

(e) $H_1: P \to Q \quad H_2: Q \quad C: P$

SOLUTION We first construct the appropriate truth table, as shown in Table 1.4.1. For (a) we observe that the first row is the only row in which both

Propositional Calculus

The Theory of inference for statement calculus:

1. Validity Using Truth Tables:

Table 1-4.1

P	Q	$P \to Q$	$\neg P$	$\neg Q$	$\neg(P \land Q)$	$P \rightleftarrows Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

- To demonstrate process of derivation we will use mostly the following 2 rules. Additional rules will be discussed later.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceding formulas in the derivation.

Rule CP If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

- Before proceeding the actual process of derivation, use the important implications and equivalences.

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

Table 1-4.2 IMPLICATIONS

I_1	$P \wedge Q \Rightarrow P$	(simplification)
I_2	$P \wedge Q \Rightarrow Q$	(simplification)
I_3	$P \Rightarrow P \vee Q$	(addition)
I_4	$Q \Rightarrow P \vee Q$	(addition)
I_5	$\neg P \Rightarrow P \rightarrow Q$	
I_6	$Q \Rightarrow P \rightarrow Q$	
I_7	$\neg(P \rightarrow Q) \Rightarrow P$	
I_8	$\neg(P \rightarrow Q) \Rightarrow \neg Q$	
I_9	$P, Q \Rightarrow P \wedge Q$	
I_{10}	$\neg P, P \vee Q \Rightarrow Q$	(disjunctive syllogism)
I_{11}	$P, P \rightarrow Q \Rightarrow Q$	(modus ponens)
I_{12}	$\neg Q, P \rightarrow Q \Rightarrow \neg P$	(modus tollens)
I_{13}	$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$	(hypothetical syllogism)
I_{14}	$P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$	(dilemma)

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

Table 1-4.3 EQUIVALENCES

E_1	$\neg\neg P \Leftrightarrow P$	(double negation)
E_2	$P \wedge Q \Leftrightarrow Q \wedge P$	(commutative laws)
E_3	$P \vee Q \Leftrightarrow Q \vee P$	
E_4	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	(associative laws)
E_5	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	
E_6	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	(distributive laws)
E_7	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	
E_8	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	(De Morgan's laws)
E_9	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	
E_{10}	$P \vee P \Leftrightarrow P$	
E_{11}	$P \wedge P \Leftrightarrow P$	
E_{12}	$R \vee (P \wedge \neg P) \Leftrightarrow R$	
E_{13}	$R \wedge (P \vee \neg P) \Leftrightarrow R$	
E_{14}	$R \vee (P \vee \neg P) \Leftrightarrow \mathbf{T}$	
E_{15}	$R \wedge (P \wedge \neg P) \Leftrightarrow \mathbf{F}$	
E_{16}	$P \to Q \Leftrightarrow \neg P \vee Q$	
E_{17}	$\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$	
E_{18}	$P \to Q \Leftrightarrow \neg Q \to \neg P$	
E_{19}	$P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$	
E_{20}	$\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$	
E_{21}	$P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$	
E_{22}	$(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$	

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

EXAMPLE 1 Demonstrate that R is a valid inference from the premises $P \to Q, Q \to R$, and P.

SOLUTION

$\{1\}$	(1)	$P \to Q$	Rule **P**
$\{2\}$	(2)	P	Rule **P**
$\{1, 2\}$	(3)	Q	Rule **T**, (1), (2), and I_{11} (modus ponens)
$\{4\}$	(4)	$Q \to R$	Rule **P**
$\{1, 2, 4\}$	(5)	R	Rule **T**, (3), (4), and I_{11}

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

EXAMPLE 2 Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$, and $(A \wedge \neg B) \rightarrow (R \vee S)$.

SOLUTION

{1}	(1)	$(C \vee D) \rightarrow \neg H$	P
{2}	(2)	$\neg H \rightarrow (A \wedge \neg B)$	P
{1, 2}	(3)	$(C \vee D) \rightarrow (A \wedge \neg B)$	T, (1), (2), and I_{13}
{4}	(4)	$(A \wedge \neg B) \rightarrow (R \vee S)$	P
{1, 2, 4}	(5)	$(C \vee D) \rightarrow (R \vee S)$	T, (3), (4), and I_{13}
{6}	(6)	$C \vee D$	P
{1, 2, 4, 6}	(7)	$R \vee S$	T, (5), (6), and I_{11}

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

EXAMPLE 3 Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

SOLUTION

$\{1\}$	(1)	$P \vee Q$	P
$\{1\}$	(2)	$\neg P \rightarrow Q$	T, (1), E_1, and E_{16}
$\{3\}$	(3)	$Q \rightarrow S$	P
$\{1, 3\}$	(4)	$\neg P \rightarrow S$	T, (2), (3), and I_{13}
$\{1, 3\}$	(5)	$\neg S \rightarrow P$	T, (4), E_{18}, and E_1
$\{6\}$	(6)	$P \rightarrow R$	P
$\{1, 3, 6\}$	(7)	$\neg S \rightarrow R$	T, (5), (6), and I_{13}
$\{1, 3, 6\}$	(8)	$S \vee R$	T, (7), E_{16}, and E_1 ////

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

EXAMPLE 4 Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$, and $\neg M$.

SOLUTION

$\{1\}$	(1)	$P \rightarrow M$	**P**	
$\{2\}$	(2)	$\neg M$	**P**	
$\{1, 2\}$	(3)	$\neg P$	**T**, (1), (2), and I_{12}	
$\{4\}$	(4)	$P \vee Q$	**P**	
$\{1, 2, 4\}$	(5)	Q	**T**, (3), (4), and I_{10}	
$\{6\}$	(6)	$Q \rightarrow R$	**P**	
$\{1, 2, 4, 6\}$	(7)	R	**T**, (5), (6), and I_{11}	
$\{1, 2, 4, 6\}$	(8)	$R \wedge (P \vee Q)$	**T**, (4), (7), and I_9	////

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

EXAMPLE 5 Show I_{12}: $\neg Q, P \to Q \Rightarrow \neg P$.

SOLUTION

$\{1\}$	(1)	$P \to Q$	**P**
$\{1\}$	(2)	$\neg Q \to \neg P$	**T**, (1), and E_{18}
$\{3\}$	(3)	$\neg Q$	**P**
$\{1, 3\}$	(4)	$\neg P$	**T**, (2), (3), and I_{11} ////

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

Let P denote the conjunction of the set of premises and let R be any formula. The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \rightarrow S$ can be derived from the premises P alone.

Rule **CP** is also called the *deduction theorem* and is generally used if the conclusion is of the form $R \rightarrow S$. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

Propositional Calculus

The Theory of inference for statement calculus:

2. Rules of Inference:

EXAMPLE 6 Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$, and Q.

SOLUTION Instead of deriving $R \to S$, we shall include R as an additional premise and show S first.

$\{1\}$	(1)	$\neg R \vee P$	**P**
$\{2\}$	(2)	R	**P** (assumed premise)
$\{1, 2\}$	(3)	P	**T**, (1), (2), and I_{10}
$\{4\}$	(4)	$P \to (Q \to S)$	**P**
$\{1, 2, 4\}$	(5)	$Q \to S$	**T**, (3), (4), and I_{11}
$\{6\}$	(6)	Q	**P**
$\{1, 2, 4, 6\}$	(7)	S	**T**, (5), (6), and I_{11}
$\{1, 4, 6\}$	(8)	$R \to S$	**CP**

////

Propositional Calculus

The Theory of inference for statement calculus:

3. Consistency of premises and Indirect Method of Proof

- A Set of formulas $H_1, H_2, \ldots, H_n$. is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in H1, H2, … , Hn. Otherwise, it is considered as inconsistent.

Alternatively, a set of formulas $H_1, H_2, \ldots, H_m$ is inconsistent if their conjunction implies a contradiction, that is,

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow R \wedge \neg R$$

- The notion of inconsistency is used in a procedure called **proof by contradiction** or **reductio ad absurdum** or **Indirect method of proof.**

Propositional Calculus

The Theory of inference for statement calculus:

3. Consistency of premises and Indirect Method of Proof

EXAMPLE 1 Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

SOLUTION We introduce $\neg\neg(P \wedge Q)$ as an additional premise and show that this additional premise leads to a contradiction.

{1}	(1)	$\neg\neg(P \wedge Q)$	**P** (assumed)
{1}	(2)	$P \wedge Q$	**T**, (1), and E_1
{1}	(3)	P	**T**, (2), and I_1
{4}	(4)	$\neg P \wedge \neg Q$	**P**
{4}	(5)	$\neg P$	**T**, (4), I_1
{1, 4}	(6)	$P \wedge \neg P$	**T**, (3), (5), I_9 ////

Propositional Calculus

The Theory of inference for statement calculus:

3. Consistency of premises and Indirect Method of Proof

EXAMPLE 2 Show that the following premises are inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books.

SOLUTION

E: Jack misses many classes.

S: Jack fails high school.

A: Jack reads a lot of books.

H: Jack is uneducated.

The premises are $E \to S$, $S \to H$, $A \to \neg H$, and $E \wedge A$.

Propositional Calculus

The Theory of inference for statement calculus:

3. Consistency of premises and Indirect Method of Proof

- We can use **conditional proof instead of proof of contradiction.**

Proof by contradiction is sometimes convenient. However, it can always be eliminated and replaced by a conditional proof (**CP**). Observe that

$$P \to (Q \land \neg Q) \Rightarrow \neg P \qquad (1)$$

In the proof by contradiction we show

$$H_1, H_2, \ldots, H_m \Rightarrow C$$

by showing

$$H_1, H_2, \ldots, H_m, \neg C \Rightarrow R \land \neg R \qquad (2)$$

Now (2) can be converted to the following by using rule **CP**

$$H_1, H_2, \ldots, H_m \Rightarrow \neg C \to (R \land \neg R) \qquad (3)$$

From (3) and (1) and E_1, we can show

$$H_1, H_2, \ldots, H_m \Rightarrow C$$

which is the required derivation.

Predicate Calculus

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Predicate Calculus

- In Propositional calculus, It is not possible to express the fact that any two atomic statement have some features in common.
- The logic based upon the analysis of predicates in any statement is called predicate logic.

Predicate:

- In a statement, a part of it can be shared by one or more individuals can be called as predicate.
- For Eg:
  - John is a bachelor
  - Smith is a bachelor
- Some other examples to be considered
  - All human beings are mortal
  - John is a human being
  - Therefore, John is a mortal.

Predicate Calculus

- Predicates are represented as Q(p) where Q is a predicate and p a subject.
- Eg: B(j), B(s)
- A predicate requiring m names ($m > 0$) is called an m-place predicate.

Statement Functions, Variables and Quantifiers:

- Let H be the predicate “is a mortal”, b the name “Mr. Brown”, c “ Canada” and S “A Shirt” then H(b), H(c) and H(s) all denote statements. If we use x as place holder instead of names of object then H(x) is called a **Statement function.**
- Thus a statement function is an expression having Predicate Symbol and one or multiple variables. This statement function gives a statement when we replaced the variables with objects. This replacement is called substitution instance of statement function.

Predicate Calculus

- A Collection of one or more simple statements with logical connectives are called compound statement functions.
- To represent a given variable x , how many values it can hold. For this we Quantified variables Using **Quantifiers.**
- There are two types of quantifier in predicate logic - Existential Quantifier and Universal Quantifier.
- To Represent the Phrase "For all x" by the symbol "$(\forall x)$"or by "(x)" are called **universal Quantifiers.**
- The Phrase like "there is at least one x such that", or there exist an x such that" or "for some x". We represent by the symbol " ($\exists x$) " , called **existential Quantifiers.**

Predicate Calculus

Predicate Formula:

- In General $P(x_1, x_2, \dots x_n)$ will be called an atomic formula in predicate calculus.
- For example: Q(x), P(x, y), A(x, y, z) are the examples of atomic formulas.
- A well defined formulas of predicate calculus is obtained by using the following rules.
  1. An Atomic formula is a well-formed formula.
  2. If A is a well-formed formula, then $\neg A$ is a well-formed formula.
  3. If A and B are well Formed Formulas, then $(A \wedge B), (A \vee B), (A \rightarrow$

Predicate Calculus

Free and Bound Variables:

- Consider a Predicate formula having a part in form of (∃ x) P(x) of (x)P(x), then such part is called x-bound part of the formula. Any occurrence of x in x-bound part is termed as bound occurrence and any occurrence of x which is not x-bound is termed as free occurrence.
- Examples

$$(x)P(x, y) \tag{1}$$

$$(x)(P(x) \rightarrow Q(x)) \tag{2}$$

$$(x)(P(x) \rightarrow (\exists y)R(x, y)) \tag{3}$$

$$(x)(P(x) \rightarrow R(x)) \vee (x)(P(x) \rightarrow Q(x)) \tag{4}$$

$$(\exists x)(P(x) \wedge Q(x)) \tag{5}$$

$$(\exists x)P(x) \wedge Q(x) \tag{6}$$

Predicate Calculus

Free and Bound Variables:

EXAMPLE 1 Let

$$P(x): x \text{ is a person.}$$

$$F(x, y): x \text{ is the father of } y.$$

$$M(x, y): x \text{ is the mother of } y.$$

Write the predicate "x is the father of the mother of y."

SOLUTION In order to symbolize the predicate, we name a person called z as the mother of y. Obviously we want to say that x is the father of z and z the mother of y. It is assumed that such a person z exists. We symbolize the predicate as

$$(\exists z)(P(z) \wedge F(x, z) \wedge M(z, y)) \qquad ////$$

Predicate Calculus

Universe of Discourse:

- We can limit the class of individuals/objects used in a statment. Here limiting means confining the input variable to a set of particular individuals/objects. **Such a restricted class is termed as Universe of Discourse/domain of individual or universe**.

EXAMPLE 1 Symbolize the statement "All men are giants."

SOLUTION Using

$$G(x): x \text{ is a giant.}$$

$$M(x): x \text{ is a man.}$$

the given statement can be symbolized as $(x)(M(x) \rightarrow G(x))$. However, if we restrict the variable x to the universe which is the class of men, then the statement is

$$(x)G(x) \qquad ////$$

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Valid Formulas and Equivalence:** The Formulas for the predicate calculus are assumed to contain statement variables, predicates and object variables
- Let A and B be any predicate formulas over a common universe denoted by the symbol E. if, for every assignment of object names from the universe of discourse E to each of the variables appearing in A and B, the resulting statements have the same truth values, then the predicate formulas A and B are said to be equivalent to each other over E. The idea is symbolized by writing A $\Leftrightarrow$ B over E. if E is arbitrary, then we say A and B are equivalent.
- Similarly a Formula A is said to be valid in E written $\vDash A \: in \: E$ if, for every assignment of object names from E to the corresponding variables in A and for every assignment of statement variables, The resulting statement have the truth value T. if a formula is valid for an arbitrary E, then it is written as $\vDash A$

Predicate Calculus

Inference Theory for Predicate Calculus:

- The determination of validity of a formula by the truth table would involve examination of all possible universe which is impossible.
- Formulas of the predicates calculus that involve quantifiers and no free variables are also formulas of statement calculus. **Therefore substation instance of all tautologies by these formulas yield any number of special tautologies.**

$$P \vee \neg P \qquad P \rightarrow Q \rightleftarrows \neg P \vee Q$$

$$((x)R(x)) \vee \neg((x)R(x))$$

$$((x)R(x)) \rightarrow ((\exists x)S(x)) \rightleftarrows \neg((x)R(x)) \vee ((\exists x)S(x))$$

Predicate Calculus

Inference Theory for Predicate Calculus:

- In general, a tautology of the statement calculus remains a valid formula of the predicate of the predicate calculus when prime formulas are substituted for statement variables through out the formula.

$$\neg\neg A(x) \rightleftarrows A(x) \qquad E_1$$

$$C(x, y) \wedge B(x) \rightleftarrows B(x) \wedge C(x, y) \qquad E_2$$

$$A(x) \to B(x) \rightleftarrows \neg A(x) \vee B(x) \qquad E_{16}$$

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Some valid Formulas and over Finite Sequence:** If in a formula A(x), we replace each free occurrence of the variable x by another variable y, then we say *y is substituted for x* in a formula, and the resulting formula is denoted by A(x).
- For such a substitution the formula A(x) must be free from y. A formula A(x) is said be free from y if no free occurrence of x in the scope of the quantifiers (y) *or* $(\exists\ y)$.
- If *A(x) is not free of y,* then it is necessary to change variable y , appearing as a bound variable, to another variable before substituting y for x. if y is substituted then it is usually a good idea to make all the bound variables different from y.

Predicate Calculus

Inference Theory for Predicate Calculus:

$A(x)$	$A(y)$
$P(x,y) \wedge (\exists y)Q(y)$	$P(y,y) \wedge (\exists y)Q(y)$ or $P(y,y) \wedge (\exists z)Q(z)$
$(S(x) \wedge S(y)) \vee (x)R(x)$	$(S(y) \wedge S(y)) \vee (x)R(x)$ or $(S(y) \wedge S(y)) \vee (z)R(z)$

The following formulas are not free for y.

$$P(x, y) \wedge (y)Q(x, y) \qquad (y)(S(y) \rightarrow S(x))$$

$A(x)$	$A(y)$
$P(x,y) \wedge (z)Q(x,z)$	$P(y,y) \wedge (z)Q(y,z)$
$(z)(S(z) \rightarrow S(x))$	$(z)(S(z) \rightarrow S(y))$

Predicate Calculus

Inference Theory for Predicate Calculus:

- Let the universe of Discourse is Finite and is denoted by S = { a_1, a_2, . . . a_n}

$$(x)A(x) \Leftrightarrow A(a_1) \wedge A(a_2) \wedge \cdots \wedge A(a_n) \qquad (1)$$

$$(\exists x)A(x) \Leftrightarrow A(a_1) \vee A(a_2) \vee \cdots \vee A(a_n) \qquad (2)$$

$$\neg((x)A(x)) \Leftrightarrow (\exists x)\neg A(x) \qquad (3)$$

$$\neg((\exists x)A(x)) \Leftrightarrow (x)\neg A(x) \qquad (4)$$

$$\begin{aligned} \neg((x)A(x)) &\Leftrightarrow \neg(A(a_1) \wedge A(a_2) \wedge \cdots \wedge A(a_n)) \\ &\Leftrightarrow \neg A(a_1) \vee \neg A(a_2) \vee \cdots \vee \neg A(a_n) \\ &\Leftrightarrow (\exists x)\neg A(x) \end{aligned}$$

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Special Valid Formulas Involving Quantifiers:**

1. **Universal Specification (US) :** $(x)\ A(x) \Longrightarrow A(x)$
2. **Universal Generalization (UG):** $A(x) \Longrightarrow (x)\ A(x)$
3. **Existential Specification (ES):** $(\exists x)\ A(x) \Longrightarrow A(y)$
4. **Existential generalization (EG):** $A(y) \Longrightarrow (\exists x)\ A(x)$

$$(\exists x)(A(x) \rightarrow B(x)) \Leftrightarrow (x)A(x) \rightarrow (\exists x)B(x) \qquad E_{33}$$

$$(\exists x)A(x) \rightarrow (x)B(x) \Leftrightarrow (x)(A(x) \rightarrow B(x)) \qquad E_{34}$$

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Special Valid Formulas Involving Quantifiers:**

Table 1-6.1

$(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$	E_{23}
$(x)(A(x) \wedge B(x)) \Leftrightarrow (x)A(x) \wedge (x)B(x)$	E_{24}
$\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$	E_{25}
$\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$	E_{26}
$(x)A(x) \vee (x)B(x) \Rightarrow (x)(A(x) \vee B(x))$	I_{15}
$(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$	I_{16}

Table 1-6.2

$(x)(A \vee B(x)) \Leftrightarrow A \vee (x)B(x)$	E_{27}
$(\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)B(x)$	E_{28}
$(x)A(x) \rightarrow B \Leftrightarrow (\exists x)(A(x) \rightarrow B)$	E_{29}
$(\exists x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$	E_{30}
$A \rightarrow (x)B(x) \Leftrightarrow (x)(A \rightarrow B(x))$	E_{31}
$A \rightarrow (\exists x)B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$	E_{32}

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Theory of Inference for Predicate Calculus:**
- Use rule P, T, CP of Statement Calculus and Rule US, ES, UG and EG.

Rule **US** (Universal Specification) From $(x)A(x)$ one can conclude $A(y)$.

Rule **ES** (Existential Specification) From $(\exists x)A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be met by choosing a new variable each time **ES** is used. (The conditions of **ES** are more restrictive than ordinarily required, but they do not affect the possibility of deriving any conclusion.)

Rule **EG** (Existential Generalization) From $A(x)$ one can conclude $(\exists y)A(y)$.

Rule **UG** (Universal Generalization) From $A(x)$ one can conclude $(y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of **ES**, then no variables introduced by that use of **ES** appear free in $A(x)$.

Predicate Calculus

Inference Theory for Predicate Calculus:

EXAMPLE 1 Show that $(x)(H(x) \to M(x)) \wedge H(s) \Rightarrow M(s)$. Note that this problem is a symbolic translation of a well-known argument known as the "Socrates argument" which is given by:

All men are mortal.

Socrates is a man.

Therefore Socrates is a mortal.

If we denote $H(x)$: x is a man, $M(x)$: x is a mortal, and s: Socrates, we can put the argument in the above form.

SOLUTION

$\{1\}$	(1)	$(x)(H(x) \to M(x))$	**P**
$\{1\}$	(2)	$H(s) \to M(s)$	**US**, (1)
$\{3\}$	(3)	$H(s)$	**P**
$\{1, 3\}$	(4)	$M(s)$	**T**, (2), (3), I_{11}

Note that in step 2 first we remove the universal quantifier. ////

Predicate Calculus

Inference Theory for Predicate Calculus:

EXAMPLE 2 Show that

$$(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$$

SOLUTION

$\{1\}$	(1)	$(x)(P(x) \to Q(x))$	P
$\{1\}$	(2)	$P(y) \to Q(y)$	US, (1)
$\{3\}$	(3)	$(x)(Q(x) \to R(x))$	P
$\{3\}$	(4)	$Q(y) \to R(y)$	US, (3)
$\{1, 3\}$	(5)	$P(y) \to R(y)$	T, (2); (4), I_{13}
$\{1, 3\}$	(6)	$(x)(P(x) \to R(x))$	UG, (5)

////

Predicate Calculus

Inference Theory for Predicate Calculus:

EXAMPLE 3 Show that $(\exists x)M(x)$ follows logically from the premises

$$(x)(H(x) \to M(x)) \quad \text{and} \quad (\exists x)H(x)$$

SOLUTION

$\{1\}$	(1)	$(\exists x)H(x)$	P
$\{1\}$	(2)	$H(y)$	ES, (1)
$\{3\}$	(3)	$(x)(H(x) \to M(x))$	P
$\{3\}$	(4)	$H(y) \to M(y)$	US, (3)
$\{1, 3\}$	(5)	$M(y)$	T, (2), (4), I_{11}
$\{1, 3\}$	(6)	$(\exists x)M(x)$	EG, (5)

Predicate Calculus

Inference Theory for Predicate Calculus:

EXAMPLE 4 Prove that

$$(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$$

SOLUTION

$\{1\}$	(1)	$(\exists x)(P(x) \wedge Q(x))$	**P**
$\{1\}$	(2)	$P(y) \wedge Q(y)$	**ES**, (1), y fixed
$\{1\}$	(3)	$P(y)$	**T**, (2), I_1
$\{1\}$	(4)	$Q(y)$	**T**, (2), I_2
$\{1\}$	(5)	$(\exists x)P(x)$	**EG**, (3)
$\{1\}$	(6)	$(\exists x)Q(x)$	**EG**, (4)
$\{1\}$	(7)	$(\exists x)P(x) \wedge (\exists x)Q(x)$	**T**, (4), (5), I_9 ////

Predicate Calculus

Inference Theory for Predicate Calculus:

EXAMPLE 5 Show that from

(a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$

(b) $(\exists y)(M(y) \wedge \neg W(y))$

the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows.

Predicate Calculus

Inference Theory for Predicate Calculus:

{1}	(1)	$(\exists y)(M(y) \land \neg W(y))$	P
{1}	(2)	$M(z) \land \neg W(z)$	ES, (1)
{1}	(3)	$\neg(M(z) \rightarrow W(z))$	T, (2), E_{17}
{1}	(4)	$(\exists y)\neg(M(y) \rightarrow W(y))$	EG, (3)
{1}	(5)	$\neg(y)(M(y) \rightarrow W(y))$	E_{26}, (4)
{6}	(6)	$(\exists x)(F(x) \land S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$	P
{1, 6}	(7)	$\neg(\exists x)(F(x) \land S(x))$	T, (5), (6), I_{12}
{1, 6}	(8)	$(x)\neg(F(x) \land S(x))$	T, (7), E_{25}
{1, 6}	(9)	$\neg(F(x) \land S(x))$	US, (8)
{1, 6}	(10)	$F(x) \rightarrow \neg S(x)$	T, (9), E_9, E_{16}, E_{17}
{1, 6}	(11)	$(x)(F(x) \rightarrow \neg S(x))$	UG, (10)

Predicate Calculus

Inference Theory for Predicate Calculus:

EXAMPLE 6 Show that

$$(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$$

SOLUTION We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \vee (\exists x)Q(x))$	**P** (assumed)
{1}	(2)	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	**T**, (1), E_9
{1}	(3)	$\neg(x)P(x)$	**T**, (2), I_1
{1}	(4)	$(\exists x)\neg P(x)$	**T**, (3), E_{26}
{1}	(5)	$\neg(\exists x)Q(x)$	**T**, (2), I_2
{1}	(6)	$(x)\neg Q(x)$	**T**, (5), E_{25}
{1}	(7)	$\neg P(y)$	**ES**, (4)
{1}	(8)	$\neg Q(y)$	**US**, (6)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	**T**, (7), (8), I_9
{1}	(10)	$\neg(P(y) \vee Q(y))$	**T**, (9), E_9
{11}	(11)	$(x)(P(x) \vee Q(x))$	**P**
{11}	(12)	$P(y) \vee Q(y)$	**US**, (11)
{1, 11}	(13)	$\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	**T**, (10), (12), I_9 contradiction

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Formulas Involving more than one Quantifier:**
- The Possibility ways P(x,y) can be represented Using Quantifiers:

following possibilities

$$(x)(y)P(x,y) \qquad (x)(\exists y)P(x,y) \qquad (\exists x)(y)P(x,y)$$

$$(\exists x)(\exists y)P(x,y) \qquad (y)(x)P(x,y) \qquad (\exists y)(x)P(x,y)$$

$$(y)(\exists x)P(x,y) \qquad (\exists y)(\exists x)P(x,y)$$

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Formulas Involving more than one Quantifier:**
- From the meaning of Quantifiers The following Formulas are derived:

$$(x)(y)P(x, y) \Leftrightarrow (y)(x)P(x, y) \qquad (1)$$

$$(x)(y)P(x, y) \Rightarrow (\exists y)(x)P(x, y) \qquad (2)$$

$$(y)(x)P(x, y) \Rightarrow (\exists x)(y)P(x, y) \qquad (3)$$

$$(\exists y)(x)P(x, y) \Rightarrow (x)(\exists y)P(x, y) \qquad (4)$$

$$(\exists x)(y)P(x, y) \Rightarrow (y)(\exists x)P(x, y) \qquad (5)$$

$$(x)(\exists y)P(x, y) \Rightarrow (\exists y)(\exists x)P(x, y) \qquad (6)$$

$$(y)(\exists x)P(x, y) \Rightarrow (\exists x)(\exists y)P(x, y) \qquad (7)$$

$$(\exists x)(\exists y)P(x, y) \Leftrightarrow (\exists y)(\exists x)P(x, y) \qquad (8)$$

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Formulas Involving more than one Quantifier:**
- Graphical Representation of the above Formulas:

…, some special care would

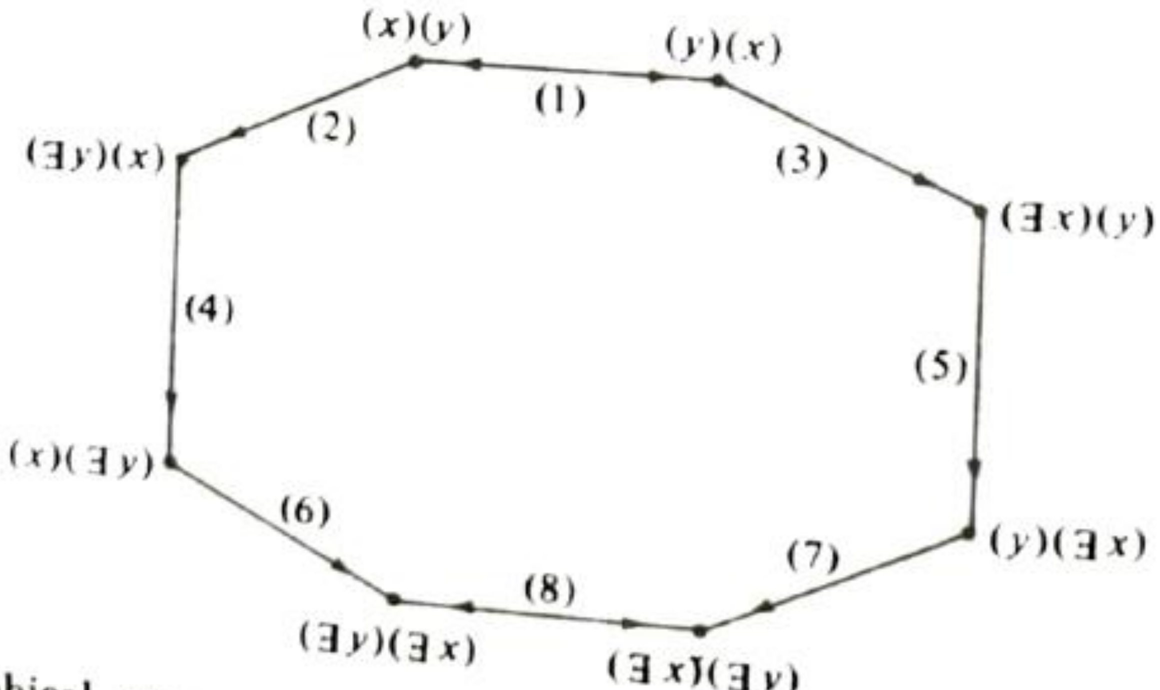

FIGURE 1-6.1 Graphical representation of relationships among formulas involving two quantifiers.

Predicate Calculus

Inference Theory for Predicate Calculus:

- **Formulas Involving more than one Quantifier:**

EXAMPLE 1 Show that $\neg P(a, b)$ follows logically from $(x)(y)(P(x, y) \rightarrow W(x, y))$ and $\neg W(a, b)$.

SOLUTION

{1}	(1)	$(x)(y)(P(x, y) \rightarrow W(x, y))$	**P**
{1}	(2)	$(y)(P(a, y) \rightarrow W(a, y))$	**US**, (1)
[illegible]	(3)	$P(a, b) \rightarrow W(a, b)$	**US**, (2)
[illegible]	(4)	$\neg W(a, b)$	**P**
{1, 4}	(5)	$\neg P(a, b)$	**T**, (3), (4), I_{12} ////

End of Unit-1