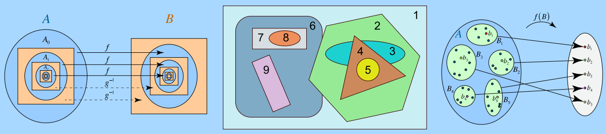
MATH24

Calculus

Set Theory



Operations on Relations

Home → Calculus → Set Theory → Operations on Relations

Since binary relations defined on a pair of sets A and B are subsets of the Cartesian product $A \times B$, we can perform all the usual set operations on them.

Let R and S be two relations over the sets A and B, respectively.

Intersection of Relations

The intersection of the relations $R \cap S$ is defined by

$$R \cap S = \{(a, b) \mid aRb \text{ and } aSb\},$$

where $a \in A$ and $b \in B$.

For example, let R and S be the relations "*is a friend of*" and "*is a work colleague of*" defined on a set of people A (assuming $A = B$). Their intersection $R \cap S$ will be the relation "*is a friend and work colleague of*".

If the relations R and S are defined by matrices $M_R = [a_{ij}]$ and $M_S = [b_{ij}]$, the matrix of their intersection $R \cap S$ is given by

$$M_{R \cap S} = M_R * M_S = [a_{ij} * b_{ij}],$$

where the product operation is performed as element-wise multiplication.

Union of Relations

Similarly, the union of the relations $R \cup S$ is defined by

$$R \cup S = \{(a, b) \mid aRb \text{ or } aSb\},$$

provided $a \in A$ and $b \in B$.

For example, the union of the relations "*is less than*" and "*is equal to*" on the set of integers will be the relation "*is less than or equal to*".

If the relations R and S are defined by matrices $M_R = [a_{ij}]$ and $M_S = [b_{ij}]$, the union of the relations $R \cup S$ is given by the following matrix:

$$M_{R \cup S} = M_R + M_S = [a_{ij} + b_{ij}],$$

where the sum of the elements is calculated by the rules

$$0 + 0 = 0, \;\; 1 + 0 = 0 + 1 = 1, \;\; 1 + 1 = 1.$$

Difference of Relations

The difference of two relations is defined as follows:

$$R \backslash S = \{(a,b) \mid aRb \text{ and not } aSb\},$$

$$S \backslash R = \{(a,b) \mid aSb \text{ and not } aRb\},$$

where $a \in A$ and $b \in B$.

Suppose $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. The relations R and S have the form

$$R = \{(a,1), (b,2), (c,3), (d,1)\}, \quad S = \{(a,1), (b,1), (c,1), (d,1)\}.$$

Then the relation differences $R \backslash S$ and $S \backslash R$ are given by

$$R \backslash S = \{(b,2), (c,3)\}, \quad S \backslash R = \{(b,1), (c,1)\}.$$

Symmetric Difference of Relations

The symmetric difference of two binary relations R and S is the binary relation defined as

$$R \,\triangle\, S = (R \cup S) \backslash (R \cap S), \quad \text{or} \quad R \,\triangle\, S = (R \backslash S) \cup (S \backslash R).$$

Let R and S be relations of the previous example. Then

$$R \,\triangle\, S = \{(b,2), (c,3)\} \cup \{(b,1), (c,1)\} = \{(b,1), (c,1), (b,2), (c,3)\}.$$

Complement of a Binary Relation

Suppose that R is a binary relation between two sets A and B. The complement of R over A and B is the binary relation defined as

$$\bar{R} = \{(a,b) \mid \text{not } aRb\},$$

where $a \in A$ and $b \in B$.

For example, let $A = \{1, 2\}$, $B = \{1, 2, 3\}$. If a relation R between sets A and B is given by

$$R = \{(1,2), (1,3), (2,2), (2,3)\},$$

then the complement of R has the form

$$\bar{R} = \{(1,1), (2,1)\}.$$

Converse of a Binary Relation

Let R be a binary relation on sets A and B. The converse relation or transpose of R over A and B is obtained by switching the order of the elements:

$$R^T = \{(b, a) \mid aRb\},$$

where $a \in A$, $b \in B$.

So, if $R = \{(1,2), (1,3), (1,4)\}$, then the converse of R is

$$R^T = \{(2,1), (3,1), (4,1)\}.$$

If a relation R is defined by a matrix M, then the converse relation R^T will be represented by the transpose matrix M^T (formed by interchanging the rows and columns). For example,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad M^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Sometimes the converse relation is also called the inverse relation and denoted by R^{-1}.

Empty, Universal and Identity Relations

A relation R between sets A and B is called an empty relation if $R = \varnothing$.

The universal relation between sets A and B, denoted by U, is the Cartesian product of the sets: $U = A \times B$.

A relation R defined on a set A is called the identity relation (denoted by I) if $I = \{(a, a) \mid \forall a \in A\}$.

Properties of Combined Relations

When we apply the algebra operations considered above we get a combined relation. The original relations may have certain properties such as reflexivity, symmetry, or transitivity. The question is whether these properties will persist in the combined relation? The table below shows which binary properties hold in each of the basic operations.

Relation	*Reflexive*	*Irreflexive*	*Symmetric*	*Antisymmetric*	*Asymmetric*	*Transitive*
R	Y	Y	Y	Y	Y	Y
S	Y	Y	Y	Y	Y	Y
$R \cap S$	Y	Y	Y	Y	Y	Y
$R \cup S$	Y	Y	Y	N	N	N
$R \setminus S$	N	Y	Y	Y	Y	Y
$S \setminus R$	N	Y	Y	Y	Y	Y
$R \Delta S$	N	Y	Y	N	N	N

Figure 1.

See solved problems on Page 2.

Recommended Pages

Recommended Pages

- Logic and Set Notation
- Introduction to Sets
- Set Operations and Venn Diagrams
- Set Identities
- Cartesian Product of Sets
- Binary Relations
- Composition of Relations

Page 1 Page 2

Legal

Alex Svirin, PhD

Privacy Policy

Contact

info@math24.net

Tools