Discrete Mathematics
R204GA05401

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Objectives

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Objectives

- This course will introduce and illustrate in the elementary discrete mathematics for computer science and engineering students.
- To equip the students with standard concepts like formal logic notation, methods of proof, induction, sets, relations, graph theory, permutations and combinations, counting principles.

Course Outcomes

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Course Outcomes

1. Illustrate discrete mathematic components like statements, logic, sets, structures, numbers and combinatorics.
2. Evaluate and simplify propositional and predicate calculus using inference theory.
3. Perform the operations on Sets, Relations and functions and their properties.
4. Identify algebraic systems and use general properties on number theory.
5. Use combinatorics solving the counting problems.
6. Use graph algorithms for representing, identifying, generating and evaluating the Graphs.

Unit II Set Theory

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Unit II Set Theory

Introduction, Operations on Binary Sets, Principle of Inclusion and Exclusion,

Relations: Properties of Binary Relations, Relation Matrix and Digraph, Operations on Relations, Partition and Covering, Transitive Closure, Equivalence, Compatibility and Partial Ordering Relations, Hasse Diagrams,

Functions: Bijective Functions, Composition of Functions, Inverse Functions, Permutation Functions, Recursive Functions, Lattice and its Properties.

Introduction on Sets

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Introduction on Sets

- A set is defined as collection on objects in any order.
- If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.
- Sets can be represented in two wavs:
  1. Roster or Tabular Form
  2. Set Builder Notation

$A = \{1, 3, 5, 7, 9\}$ and $B = \{x \mid x \text{ is an even integer}, x > 0\}$

Note : If an element x is a member of any set S, it is denoted by x∈S and if an element y is not a member of set S, it is denoted by y∉S.

- **Cardinality of a Set** : The number of elements in a set is called cardinality of a set.
- It is denoted by $|S|$

1. Let P = {k, l, m, n}

The cardinality of the set P is 4.

2. Let A is the set of all non-negative even integers, i.e.

A = {0, 2, 4, 6, 8, 10......}.

Introduction on Sets

Types of Sets:

1. Finite Set
2. Infinite Set
3. Subset
4. Proper Subset
5. Universal set
6. Empty or null set
7. Singleton or Unit set
8. Equal set
9. Equivalent set
10. Overlapping set

Introduction on Sets

Types of Sets:

Finite Set

A set which contains a definite number of elements is called a finite set.

Example – $S = \{x \mid x \in N$ and $70 > x > 50\}$

Infinite Set

A set which contains infinite number of elements is called an infinite set.

Example – $S = \{x \mid x \in N$ and $x > 10\}$

Subset

A set X is a subset of set Y (Written as $X \subseteq Y$) if every element of X is an element of set Y.

Example 1 – Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Example 2 – Let, $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3\}$. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Introduction on Sets

Types of Sets:

Proper Subset

The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X| < |Y|$.

Example – Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2\}$. Here set $Y \subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y .

Universal Set

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U .

Example – We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U , set of all fishes is a subset of U , set of all insects is a subset of U , and so on.

Introduction on Sets

Types of Sets:

Empty Set or Null Set

An empty set contains no elements. It is denoted by $\emptyset$. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example − $S = \{x \mid x \in N$ and $7 < x < 8\} = \emptyset$

Singleton Set or Unit Set

Singleton set or unit set contains only one element. A singleton set is denoted by $\{s\}$.

Example − $S = \{x \mid x \in N,\ 7 < x < 9\}$ = $\{8\}$

Equal Set

If two sets contain the same elements they are said to be equal.

Example − If $A = \{1, 2, 6\}$ and $B = \{6, 1, 2\}$, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Introduction on Sets

Types of Sets:

Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example − If $A = \{1, 2, 6\}$ and $B = \{16, 17, 22\}$, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A| = |B| = 3$

Overlapping Set

Two sets that have at least one common element are called overlapping sets.

In case of overlapping sets −

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
- $n(A) = n(A - B) + n(A \cap B)$
- $n(B) = n(B - A) + n(A \cap B)$

Example − Let, $A = \{1, 2, 6\}$ and $B = \{6, 12, 42\}$. There is a common element '6', hence these sets are overlapping sets.

Introduction on Sets

Types of Sets:

Disjoint Set

Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties −

- $n(A \cap B) = \emptyset$
- $n(A \cup B) = n(A) + n(B)$

Example − Let, $A = \{1,2,6\}$ and $B = \{7,9,14\}$, there is not a single common element, hence these sets are overlapping sets.

Introduction on Sets

Operations on Binary sets:

1. **Union**
2. **Intersection**
3. **Difference**
4. **Complement**
5. **Cartesian product**

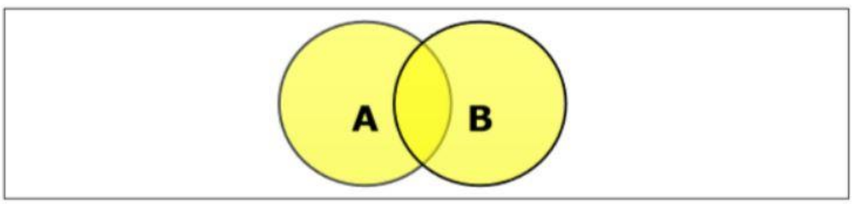
Introduction on Sets

Operations on Binary sets:

Set Union

The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A, in B, or in both A and B. Hence, $A \cup B = \{x \mid x \in A \; OR \; x \in B\}$.

Example – If $A = \{10, 11, 12, 13\}$ and B = $\{13, 14, 15\}$, then $A \cup B = \{10, 11, 12, 13, 14, 15\}$. (The common element occurs only once)



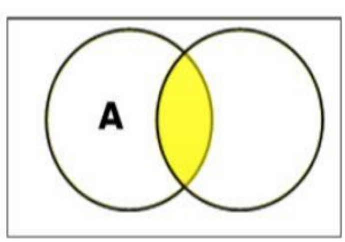
Introduction on Sets

Operations on Binary sets:

Set Intersection

The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B. Hence, $A \cap B = \{x \mid x \in A \; AND \; x \in B\}$.

Example – If $A = \{11, 12, 13\}$ and $B = \{13, 14, 15\}$, then $A \cap B = \{13\}$.



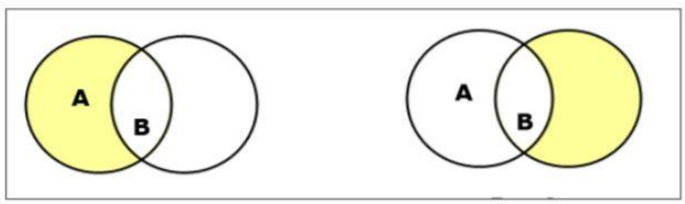
Introduction on Sets

Operations on Binary sets:

Set Difference/ Relative Complement

The set difference of sets A and B (denoted by $A-B$) is the set of elements which are only in A but not in B. Hence, $A-B=\{x \mid x \in A \; AND \; x \notin B\}$.

Example – If $A=\{10,11,12,13\}$ and $B=\{13,14,15\}$, then $(A-B)=\{10,11,12\}$ and $(B-A)=\{14,15\}$. Here, we can see $(A-B) \neq (B-A)$



Introduction on Sets

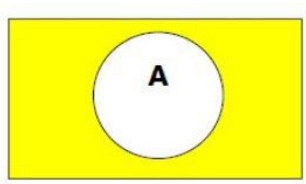
Operations on Binary sets:

Complement of a Set

The complement of a set A (denoted by A') is the set of elements which are not in set A. Hence, $A'=\{x|x \notin A\}$.

More specifically, $A'=(U-A)$ where U is a universal set which contains all objects.

Example – If $A=\{x \mid x \; belongs \; to \; set \; of \; odd \; integers\}$ then $A'=\{y \mid y \; does \; not \; belong \; to \; set \; of \; odd \; integers\}$



Introduction on Sets

Operations on Binary sets:

Cartesian Product / Cross Product

The Cartesian product of n number of sets $A_1, A_2, \dots A_n$ denoted as $A_1 \times A_2 \cdots \times A_n$ can be defined as all possible ordered pairs $(x_1, x_2, \dots x_n)$ where $x_1 \in A_1, x_2 \in A_2, \dots x_n \in A_n$

Example – If we take two sets $A = \{a, b\}$ and $B = \{1, 2\}$,

The Cartesian product of A and B is written as – $A \times B = \{(a,1), (a,2), (b,1), (b,2)\}$

The Cartesian product of B and A is written as – $B \times A = \{(1,a), (1,b), (2,a), (2,b)\}$

Introduction on Sets

Principle of Inclusion and Exclusion:

Inclusion–Exclusion Principle

There is a formula for $n(A \cup B)$ even when they are not disjoint, called the Inclusion–Exclusion Principle. Namely:

Theorem (Inclusion–Exclusion Principle) 1.9: Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

We can apply this result to obtain a similar formula for three sets:

Corollary 1.10: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.8) may be used to further generalize this result to any number of finite sets.

Introduction on Sets

Principle of Inclusion and Exclusion:

EXAMPLE 1.8 Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students: (a) only on list A, (b) only on list B, (c) on list A or B (or both), (d) on exactly one list.

(a) List A has 30 names and 20 are on list B; hence $30 - 20 = 10$ names are only on list A.

(b) Similarly, $35 - 20 = 15$ are only on list B.

(c) We seek $n(A \cup B)$. By inclusion–exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$$

In other words, we combine the two lists and then cross out the 20 names which appear twice.

(d) By (a) and (b), $10 + 15 = 25$ names are only on one list; that is, $n(A \oplus B) = 25$.

Introduction on Sets

Relations:

- A relation is mostly suitable in comparing the objects which are related to one another.
- Examples : father to son, mother to son, less than , greater than, parents and child, coincidence of three lines.

Definition and Properties:

- Let P and Q be two non- empty sets. A binary relation R is defined to be a subset of P x Q from a set P to Q. If (a, b) ∈ R and R ⊆ P x Q then a is related to b by R i.e., aRb. If sets P and Q are equal, then we say R ⊆ P x P is a relation on P.

Introduction on Sets

Examples:

(i) Let A = {a, b, c}

B = {r, s, t}

Then R = {(a, r), (b, r), (b, t), (c, s)}

is a relation from A to B.

(ii) Let A = {1, 2, 3} and B = A

R = {(1, 1), (2, 2), (3, 3)}

is a relation (equal) on A.

Introduction on Sets

Domain and Range of Relation:

- The Domain of relation R is the set of elements in P which are related to some elements in Q, or it is the set of all first entries of the ordered pairs in R. It is denoted by DOM (R).
- The range of relation R is the set of elements in Q which are related to some element in P, or it is the set of all second entries of the ordered pairs in R. It is denoted by RAN (R).

Introduction on Sets

Domain and Range of Relation:

Let, $A = \{1, 2, 9\}$ and $B = \{1, 3, 7\}$

- Case 1 – If relation R is 'equal to' then $R = \{(1, 1), (3, 3)\}$

  Dom(R) = $\{1, 3\}, Ran(R) = \{1, 3\}$

- Case 2 – If relation R is 'less than' then $R = \{(1, 3), (1, 7), (2, 3), (2, 7)\}$

  Dom(R) = $\{1, 2\}, Ran(R) = \{3, 7\}$

- Case 3 – If relation R is 'greater than' then $R = \{(2, 1), (9, 1), (9, 3), (9, 7)\}$

  Dom(R) = $\{2, 9\}, Ran(R) = \{1, 3, 7\}$

Introduction on Sets

Properties of a binary Relation:

- A relation R on set A is called **Reflexive** if ∀a ∈A is related to a (aRa holds).

Example – The relation $R = \{(a, a), (b, b)\}$ on set $X = \{a, b\}$ is reflexive.

- A relation R on set A is called **Irreflexive** if no a∈A is related to a (aRa does not hold).

Example – The relation $R = \{(a, b), (b, a)\}$ on set $X = \{a, b\}$ is irreflexive.

- A relation R on set A is called **Symmetric** if xRy implies yRx, ∀ x ∈ A and ∀y ∈ A.

Example – The relation $R = \{(1, 2), (2, 1), (3, 2), (2, 3)\}$ on set $A = \{1, 2, 3\}$ is symmetric.

Introduction on Sets

Properties of a binary Relation:

- A relation R on set A is called **Anti-Symmetric** if xRy and yRx implies x=y ∀ x∈ A and ∀y ∈ A.

Example – The relation $R = \{(x, y) \to N | \, x \leq y\}$ is anti-symmetric since $x \leq y$ and $y \leq x$ implies $x = y$.

- An **asymmetric binary relation** is similar to antisymmetric relation. The difference is that an asymmetric relation never has both elements aRb and bRa even if a=b.
- Every asymmetric relation is also antisymmetric. The converse is not true. If an antisymmetric relation contains an element of kind (a,a), it cannot be asymmetric. Thus, a binary relation R is asymmetric if and only if it is both antisymmetric and irreflexive.

Introduction on Sets

Properties of a binary Relation:

- Examples of asymmetric relations:

1. The relation $>$ ("is greater than") on the set of real numbers.
2. The family relation "is father of".
3. The relation $R = \{(2, 1), (2, 3), (3, 1)\}$ on the set $A = \{1, 2, 3\}$.

- A relation R on set A is called **Transitive** if xRy and yRz implies xRz, ∀ x, y, z ∈ A.

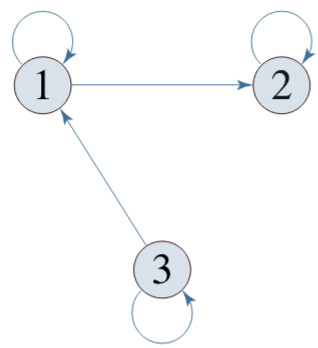
Example – The relation $R = \{(1,2),(2,3),(1,3)\}$ on set $A = \{1,2,3\}$ is transitive.

Introduction on Sets

Examples of reflexive relations:

1. The relation $\geq$ ("is greater than or equal to") on the set of real numbers.
2. Similarity of triangles.
3. The relation $R = \{(1,1), (1,2), (2,2), (3,3), (3,1)\}$ on the set $A = \{1,2,3\}$.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

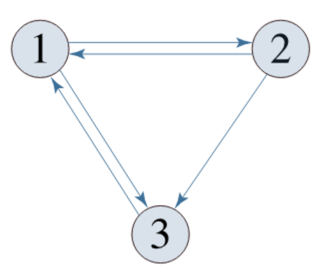


Introduction on Sets

Examples of irreflexive relations:

1. The relation $<$ ("is less than") on the set of real numbers.
2. Relation of one person being son of another person.
3. The relation $R = \{(1,2), (2,1), (1,3), (2,3), (3,1)\}$ on the set $A = \{1,2,3\}$.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

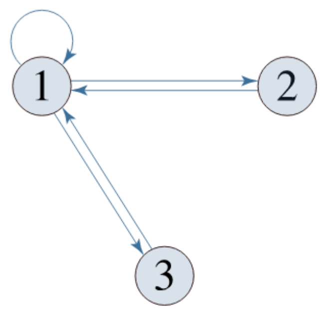


Introduction on Sets

- **Examples of symmetric relations:**

1. The relation $=$ ("is equal to") on the set of real numbers.
2. The relation "is perpendicular to" on the set of straight lines in a plane.
3. The relation $R = \{(1,1), (1,2), (2,1), (1,3), (3,1)\}$ on the set $A = \{1,2,3\}$.

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

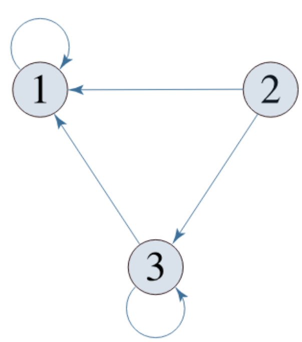


Introduction on Sets

- **Examples of antisymmetric relations:**

1. The relation $\geq$ ("is greater than or equal to") on the set of real numbers.
2. The subset relation $\subseteq$ on a power set.
3. The relation $R = \{(1,1), (2,1), (2,3), (3,1), (3,3)\}$ on the set $A = \{1,2,3\}$.

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

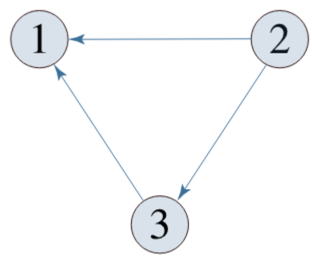


Introduction on Sets

- **Examples of asymmetric relations:**

1. The relation $>$ ("is greater than") on the set of real numbers.
2. The family relation "is father of".
3. The relation $R = \{(2,1),(2,3),(3,1)\}$ on the set $A = \{1,2,3\}$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

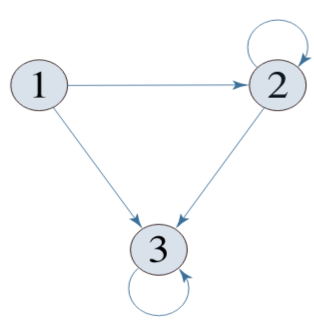


Introduction on Sets

- **Examples of transitive relations:**

1. The relation $>$ ("is greater than") on the set of real numbers.
2. The relation "is parallel to" on the set of straight lines.
3. The relation $R = \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$ on the set $A = \{1,2,3\}$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



Introduction on Sets

Relation matrix and Digraph:

Let P = [a_1,a_2,a_3, . . . a_m] and Q = [b_1,b_2,b_3, . . . b_n] are finite sets, containing m and n number of elements respectively. R is a relation from P to Q. The relation R can be represented by m x n matrix M = [M_{ij}], defined as

M_{ij} = 0 if $(a_i,b_j) \notin R$

1 if $(a_i,b_j) \in R$

Let P = {1, 2, 3, 4}, Q = {a, b, c, d}

and R = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}.

Introduction on Sets

Relation matrix and Digraph:

- The Relation matrix reflects some properties of a relation in a set.

1. Reflexive = if all diagonal entries is ‘1’.
2. Irreflexive = if all diagonal entries is ‘0’.
3. Symmetric = M_{ij} = 1 and M_{ji} = 1 for all i, j i is not equal to j.
4. Anti-Symmetric = M_{ij} = 1 and M_{ji} = 0 for all i, j i is not equal to j.

Introduction on Sets

Relation matrix and Digraph:

Transitive Algorithm:

Given a relation matrix R which represent a Symmetric and reflexive relation, it is required to determine if this relation is transitive.

The relation is in the set {1, 2. ... , n}. If it is transitive then FLAG, which is initially false, is set to true.

1. [Scan each row.] Repeat steps 2 and 3 for i = 1,2, . . . , n-2.
2. [Scan to right of diagonal] Repeat step 3 for i = i+1, i+2, . . . , n-1.
3. [Transitive?] If R[i,j] = T then repeat for k = j+ 1, j+2, . . . , n.
   If R[i, k] = T and R[j, k] = F then Exit.

4 [Successful test.] Set FLAG $\leftarrow$ T and Exit.

Introduction on Sets

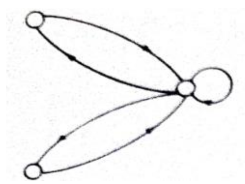
Relation matrix and Digraph:

- A relation can also be represented pictorially by drawing the graph.
- Let R be a relation in a set X = { x_1, x_2, . . . , x_m }. The elements of X are represented by points or circles called nodes. The nodes corresponds to x_i and x_j are labelled x_i and x_j .
- If x_i R x_j , that is , if $(x_i, x_j) \in R$, then we connect node x_i and x_j by means of an arc and put an arrow on the arc in the direction from x_i to x_j.
- If x_i R x_j and x_jR x_i then we have to draw two arc between x_i and x_j .
- For the sake of simplicity, we may replace the two arcs by one arc with arrows pointing in both directions.

Introduction on Sets

Relation matrix and Digraph:

- **From the graph,** It is possible to observe some of its properties.
- **Reflexive:** a loop at each node.
- **Irreflexive:** no loop at any node.
- **Symmetric:** if one node is connected to another, then there is return arc from second node to first node.

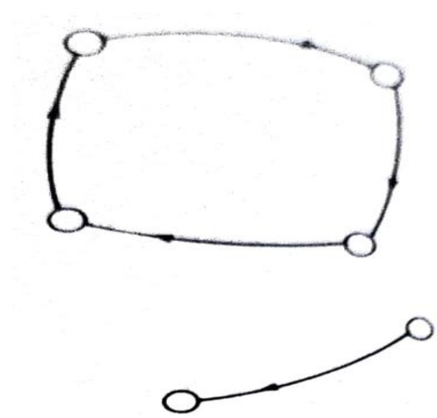


- **Transitive:** the graph contain loops of the following

Introduction on Sets

Relation matrix and Digraph:

- **Anti-Symmetric:** No direct path exist from one node to another node. Refer the following diagram.

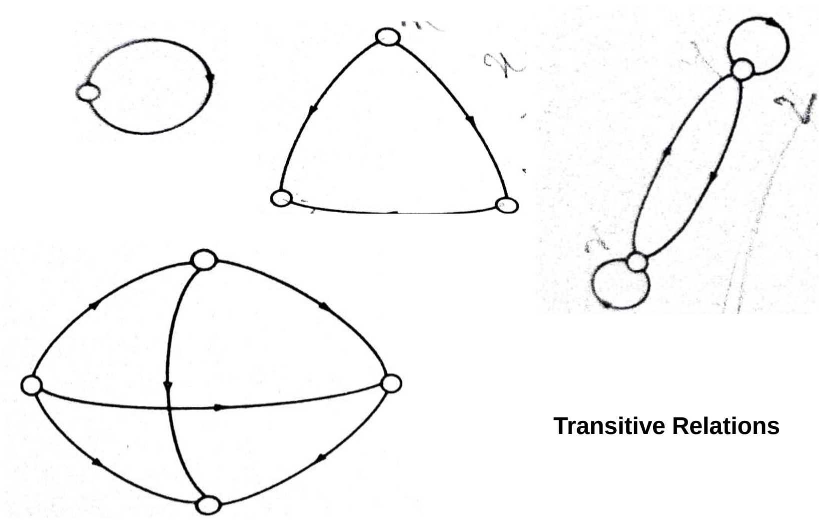


Anti-Symmetric

Introduction on Sets

Relation matrix and Digraph:

- **Transitive:** the graph contain loops of the following



Transitive Relations

Introduction on Sets

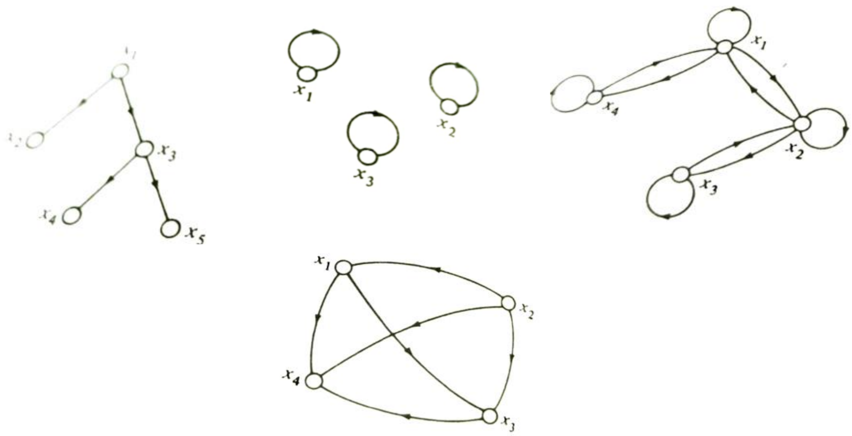
Relation matrix and Digraph:

- **Example 1:** Let X = { 1, 2, 3, 4 } and R = { (x, y) | x > y }. Draw the graph of R and also give its matrix. Also specify the type of relation.
- **Example 2:** Determine the properties of the relation given by the following graphs and also write the corresponding relation matrices.

Introduction on Sets

Relation matrix and Digraph:

- **Example 2:** Determine the properties of the relation given by the following graphs and also write the corresponding relation matrices.



Introduction on Sets

Operations on Relations:

1. Intersection of Relations
2. Union of Relations
3. Difference of Relations
4. Symmetric Difference of Relations
5. Complement of a Binary Relation
6. Converse of a Binary Relation

Introduction on Sets

Intersection of Relations:

- The intersection of the relations $R \cap S$ is defined by $R \cap S = \{(a, b) \mid aRb \text{ and } aSb\}$, where $a \in A$ and $b \in B$.
- If the relations and are defined by matrices $M_R = [a_{ij}]$ and $M_S = [b_{ij}]$, the matrix o f their intersection is given by

$$M_{R \cap S} = M_R * M_S = [a_{ij} * b_{ij}]$$

- Where the product operation is performed as element-wise multiplication.

Union of Relations:

- The intersection of the relations $R \cup S$ is defined by $R \cup S = \{(a, b) \mid aRb \text{ or } aSb\}$, where $a \in A$ and $b \in B$.

Introduction on Sets

Union of Relations:

- The intersection of the relations $R \cup S$ is defined by $R \cup S = \{(a, b) \mid aRb \text{ or } aSb\}$, where $a \in A$ and $b \in B$.
- If the relations R and S are defined by matrices $M_R = [a_{ij}]$ and $M_S = [b_{ij}]$ the union of the relations is given by the following matrix:

$$M_{R \cup S} = M_R + M_S = [a_{ij} + b_{ij}],$$

where the sum of the elements is calculated by the rules

$$0 + 0 = 0, \quad 1 + 0 = 0 + 1 = 1, \quad 1 + 1 = 1.$$

Introduction on Sets

Difference of Relations:

The difference of two relations is defined as follows:

$$R \backslash S = \{(a,b) \mid aRb \text{ and not } aSb\},$$

$$S \backslash R = \{(a,b) \mid aSb \text{ and not } aRb\},$$

where $a \in A$ and $b \in B$.

Suppose $A = \{a,b,c,d\}$ and $B = \{1,2,3\}$. The relations R and S have the form

$$R = \{(a,1),(b,2),(c,3),(d,1)\}, \quad S = \{(a,1),(b,1),(c,1),(d,1)\}.$$

Then the relation differences $R \backslash S$ and $S \backslash R$ are given by

$$R \backslash S = \{(b,2),(c,3)\}, \quad S \backslash R = \{(b,1),(c,1)\}.$$

Introduction on Sets

Symmetric Difference of Relations

The symmetric difference of two binary relations R and S is the binary relation defined as

$$R \triangle S = (R \cup S) \backslash (R \cap S), \quad \text{or} \quad R \triangle S = (R \backslash S) \cup (S \backslash R).$$

Let R and S be relations of the previous example. Then

$$R \triangle S = \{(b,2),(c,3)\} \cup \{(b,1),(c,1)\} = \{(b,1),(c,1),(b,2),(c,3)\}.$$

Introduction on Sets

Complement of a Binary Relation

Suppose that R is a binary relation between two sets A and B. The complement of R over A and B is the binary relation defined as

$$\bar{R} = \{(a, b) \mid \text{not } aRb\},$$

where $a \in A$ and $b \in B$.

For example, let $A = \{1, 2\}$, $B = \{1, 2, 3\}$. If a relation R between sets A and B is given by

$$R = \{(1, 2), (1, 3), (2, 2), (2, 3)\},$$

then the complement of R has the form

$$\bar{R} = \{(1, 1), (2, 1)\}.$$

Introduction on Sets

Converse of a Binary Relation

Let R be a binary relation on sets A and B. The converse relation or transpose of R over A and B is obtained by switching the order of the elements:

$$R^T = \{(b, a) \mid aRb\},$$

where $a \in A, b \in B$.

So, if $R = \{(1, 2), (1, 3), (1, 4)\}$, then the converse of R is

$$R^T = \{(2, 1), (3, 1), (4, 1)\}.$$

If a relation R is defined by a matrix M, then the converse relation R^T will be represented by the transpose matrix M^T (formed by interchanging the rows and columns). For example,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad M^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Sometimes the converse relation is also called the inverse relation and denoted by R^{-1}.

Introduction on Sets

Other Types of Relations :
Empty, Universal and Identity Relations:

A relation R between sets A and B is called an empty relation if $R = \varnothing$.

The universal relation between sets A and B, denoted by U, is the Cartesian product of the sets: $U = A \times B$.

A relation R defined on a set A is called the identity relation (denoted by I) if $I = \{(a, a) \mid \forall a \in A\}$.

Introduction on Sets

Properties of Combined Relations:

Relation	*Reflexive*	*Irreflexive*	*Symmetric*	*Antisymmetric*	*Asymmetric*	*Transitive*
R	Y	Y	Y	Y	Y	Y
S	Y	Y	Y	Y	Y	Y
$R \cap S$	Y	Y	Y	Y	Y	Y
$R \cup S$	Y	Y	Y	N	N	N
$R \setminus S$	N	Y	Y	Y	Y	Y
$S \setminus R$	N	Y	Y	Y	Y	Y
$R \Delta S$	N	Y	Y	N	N	N

Introduction on Sets

Partition and Covering of a set:

Partition on Set A:

It is defined to be a set of non-empty subsets Ai, which are pairwise disjoint as there is no intersection and whose union yields to original set A. This means that the two condition that are to be satisfied are:

$$A_i \cap A_j = \phi \text{ for each } (i, j) \in n ; i \neq j$$

$$A_i \cup A_j = A \ (\text{or}) \bigcup_{i \in n} A_i = A$$

- Partition on Set A is indicated as given below:

$$\pi(\,A)$$

Introduction on Sets

Partition and Covering of a set:

<u>Covering on Set A:</u>

It is defined as a set on non-empty subsets A_i, whose union leads to the original set A and which are need not be pairwise disjoint. Here are the two conditions that are to be satisfied:

$$\bigcup_{i \in n} A_i = A$$

$$A_i \cap A_j \neq \phi \text{ for each } (i, j) \in n ; i \neq j$$

Introduction on Sets

Transitive Closure:

- Before to discuss about Transitive closure. Let we discuss first **what is closure of a relation.**
- The closure of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain **property P.**
- In terms **of the digraph representation of R:**

1. To find the reflexive closure - add loops
2. To find the symmetric closure - **add arcs in the opposite direction.**
3. To find the transitive closure - **if there is a path from a to b,** add an arc from a to b.

Introduction on Sets

Transitive Closure:

- To find the transitive closure of a relation is equivalent to determining which pairs of vertices in the associated directed graph are connected by a path.
- Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R. **Therefore the transitive closure of a relation R equals the connectivity relation R^*.**

Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

Introduction on Sets

Transitive Closure:

Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: By Theorem 3, it follows that the zero–one matrix of R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

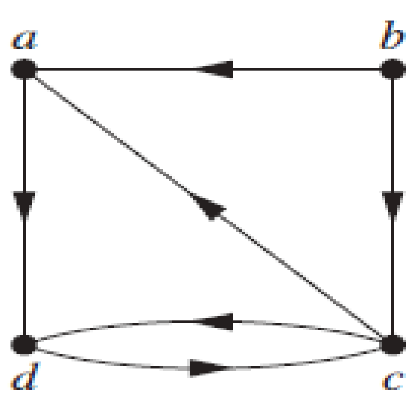
it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Introduction on Sets

Transitive Closure:

Calculate Transitive Closure of the following digraph.



Introduction on Sets

Transitive Closure:

- find the transitive closures of these relations on {1, 2, 3, 4}.

a) $\{(1,2),(2,1),(2,3),(3,4),(4,1)\}$

b) $\{(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$

c) $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

d) $\{(1,1),(1,4),(2,1),(2,3),(3,1),(3,2),(3,4),(4,2)\}$

Introduction on Sets

Transitive Closure:

- find the transitive closures of these relations on {a, b, c, d, e}.

a) $\{(a,c),(b,d),(c,a),(d,b),(e,d)\}$

b) $\{(b,c),(b,e),(c,e),(d,a),(e,b),(e,c)\}$

c) $\{(a,b),(a,c),(a,e),(b,a),(b,c),(c,a),(c,b),(d,a),(e,d)\}$

d) $\{(a,e),(b,a),(b,d),(c,d),(d,a),(d,c),(e,a),(e,b),(e,c),(e,e)\}$

Introduction on Sets

Equivalence Relation:

- A Relation R in a set X is called an equivalence relation if it is reflexive, symmetric and transitive.
- Examples:

1. Equality of numbers on a set of real numbers.
2. Equality of subsets of a universal set.
3. Similarity of lines being parallel on a set of lines in a plane.
4. Similarity of triangles on the set of triangles.

Example 1:

Let X = { 1, 2, 3, 4 } and R = { (1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3) } find the relation is equivalence or not.

Example 2:

Let X = { 1, 2, … , 7} and R = { (x, y) | x – y is divisible by 3 } find the relation is equivalence or not.

Introduction on Sets

Compatibility Relation:

- A Relation R in a set X is called an Compatibility relation if it is reflexive and symmetric.
- All Equivalence Relations are compatibility relations.

Example 1:

Let X = { ball, bed, dog, let, egg} and let the relation R be given by
$\mathbf{R} = \{ \langle \mathbf{x}, \mathbf{y} \rangle \mid \mathbf{x}, \mathbf{y} \in \mathbf{X} \ \wedge \mathbf{x} \mathbf{R} \mathbf{y}$ **if x and y contain some common letter**$\}$

- R is a compatibility relations, and x, y are called compatible if x R y. A compatibility relation is sometime denoted by $\approx$.

Introduction on Sets

Partial Ordering:

- A binary Relation R in a set P is called a partial order relation or a partial ordering in P if R is Reflexive, antisymmetric and transitive.
- It is conventional to denote a partial ordering by the symbol $\leq$.
- If $\leq$ is a partial ordering on P, then the Ordered pair $\langle P, \leq \rangle$ is called a partial ordered set or a poset.
- Let $\langle P, \leq \rangle$ be a partially ordered set. If for every
- $x, y \in P$ we have $x \leq y \vee y \leq x$ then $\leq$ is called a simple ordering on P, and $\langle P, \leq \rangle$ is called totally ordered or simply ordered set or a chain.
- Converse of a partial ordered set is also a partially ordered set.

Introduction on Sets

Partial Ordering:

- Examples:

1. Less than or Equal to, Greater than or equal to
2. Inclusion
3. Divides and Integral multiple.
4. Lexicographic Odering

Introduction on Sets

Hasse Diagram:

- A partial ordering set $\leq$ on a set P can be represented by means of a diagram known as a Hasse Diagram or a Partially ordered set diagram of $\langle P, \leq \rangle$.
- In such a diagram, Such element is represented by a small circle or a dot.
- The circle for x ∈ P is drawn below the circle for y ∈ P if x < y, and a line is drawn between x and y, if y covers x. if x < y but does not cover x then x and y are not connected directly by a single line. However they are connected through one or more elements of P.

Example 1: Let X = { 2, 3, 6, 12, 24, 36 } and the relations $\leq$ be such that $x \leq y \; if \; x \; divides \; y$. Draw the Hasse diagram of $\langle X, \leq \rangle$.

Introduction on Sets

Hasse Diagram:

1. Draw the Hasse diagram of the following sets under the partial ordering relation "divides" and indicate those which are totally ordered.

   {2, 6, 24} { 3, 5, 15} {1,2,3,6, 12} {2, 4, 8, 16} {3, 9, 27, 54}

Functions

Introduction:

- Let X and B any two sets be any two sets. A relation f from X to Y is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $\langle x, y \rangle \in f$.
- Terms such as "transformation", "map" (or "mapping"), "correspondence", and "operations" are used as synonyms for "functions".
- The notatons $f: X \rightarrow Y$ *or* $X \xrightarrow{f} Y$ are used to express f as a function from X to Y..
- If $\langle x, y \rangle \in f$, then x is called an **argument** and y is called an **image** of x under *f.*
- The range of y is defined as follows:

$$\{\, y \mid \exists x \in X \wedge y = f(x)\}$$

Functions

Introduction: (Bijective Function)

- If $f: X \rightarrow Y$ and $A \subseteq X$, then $f \cap (A\ X\ Y)$ is a function from $A \rightarrow Y$ called the restriction of f *to* A and some times written as *f/A.*
- *If g is a restriction of f, then f is called the extension of g.*
- We know that not all possible subsets of $X \times Y$ are functions from X to Y. The collection of all those subsets of $X \times Y$ which define a function *by* Y^x.
- A mapping of $f: X \rightarrow Y$ is called **onto** (surjective, a surjection) if the range $R_f = Y$ *other wise it is called* **into.**

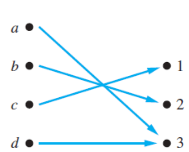


FIGURE 4 An onto function.

Functions

Introduction:

- A mapping $of\ f: X \rightarrow Y$ is called **one-to-one function** (injective , 1-1) if distinct elements of X are mapped into distinct elements of Y.

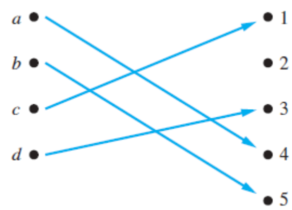


FIGURE 3 **A one-to-one function.**

- A mapping $of\ f: X \rightarrow Y$ is called **one-to-one onto function (bijective)** if it is both one-to-one and onto. Such a mapping is also called one-to-one correspondence between X and Y.

Functions

Introduction: Bijective

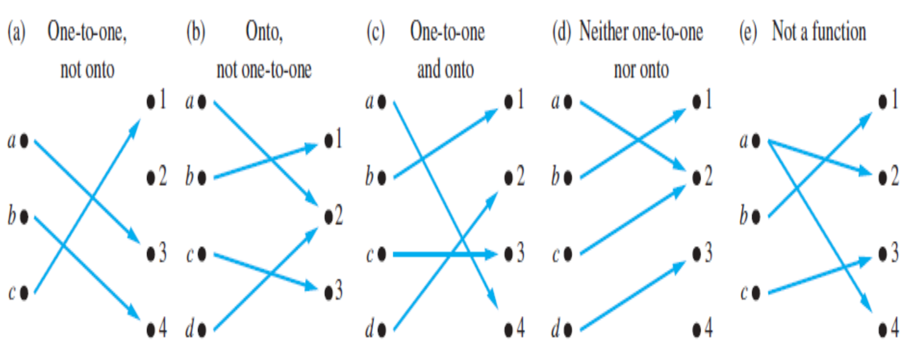


FIGURE 5 **Examples of different types of correspondences.**

Functions

Composition of functions:

Two functions $f : A \to B$ and $g : B \to C$ can be composed to give a composition gof .

This is a function from A to C defined by $(gof)(x) = g(f(x))$

Example

Let $f(x) = x + 2$ and $g(x) = 2x + 1$, find $(fog)(x)$ and $(gof)(x)$.

Solution

$$(fog)(x) = f(g(x)) = f(2x+1) = 2x+1+2 = 2x+3$$

$$(gof)(x) = g(f(x)) = g(x+2) = 2(x+2)+1 = 2x+5$$

Hence, $(fog)(x) \neq (gof)(x)$

Functions

Composition of functions:

Some Facts about Composition

- If f and g are one-to-one then the function (gof) is also one-to-one.
- If f and g are onto then the function (gof) is also onto.
- Composition always holds associative property but does not hold commutative property.

Functions

Inverse Function:

The **inverse** of a one-to-one corresponding function $f : A \rightarrow B$, is the function $g : B \rightarrow A$, holding the following property –

$f(x) = y \Leftrightarrow g(y) = x$

The function f is called **invertible**, if its inverse function g exists.

Example

- A Function $f : Z \rightarrow Z, f(x) = x + 5$, is invertible since it has the inverse function $g : Z \rightarrow Z, g(x) = x - 5$.
- A Function $f : Z \rightarrow Z, f(x) = x^2$ is not invertiable since this is not one-to-one as $(-x)^2 = x^2$.

Functions

Recursive functions:

- A function is said to be recursively defined if the function definition refers to itself. In order for the definition not to be circular, the function definition must have the following two properties.

1. There must be certain arguments, called base values, for which the function does not refer to itself.
2. Each time the function does refer to itself, the argument of the function must be closer to a base value.

Functions

Recursive functions:

- **Factorial Function**

Definition 3.1 (Factorial Function):

(a) If $n = 0$, then $n! = 1$.

(b) If $n > 0$, then $n! = n \cdot (n-1)!$

Observe that the above definition of $n!$ is recursive, since it refers to itself when it uses $(n-1)!$. However:

(1) The value of $n!$ is explicitly given when $n = 0$ (thus 0 is a base value).

(2) The value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0.

Functions

Permutation Function:

- A Bijection from a Set A to itself is called a **permutation of A**

Example 1. Let $A = \mathbb{R}$ and let $f: A \to A$ be defined by $f(a) = 2a + 1$. Since f is one to one and onto (verify), it follows that f is a permutation of A. ♦

If $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set and p is a bijection on A, we list the elements of A and the corresponding function values $p(a_1), p(a_2), \ldots, p(a_n)$ in the following form:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ p(a_1) & p(a_2) & \cdots & p(a_n) \end{pmatrix}. \qquad (1)$$

Functions

Permutation Function:

- A Bijection from a Set A to itself is called a **permutation of A**

Example 2. Let $A = \{1, 2, 3\}$. Then all the permutations of A are

$$1_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}. \quad \blacklozenge$$

Example 3. Using the permutations of Example 2, compute (a) p_4^{-1}; (b) $p_3 \circ p_2$.

Functions

Permutation Function:

- A Bijection from a Set A to itself is called a **permutation of A**

$$p_3 \circ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Functions

Permutation Function:

- **Cycle Permutation**

Figure 5.4

Let $b_1, b_2, \ldots, b_r$ be r distinct elements of the set $A = \{a_1, a_2, \ldots, a_n\}$. The permutation $p : A \to A$ defined by

$$\begin{aligned} p(b_1) &= b_2 \\ p(b_2) &= b_3 \\ &\vdots \\ p(b_{r-1}) &= b_r \\ p(b_r) &= b_1 \\ p(x) &= x, \qquad \text{if } x \in A, \quad x \notin \{b_1, b_2, \ldots, b_r\}. \end{aligned}$$

is called a **cyclic permutation** of length r, or simply a **cycle** of length r, and will be denoted by $(b_1, b_2, \ldots, b_r)$. Do not confuse this terminology with that used for

Functions

Permutation Function:

- **Cycle Permutation**

Example 4. Let $A = \{1, 2, 3, 4, 5\}$. The cycle $(1, 3, 5)$ denotes the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}. \quad \blacklozenge$$

Functions

Permutation Function:

- **Cycle Permutation**

Example 5. Let $A = \{1, 2, 3, 4, 5, 6\}$. Compute $(4, 1, 3, 5) \circ (5, 6, 3)$ and $(5, 6, 3) \circ (4, 1, 3, 5)$.

Solution: We have

$$(4, 1, 3, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$$

and

$$(5, 6, 3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned}(4, 1, 3, 5) \circ (5, 6, 3) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}(5, 6, 3) \circ (4, 1, 3, 5) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix}.\end{aligned}$$

Functions

Permutation Function:

Note: Two cycles of a set A are said to be disjoint if no element of A appears in both cycles.

Example 6. Let $A = \{1, 2, 3, 4, 5, 6\}$. Then the cycles $(1, 2, 5)$ and $(3, 4, 6)$ are disjoint, whereas the cycles $(1, 2, 5)$ and $(2, 4, 6)$ are not. ♦

Functions

Permutation Function:

Theorem 2. *A permutation of a finite set that is not the identity or a cycle can be written as a product of disjoint cycles of length* ≥ 2. ◆

Example 7. Write the permutation

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$$

of the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles.

Functions

Permutation Function:

In Exercises 3 and 4, let $A = \{1, 2, 3, 4, 5, 6\}$ *and*

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix}.$$

3. Compute

(a) p_1^{-1} (b) $p_3 \circ p_1$

(c) $(p_2 \circ p_1) \circ p_2$ (d) $p_1 \circ (p_3 \circ p_2^{-1})$

4. Compute

(a) p_3^{-1} (b) $p_1^{-1} \circ p_2^{-1}$

(c) $(p_3 \circ p_2) \circ p_1$ (d) $p_3 \circ (p_2 \circ p_1)^{-1}$

Functions

Permutation Function:

In Exercises 5 and 6, let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$*. Compute the products.*

5. (a) $(3, 5, 7, 8) \circ (1, 3, 2)$
(b) $(2, 6) \circ (3, 5, 7, 8) \circ (2, 5, 3, 4)$

6. (a) $(1, 4) \circ (2, 4, 5, 6) \circ (1, 4, 6, 7)$
(b) $(5, 8) \circ (1, 2, 3, 4) \circ (3, 5, 6, 7)$

7. Let $A = \{a, b, c, d, e, f, g\}$. Compute the products.
(a) $(a, f, g) \circ (b, c, d, e)$
(b) $(f, g) \circ (b, c, f) \circ (a, b, c)$

Functions

Permutation Function:

In Exercises 8 and 9, let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$*. Write each permutation as the product of disjoint cycles.*

8. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 5 & 1 & 8 & 7 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 7 & 5 & 8 & 6 \end{pmatrix}$

9. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 7 & 8 & 4 & 3 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 4 & 6 & 7 & 8 & 5 \end{pmatrix}$

Functions

Permutation Function:

10. Let $A = \{a, b, c, d, e, f, g\}$. Write each permutation as the product of disjoint cycles.

(a) $\begin{pmatrix} a & b & c & d & e & f & g \\ g & d & b & a & c & f & e \end{pmatrix}$

(b) $\begin{pmatrix} a & b & c & d & e & f & g \\ d & e & a & b & g & f & c \end{pmatrix}$

Functions

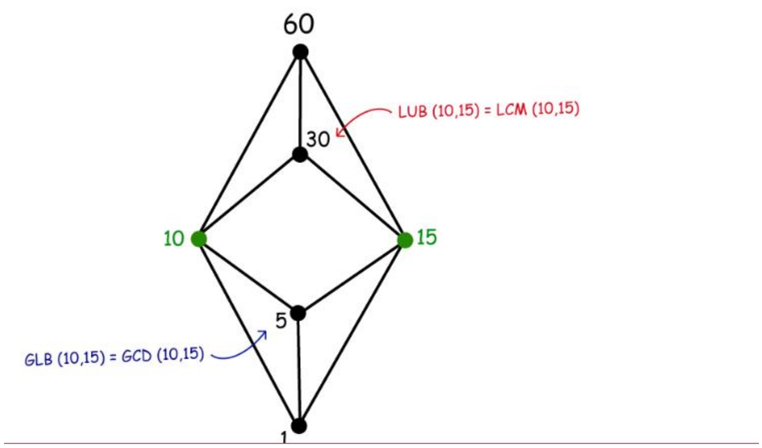
Lattice and its properties:

- Formally, a lattice is a poset, a partially ordered set, in which every pair of elements has both a least upper bound and a greatest lower bound. In other words, it is a structure with two binary operations: **Join, Meet**

Join (or sum): "a join b"	$LUB(a,b) = a \vee b$	$LUB(a,b) = LCM(a,b)$	$\oplus, +, \cup$
Meet (or product): "a meet b"	$GLB(a,b) = a \wedge b$	$GLB(a,b) = GCD(a,b)$	$*, \bullet, \cap$

Functions

Lattice and its properties:



Functions

Lattice and its properties:

- It is important to note that not all partially ordered sets are lattices. So, how do we determine whether or not a poset is a lattice?
- There are three ways we can show that a poset is or is not a lattice:

1. Construct a table for each pair of elements and confirm that each pair has a LUB and GLB.
2. Use the "join and "meet method for each pair of elements.
3. Draw a Hasse diagram and look for comparability.

Functions

Lattice and its properties:

- **Moreover, several types of lattices are worth noting:**
- **Complete Lattice –** all subsets of a poset have a join and meet, such as the divisibility relation for the natural numbers or the power set with the subset relation.
- **Bounded Lattice –** if the lattice has a least and greatest element, denoted 0 and 1 respectively.
- **Complemented Lattice –** a bounded lattice in which every element is complemented. Namely, the complement of 1 is 0, and the complement of 0 is 1.
- **Distributive Lattice –** if for all elements in the poset the distributive property holds.
- **Boolean Lattice –** a complemented distributive lattice, such as the power set with the subset relation.

Functions

Lattice and its properties:

Lattice $(L, \wedge, \vee)$, for $a, b, c \in L$

$$a \vee b = b \Leftrightarrow a \le b$$
$$a \wedge b = a \Leftrightarrow a \le b$$
$$a \wedge b = a \Leftrightarrow a \vee b = b$$

$a \vee a = a$ $a \wedge a = a$	Idempotent Law
$a \vee b = b \vee a$ $a \wedge b = b \wedge a$	Commutative Law
$a \vee (b \vee c) = (a \vee b) \vee c$ $a \wedge (b \wedge c) = (a \wedge b) \wedge a$	Associative Law
$a \vee (a \wedge b) = a$ $a \wedge (a \vee b) = a$	Absorption Law
$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$ $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$	Distributive Law (not all lattices are distributive)

Lattice Properties — Identity Laws