CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 3a: Introduction: Random variables

Prof. Oke

UMass Amherst

College of Engineering

September 23, 2025

Overview of Module 3

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- Lecture 3a: Introduction: Random Variables

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- Lecture 3a: Introduction: Random Variables
- Lecture 3b: **Normal Distribution**

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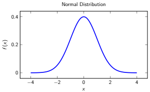
- Lecture 3b: **Normal Distribution** 

- Lecture 3c: **Lognormal and Exponential Distributions**

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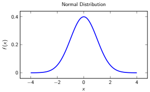
- Lecture 3b: **Normal Distribution** 

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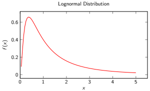

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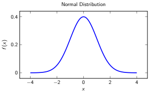

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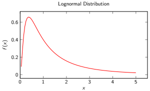

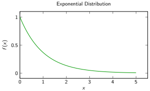

- Lecture 3d: **Binomial Distribution**

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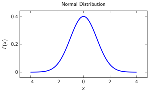

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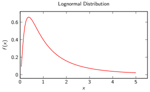

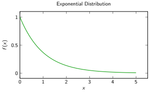

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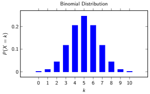

- Lecture 3e: **Poisson Distribution**

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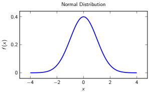

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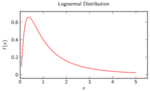

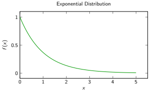

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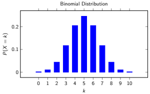

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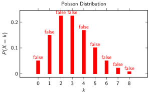

- Lecture 3f: Joint Distributions and further topics

Objectives and outline of today's lecture

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- Understand random variables

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- Distinguish between discrete and continuous random variables

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A random variable X may be:

- *Discrete*
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Describing random variables

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Central values

- Mean

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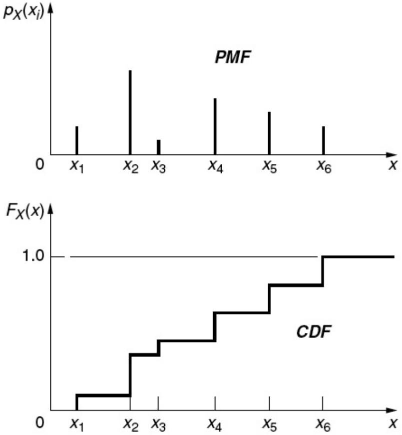

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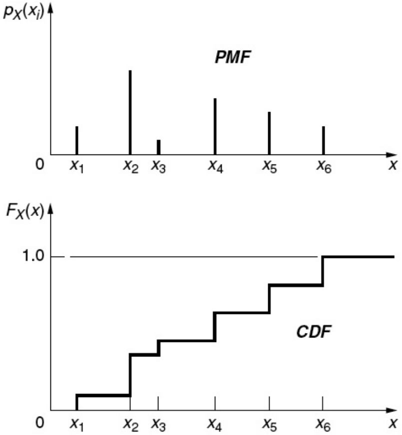

The probability masses in a PMF sum up to 1.

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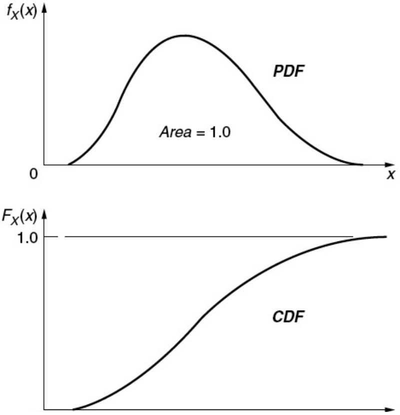

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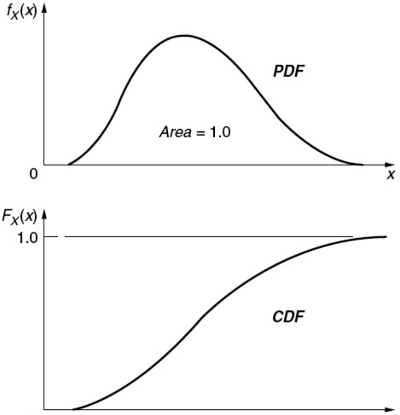

The total area under a PDF is 1.

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3. $F_X(x)$ is continuous to the right with x.

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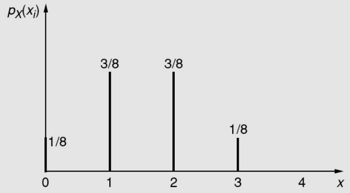

Figure: PMF

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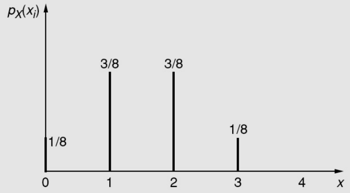

Figure: PMF

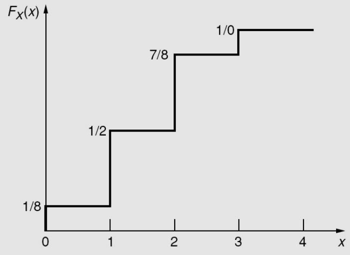

Figure: CDF

Further derivations

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3. For all random variables:

$$P(a < X \leq b) = F_X(b) - F_X(a) \tag{7}$$

Mean and variance

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Mean

Weighted average or expected value

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Mean and variance

Mean

Weighted average or expected value

$$\mathbb{E}(X) = \sum_i x_i p_X(x_i) \quad \text{discrete case} \tag{8}$$

Variance

In the discrete case:

$$\mathbb{V}(X) = \sum_i (x_i - \mu_X)^2 p_X(x_i) \tag{9}$$

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Measures of dispersion (cont.)

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$$\delta_X = \frac{\sigma_X}{\mu_X} \tag{12}$$

Example 2: Bulldozers revisited

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You are given the PMF (probability mass function) of the operating condition X of bulldozers after 6 months.

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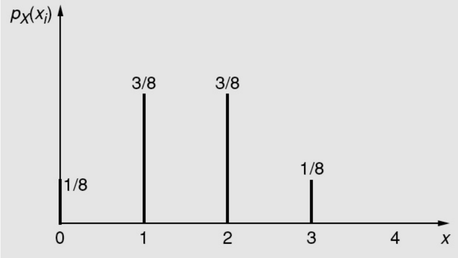

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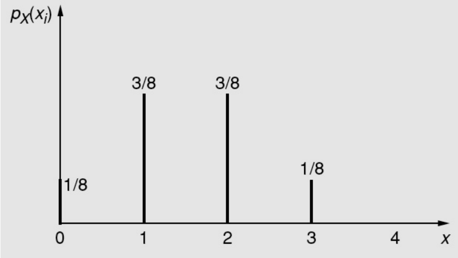

Find the mean, variance, standard deviation and coefficient of variation of X.

Example 2: Bulldozers revisited (cont.)

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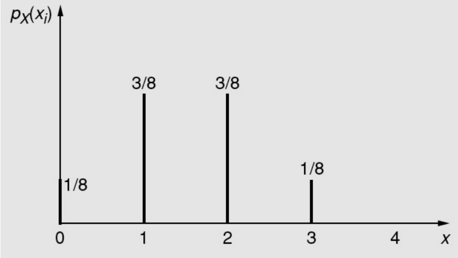

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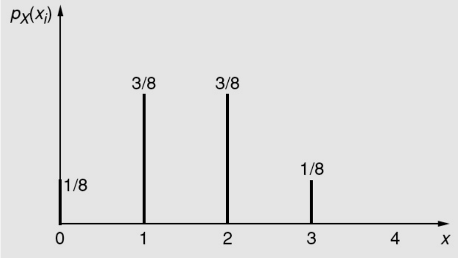

(a) Mean: $\mu_X = \mathbb{E}(X) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 1.5.$

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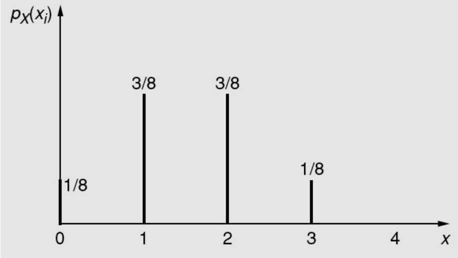

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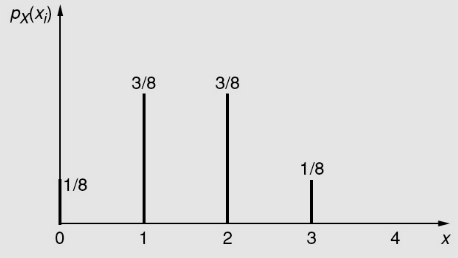

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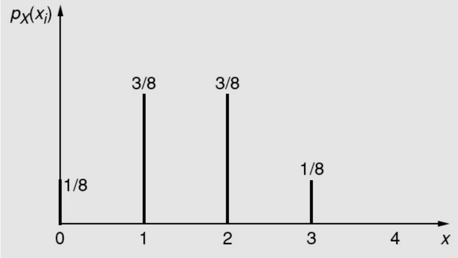

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Mean and variance

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These include the mean, median and mode.

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Example 3: Loaded beam

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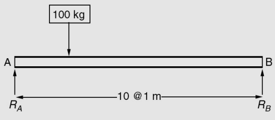
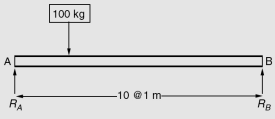


Figure E2.5a Beam AB.

Consider the beam under a 100-kg load. If the load is equally likely to be placed anywhere along the 10m span of the beam,

Example 3: Loaded beam



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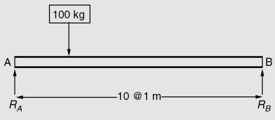

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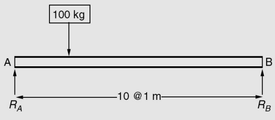

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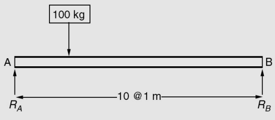

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(a) Plot the PDF of X.

(b) Solve the integral for the CDF and plot.

(c) Find $P(2 < X \leq 5)$.

Example 3: Loaded beam (cont.)

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(a) The area under the PDF must be 1. Thus, c must be $\frac{1}{10}$.

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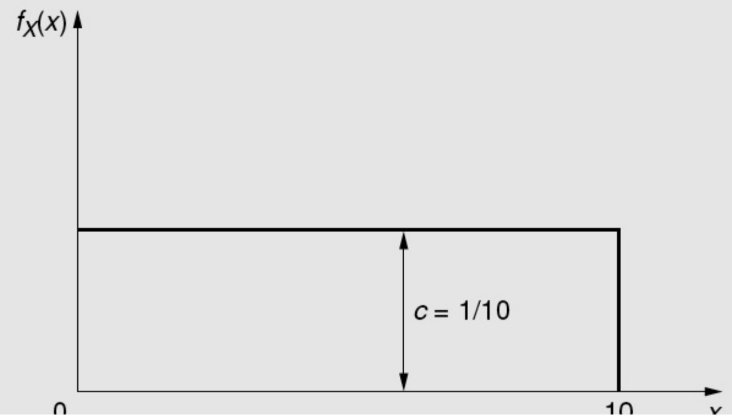

Example 3: Loaded beam (cont.)

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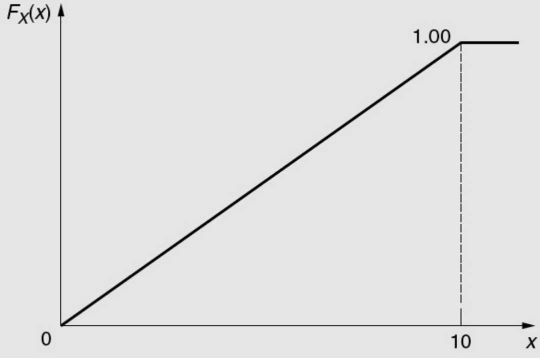

Figure E3.2b CDF of X.

Example 3: Loaded beam (cont.)

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(c) To compute $P(2 < X \leq 5)$, we use the CDF:

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$$P(2 < X \leq 5) \quad = \quad F_X(5) - F_X(2)$$

Example 3: Loaded beam (cont.)

(c) To compute $P(2 < X \leq 5)$, we use the CDF:

$$\begin{aligned} P(2 < X \leq 5) &= F_X(5) - F_X(2) \\ &= \frac{5-2}{10} = \end{aligned}$$

Example 3: Loaded beam (cont.)

(c) To compute $P(2 < X \leq 5)$, we use the CDF:

$$\begin{aligned} P(2 < X \leq 5) &= F_X(5) - F_X(2) \\ &= \frac{5-2}{10} = 0.3 \end{aligned}$$

Example 4: Useful life of machines

The useful life T of welding machines is a random variable with an exponential distribution. The PDF and CDF are:

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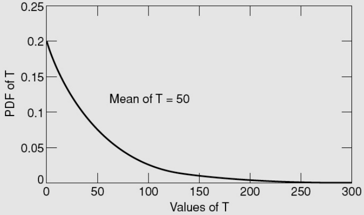

Figure E3.3a Exponential PDF of useful life T.

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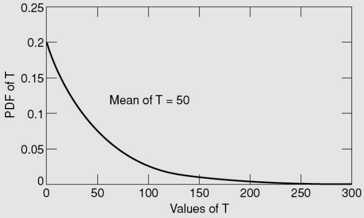

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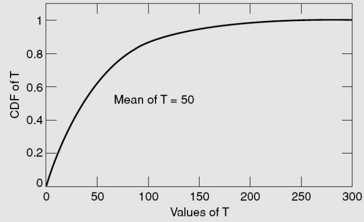

Figure E3.3b CDF of useful life T.

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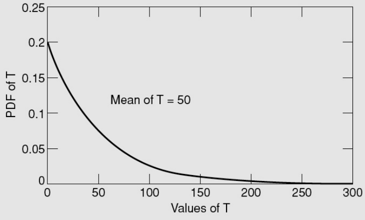

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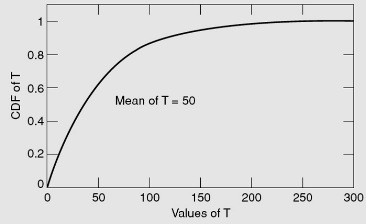

Figure E3.3b CDF of useful life T.

(a) Find the mean of this distribution
(b) Find the median
(c) Show that the variance is $\frac{1}{\lambda^2}$

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Recap

- Random variables

Reading

- Open Intro Statistics Section 3.4 (Random variables)
- Open Intro Statistics Section 3.5 (Continuous distributions)

Recap

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- Probability mass/density function (PMF/PDF) and cumulative distribution function (CDF)

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$$\theta = \frac{\mathbb{E}(X-\mu_X)^3}{\sigma^3} \tag{19}$$

Skewness (cont.)

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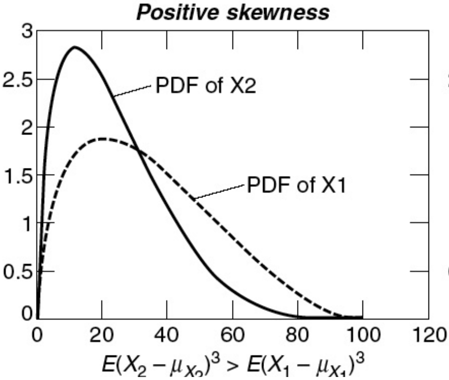

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Skewness vs. kurtosis

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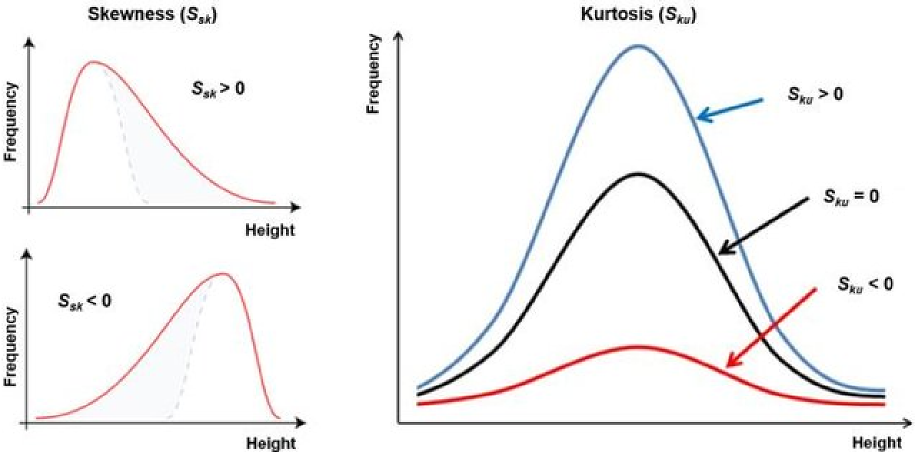

Source: Bonyar, A (2015) "Application of localization factor for the detection of tin oxidation with AFM" DOI: 10.1109/SIITME.2015.7342289

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