CEE 697M: Probabilistic Machine Learning

M2 Linear Methods: Splines, GAMs and GLMs

Jimi Oke

UMass Amherst

College of Engineering

October 7, 2025

Outline

1. Exponential family
2. GLMs
3. Fitting a GLM

The exponential family

The exponential family

A probability distribution belongs to the exponential family if its density can be modeled as:

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Commonly used GLM models and their components

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Linear regression	Gaussian	Identity: $g(\mu_n) = \mu_n = \boldsymbol{w}^\top \boldsymbol{x}_n$
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Probit regression	Bernoulli	Probit: $g(\mu_n) = \Phi^{-1}(\mu_n)$
Multinomial logit/logistic	Categorical	Multinomial logit: $g(\mu_{nc}) = \log\left(\frac{\mu_{nc}}{\mu_{nC}}\right)$
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$$\mathrm{NLL}(\boldsymbol{w}) = -\log p(\mathcal{D}|\boldsymbol{w}) = -\sum_{n=1}^{N} \log p(y_n|\boldsymbol{x}_n, \boldsymbol{w}) = \sum_{n=1}^{N} \frac{A(\eta_n)}{\sigma^2} - \frac{y_n \eta_n}{\sigma^2} \tag{26}$$

If we set $\ell_n = \eta_n y_n - A(\eta_n)$, then the NLL can be written as:

$$\mathrm{NLL}(\boldsymbol{w}) = -\sum_{n=1}^{N} \frac{\ell_n}{\sigma^2} \tag{27}$$

where $\eta_n = \boldsymbol{w}^\top \boldsymbol{x}_n$.
The gradient of the NLL is then given by:

$$\boldsymbol{g}_n = \sum_{n=1}^{N} \frac{y_n - \mu_n}{\sigma^2} \boldsymbol{x}_n \tag{28}$$

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where $\mu_n = A'(\eta_n) = \ell^{-1}(\eta_n)$ is the mean function.