CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 2b: Theory of Probability

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UMass Amherst

College of Engineering

September 16, 2025

Outline

Recap from Lecture 2a

- Set theory and operations: union, intersection, complement

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- De Morgan's Rules are useful for expressing complements of unions or of intersections:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
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- Perform data input, set operations, permutations and combinations in MATLAB

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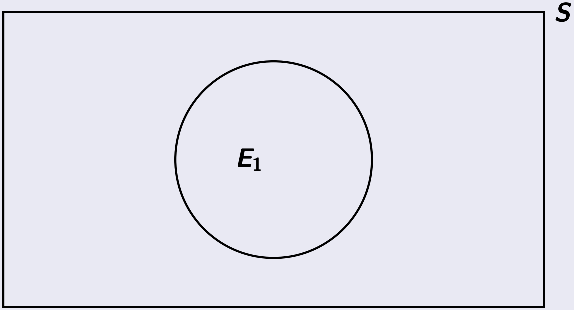

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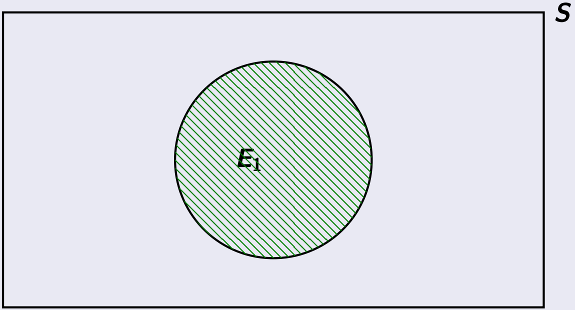

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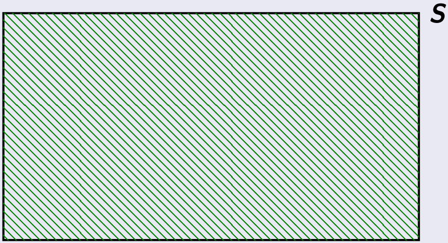

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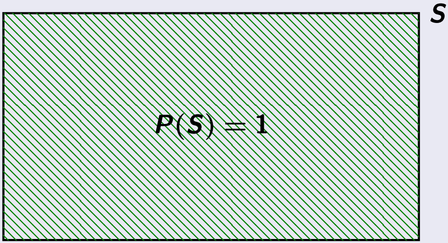

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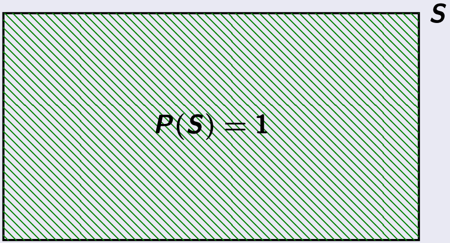

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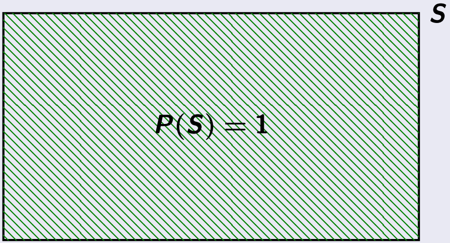

In other words, the probability of the sample space is unity (or 1).

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- $\sum_{i=1}^{n}$ denotes the summation of quantities (in this case, probabilities $P(E_i)$) indexed from 1 through n

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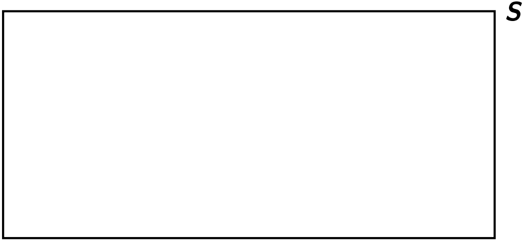

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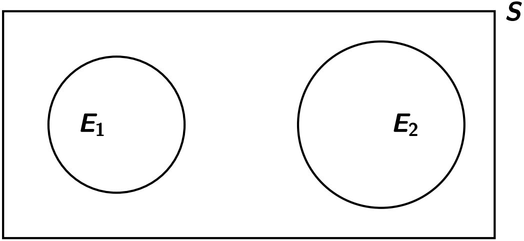

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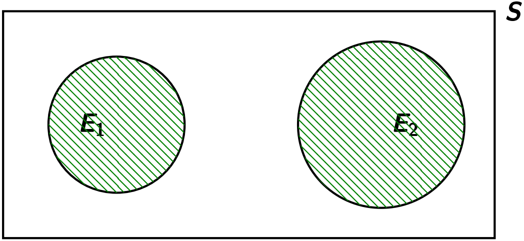

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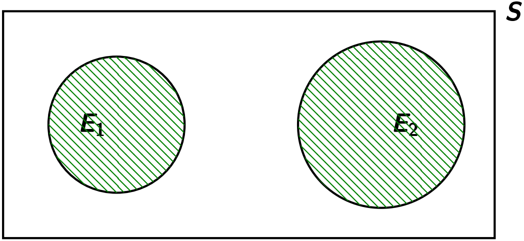

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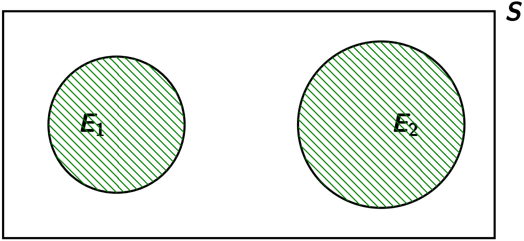

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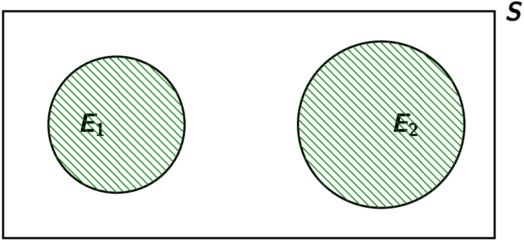

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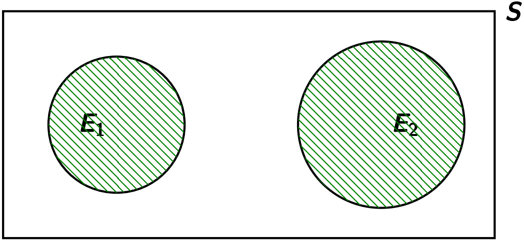

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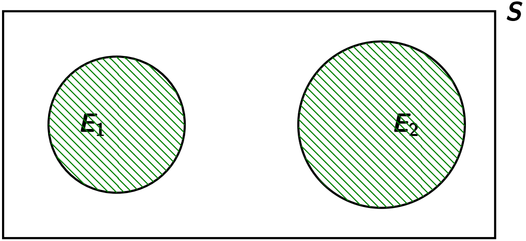

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$$0 \leq P(E) \leq 1 \tag{5}$$

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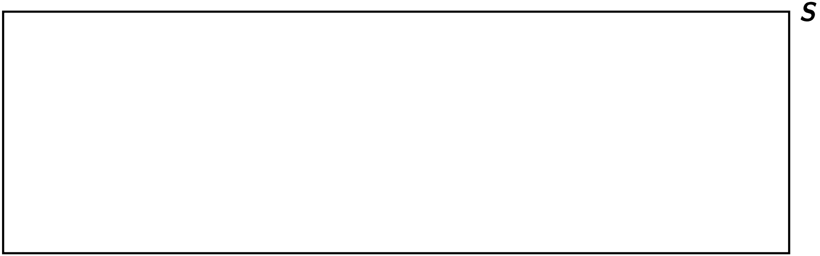

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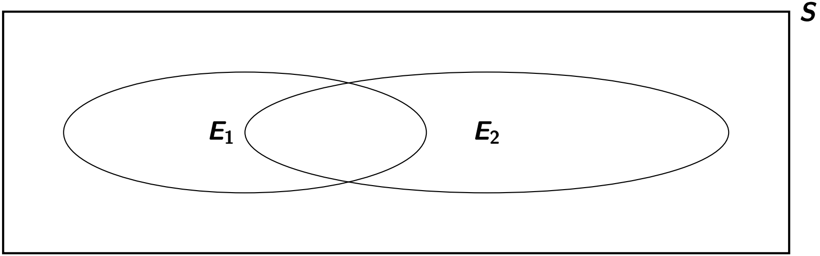

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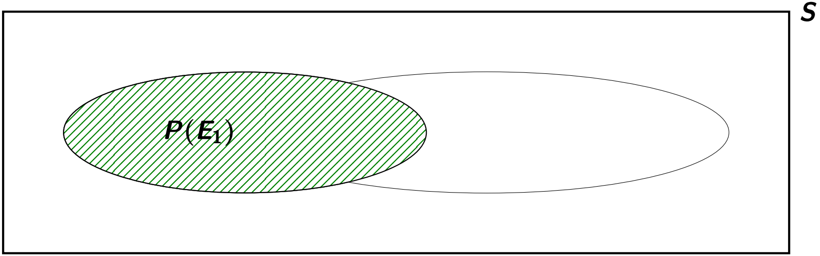

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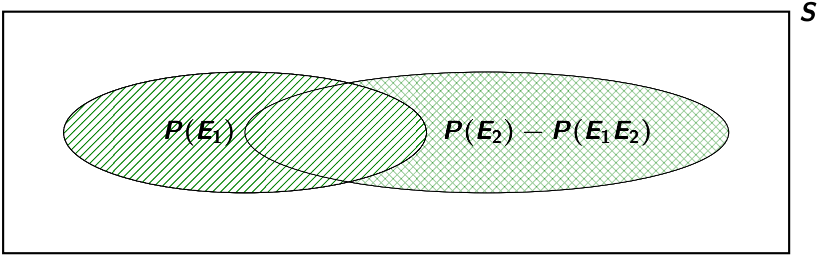

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which yields Axiom 3.

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Equivalently, $P(E) = 1 - P(\overline{E})$.

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Thus, we have

$$P(E \cup \overline{E}) = P(S) = 1 \tag{10}$$

Given that E and $\overline{E}$ are also mutually exclusive, from Eq. 8 we obtain:

$$P(E \cup \overline{E}) = P(E) + P(\overline{E}) = 1 \tag{11}$$

$$\therefore P(\overline{E}) = 1 - P(E) \tag{12}$$

Equivalently, $P(E) = 1 - P(\overline{E})$. *Complement Rule*

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$$E_1 = > 2 \text{ vehicles waiting for left turns}$$

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Find the following probabilities:

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(b) $P(E_2)$
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Fundamental principle of counting

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Given $1, 2, \ldots, k$ operations are to be performed, and $n_1, n_2, \ldots, n_k$ ways of performing each respective operation.

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The total number of possibilities is given by

$$n_1 \times n_2 \times \cdots \times n_k$$

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How many distinct ice cream orders are possible?

$$3 \times 10 \times (4 + 1) = \boxed{150}$$

(Note: you can order an ice cream without any topping, hence 5 topping possibilities

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If in 2021, the state releases a new series of plate numbers that must now end with a letter in addition to 5 preceding digits, how many distinct plates can be manufactured? Answer: $26 \times 10^5 = 2,600,000$

Permutations

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Permutations

A permutation is an arrangement (ordering) of objects in a collection.

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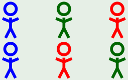

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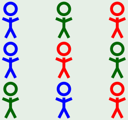

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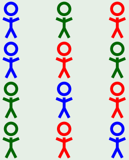

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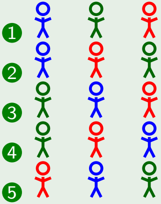

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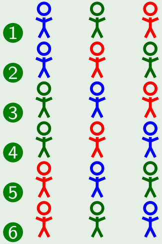

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Permutations with repetition

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Without repetition, the number of permutations of n objecst is $n!$.

However, if there are n_1 identical (or repeated) items of type 1, n_2 identical items of type 2, . . . , and n_k identical items of type k, then the number of permutations is given by

$$\frac{n!}{n_1!n_2! \times \cdots \times n_k!} \tag{15}$$

How many ways can the letters in the word "MASS" be arranged?

The letter "S" is repeated. Thus, the number of permutations are:

$$\frac{4!}{2!} = 4 \times 3 = 12$$

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With combinations, the order is not important (in constrast to permutations).

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You can form 56 different teams.

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&\quad - [P(E_1 E_3 \cup E_2 E_3) \quad \text{addition rule; distributive} \\
&= P(E_1) + P(E_2) - P(E_1 E_2) + P(E_3) \\
&\quad - [P(E_1 E_3) + P(E_2 E_3) - P(E_1 E_3 E_2 E_3) \\
&= P(E_1) + P(E_2) - P(E_1 E_2) + P(E_3) \\
&\quad - P(E_1 E_3) - P(E_2 E_3) + P(E_1 E_2 E_3) \\
\therefore \boldsymbol{P(E_1 \cup E_2 \cup E_3)} &= \boldsymbol{P(E_1) + P(E_2) + P(E_3)} \\
&\quad \boldsymbol{- P(E_1 E_2) - P(E_1 E_3) - P(E_2 E_3) + P(E_1 E_2 E_3)}
\end{aligned} \tag{17}
$$

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Apply de Morgan's rule:

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Apply de Morgan's rule:

$$\therefore P(E_1 \cup E_2 \cup \cdots E_n) = 1 - P(\overline{E_1}\,\overline{E_2} \cdots \overline{E_n}) \tag{19}$$