CEE 697M: Big Data and Machine Learning for Engineers
Lecture 1d: Decision and Information Theories

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Outline

Basics

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or maximizing expected utility $\mathbb{U}(h, a) = -\ell(h, a)$:

$$\pi^*(\boldsymbol{x}) = \arg\max_{a \in \mathcal{A}} \mathbb{E}_h[U(h, a)] \tag{3}$$

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(This is the error rate)

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$$\hat{y}(\boldsymbol{x}) = \mathbb{I}(p(y = 1|\boldsymbol{x}) \geq 1 - \tau) \tag{6}$$

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Setting $\beta = 1$ gives the harmonic mean of precision and recall F_1.

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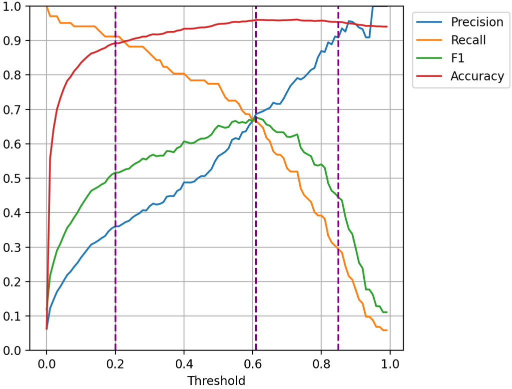
Here we choose τ to maximize $\mathcal{P}$, $\mathcal{R}$ and F_1:



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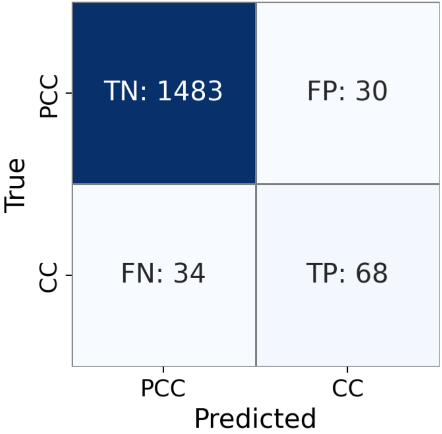

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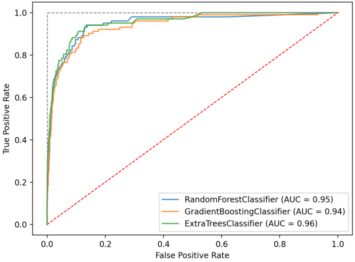

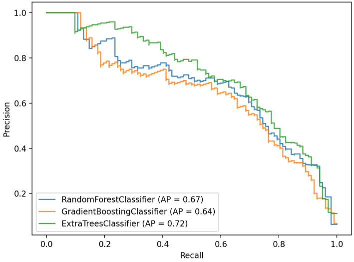

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$$\ell_\delta(h, a) = \begin{cases} \frac{(h-a)^2}{2}, & |h - a| \leq \delta \\ \delta|h - a| - \delta^2/2, & |h - a| > \delta \end{cases} \tag{11}$$

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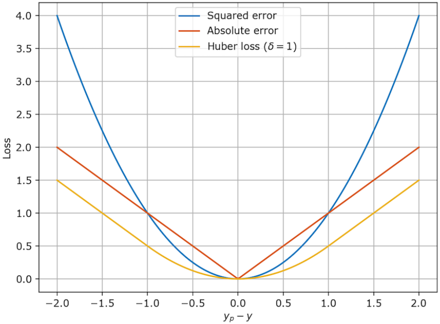

Source: `https://www.evergreeninnovations.co/blog-machine-learning-loss-functions/`

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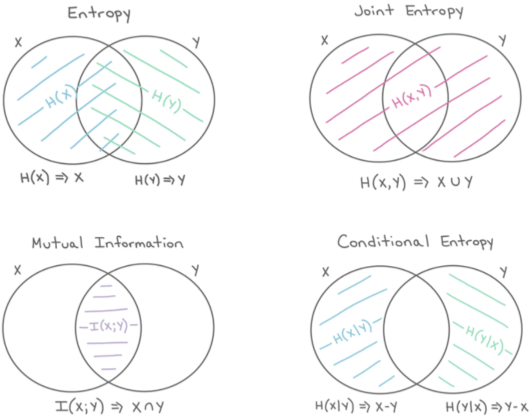

Source: PMLI Figure 6.4, page 211

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$$MIC(X,Y) = \max_{G} \frac{\mathbb{I}((X,Y)|_G)}{\log ||G||} \tag{25}$$

where G is the set of 2d grids

Reading assignments

- **PMLI** 5.1–5.4; 6.1–6.3
- **ESL** 7.1–7.7, 7.10–12