CEE 616: Probabilistic Machine Learning

M2 Linear Methods: L2b Logistic Regression

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College of Engineering

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Outline

Logistic regression

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where $\boldsymbol{W} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_C]$ is a $C \times D$ weight matrix

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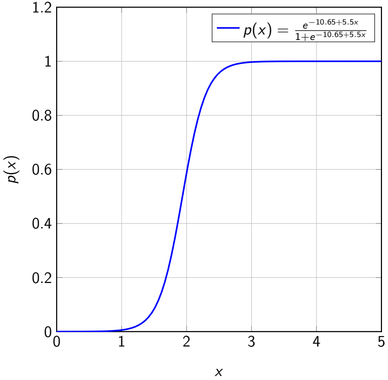

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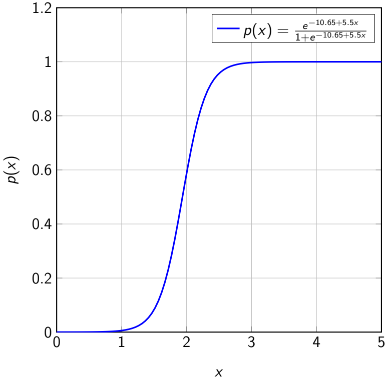

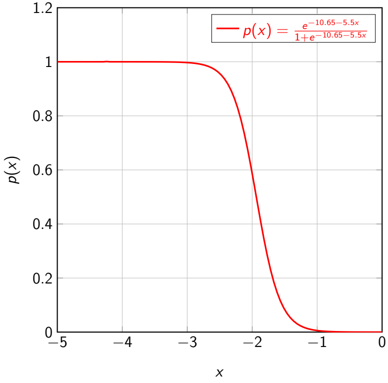

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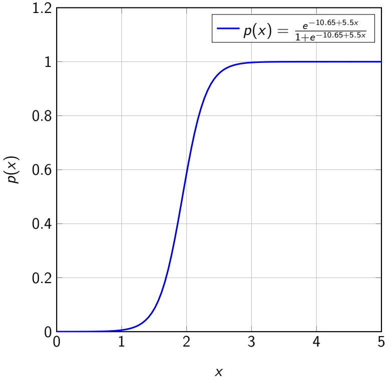

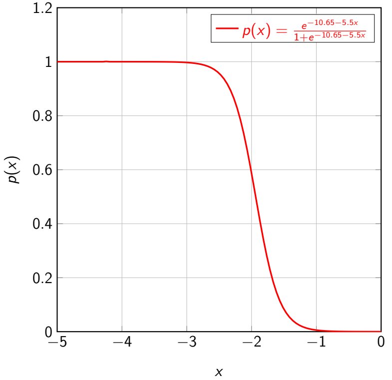

What happens when b is increased or decreased?

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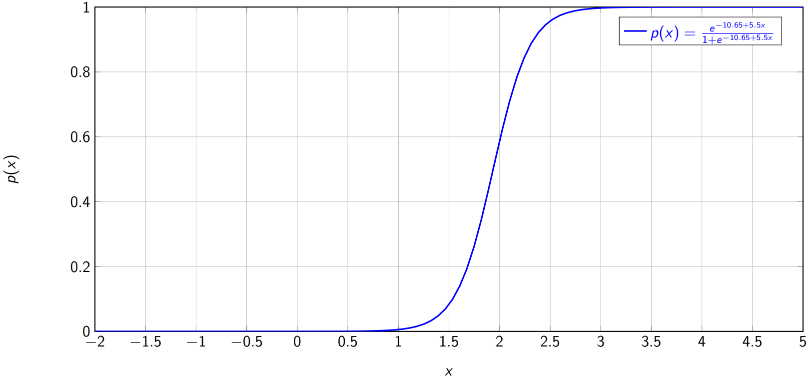

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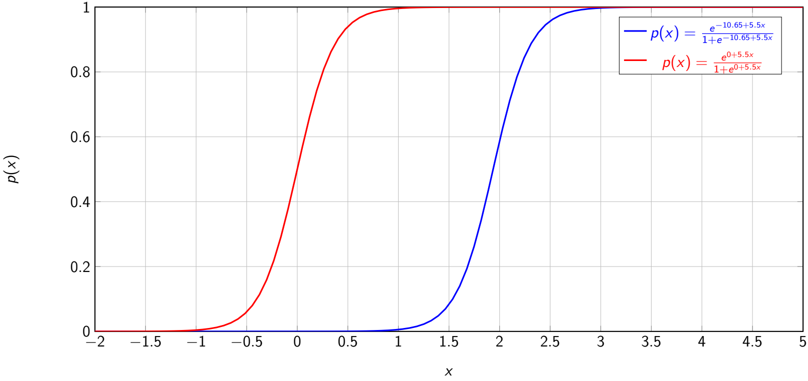

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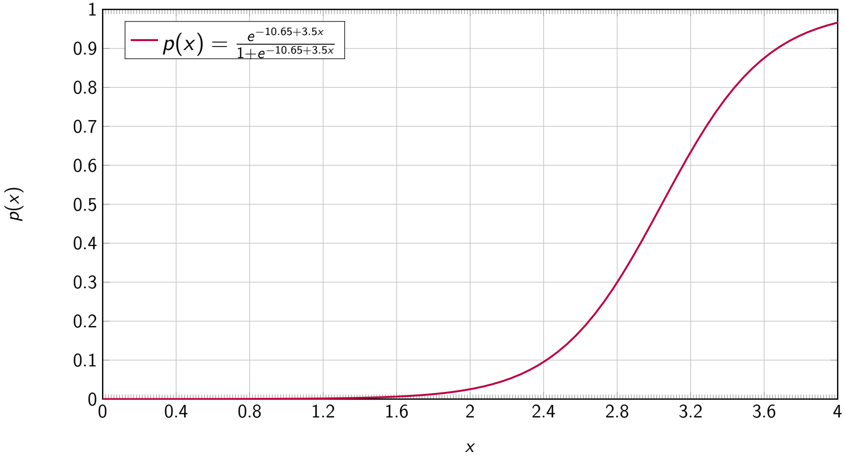

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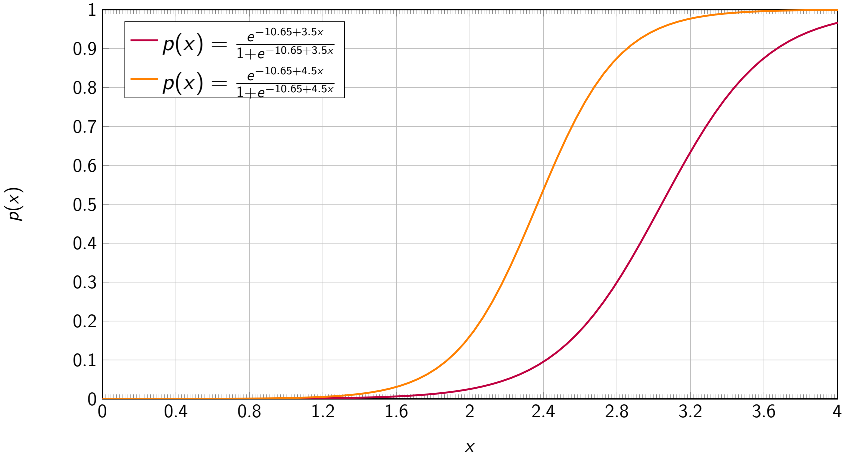

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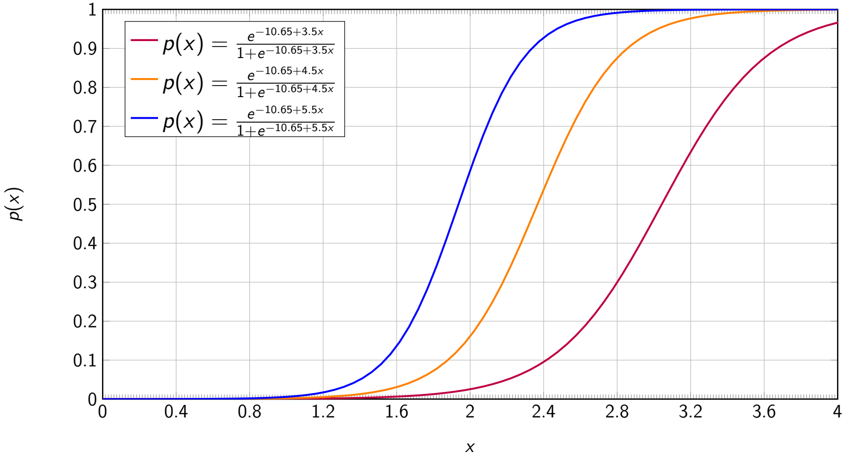

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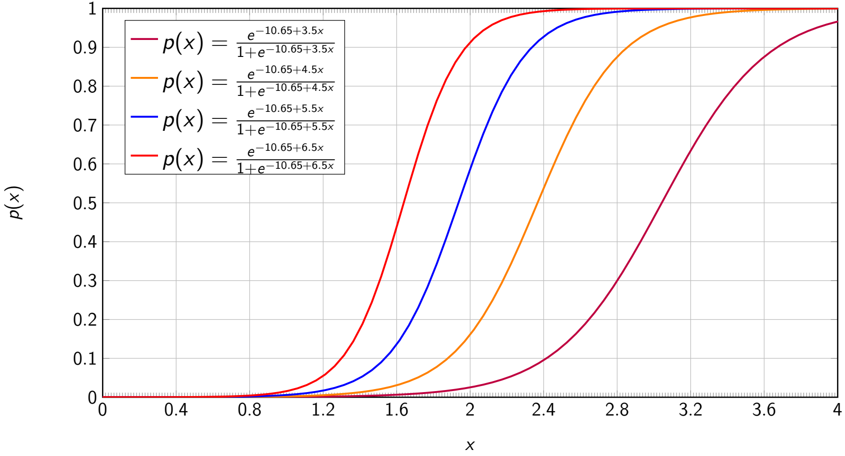

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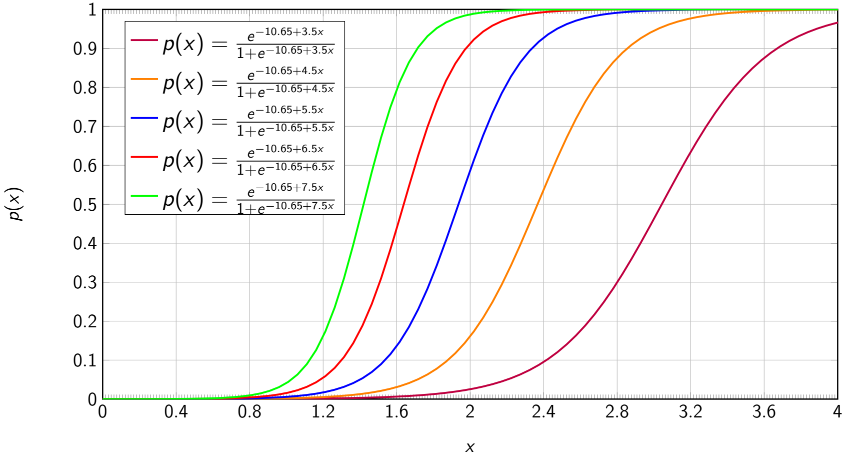

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- In the generalized linear framework, logit is the *link function* between the predictors and the mean response

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- `balance`: The average balance that the customer has remaining on their credit card after making their monthly payment
- `income`: Income of customer

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Note that the null hypothesis for the tests is: $H_0 : \boldsymbol{w}_i = 0$ (i.e. no dependence on the corresponding predictor)

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$$\hat{p}(y = 1|\boldsymbol{x}; \boldsymbol{\theta}) = \frac{e^{(-10.65+0.0055x)}}{1 + e^{(-10.65+0.0055x)}} =$$

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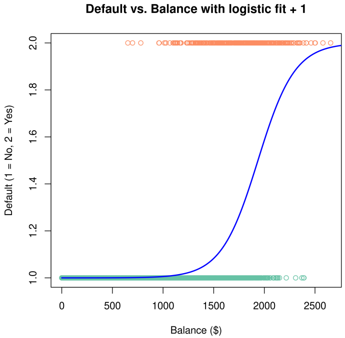

Figure: Logistic regression model predicting the probability of `default` based on

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3. There are 333 defaults out of 10000 observations. What is the impact of b?

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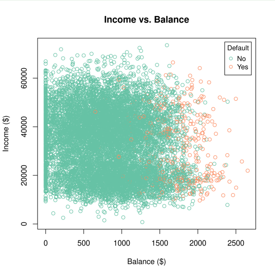
Activity: Binomial logistic regression with multiple predictors

Using the `Default` dataset, predict the probability of `default` based on `balance` and `student`. Comment on your results and interpret the coefficient estimates.



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Typically, $\tau = 0.5$

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Alternatively, we can use the **gradient descent** approach to directly minimize NLL.

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We can use either Newton-Raphson or gradient *descent* to *minimize* NLL.

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Note that $p_i = \boldsymbol{\sigma}(\boldsymbol{w}^\top \boldsymbol{x}_i)$.

- Binary cross-entropy quantifies how far your predicted probabilities are from the actual binary labels.

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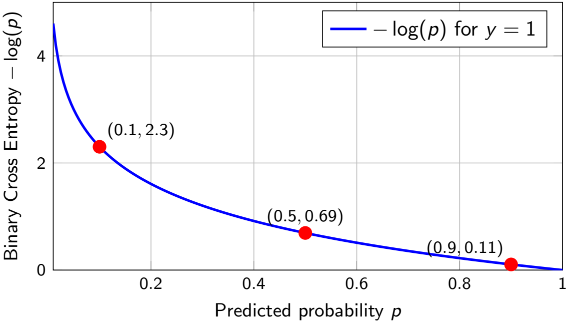

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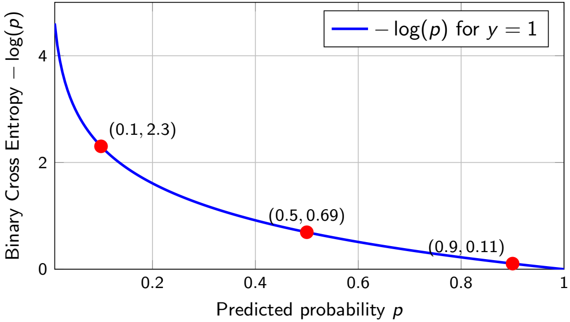

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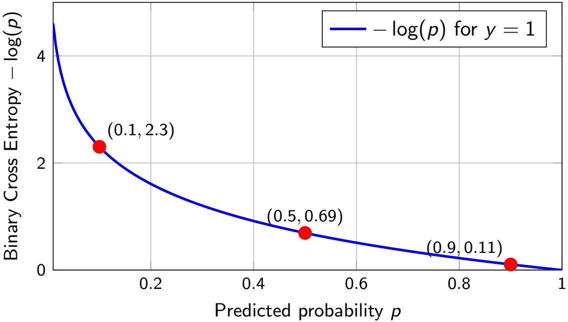

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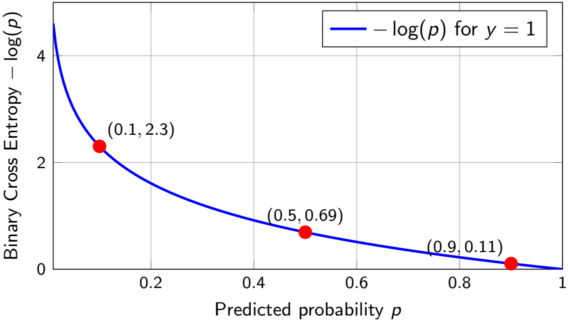

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  - Note that the update step is identical in form to the WLS estimator
  - However, $\boldsymbol{W}$ and $\boldsymbol{z}$ change in each iteration, hence the name iteratively reweighted least squares (IRLS)

Summary

- The binary logistic regression model is given by:

$$p(y=1|\boldsymbol{x};\boldsymbol{\theta}) = \frac{1}{1+e^{-\boldsymbol{w}^\top \boldsymbol{x}}} \tag{45}$$

- The negative log-likelihood of a sample of N observations in the binomial response case is:

$$\text{NLL}(\boldsymbol{w}) = \sum_i \left[y_i\left(b + w_1 x_i\right) - \log\left(1 + e^{b+w_1 x_i}\right)\right] \tag{46}$$

- Based on the principle of maximum likelihood, the estimate $\hat{\boldsymbol{w}}$ is given by the minimizing NLL.
- This can be solved via gradient descent or Newton-Raphson.

Summary (cont.)

Gradient descent update for logistic regression

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k - \lambda \nabla \text{NLL}(\boldsymbol{w}_k) \tag{47}$$

Newton-Raphson update for logistic regression

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k - \boldsymbol{H}_{\boldsymbol{w}_k}^{-1}(\text{NLL}) \nabla_{\boldsymbol{w}_k} \text{NLL}(\boldsymbol{w}_k) \tag{48}$$

This can be rewritten as:

$$\boldsymbol{w}_{k+1} = (\boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{W} \boldsymbol{z} \tag{49}$$

where the adjusted response $\boldsymbol{z}$ is given as:

$$\boldsymbol{z} = \boldsymbol{X} \boldsymbol{w}_k + \boldsymbol{W}^{-1}(\boldsymbol{y} - \boldsymbol{p}) \tag{50}$$

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In this form, the estimation is also called iteratively reweighted least squares (IRLS).

Other considerations

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- MAP estimation: weight decay/regularization to make NLL convex (have unique solution). We define the **penalized negative log-likelihood** PNLL as:

$$\text{PNLL}(\boldsymbol{w}) = \text{NLL}(\boldsymbol{w}) + \lambda \boldsymbol{w}^\top \boldsymbol{w} \tag{51}$$

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Reading assignments

- **PMLCE** 9.2
- **PMLI** 10.1-3
- **ESL** 4.4