CEE 616: Probabilistic Machine Learning
M3 Deep Neural Networks:
Lecture 3D: Neural Networks for Sequences

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Outline

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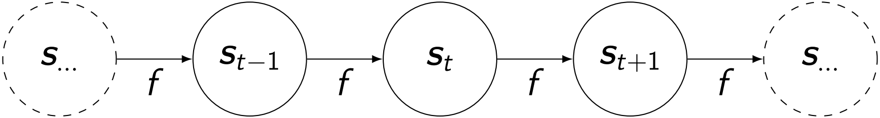

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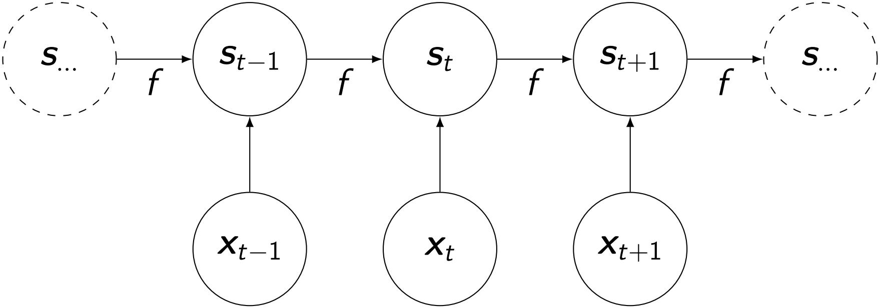

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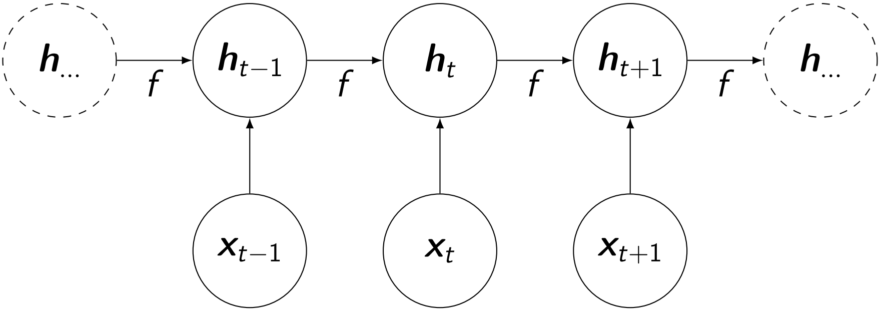

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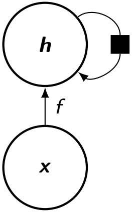

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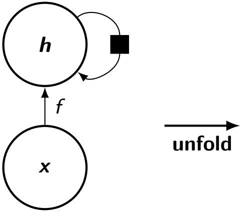

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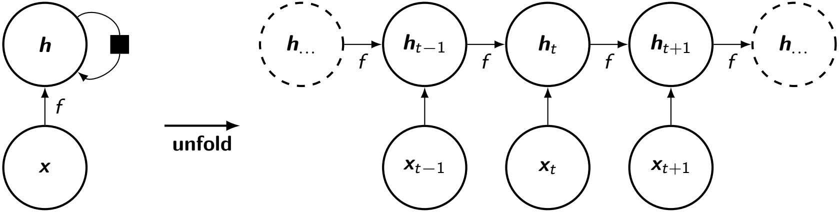

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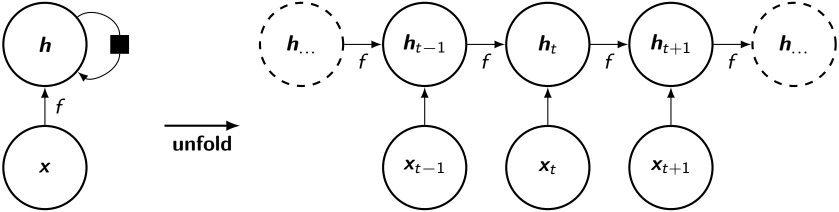

Figure: Recurrent network with no outputs. L: recurrent graph; R: time-unfolded computational graph. (Black square indicates time-step-delayed interaction)

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$$p(\boldsymbol{y}_t|\boldsymbol{h}_t) = \begin{cases} \text{Cat}(\boldsymbol{y}_t|\mathcal{S}(\boldsymbol{W}_{ho}\boldsymbol{h}_t + \boldsymbol{b}_o)), & \text{(qualitative outputs)} \\ \mathcal{N}(\boldsymbol{y}_t|\boldsymbol{W}_{ho}\boldsymbol{h}_t + \boldsymbol{b}_o, \sigma^2\boldsymbol{I}), & \text{(real-valued outputs)} \end{cases} \tag{6}$$

- $\boldsymbol{W}_{ho}$: matrix of hidden–output weights
- $\boldsymbol{b}_o$: bias term
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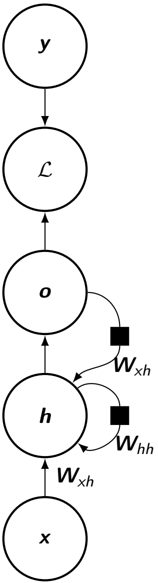

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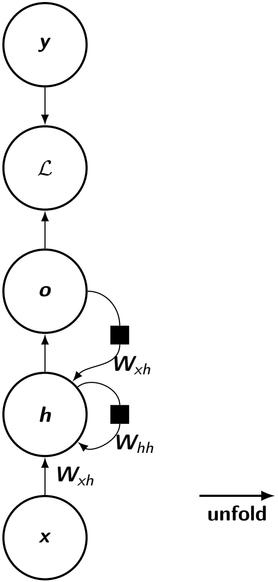

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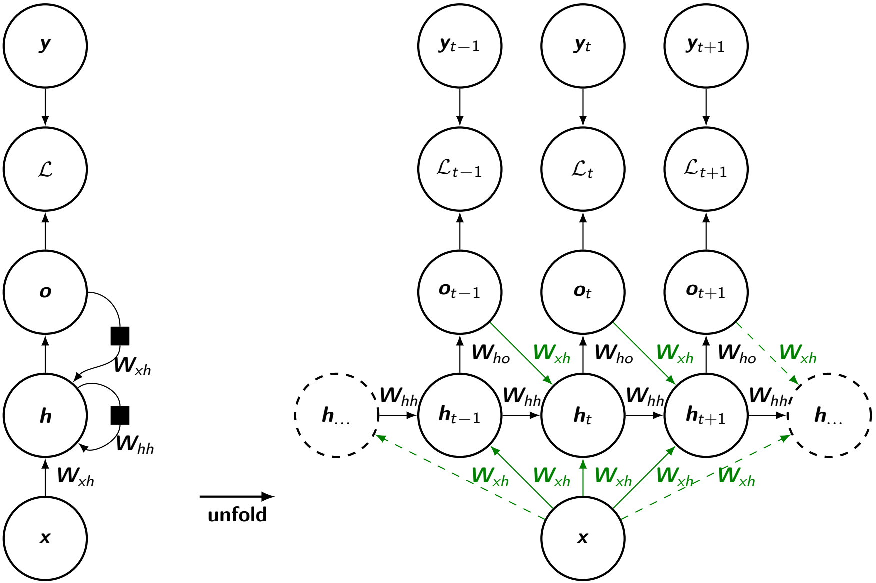

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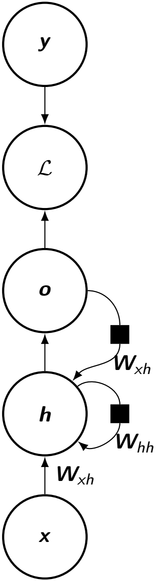

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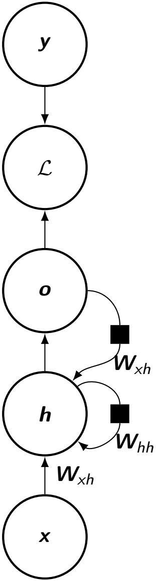

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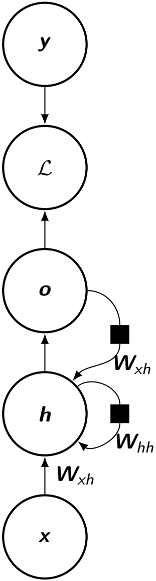

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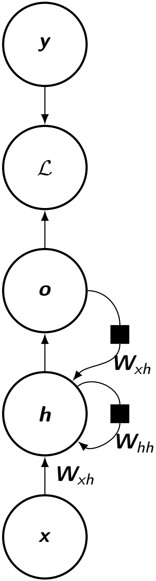

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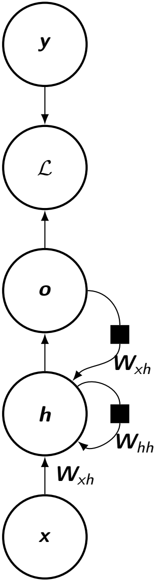

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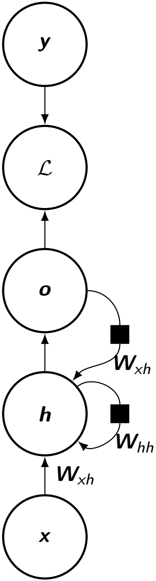

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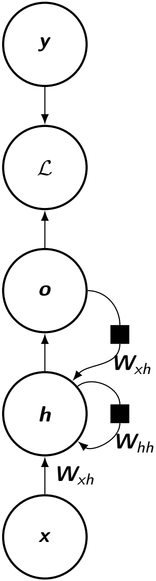

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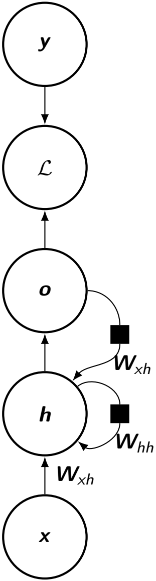

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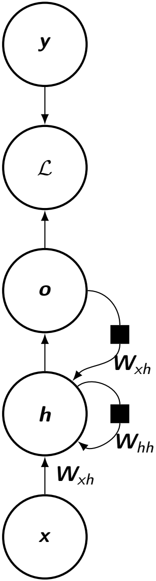

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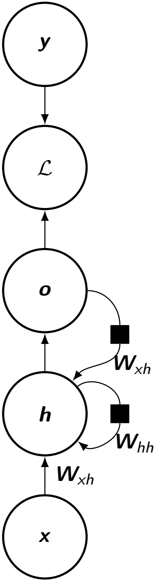

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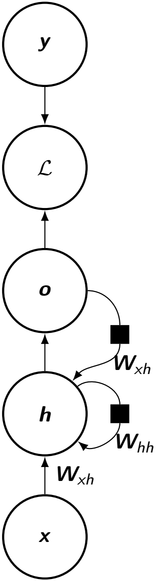

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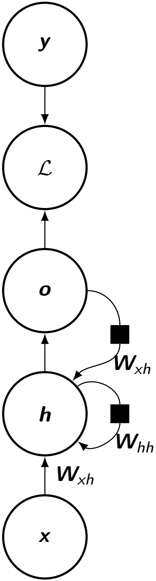

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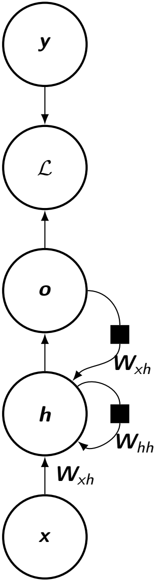

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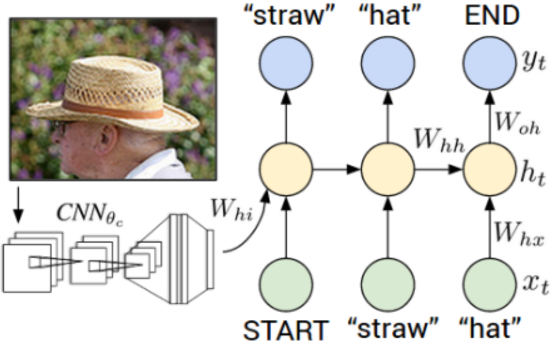

Source: https://towardsdatascience.com/image-captioning-in-deep-learning-9cd23fb4d8d2

Seq2Seq model: aligned case

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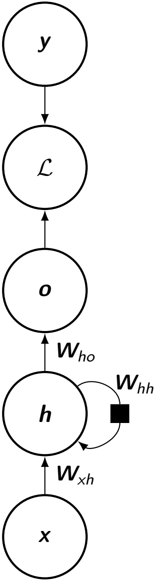

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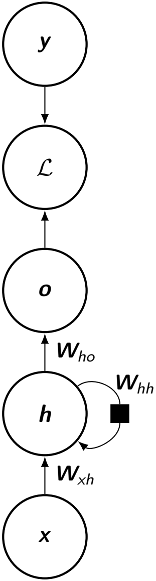

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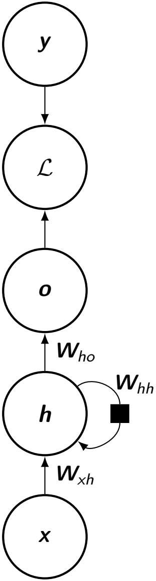

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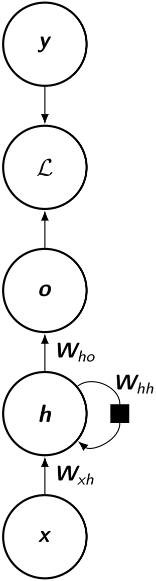

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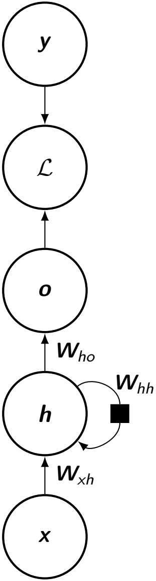

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- $\boldsymbol{h}$: hidden units
- $\boldsymbol{b}_h$, $\boldsymbol{b}_o$: bias vectors
- $\boldsymbol{W}_{xh}$: weight matrix of input–hidden unit connections
- $\boldsymbol{W}_{hh}$: weight matrix of hidden–hidden unit connections
- $\boldsymbol{W}_{ho}$: weight matrix of hidden–output unit connections
- $\boldsymbol{o}$: output vector
- $\boldsymbol{y}$ target sequence
- $\mathcal{L}$: loss function measuring error between $\hat{\boldsymbol{y}}$ and $\boldsymbol{y}$

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Seq2seq model: unaligned case

To map a sequence of length T to another of length T', we use an **encoder-decoder architecture**:

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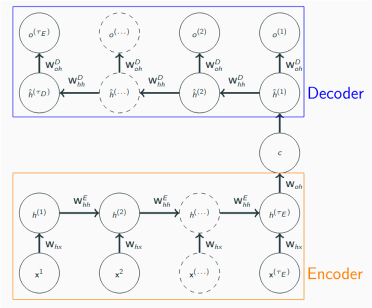

Source: https://www.inf.ed.ac.uk/teaching/courses/mlp/2019-20/lectures/mlp09-rnn.pdf

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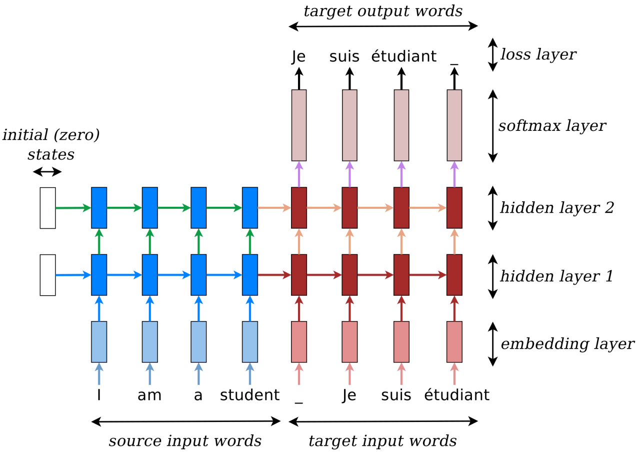

Illustration of seq2seq model for English-to-French translation.

Source: `https://github.com/probml/pml-book/blob/main/book1-figures/Figure_15.8_A.png`

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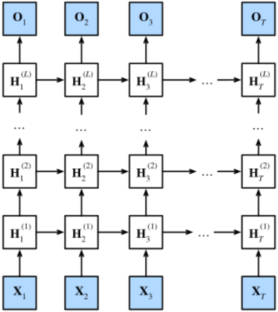

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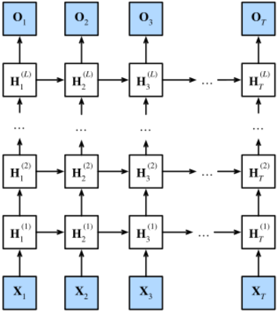

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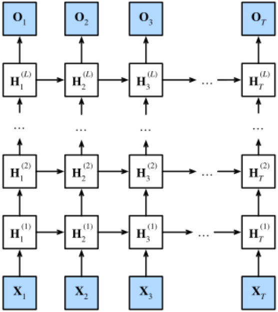

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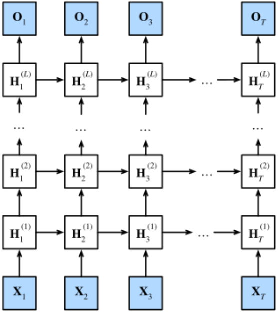

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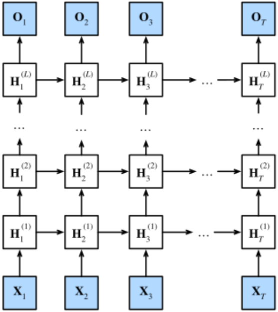

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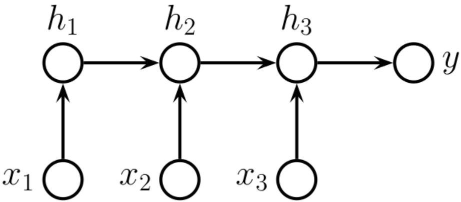

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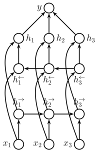

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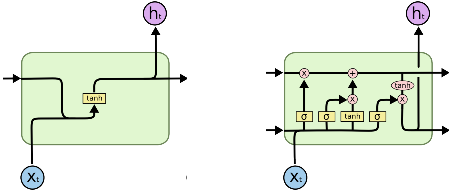

Figure: L: hidden unit in standard RNN; R: hidden unit in LSTM.

Source: https://colah.github.io/posts/2015-08-Understanding-LSTMs/

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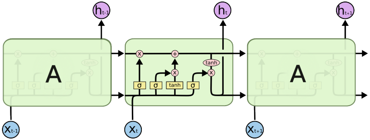

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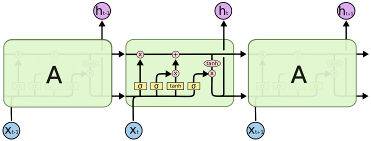

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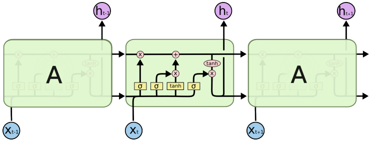

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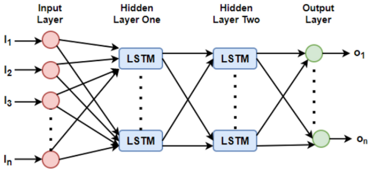

△ LSTM network with 2 blocks of LSTM cells

(https://link.springer.com/article/10.1007/s42452-021-04421-x)

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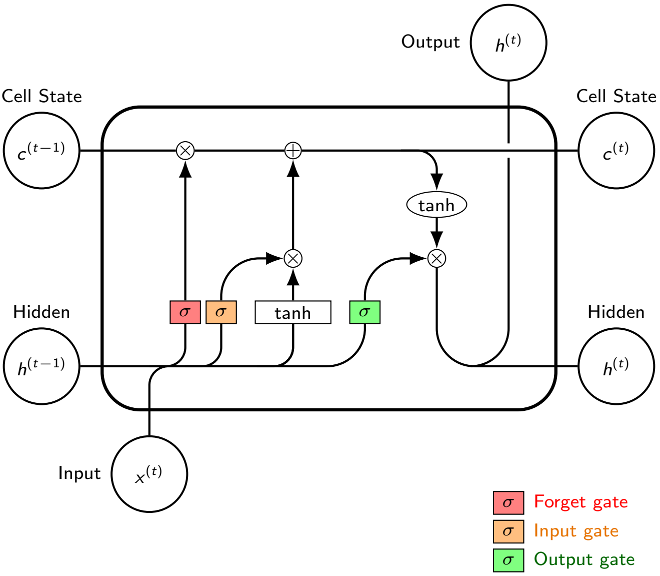
- **Forget gate:** Decides what information will be discarded from cell state. Contained in sigmoid layer that outputs number between 0 and 1 for each number in cell state $\boldsymbol{c}_{t-1}$
- **Input gate:** Decides what new information to store in cell state. Comprises
  - a sigmoid layer which decides values to update
  - b tanh layer to create new candidate values for the state
- **Output gate:** Decides what information is passed from cell state to output. Comprises
  - a sigmoid layer to decide portion of cell state to retain
  - b tanh later which admits state values in $[-1, 1]$

LSTM cell anatomy



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The cell state update is given by:

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- $\boldsymbol{b}$, $\boldsymbol{U}$, $\boldsymbol{W}$ are biases, input weights and recurrent weights, respectively, into the LSTM cell
- $c_{t-1,i}$ is the cell state for the prior timestep
- The term in purple represents the signal from new information $\boldsymbol{x}_t$ that may be included in the cell state update. Let us call it $\tilde{\boldsymbol{c}}_t$
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- In this form, we explicitly how the cell state is composed of a weighted sum of the prior cell state (accumulated information) and new inputs

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And the final output (hidden state) of the LSTM cell is:

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- $\odot$: element-wise (Hadamard) product

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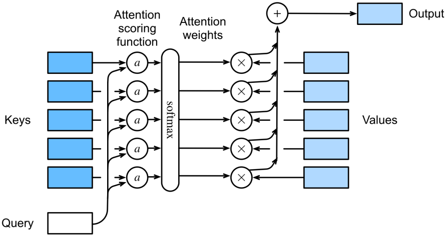

△ Attention as weighted sum of values (Source: https://github.com/probml/pml-book/blob/main/book1-figures/Figure_15.16.pdf)

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- Popular transformer models include BERT, GPT, and T5.

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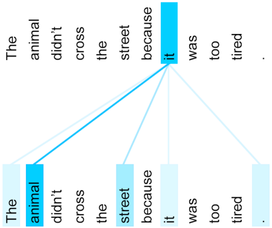
Self-attention can allow for improved representation of context.



△ Self-attention for context representation (Source: `https://github.com/probml/pml-book/blob/main/book1-figures/Figure_15.23.png`)

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where x_t is the t-th word in a sequence of T words.
Examples include:

- Embeddings from Language Model (ELMo)
- Bidirectional Encoder Representations from Transformers (BERT)
- Generative Pre-trained Transformer (GPT) models
- Text-to-text Transfer Transformer (T5) models

Summary

- Recurrent neural networks are designed to learn from sequential (temporal/spatial) data
- The recurrence structure in RNNs renders them susceptible to the vanishing/exploding gradient problem
  - It also makes it challenging for the standard RNN to learn long-term dependencies
- Several approaches have been proposed to address these issues, including the use of gated RNNs
- LSTMs in particular are able to learn when and how much of prior information to include or forget in generating the output at each timestep
  - This is done via the use of gates to control the flow of information
- LSTMs have been successfully applied to handwriting recognition/generation, speech recognition, machine translation, image captioning, among others.

Reading

- Text:**PMLI** 15, DL10
- Note that in DL10, the following symbology used in describing LSTMs (10.1.1.): (Commonly used counterparts that you may find in other literature are parenthesized.)
  - cell state: $\boldsymbol{s}_t$ (alternative: $\boldsymbol{c}_t$)
  - input gate: $\boldsymbol{g}_t$ (alternative: $\boldsymbol{i}_t$)
  - output gate: $\boldsymbol{q}_t$ (alternative: $\boldsymbol{o}_t$)
- I think the $f/c/i/o$ notation is easier to follow than the $f/s/g/q$ used in the DL text
  - However, given that DL uses i as the index for each cell, it is probably less confusing to have i representing two different things.
- An excellent resource for further explanations on how LSTMs work is available on Chris Olah's blog: `https://colah.github.io/posts/2015-08-Understanding-LSTMs/`