CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 3B: The Normal Distribution

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College of Engineering

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Outline

1. The normal distribution
2. Standard normal distribution
3. Computing normal probabilities
4. More Examples
5. Outlook

Reading: OpenIntro Statistics 4.1

Normal (Gaussian) distribution

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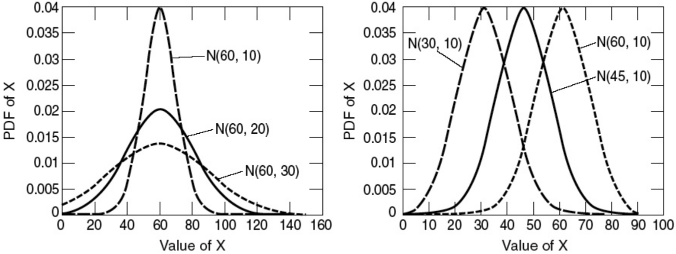

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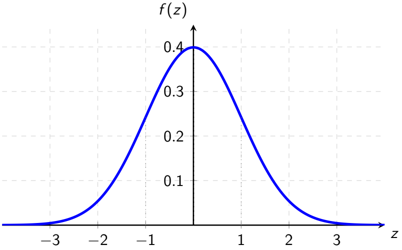

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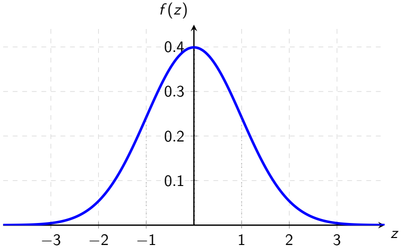

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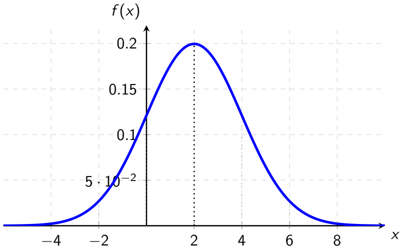

Figure: Normal distribution with $\mu = 2$, $\sigma = 2$

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where

$$z = \frac{x-\mu}{\sigma} \tag{4}$$

- The standardized normal variable z is often referred to as the Z-score

Standard normal distribution

If a random variable X has a normal distribution $\mathcal{N}(\mu, \sigma^2)$, then the r.v. Z has a standard normal distribution if

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The CDF Φ of the standard normal variate Z is given by:

$$\Phi(z) = F_Z(z) \equiv P(Z \leq z) \tag{7}$$

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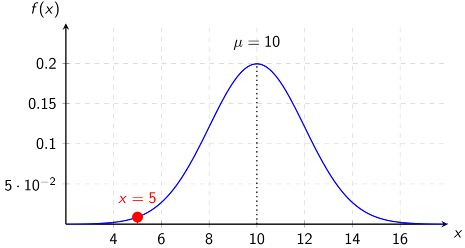

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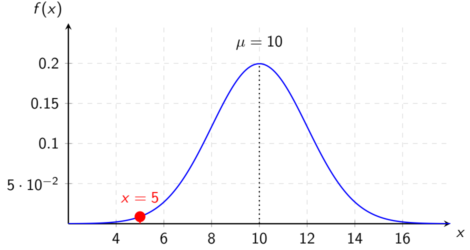

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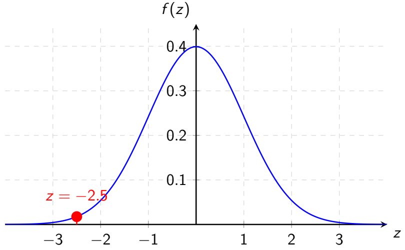

Figure: Standard normal distribution with $z = -2.5$ indicated

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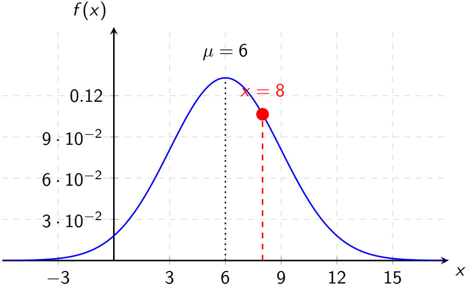

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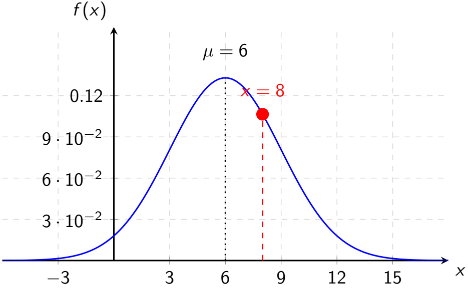

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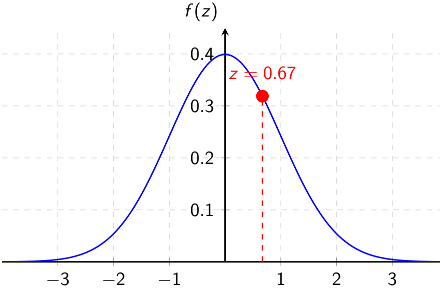

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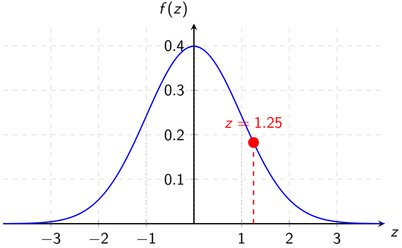

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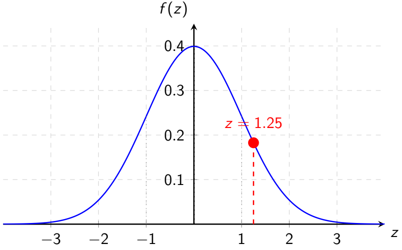

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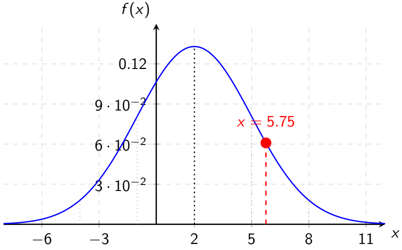

Figure: Normal distribution with $\mu = 2$, $\sigma = 3$, and $x = 5.75$ indicated

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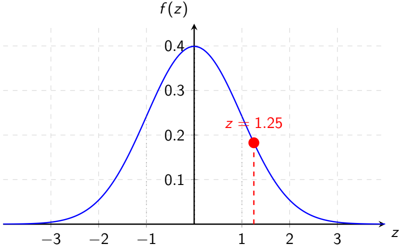

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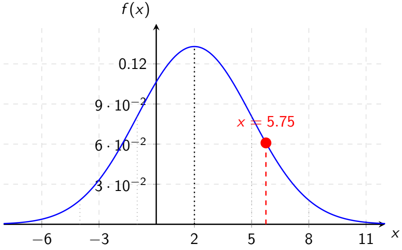

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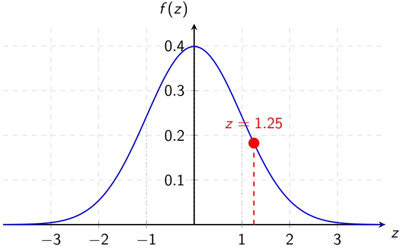

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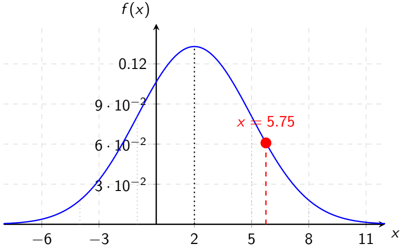

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$$
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z &= \frac{x - \mu}{\sigma} \\
x &= \mu + z\sigma = 2 + 1.25(3) = \boxed{5.75}
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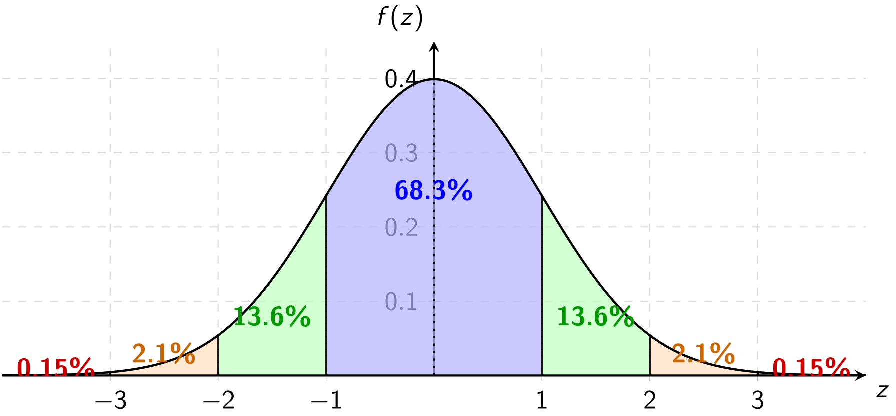

Figure: Standard normal distribution with color-coded regions showing probability percentages

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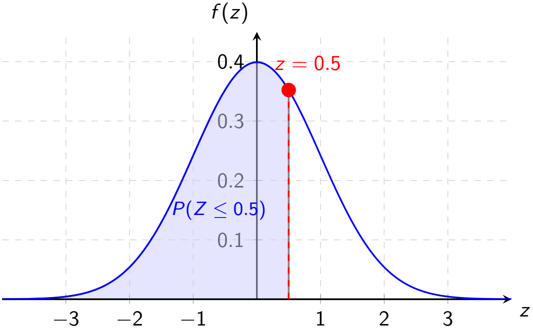

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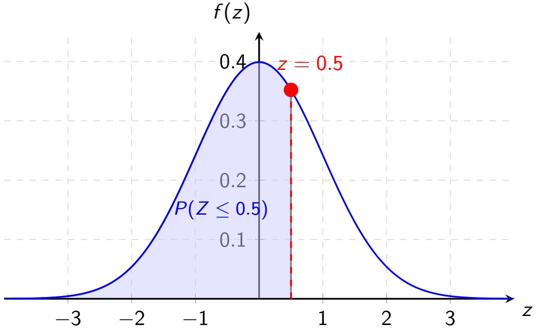

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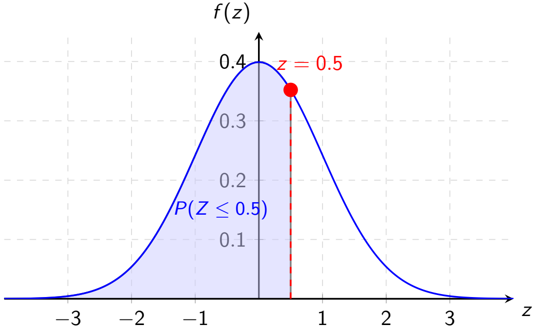

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- Recall that the area under the PDF within a given interval is the CDF evaluated in that range
- Thus, in the above figure: $p = P(Z \leq z_p) = \Phi(z_p)$

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Substituting $z = \frac{x-\mu}{\sigma}$ and $dx = \sigma dz$, we obtain:

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$$P(a < X \leq b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \tag{8}$$

Substituting $z = \frac{x-\mu}{\sigma}$ and $dx = \sigma dz$, we obtain:

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Recognizing that the integrand is the PDF of a standard normal distribution, we have:

$$P(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \tag{10}$$

Example 5a: Normal probabilities

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In Python, you can compute this probability using the `scipy.stats` library:

```
import scipy.stats as stats
p = 1 - stats.norm.cdf(1200, 1100, 200)
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The first 3 arguments of `stats.norm.cdf` are the value, mean (texttloc), and standard deviation (texttscale), respectively.

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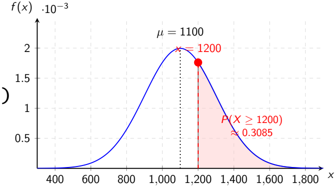

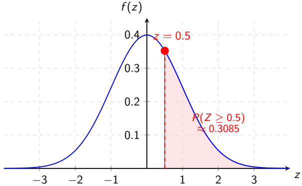

Example 5b: Normal probabilities

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SAT scores are normally distributed as $X \sim \mathcal{N}(1100, \sigma = 200)$.

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In Python:

```
from scipy.stats import norm
p = norm.cdf(1200, 1100, 200)
    - norm.cdf(600, 1100, 200)
```

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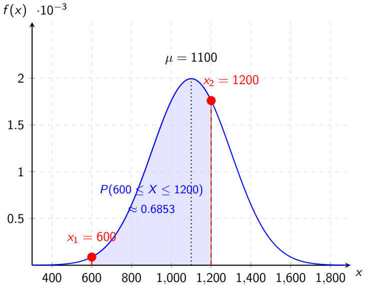

Example 5b: Normal probabilities (cont)

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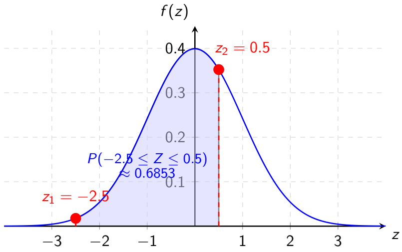

Figure: Standard normal PDF with probability area shaded

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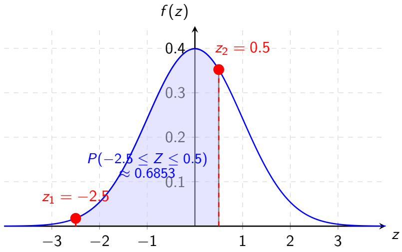

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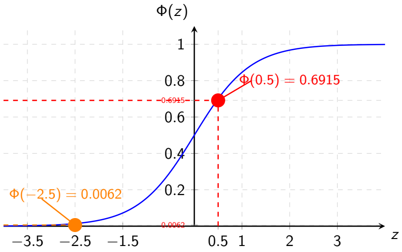

Figure: Standard normal CDF with corresponding probability values marked

Example 6: Inverse normal probabilities

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SAT scores are normally distributed as $X \sim \mathcal{N}(1100, \sigma = 200)$.

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$$\begin{aligned} z &= \Phi^{-1}(.4) = -.2533 = \frac{x-\mu}{\sigma} \\ \therefore x &= z\sigma + \mu = -.2533(200) + 1100 = \boxed{1049} \end{aligned}$$

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```python
from scipy.stats import norm
p = norm.ppf(0.4, 1100, 200)
```

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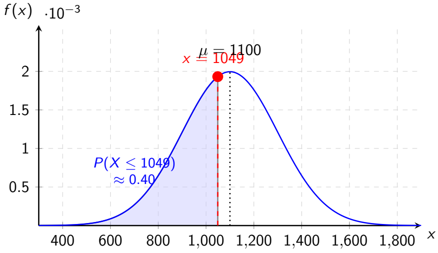

Figure: Normal distribution with $\mu = 1100$, $\sigma = 200$, and $P(X \leq 1049)$ shaded

Example 6: Inverse normal probabilities (cont.)

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(c) You can think of the inverse CDF as finding the x value that corresponds to a given percentile.

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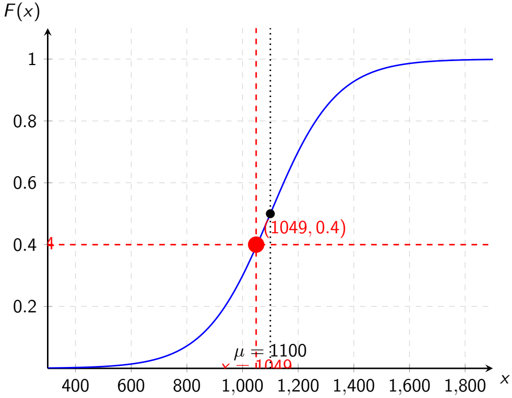

Figure: Normal distribution CDF showing the 40th percentile at $x = 1049$

More on the normal CDF

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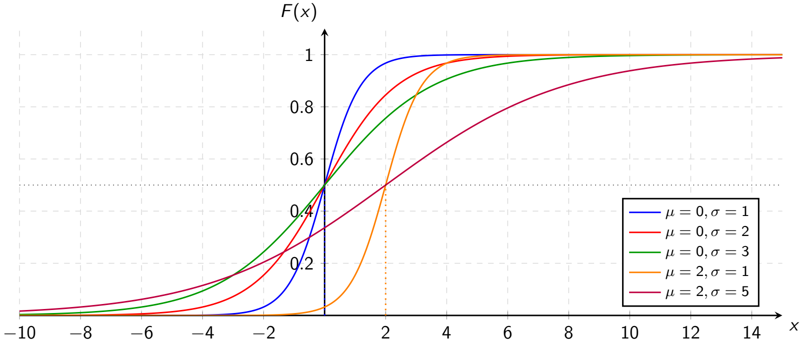
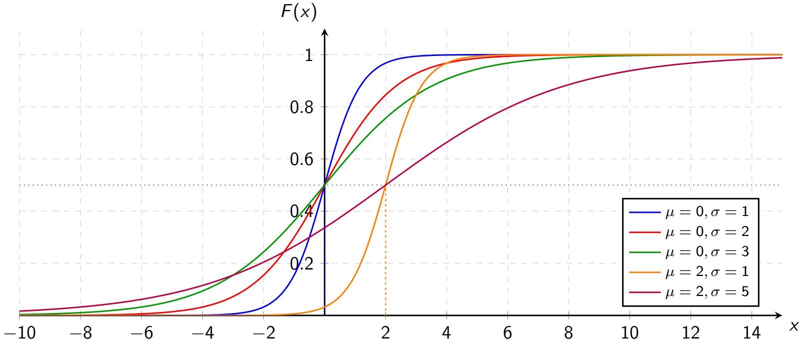


Figure: Comparison of normal distribution CDFs with different parameters

More on the normal CDF



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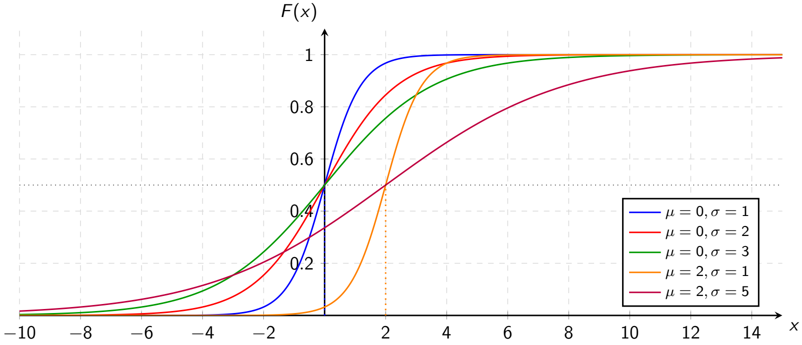

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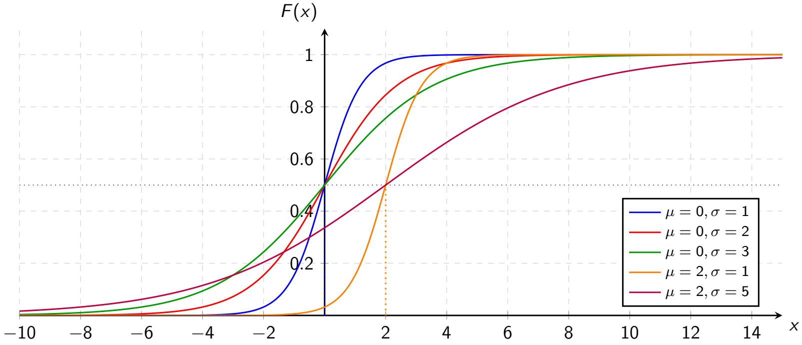

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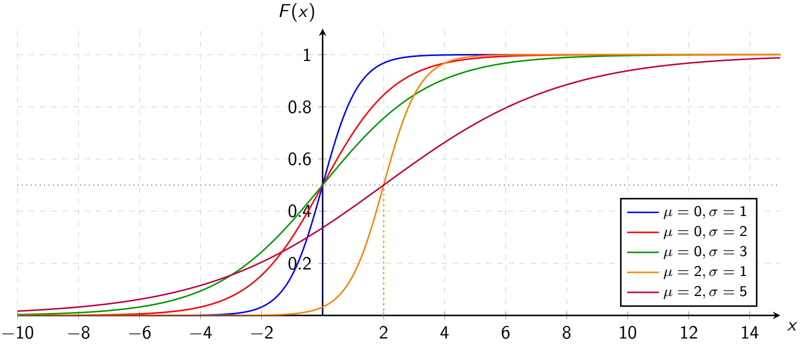

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- $z = \Phi^{-1}(p) = -\Phi^{-1}(1 - p)$

Example 7: Probability of flooding

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The drainage from a community during a storm is a normal random variable estimated to have a mean of 1.2 million gallons per day (mgd) and an SD of 0.4 mgd. If the storm drain system is designed with a maximum drainage capacity of 1.5 mgd:

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(a) What is the underlying probability of flooding during a storm that is assumed in the design of the drainage system?

(b) Find $P(1.0 < X \leq 1.6)$.

(c) Find the 90th-percentile drainage load from the community during a storm.

Example 7: Probability of flooding (cont.)

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Given $\mu = 1.2$ and $\sigma = 0.4$.

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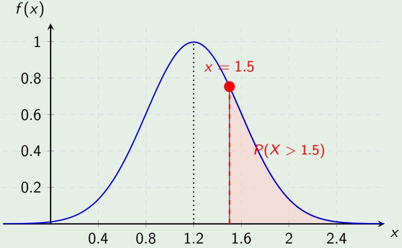

Figure: Normal distribution with $\mu = 1.2$, $\sigma = 0.4$, and $P(X > 1.5)$ shaded

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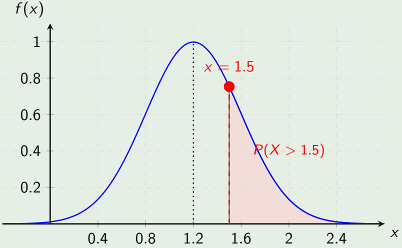

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$$P(X > 1.5) \quad = \quad 1 - P(X \leq 1.5)$$

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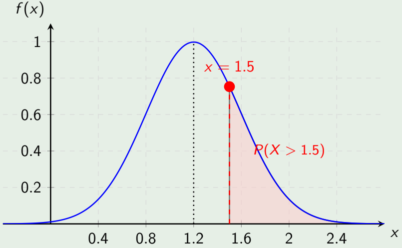

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$$
\begin{aligned}
P(X > 1.5) &= 1 - P(X \leq 1.5) \\
&= 1 - \Phi\left(\frac{1.5 - 1.2}{0.4}\right)
\end{aligned}
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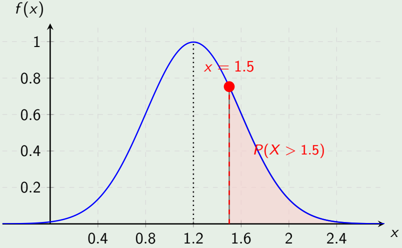

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\begin{aligned}
P(X > 1.5) &= 1 - P(X \le 1.5) \\
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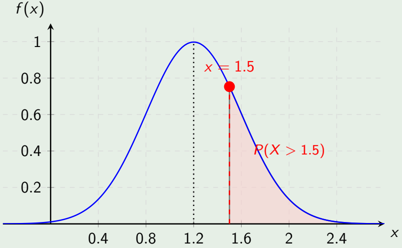

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$$\begin{aligned} P(X > 1.5) &= 1 - P(X \leq 1.5) \\ &= 1 - \Phi\left(\frac{1.5 - 1.2}{0.4}\right) \\ &= 1 - \Phi(0.75) \\ &= 1 - 0.7734 = \boxed{0.227} \end{aligned}$$

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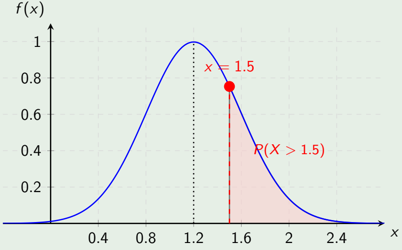

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$$

In Python: `1 - norm.cdf(1.5, 1.2, 0.4)`

Example 7: Probability of flooding (cont.)

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(b) Find $p = P(1.0 < X \leq 1.6)$:

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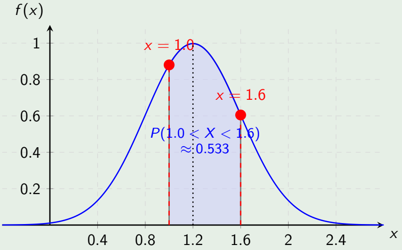

Figure: Normal distribution with $\mu = 1.2$, $\sigma = 0.4$, and $P(1.0 < X < 1.6)$ shaded

Example 7: Probability of flooding (cont.)

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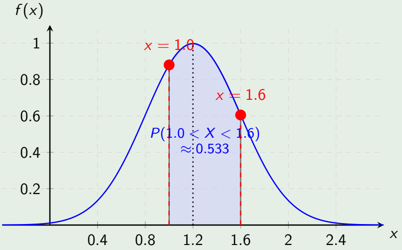

Figure: Normal distribution with $\mu = 1.2$, $\sigma = 0.4$, and $P(1.0 < X < 1.6)$ shaded

$$p = \Phi\left(\frac{1.6 - 1.2}{0.4}\right) - \Phi\left(\frac{1.0 - 1.2}{0.4}\right)$$

Example 7: Probability of flooding (cont.)

(b) Find $p = P(1.0 < X \leq 1.6)$:

Solution

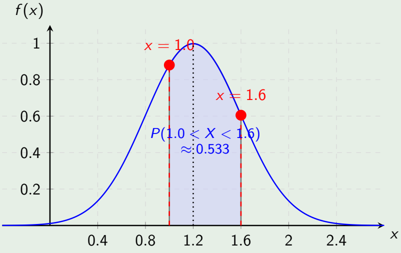

Figure: Normal distribution with $\mu = 1.2$, $\sigma = 0.4$, and $P(1.0 < X < 1.6)$ shaded

$$\begin{aligned} p &= \Phi\left(\frac{1.6 - 1.2}{0.4}\right) \\ &\qquad -\Phi\left(\frac{1.0 - 1.2}{0.4}\right) \\ &= \Phi(1.0) - \Phi(-0.5) \end{aligned}$$

Example 7: Probability of flooding (cont.)

(b) Find $p = P(1.0 < X \leq 1.6)$:

Solution

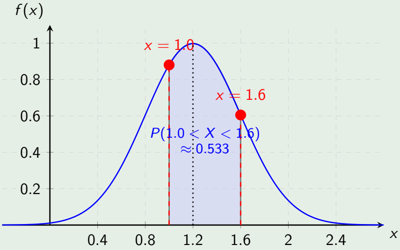

Figure: Normal distribution with $\mu = 1.2$, $\sigma = 0.4$, and $P(1.0 < X < 1.6)$ shaded

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p &= \Phi\left(\frac{1.6 - 1.2}{0.4}\right) \\
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\end{aligned}
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Example 7: Probability of flooding (cont.)

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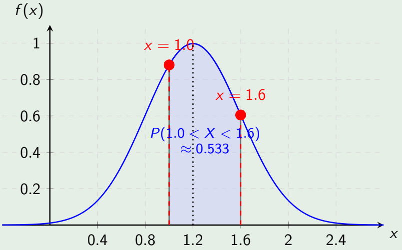

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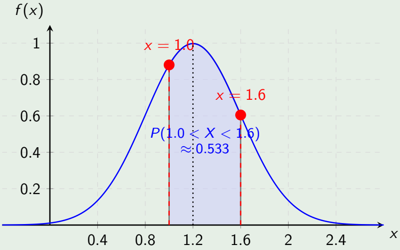

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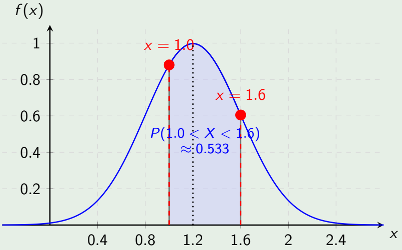

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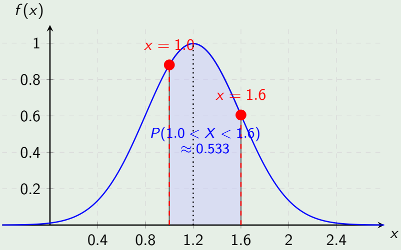

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In Python: `norm.cdf(1.6, 1.2, 0.4) - norm.cdf(1.0, 1.2, 0.4)`

Example 7: Probability of flooding (cont.)

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(c) Find the 90th-percentile drainage load from the community during a storm.

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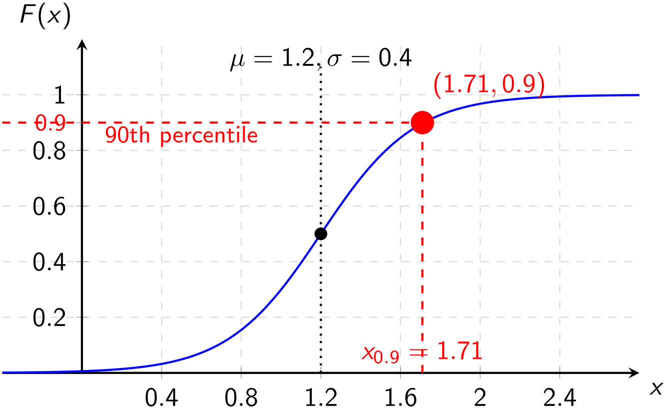

Figure: Normal CDF showing the 90th percentile at $x = 1.71$

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In Python:

```python
from scipy.stats import norm
p90 = norm.ppf(0.9, 1.2, 0.4)
```

gives 1.7095 mgd

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Assume the variability E in the lengths of steel beams is normally distributed. What is the precision (in terms of σ) required for a reliability of 99.7%, given that the specified tolerance for a construction project is ± 5 mm?

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Definitions:

- **Precision**: in physical terms is the inverse of the variance (i.e. higher precision means lower variance). In this context, all you need to do is find the standard deviation σ.
- **Reliability**: probability that the deviation in the length of a beam meets (falls within) the specified tolerance

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- The symbol Φ ("phi") is used to represent the CDF of the *standard* normal distribution, whose values can be looked up in a table.
- In Python, the `scipy.stats.norm.cdf(x, mu, sigma)` and `scipy.stats.norm.ppf(p, mu, sigma)` can be used to compute probabilities and inverse CDFs of the normal distribution, respectively.