CEE 616: Probabilistic Machine Learning
M4 Nonparametric Methods:
L4b: Gaussian Processes

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Outline

Gaussian processes

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Given a domain $\mathcal{X}$, a Gaussian process (GP) defines a joint distribution over functions of the form $f : \mathcal{X} \in \mathbb{R}$.

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$$\boldsymbol{f}(\boldsymbol{x}) = \mathcal{GP}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{ij}) \tag{1}$$

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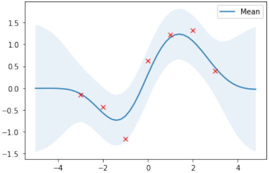

Source: http://krasserm.github.io/2018/03/19/gaussian-processes/

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- Thus, a Mercer kernel is also known as a positive definite kernel

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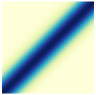

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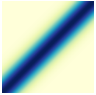

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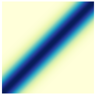

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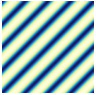

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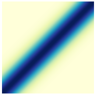

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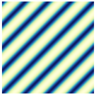

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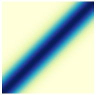

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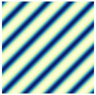

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- **Automatic relevancy determination** (ARD) kernel
- Kernels can be composed by addition, multiplication and other operations

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- SE is also referred to as Gaussian, RBF or exponentiated quadratic

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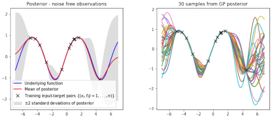

Source: https://www.aidanscannell.com/post/gaussian-process-regression/

Noise-free observations

In standard GP, we assume that observations in the training set are noise-free (exact output based on a known function f):

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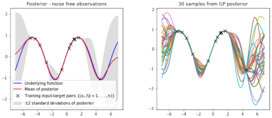

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To predict function outputs for new inputs $\boldsymbol{x}^*$, we estimate the *posterior conditional distribution*

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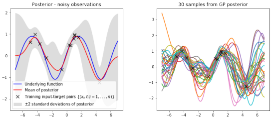

GPR example with noisy inputs. Source: https://www.aidanscannell.com/post/gaussian-process-regression/

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- Hyperparameters can be learned via **maximum marginal likelihood**

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Reading

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- https://github.com/aidanscannell/probabilistic-modelling/blob/master/notebooks/gaussian-process-regression.ipynb