CEE 616: Probabilistic Machine Learning

Lecture 1e: Foundations—Linear Algera

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Outline

1. Inroduction
2. Vectors
3. Matrices
4. Special matrices
5. EVD
6. Linear systems
7. Outlook

Scalars, vectors and matrices

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- A common application is the representation of an RGB image, e.g. a square 256-pixel image can be denoted by $\boldsymbol{A} \in \mathbb{R}^{256\times256\times3}$

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Each element $[\boldsymbol{C}]_{md}$ is obtained as the dot product between the mth row of $\boldsymbol{A}$ and the d-th column of $\boldsymbol{B}$.

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$$\boldsymbol{AB} \neq \boldsymbol{BA} \tag{6}$$

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$$\nexists \boldsymbol{\alpha} \in \mathbb{R}^n \quad \text{s.t.} \quad \alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_n \boldsymbol{x}_n = \mathbf{0} \tag{8}$$

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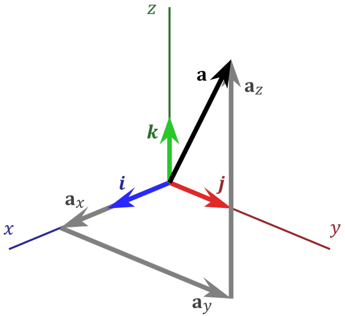

Standard basis vectors in $\mathbb{R}^3$: $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ or $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$.

Source: `https://en.wikipedia.org/wiki/Standard_basis`

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$$\boldsymbol{e}_4 \in \mathbb{R}^5 = (0, 0, 0, 1, 0) \tag{14}$$

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where $\lambda_{\max}$ is the greatest eigenvalue and σ_n is n-th singular value.

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$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{28}$$

- For block diagonal matrix:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{A}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}^{-1} \end{pmatrix} \tag{29}$$

Other important results

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Read **PMLI** 7.3 for details

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- Schur complement: Given a partitioned matrix $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{G} & \boldsymbol{H} \end{pmatrix}$ then:

$$\boldsymbol{M}/\boldsymbol{H} = \boldsymbol{E} - \boldsymbol{F}\boldsymbol{H}^{-1}\boldsymbol{G} \quad \text{(Schur complement of } \boldsymbol{M} \text{ wrt } \boldsymbol{H}) \tag{30}$$

$$\boldsymbol{M}/\boldsymbol{E} = \boldsymbol{H} - \boldsymbol{G}\boldsymbol{E}^{-1}\boldsymbol{F} \tag{31}$$

- Matrix inversion lemma (Sherman-Morrison formula)
- Matrix determinant lemma:

$$|\boldsymbol{A} + \boldsymbol{u}\boldsymbol{v}^\top| = (1 + \boldsymbol{v}^\top \boldsymbol{A}^{-1})\boldsymbol{u}|\boldsymbol{A}| \tag{32}$$

Eigenvectors and eigenvalues (review)

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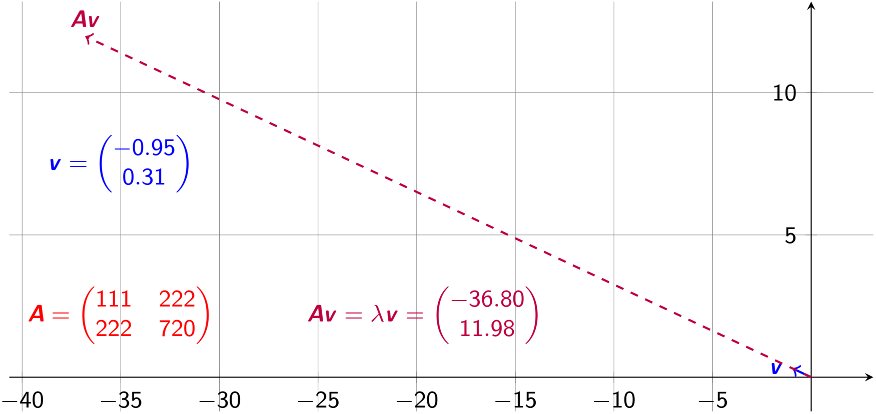

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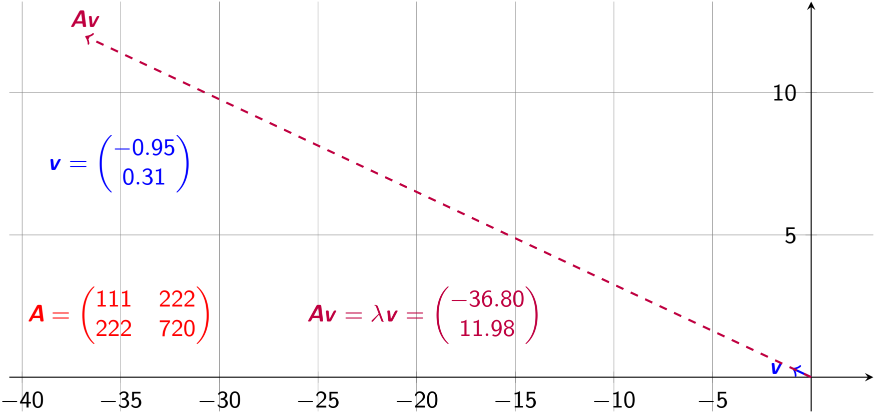

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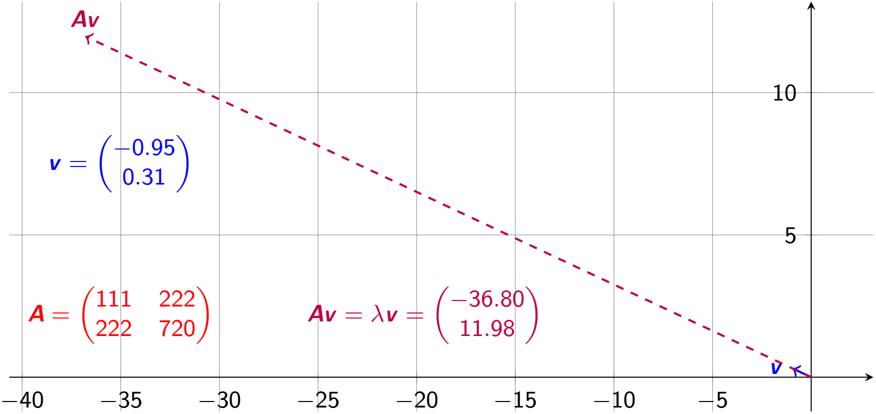

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The singular value decomposition of $\boldsymbol{X}$ generalizes EVD to rectangular matrices:

$$\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T \tag{43}$$

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- The columns of $\boldsymbol{U\Gamma}$ are called the **principal components** of $\boldsymbol{X}$.

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SVD (cont.)

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- If $m < n$, the system is **underdetermined** (no unique solution)
- If $m > n$, the system is **overdetermined** (no exact solution; solve via least squares)

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is the ordinary least squares (OLS) solution. Checking that the Hessian $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{A}^\top \boldsymbol{A}$ is pd confirms the solution is unique.

Reading assignments

- **PMLI** 7
- **PMLCE** 2