CEE 616: Probabilistic Machine Learning

M5 Unsupervised Learning: 5B: Factor Analysis and Autoencoders

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Outline

Factor analysis model

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- Basic idea: there are latent (hidden) **common factors** $\boldsymbol{z}$ underlying some multivariate observations $\boldsymbol{x}_n \in \mathbb{R}^D$

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- Posterior has closed-form solution under Gaussian distribution

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- Non-Gaussian priors for latent factors

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To introduce flexibility, we can specify f_e and f_e are nonlinear/more complex functions. This is best accomplished via neural network, resulting in an **autoencoder** (AE).

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Basic autoencoder (AE) architecture

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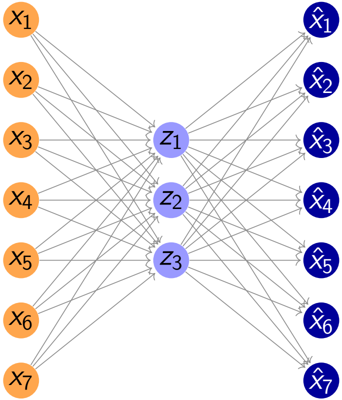
Autoencoder with 2 single-layer MLPS: input layer, hidden layer (latent representation) and output layer (reconstruction)



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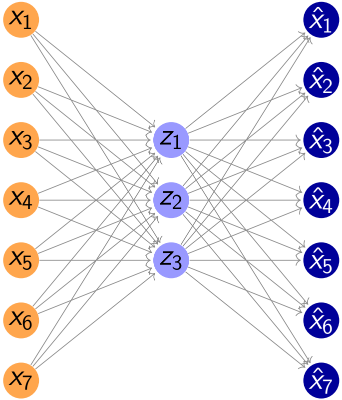

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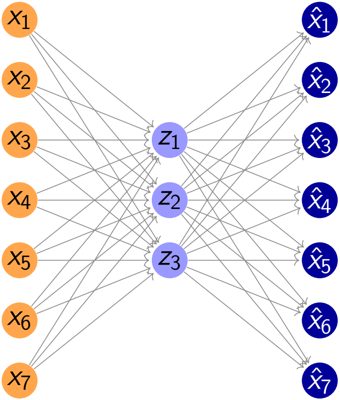

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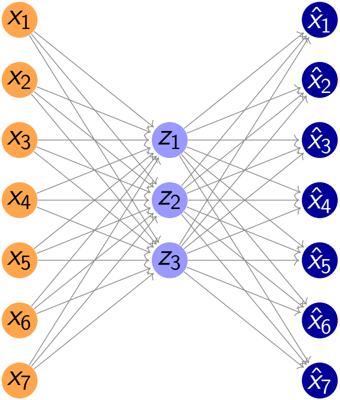

- Hidden layer (size L) is a low-dimensional **bottleneck** between input and reconstruction
- $L \ll D$: undercomplete representation
- $L \gg D$: overcomplete representation (regularize to prevent identity learning)

Denoising autoencoders

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In denoising autoencoders (DAEs), the input is corrupted ($\tilde{\boldsymbol{x}}$) by:

- Gaussian noise: $p_c(\tilde{\boldsymbol{x}}|\boldsymbol{x}) = \mathcal{N}(\tilde{\boldsymbol{x}}|\boldsymbol{x}, \sigma^2 \boldsymbol{I})$

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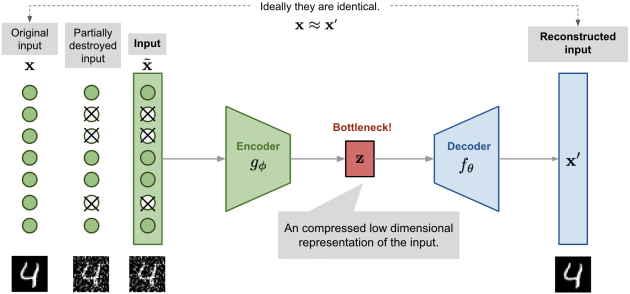
- Gaussian noise: $p_c(\tilde{\boldsymbol{x}}|\boldsymbol{x}) = \mathcal{N}(\tilde{\boldsymbol{x}}|\boldsymbol{x}, \sigma^2 \boldsymbol{I})$
- Bernoulli dropout: randomly setting a proportion of input nodes to zero



Schematic of a DAE.

Source: https://lilianweng.github.io/posts/2018-08-12-vae/

The model is then trained to minimize the loss between the reconstructed input $r(\tilde{\boldsymbol{x}})$ and its uncorrupted version $\boldsymbol{x}$

Uses of DAE

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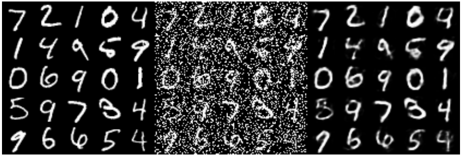

Original, corrupted and reconstructed images from MNIST dataset.

Source: http://www.opendeep.org/v0.0.5/docs/tutorial-your-first-model

- They can also learn vector fields of input data

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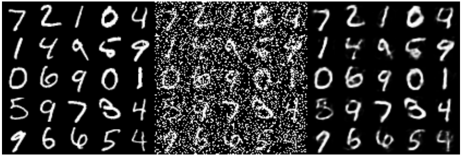

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- They can also learn vector fields of input data
- Popular for their simplicity

Sparse autoencoder (SAE)

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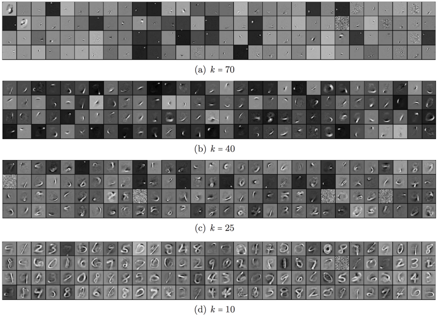
$$\Omega(\boldsymbol{z}) = \lambda||\boldsymbol{z}||_1 \tag{23}$$

- *k*-Sparse autoencoder: use only k largest activations in training



Filters of the k -sparse autoencoder for different sparsity levels k, learnt from MNIST with 1000 hidden units.

Source: `https://arxiv.org/pdf/1312.5663.pdf`

Useful for interpretability

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- Variational autoencoder (VAE): probablistic version of AE/generative model

Reading

- **PMLI** 20.3
- **DL** 20