CEE 616: Probabilistic Machine Learning

# M5 Unsupervised Learning:
# 5B: Factor Analysis and Autoencoders

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# Outline

# Factor analysis model

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- Basic idea: there are latent (hidden) **common factors** $\boldsymbol{z}$ underlying some multivariate observations $\boldsymbol{x}_n \in \mathbb{R}^D$

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- Posterior has closed-form solution under Gaussian distribution

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- **Decoder** $f_d$: mapping from $\boldsymbol{z} \rightarrow \boldsymbol{x}$

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- **Encoder** $f_e$: mapping from $\boldsymbol{x} \rightarrow \boldsymbol{z}$
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And $f_d$ is given by:

$$\hat{\boldsymbol{x}} = \boldsymbol{W}\boldsymbol{z} \equiv f_d(\boldsymbol{z}) \tag{19}$$

To introduce flexibility, we can specify $f_e$ and $f_e$ are nonlinear/more complex functions. This is best accomplished via neural network, resulting in an **autoencoder** (AE).

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$$\mathcal{L}(\boldsymbol{\theta}) = -\log p(\boldsymbol{x}|r(\boldsymbol{x})) \tag{22}$$

# Basic autoencoder (AE) architecture

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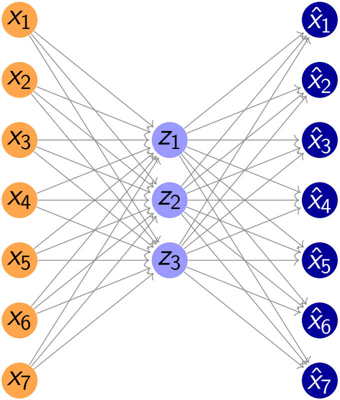
Autoencoder with 2 single-layer MLPS: input layer, hidden layer (latent representation) and output layer (reconstruction)



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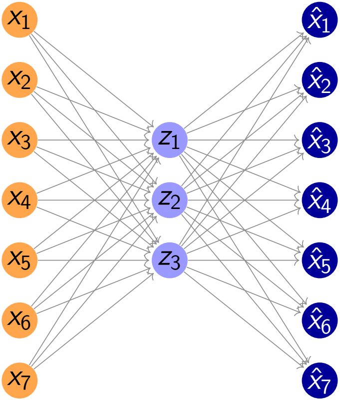


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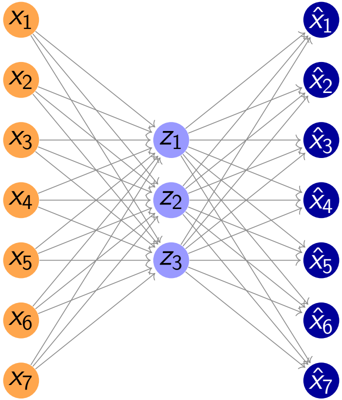


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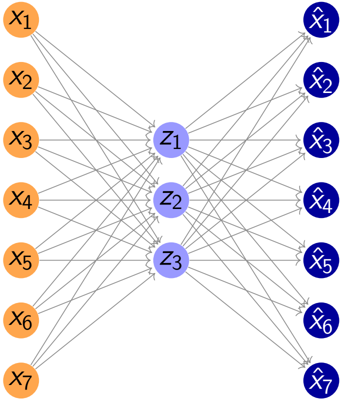


- Hidden layer (size $L$) is a low-dimensional **bottleneck** between input and reconstruction
- $L \ll D$: undercomplete representation
- $L \gg D$: overcomplete representation (regularize to prevent identity learning)

# Denoising autoencoders

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In denoising autoencoders (DAEs), the input is corrupted ($\tilde{\boldsymbol{x}}$) by:

- Gaussian noise: $p_c(\tilde{\boldsymbol{x}}|\boldsymbol{x}) = \mathcal{N}(\tilde{\boldsymbol{x}}|\boldsymbol{x}, \sigma^2\boldsymbol{I})$

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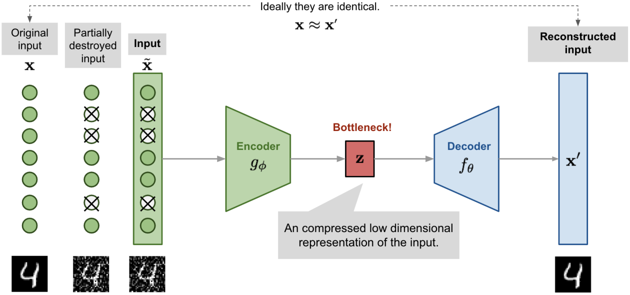
- Gaussian noise: $p_c(\tilde{\boldsymbol{x}}|\boldsymbol{x}) = \mathcal{N}(\tilde{\boldsymbol{x}}|\boldsymbol{x}, \sigma^2 \boldsymbol{I})$
- Bernoulli dropout: randomly setting a proportion of input nodes to zero




Schematic of a DAE.

Source: https://lilianweng.github.io/posts/2018-08-12-vae/

The model is then trained to minimize the loss between the reconstructed input $r(\tilde{\boldsymbol{x}})$ and its uncorrupted version $\boldsymbol{x}$

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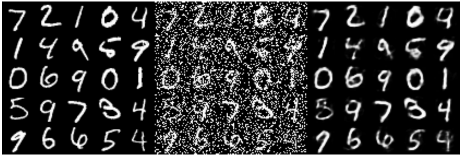

Original, corrupted and reconstructed images from MNIST dataset.

Source: http://www.opendeep.org/v0.0.5/docs/tutorial-your-first-model

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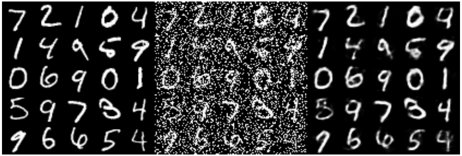

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- They can also learn vector fields of input data
- Popular for their simplicity

# Sparse autoencoder (SAE)

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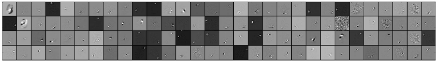
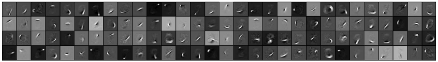
$$\Omega(\boldsymbol{z}) = \lambda ||\boldsymbol{z}||_1 \quad (23)$$

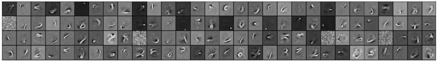
- *k*-Sparse autoencoder: use only $k$ largest activations in training



(a) $k = 70$



(b) $k = 40$



(c) $k = 25$

(d) $k = 10$

Filters of the k -sparse autoencoder for different sparsity levels k, learnt from MNIST with 1000 hidden units.

Source: https://arxiv.org/pdf/1312.5663.pdf

Useful for interpretability

# Other AEs

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$$\Omega(\boldsymbol{z}, \boldsymbol{x}) = \lambda \left\| \frac{\partial f_e(\boldsymbol{x})}{\partial \boldsymbol{x}} \right\|_F^2 \tag{24}$$

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- Variational autoencoder (VAE): probablistic version of AE/generative model

# Reading

- **PMLI** 20.3
- **DL** 20