CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture M4a: Point Estimates, Sampling Variability and Central Limit Theorem

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Outline

Statistical inference

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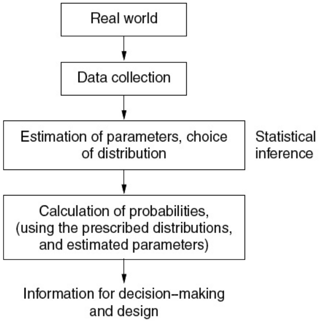
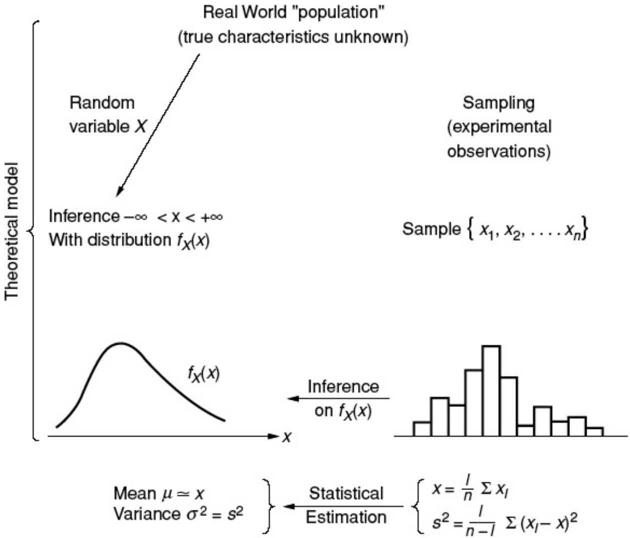


Role of sampling in statistical inference



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- Hypothesis testing for a proportion (M4c)

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A **point estimate** of a parameter θ (e.g. proportion p, or mean value μ) is a single number that can be regarded as a sensible value for θ and is obtained by computing the value of a suitable statistic (e.g. sample mean, sample standard deviation, etc) from given sample data.

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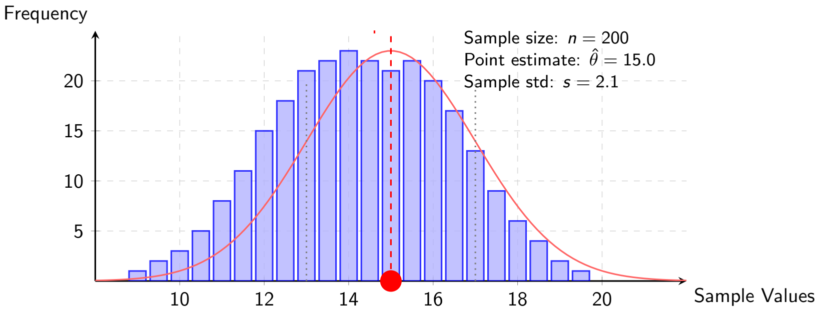

Figure: Sample histogram with point estimate $\hat{\theta}$ showing the center of the distribution

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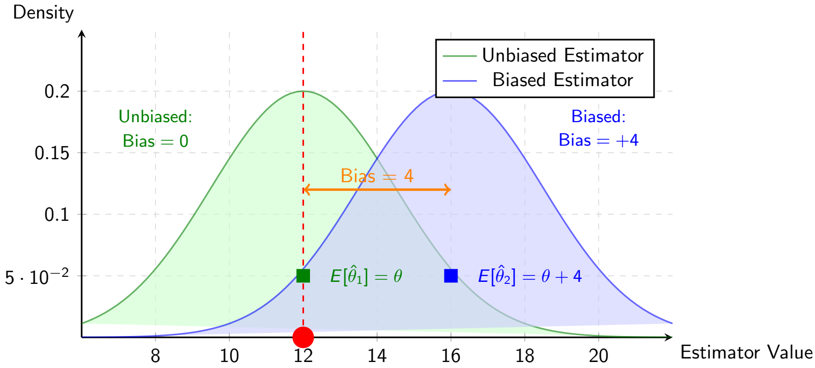

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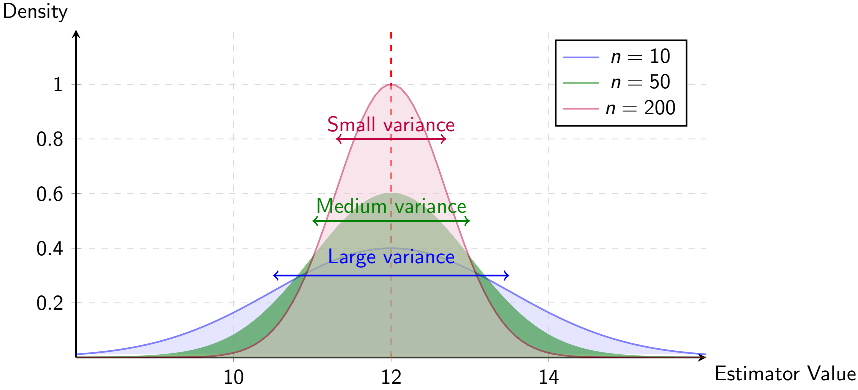

Figure: As sample size increases, the sampling distribution becomes more concentrated around the true parameter

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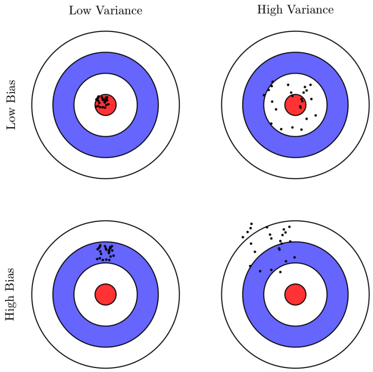

Image source: https://tex.stackexchange.com/a/307285/2269

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$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \tag{5}$$

Unbiasedness of s^2

From Equation (5), you can show (as an exercise) that:

$$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right] \tag{6}$$

You may be wondering why the sample variance is not just the average of the sum of squared deviations from the sample mean.

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(c) Now, estimate the variance replacing the denominator $(n-1)$ with n in the estimator s^2. What do you notice?

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$\hat{\sigma}^2$ underestimates σ^2 by 0.031 squared units.

Variability of a point estimate

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Example 2: Solar energy expansion

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- **(a)** Create a set of a large number of entries (e.g. 30,000) where 88% are in support and 12% are not.
- **(b)** Sample $n = 1000$ entries without replacement
- **(c)** Plot the histogram of the sampling distribution of $\hat{p}$
- **(d)** Compute the sample mean $x_{\hat{p}}$
- **(e)** Compute the standard deviation $s_{\hat{p}}$ (called the **standard error** $SE_{\hat{p}}$).
- **(f)** Investigate what happens as n increases.

The Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. If n is sufficiently large, then the sample mean $\overline{X}$ has approximately a **normal distribution** with

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- The product of large number of random components approaches the lognormal distribution

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Sample proportion and the CLT

If the observations in a given sample are a Bernoulli sequence with a constant proportion (or probability) p, then if n is large, the sample proportion $\hat{p}$ follows a normal distribution (according to the CLT):

$$\hat{p} \sim \mathcal{N}(\mu_{\hat{p}}, SE_{\hat{p}}^2) = \mathcal{N}\left(p, \frac{p(1-p)}{n}\right) \tag{17}$$

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$$\begin{aligned} \text{Sample mean proportion: } \mu_{\hat{p}} &= p \\ \text{Sampling error/standard error of } \hat{p}\text{: } SE_{\hat{p}} &= \sqrt{\frac{p(1-p)}{n}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \end{aligned}$$

One rule of thumb for determining whether n is large enough is to check that both np and $n(1-p)$ are ≥ 10 (also known as the success-failure condition).

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(a) According to the CLT, what is the distribution of $\hat{p}$?

(b) According to the CLT, what are $\mu_{\hat{p}}$ and $SE_{\hat{p}}$, respectively?

CLT application: sample proportion (cont.)

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(a) First, we note that the response of each American adult in the entire population is part of a Bernoulli sequence with $p = 0.88$. According to the CLT, the distribution of $\hat{p}$ (sample proportion) is normal/Gaussian. We can denote this as:

$$\hat{p} \sim \mathcal{N}\left(p, \frac{\sigma^2}{n}\right) \text{ OR } \mathcal{N}\left(\mu_p, \frac{\sigma_p^2}{n}\right) \tag{20}$$

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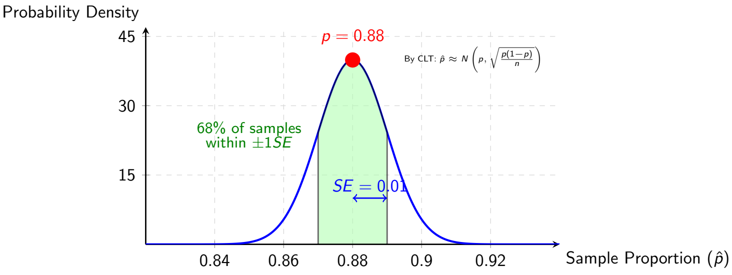

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(b) $\mu_{\hat{p}}$ denotes the mean estimate of p, which is 0.88 (according to the CLT, the mean of the sample is the population mean if n is large).
$SE_{\hat{p}}$ denotes the sampling error, which is the the square root of the variance of the sample mean: $\sqrt{\sigma^2/n}$. Given that the sample is governed by the Binomial distribution with $\sigma^2 = p(1-p)$. Thus:

$$
\begin{aligned}
SE_{\hat{p}}^2 &= \frac{\sigma^2}{n} = \frac{p(1-p)}{n} = \frac{0.88(0.12)}{1000} \\
SE_{\hat{p}} &= \sqrt{\frac{0.88(0.12)}{1000}} = \boxed{0.01}
\end{aligned}
$$

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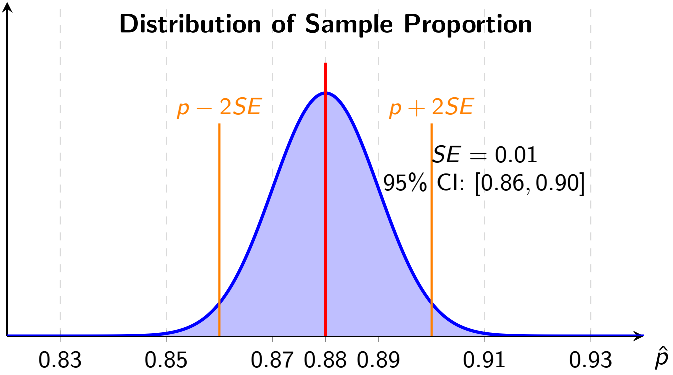

Figure: Sample proportion distribution: most samples fall within $\pm 2SE$ of the true proportion

Another application of the CLT

Example 4: Mean batch weight

A certain brand of cement is shipped in batches of 40 bags. Previous records indicate the weight of a randomly selected bag of this brand has a mean of 2.5 kg and an SD of 0.1 kg. The exact distribution is unknown.

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(b) If the shipping company charges an overweight fee if a batch exceeds the mean batch weight by more than 1 kg, what is the probability that a batch will be charged?

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Summary

- Desired properties of point estimates: unbiasedness and efficiency
- Distribution of sample proportions (or other parameters) is called a sampling distribution
- When n is sufficiently large and observations are independent, the sample proportion (or other parameter) follows a normal distribution
- The success-failure condition can be used to determine if n is large enough for the CLT to hold (for a sample proportion)