CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 3f: Joint Distributions

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College of Engineering

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Outline

Today's objectives

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Joint distributions:

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- Understand the concept of jointly distributed random variables

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- Differentiate between marginal and conditional distributions

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- Understand the concept of jointly distributed random variables
- Differentiate between marginal and conditional distributions
- Manipulate joint distributions to compute probabilities

Joint distributions

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Given two random variables X and Y:

Discrete case

The joint PMF is:

$$p_{X,Y}(x_i, y_j) = P(X = x_i, Y = y_j) \tag{1}$$

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Continuous case

The joint probability is given by:

$$P(a < X \leq b, c < Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y) dydx \tag{3}$$

Joint and marginal distributions

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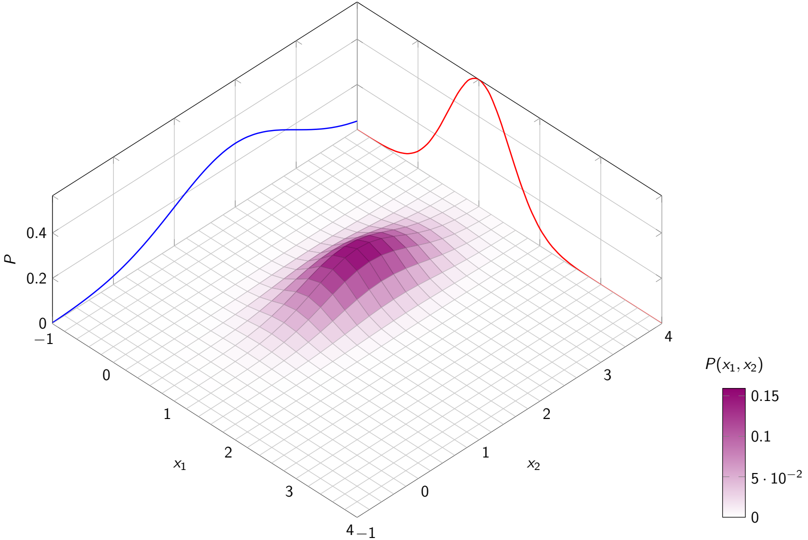

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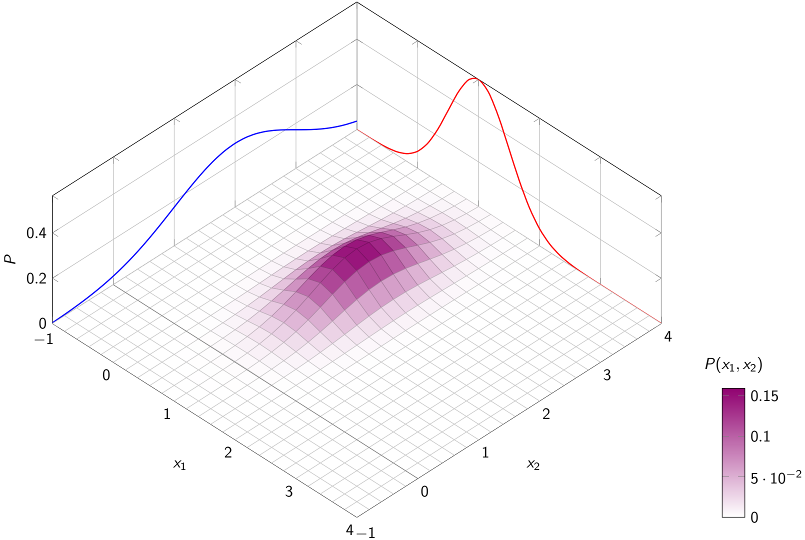

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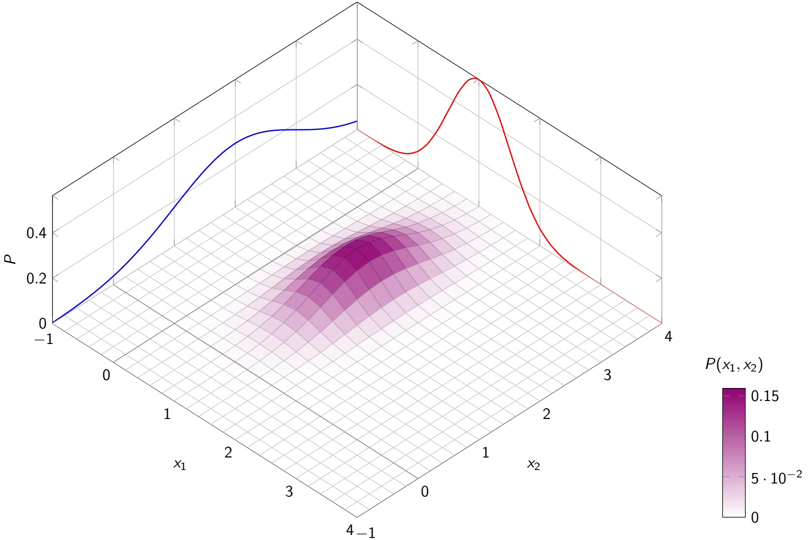

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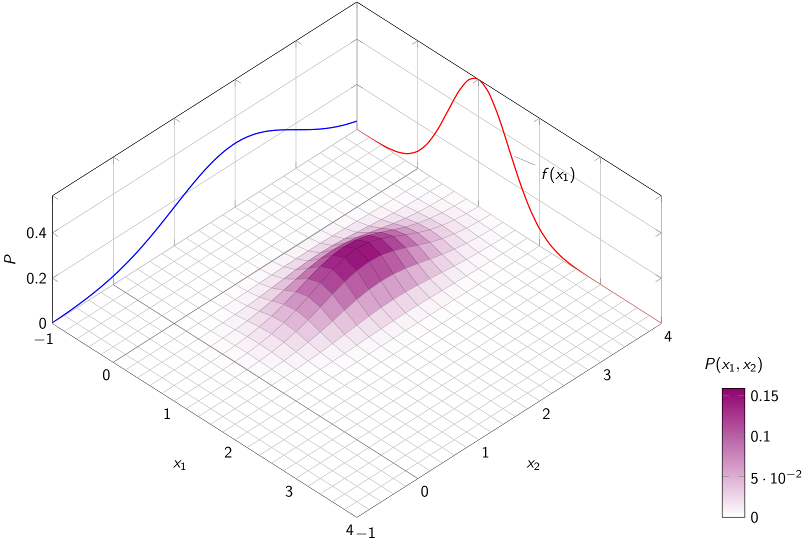

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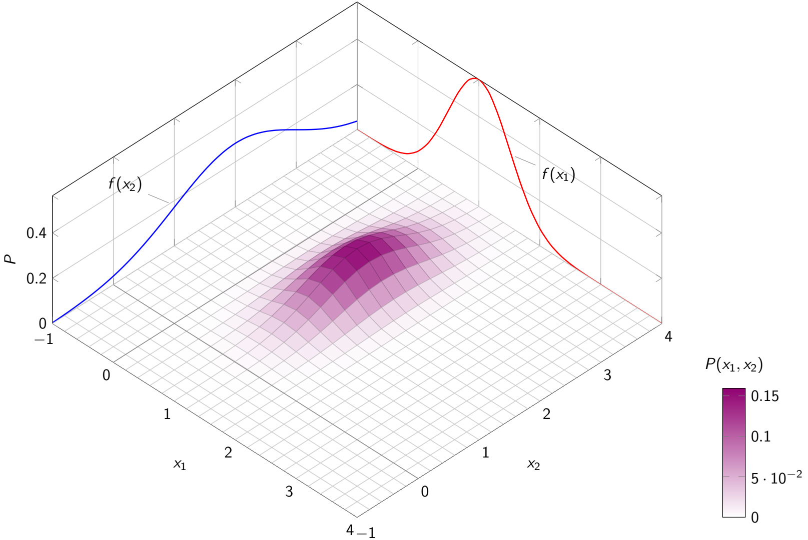

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	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	?
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$p_Y(y)$	?	?	?	

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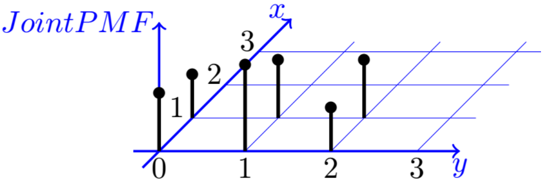

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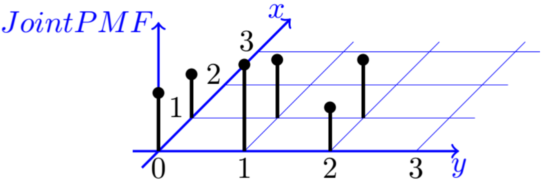

(a) Find $P(X=0, Y\leq 1)$

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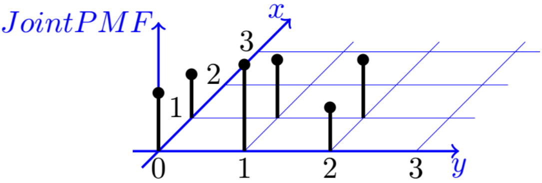

(a) Find $P(X=0, Y \leq 1)$

(b) Find the marginal PMFs of X and Y

(c) Find $P(Y=1|X=0)$

(d) Are X and Y independent?

Example 1: Joint PMF of two random variables (cont.)

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Solution

(a) To find the probability $P(X = 0, Y \leq 1)$, simply add up the cells *jointly* satisfying the conditions.

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$p_Y(y)$	?	?	?	

$$P(X = 0, Y \leq 1) \quad =$$

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$p_Y(y)$	?	?	?	

$$P(X = 0, Y \leq 1) = \frac{1}{6} + \frac{1}{4}$$

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$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	?
$p_Y(y)$	?	?	?	

$$
\begin{aligned}
P(X = 0, Y \leq 1) &= \frac{1}{6} + \frac{1}{4} \\
&= \frac{4 + 6}{24} = \frac{10}{24}
\end{aligned}
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$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	?
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$$\begin{aligned} P(X = 0, Y \leq 1) &= \frac{1}{6} + \frac{1}{4} \\ &= \frac{4+6}{24} = \frac{10}{24} = \frac{5}{12} \end{aligned}$$

Example 1: Joint PMF of two random variables (cont.)

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(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

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$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8}$

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Thus, we obtain:

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Thus, we obtain:

$$p_X(x) =$$

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Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ & \\ & \end{cases}$$

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Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ & \end{cases}$$

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	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

Solution

(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

Solution

(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases} \qquad p_Y(y) =$$

Example 1: Joint PMF of two random variables (cont.)

Solution

(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases} \qquad p_Y(y) = \begin{cases} \frac{7}{24}, & y = 0 \\ & \\ & \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

Solution

(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{7}{24}, & y = 0 \\ \frac{5}{12}, & y = 1 \\ & \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

Solution

(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases} \qquad p_Y(y) = \begin{cases} \frac{7}{24}, & y = 0 \\ \frac{5}{12}, & y = 1 \\ \frac{7}{24}, & y = 2 \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

Solution

(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases} \qquad p_Y(y) = \begin{cases} \frac{7}{24}, & y = 0 \\ \frac{5}{12}, & y = 1 \\ \frac{7}{24}, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8}$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}$

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} =$			

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=$		

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} =$	

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$P(Y=1|X=0) \quad =$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$P(Y=1|X=0) \quad = \quad \frac{P(Y=1 \cap X=0)}{P(X=0)}$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$P(Y=1|X=0) \quad = \quad \frac{P(Y=1\cap X=0)}{P(X=0)} = \frac{P(Y=1, X=0)}{P(X=0)}$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$\begin{aligned} P(Y=1|X=0) &= \frac{P(Y=1\cap X=0)}{P(X=0)} = \frac{P(Y=1,X=0)}{P(X=0)} \\ &= \frac{p_{XY}(0,1)}{p_X(0)} = \end{aligned}$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$
\begin{aligned}
P(Y=1|X=0) &= \frac{P(Y=1\cap X=0)}{P(X=0)} = \frac{P(Y=1, X=0)}{P(X=0)} \\
&= \frac{p_{XY}(0,1)}{p_X(0)} = \frac{\frac{1}{4}}{\frac{13}{24}} =
\end{aligned}
$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$\begin{aligned} P(Y=1|X=0) &= \frac{P(Y=1\cap X=0)}{P(X=0)} = \frac{P(Y=1,X=0)}{P(X=0)} \\ &= \frac{p_{XY}(0,1)}{p_X(0)} = \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13} \end{aligned}$$

Example 1: Joint PMF of two random variables (cont.)

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}$

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6}$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=$			

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=$		

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=$	

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(d) Two r.v.'s are independent if $P(Y=y_j|X=x_i)=P(X=x_i)$ or $P(X=x_i|Y=y_j)=P(Y=y_j)$ for all i and j.

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(d) Two r.v.'s are independent if $P(Y=y_j|X=x_i)=P(X=x_i)$ or $P(X=x_i|Y=y_j)=P(Y=y_j)$ for all i and j. Here,

$$P(Y=1|X=0)$$

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Solution

(d) Two r.v.'s are independent if $P(Y = y_j | X = x_i) = P(X = x_i)$ or $P(X = x_i | Y = y_j) = P(Y = y_j)$ for all i and j. Here,

$$P(Y = 1 | X = 0) = \frac{6}{13}$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
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(d) Two r.v.'s are independent if $P(Y=y_j|X=x_i)=P(X=x_i)$ or $P(X=x_i|Y=y_j)=P(Y=y_j)$ for all i and j. Here,

$$P(Y=1|X=0)=\frac{6}{13}\neq P(Y=1)$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
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The independence condition fails. Hence X and Y are not independent.

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The daily water levels of two reservoirs A and B are denoted by two r.v.'s X and Y having the following joint PDF:

$$f(x, y) = \frac{6}{5}\left(x + y^2\right), \quad 0 < x < 1; 0 < y < 1$$

(a) Determine the marginal density function of the daily water level for reservoir A.

(b) If reservoir A is half full on a given day, what is the probability that the water level will be more than half full?

Example 2: Water levels (cont.)

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$$f(x,y) = \frac{6}{5}\left(x + y^2\right), \quad 0 < x < 1; 0 < y < 1$$

Solution

(a) We obtain the marginal density function $f_X(x)$ by integrating the joint PDF over y:

Example 2: Water levels (cont.)

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(a) We obtain the marginal density function $f_X(x)$ by integrating the joint PDF over y:

$$f_X(x) = \int_0^1 \frac{6}{5}\left(x + y^2\right) dy$$

Example 2: Water levels (cont.)

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(a) We obtain the marginal density function $f_X(x)$ by integrating the joint PDF over y:

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{6}{5}\left(x + y^2\right) dy \\ &= \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_0^1 \end{aligned}$$

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This is the **marginal distribution**

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Thus: $P(Y>0.5|X=0.5) =$

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Reading

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Reading

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- Joint distributions: Section 2.6 (Navidi)