

# CEE 260/MIE 273: Probability and Statistics in Civil Engineering

## Lecture 6C: Analysis of Variance (ANOVA)

**Jimi Oke**

UMassAmherst

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College of Engineering

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# Outline

- ① Introduction
- ② One-way ANOVA
- ③ ANOVA table
- ④ Python
- ⑤ Outlook

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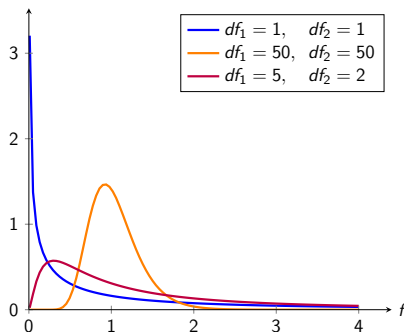
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- $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$
- $MSE$  is an unbiased estimate of  $\sigma^2$  whether or not  $H_0$  is true
- $MSG$  is an unbiased estimate of  $\sigma^2$  ONLY when  $H_0$  is true

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The test procedure compares a **measure of differences among the sample means** (“between-samples” variation) to a **measure of variation calculated from within each of the samples**.

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# The ANOVA F-test

---

<sup>1</sup>Use `f.ppf(1 - alpha, df1, df2)` in Python

# The ANOVA F-test

**Assumption** All  $k$  populations are normally distributed with means  $\mu_i$  and equal variance  $\sigma^2$ .

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# The ANOVA F-test

**Assumption** All  $k$  populations are normally distributed with means  $\mu_i$  and equal variance  $\sigma^2$ . The samples are independent and random.

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$H_1$  : at least one the other  $\mu_i$ 's differs from the others

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**Compare** (a) Find critical value  $F_{(1-\alpha, df_1, df_2)}^1$  (also called  $f^*$ ).

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(b) Find  $p$ -value.

If  $p\text{-value} \geq \alpha$ , **fail to reject  $H_0$**

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# The ANOVA F-test

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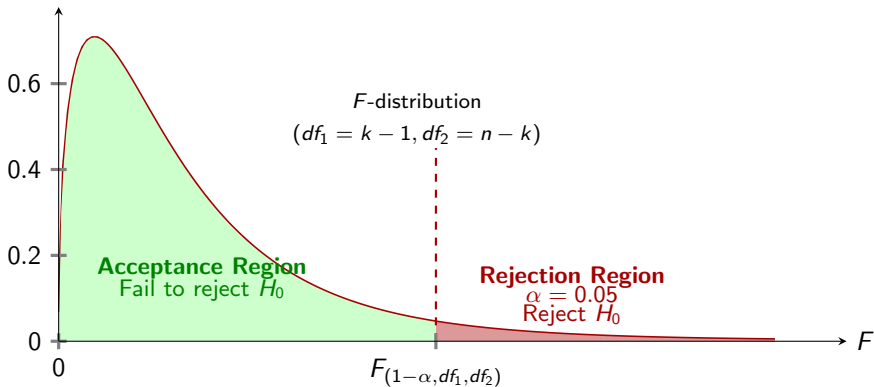
If  $p\text{-value} \geq \alpha$ , **fail to reject  $H_0$**

If  $p\text{-value} < \alpha$ , **reject  $H_0$**

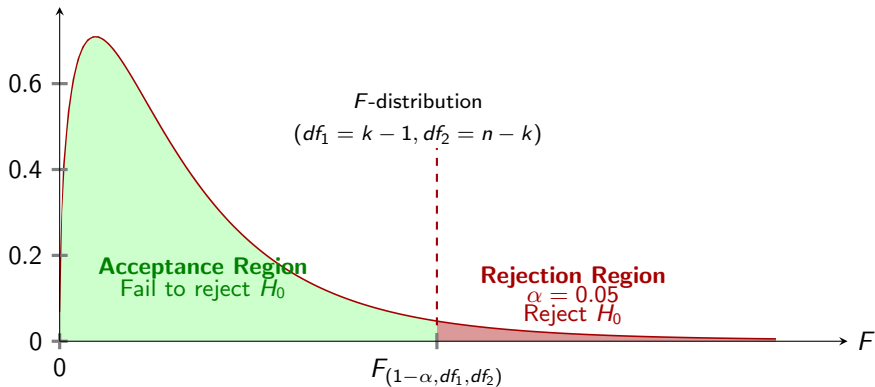
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# ANOVA F-test visualization

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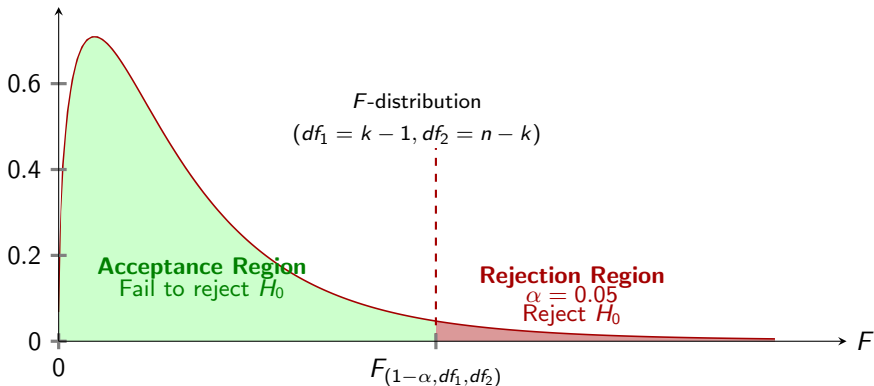


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## Decision Rule

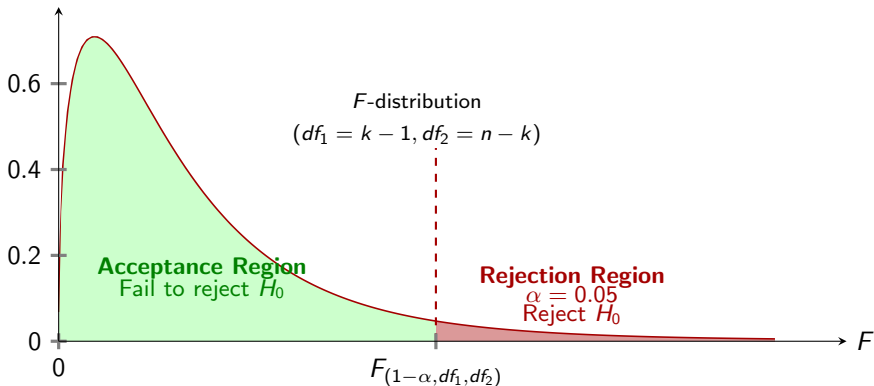
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## Decision Rule

- If  $f \leq F_{(1-\alpha, df_1, df_2)}$ : Fail to reject  $H_0$  (means are equal)

# ANOVA F-test visualization



## Decision Rule

- If  $f \leq F_{(1-\alpha, df_1, df_2)}$ : Fail to reject  $H_0$  (means are equal)
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# ANOVA table

When conducting one-way ANOVA, computations are usually summarized in an ANOVA table

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Source of variation	d.o.f.	Sum of Squares	Mean Square	$f$
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Source of variation	d.o.f.	Sum of Squares	Mean Square	$f$
Groups	$k - 1$	$SSG$	$MSG = SSG / (k - 1)$	$MSG / MSE$

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Source of variation	d.o.f.	Sum of Squares	Mean Square	$f$
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Error	$n - k$	$SSE$	$MSE = SSE / (n - k)$	

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Error	$n - k$	$SSE$	$MSE = SSE / (n - k)$	
Total	$n - 1$	$SST$		

# Using the ANOVA table

## Example 1: Elastic moduli of alloys

# Using the ANOVA table

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An experiment was performed to measure the elastic modulus (GPa) of an alloy produced using three different casting processes. Let  $\mu_1, \mu_2, \mu_3$  denote the true average elastic moduli for the 3 different processes. Using the ANOVA table given below, test the null hypothesis that all three means are equal (using the  $p$ -value approach).

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Error	19	6.00	.3158	

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Source of variation	d.o.f.	Sum of Squares	Mean Square	$f$
Treatments	2	7.93	3.965	?
Error	19	6.00	.3158	
Total	21	13.93		

# Using the ANOVA table

## Example 1: Elastic moduli of alloys (cont.)

# Using the ANOVA table

## Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute  $f$ :

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# Using the ANOVA table

## Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute  $f$ :

$$f = \frac{MSG}{MSE} = \frac{3.965}{.3158} = 12.56$$

# Using the ANOVA table

## Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute  $f$ :

$$f = \frac{MSG}{MSE} = \frac{3.965}{.3158} = 12.56$$

Step 2. Second, we find the  $p$ -value<sup>a</sup>:

# Using the ANOVA table

## Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute  $f$ :

$$f = \frac{MSG}{MSE} = \frac{3.965}{.3158} = 12.56$$

Step 2. Second, we find the  $p$ -value<sup>a</sup>:

$$p\text{-value} = 1 - F(12.56, 2, 19) = 0.0003$$

Step 3. We conclude: since the  $p$ -value is very small, we can **reject the null hypothesis** at any reasonable significance level.

---

<sup>a</sup>In MATLAB, use: `1 - f.cdf(12.56, 2, 19)` OR `f.sf(12.56, 2, 19)`

# One-way ANOVA in practice

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## Example 2: Recording tape quality

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In a effort to improve the quality of recording tapes, the effects of four kinds of coatings A, B,C, D on the reproducing quality of sound are compared. Independent samples are obtained foreach kind of coating.

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## Example 2: Recording tape quality

In a effort to improve the quality of recording tapes, the effects of four kinds of coatings A, B,C, D on the reproducing quality of sound are compared.

Independent samples are obtained foreach kind of coating.

The following values on distortion are measured:

Coating	Observations	Sample Sizes
A	10, 15, 8, 12, 15	5
B	14, 18, 21, 15	4
C	17, 16, 14, 15, 17, 15, 18	7
D	12, 15, 17, 15, 16, 15	6

With the help of such a sample we want to decide if the four different coatings result in different mean distortions, i.e. test if  $\mu_A = \mu_B = \mu_C = \mu_D$  (use  $\alpha = 0.05$ ).

# One-way ANOVA in practice

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## Example 2: Recording tape quality (cont.)

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Step 1. Assemble the parameters:

# One-way ANOVA in practice

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$$k = 4$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

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$$k = 4$$

$$n_1 = 5, n_2 = 4, n_3 = 7, n_4 = 6$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 1. Assemble the parameters:

$$k = 4$$

$$n_1 = 5, n_2 = 4, n_3 = 7, n_4 = 6$$

$$T_1 = 60, T_2 = 68, T_3 = 112, T_4 = 90$$

# One-way ANOVA in practice

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$$n = \sum n_i = 22$$

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$$T_1 = 60, T_2 = 68, T_3 = 112, T_4 = 90$$

$$n = \sum n_i = 22$$

$$\sum x_{ij} = \sum T_i = 330$$

# One-way ANOVA in practice

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## Example 2: Recording tape quality (cont.)

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Step 2. Compute the sums:

$$CM = \frac{1}{n} \left( \sum x_{ij} \right)^2 = \frac{330^2}{22} = 4950$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

$$CM = \frac{1}{n} \left( \sum x_{ij} \right)^2 = \frac{330^2}{22} = 4950$$

$$SST = \sum x_{ij}^2 - CM = (10^2 + 15^2 + 8^2 + \cdots + 15^2 + 5^2) - CM$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

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$$\begin{aligned} SST &= \sum x_{ij}^2 - CM = (10^2 + 15^2 + 8^2 + \cdots + 15^2 + 5^2) - CM \\ &= 5112 - 4950 = 162 \end{aligned}$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

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$$SSG = \sum \frac{T_i^2}{n_i} - CM$$

# One-way ANOVA in practice

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$$\begin{aligned} SSG &= \sum \frac{T_i^2}{n_i} - CM \\ &= \left( \frac{60^2}{5} + \frac{68^2}{4} + \frac{112^2}{7} + \frac{90^2}{6} \right) - CM = 68 \end{aligned}$$

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## Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

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$$\begin{aligned} SSG &= \sum \frac{T_i^2}{n_i} - CM \\ &= \left( \frac{60^2}{5} + \frac{68^2}{4} + \frac{112^2}{7} + \frac{90^2}{6} \right) - CM = 68 \end{aligned}$$

$$SSE = SST - SSG = 162 - 68 = 94$$

# One-way ANOVA in practice

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 3. Create the ANOVA table:

Source	d.o.f.	SS	MS	$f$
Coating	3	68	$MSG = 22.67$	$MSG/MSE = 4.343$
Error	18	94	$MSE = 5.22$	
Total	21	162		

# One-way ANOVA in practice

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad (\text{from Python})$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad (\text{from Python})$$

Notes:

- Use `f.ppf(0.95, 3, 18)` to find the critical value.

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

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$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad (\text{from Python})$$

Notes:

- Use `f.ppf(0.95, 3, 18)` to find the critical value.
- The  $F$  curve is NOT symmetric. So, `-f.ppf(0.05, 3, 18)` will NOT give the same answer.

# One-way ANOVA in practice

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- However, the  $F$ -curve has the property

# One-way ANOVA in practice

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- However, the  $F$ -curve has the property

$$F_{(\alpha, df_1, df_2)} = \frac{1}{F_{(1-\alpha, df_2, df_1)}} \quad (16)$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad (\text{from Python})$$

Notes:

- Use `f.ppf(0.95, 3, 18)` to find the critical value.
- The  $F$  curve is NOT symmetric. So, `-f.ppf(0.05, 3, 18)` will NOT give the same answer.
- However, the  $F$ -curve has the property

$$F_{(\alpha, df_1, df_2)} = \frac{1}{F_{(1-\alpha, df_2, df_1)}} \quad (16)$$

- So, in this case, `f.ppf(0.05, 18, 3)` will give the same answer as `f.ppf(0.95, 3, 18)`.

# One-way ANOVA in practice

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 5. Compare:

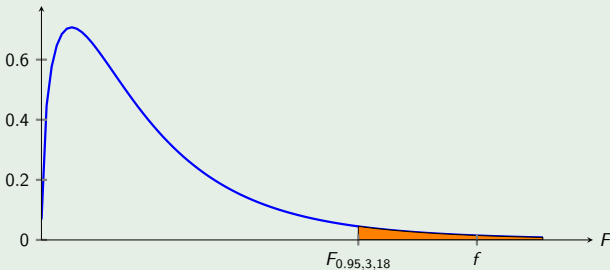
$$f = 4.343 > F_{0.95,3,18} = 3.1599$$

# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 5. Compare:

$$f = 4.343 > F_{0.95,3,18} = 3.1599$$

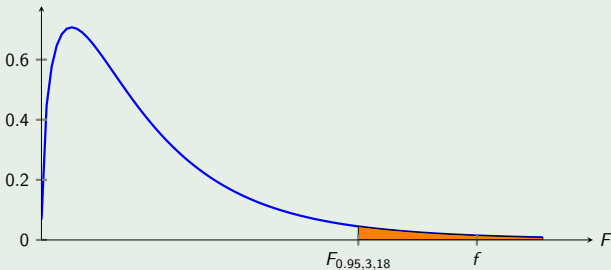


# One-way ANOVA in practice

## Example 2: Recording tape quality (cont.)

Step 5. Compare:

$$f = 4.343 > F_{0.95,3,18} = 3.1599$$



Thus,  $f$  lies in the critical region, and we **reject the null hypothesis** that the mean distortions from the four coatings are equal.

# ANOVA in MATLAB/Python

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## Example 3: Investigating fuel efficiency of cars manufactured in different countries

The `carbig` dataset in MATLAB contains variables<sup>a</sup> for 400 cars from the 70s and 80s. The `carsmall` dataset is a subset of `carbig` (years '70, '76 and '82). We want to check if the true average fuel efficiency (MPG) of vehicles from different countries (USA/Europe/Japan) are equal.

<sup>a</sup>For more on visualizing some of these variables, visit

<https://www.mathworks.com/help/stats/examples/visualizing-multivariate-data.html>

# ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

# ANOVA in MATLAB (cont.)

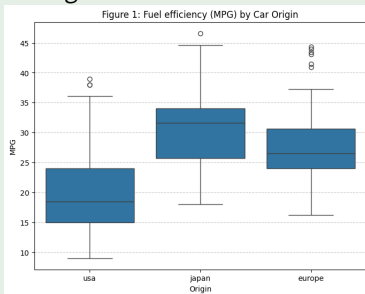
## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

**Step 1.** Examine the contents of the `6C-anova-cars.ipynb` script. Run the script.

# ANOVA in MATLAB (cont.)

## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

- Step 1.** Examine the contents of the `6C-anova-cars.ipynb` script. Run the script.
- Step 2.** Figure 1 is a boxplot of the MPG across the different origins. What can you say about the variance *within* each of the countries/regions, and *among* them?



# ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

# ANOVA in MATLAB (cont.)

## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

- Step 3. The function `scipy.stats.f_oneway` generates the ANOVA results (though not formatted as nicely as in MATLAB—see Appendix). More detail is provided by `statsmodels.stats.oneway.anova_oneway`.
- Step 4. What is the p-value for this test?

# ANOVA in MATLAB (cont.)

## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

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# ANOVA in MATLAB (cont.)

## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

- Step 3.** The function `scipy.stats.f_oneway` generates the ANOVA results (though not formatted as nicely as in MATLAB—see Appendix). More detail is provided by `statsmodels.stats.oneway.anova_oneway`.
- Step 4.** What is the p-value for this test? Based on this, do we **reject/fail to reject** the null hypothesis at any reasonable significance level?
- Step 5.** In Figure 3, we have a notched boxplot, which denotes the confidence interval of the median. What are your observations?

# Multiple comparisons in ANOVA

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# Multiple comparisons in ANOVA

- If after performing a one-way ANOVA, we reject the null hypothesis, naturally we might want to find out which populations significantly differ from each other
- Various methods have been developed for multiple pairwise comparisons (out of scope of this lecture):
  - One option: perform pairwise  $t$ -tests between group means using the Bonferroni correction for  $\alpha$
  - Use Tukey's range test, which performs a significance test based on the studentized range distribution

# Multiple comparisons in ANOVA (cont.)

# Multiple comparisons in ANOVA (cont.)

## Example 3: Investigating fuel efficiency of cars (cont.)

# Multiple comparisons in ANOVA (cont.)

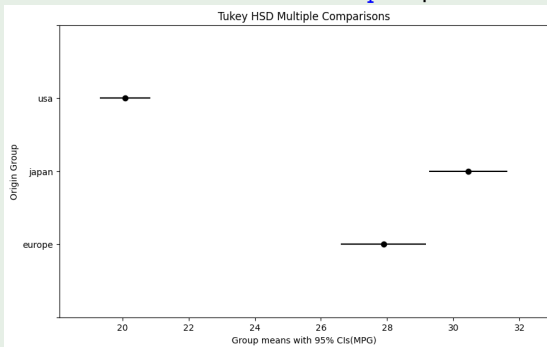
## Example 3: Investigating fuel efficiency of cars (cont.)

Step 5. Use the `pairwise_tukeyhsd` function from `statsmodels.stats.multicomp` to perform multiple comparisons.

# Multiple comparisons in ANOVA (cont.)

## Example 3: Investigating fuel efficiency of cars (cont.)

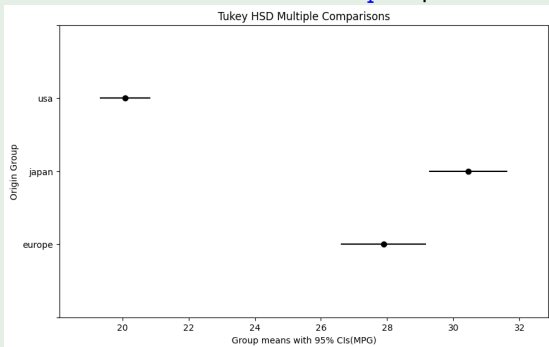
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Which regions have significantly different average fuel efficiency?

# Summary

- The analysis of variance (ANOVA) test is used to determine whether the population means across multiple samples are equal
- It uses the test statistic ( $f = MSG/MSE$ ) which is  $F$ -distributed
- Reading: Section 7.5 (Open Intro Statistics)

# Summary

- The analysis of variance (ANOVA) test is used to determine whether the population means across multiple samples are equal
- It uses the test statistic ( $f = MSG/MSE$ ) which is  $F$ -distributed
- Reading: Section 7.5 (Open Intro Statistics)
- If interested, read Section 7.4 on “power” analysis (but we will consider this out of scope for the course)

## ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

# ANOVA in MATLAB (cont.)

## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

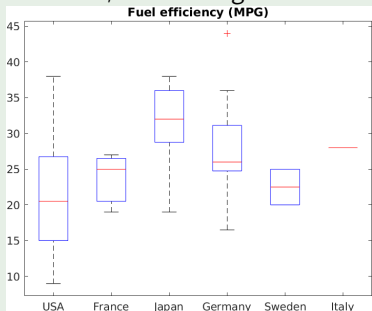
**Step 1.** Examine the contents of the `example3.m` script. Run the script.

# ANOVA in MATLAB (cont.)

## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

**Step 1.** Examine the contents of the `example3.m` script. Run the script.

**Step 2.** Figure 1 is a boxplot of the MPG across the different origins (6 countries). What can you say about the variance *within* each of the countries, and *among* them?



# ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

# ANOVA in MATLAB (cont.)

## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

**Step 3.** The function `anova1` generates 2 figures: the first is the ANOVA table (Figure 2) and the second a notched boxplot (Figure 3)

ANOVA Table					
Source	SS	df	MS	F	Prob>F
Groups	1629.87	5	325.974	6.56	3.14631e-05
Error	4375.41	88	49.721		
Total	6005.28	93			

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# ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

# ANOVA in MATLAB (cont.)

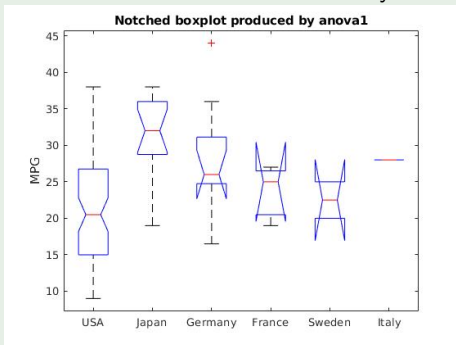
## Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

**Step 5.** In Figure 3, we have a notched boxplot, which denotes the confidence interval of the median. What are your observations?

# ANOVA in MATLAB (cont.)

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## Multiple comparisons in ANOVA (cont.)

### Example 3: Investigating fuel efficiency of cars (cont.)

## Multiple comparisons in ANOVA (cont.)

### Example 3: Investigating fuel efficiency of cars (cont.)

Step 5. Uncomment lines 13 and 14 and rerun the `example3.m` script.

# Multiple comparisons in ANOVA (cont.)

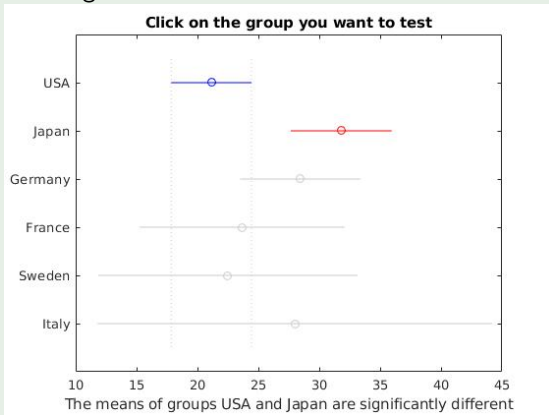
## Example 3: Investigating fuel efficiency of cars (cont.)

**Step 5.** Uncomment lines 13 and 14 and rerun the `example3.m` script. Interact with Figure 5.

# Multiple comparisons in ANOVA (cont.)

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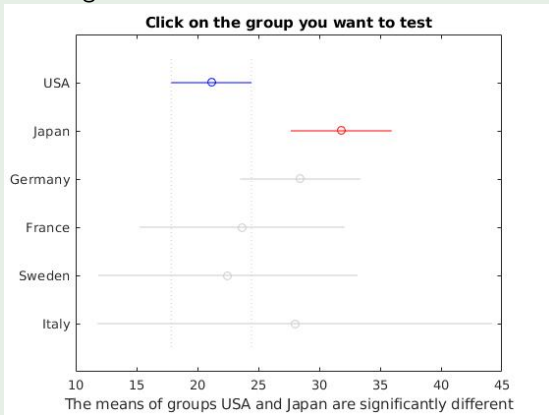
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## Example 3: Investigating fuel efficiency of cars (cont.)

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Which countries have significantly different average fuel efficiency?