CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 6B: Inference for Two Samples

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Outline

Confidence intervals for difference of two means

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- You can easily derive the formulas for the upper/lower confidence bounds.

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Example 1: Permeability of textile fabrics

The void volume within a textile fabric affects comfort, flammability and insulation properties. Permeability of a fabric refers to the accessibility of void space to the flow of a gas or liquid. Consider the following permeability ($cm^3/cm^2/sec$) summary data on two different types of plain-weave fabric

Microscopic images of cotton fiber arrangements. Source: https://journals.sagepub.com/doi/full/10.1177/15589250211024225

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Fabric type	Sample size	Sample mean	Sample SD
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Triacetate	10	136.14	3.59

Assuming that the permeability distributions for both types of fabric are normal, calculate a CI for the difference between true average permeability for the cotton fabric and that for the triacetate fabric using a 95% confidence level.

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$$df \approx \min(n_1 - 1, n_2 - 1) \tag{6}$$

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In this case, we can see that since $n_1 = n_2$, we ended up with $df = 10 - 1 = 9$.

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With a high degree of confidence, we can say that true average permeability for triacetate fabric specimens exceeds that for cotton specimens by between 81.80 and 87.06 $cm^3/cm^2/sec$.

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Key assumptions

- Normality: X_1 is a random normal sample; X_2 is a random normal sample

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- Is there any significant difference between the performances in the 2018 NY marathon and that of 2019?
- Is the hardness of heat-treated steel similar to that of cold-rolled steel?
- How can an engineer test if the proportion of defective batteries in a production batch is similar to that of another batch?

Key assumptions

- Normality: X_1 is a random normal sample; X_2 is a random normal sample
- Independence: X_1 and X_2 are independent of each other

Testing normal populations with known variances

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Null hypothesis:

Testing normal populations with known variances

Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0$.

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Often Δ_0 is often 0, in which case, $H_0 : \mu_1 = \mu_2$.

Test statistic:

$$z = \frac{\overline{x_1} - \overline{x_2} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \tag{9}$$

Testing normal populations with known variances

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where $\overline{x}_1$, $\overline{x}_2$ are the sample means, σ_1^2 σ_2^2 are the respective population variances, and n_1, n_2, the respective sample sizes.

Alternative Hypothesis	Rejection Region for level α test
$H_1 : \mu_1 - \mu_2 > \Delta_0$	$z \geq z_{1-\alpha}$ (upper-tailed)
$H_1 : \mu_1 - \mu_2 < \Delta_0$	$z \leq z_{\alpha}$ (lower-tailed)
$H_1 : \mu_1 - \mu_2 \neq \Delta_0$	$z \leq z_{\alpha/2}$ or $z \geq z_{(1-\alpha/2)}$ (both tails)

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Difference between two population means

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Analysis of a random sample consisting of $n_1 = 20$ specimens of cold-rolled steel to determine yield strengths resulted in a sample average strength of $\overline{x} = 29.8$ ksi.

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Analysis of a random sample consisting of $n_1 = 20$ specimens of cold-rolled steel to determine yield strengths resulted in a sample average strength of $\overline{x} = 29.8$ ksi.

A second random sample of $n_2 = 25$ two-sided galvanized steel specimens gave a sample average strength of $\overline{y} = 34.7$ ksi.

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Assuming that the two yield-strength distributions are normal with $\sigma_1 = 4.0$ and $\sigma_2 = 5.0$, do the data indicate that the corresonding true average yield strengths μ_1 and μ_2 are different?

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Assuming that the two yield-strength distributions are normal with $\sigma_1 = 4.0$ and $\sigma_2 = 5.0$, do the data indicate that the corresonding true average yield strengths μ_1 and μ_2 are different?

Carry out a test at significance level $\alpha = 0.01$.

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Step 1. Parameter of interest: $\mu_1 - \mu_2$ (difference between the true average strengths)

Step 2. Null hypothesis:

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Step 1. Parameter of interest: $\mu_1 - \mu_2$ (difference between the true average strengths)

Step 2. Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0 = 0$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Step 1. Parameter of interest: $\mu_1 - \mu_2$ (difference between the true average strengths)

Step 2. Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0 = 0$

Step 3. Alternative hypothesis:

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Step 1. Parameter of interest: $\mu_1 - \mu_2$ (difference between the true average strengths)

Step 2. Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0 = 0$

Step 3. Alternative hypothesis: $H_1 : \mu_1 - \mu_2 \neq 0$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Step 1. Parameter of interest: $\mu_1 - \mu_2$ (difference between the true average strengths)

Step 2. Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0 = 0$

Step 3. Alternative hypothesis: $H_1 : \mu_1 - \mu_2 \neq 0$

Step 4. Formula for test statistic value:

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel

Step 1. Parameter of interest: $\mu_1 - \mu_2$ (difference between the true average strengths)

Step 2. Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0 = 0$

Step 3. Alternative hypothesis: $H_1 : \mu_1 - \mu_2 \neq 0$

Step 4. Formula for test statistic value:

$$z = \frac{\overline{x} - \overline{y} - (0)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} =$$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} = \frac{-4.90}{1.34} =$$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} = \frac{-4.90}{1.34} = -3.66$$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} = \frac{-4.90}{1.34} = -3.66$$

Step 6. Find the critical values (two-tailed test):

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

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$$\alpha/2 = 0.01/2 = 0.005$$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

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$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} = \frac{-4.90}{1.34} = -3.66$$

Step 6. Find the critical values (two-tailed test):

$$\begin{aligned} \alpha/2 &= 0.01/2 = 0.005 \\ z^* = z_{(1-\alpha/2)} &= z_{0.995} = \end{aligned}$$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} = \frac{-4.90}{1.34} = -3.66$$

Step 6. Find the critical values (two-tailed test):

$$\alpha/2 = 0.01/2 = 0.005$$

$$z^* = z_{(1-\alpha/2)} = z_{0.995} = 2.58 \quad (\texttt{norm.ppf(0.995)})$$

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} = \frac{-4.90}{1.34} = -3.66$$

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Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{29.8 - 34.7}{\sqrt{\frac{16.0}{20} + \frac{25.0}{25}}} = \frac{-4.90}{1.34} = -3.66$$

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Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 7. Conclude:

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 7. Conclude:
Since $-3.66 < -2.58$, z falls in the lower tail of the rejection region.

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

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Since $-3.66 < -2.58$, z falls in the lower tail of the rejection region. H_0 is therefore rejected in favor of the conclusion that $\mu_1 \neq \mu_2$.

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 7. Conclude:

Since $-3.66 < -2.58$, z falls in the lower tail of the rejection region. H_0 is therefore rejected in favor of the conclusion that $\mu_1 \neq \mu_2$.

Furthermore, the p-value is $2(1 - \Phi(3.66)) \approx 2(1 - 1) = 0$.

Difference between two population means

Example 1: Cold-rolled vs. galvanized steel (cont.)

Step 7. Conclude:
Since $-3.66 < -2.58$, z falls in the lower tail of the rejection region. H_0 is therefore rejected in favor of the conclusion that $\mu_1 \neq \mu_2$.

Furthermore, the p-value is $2(1 - \Phi(3.66)) \approx 2(1 - 1) = 0$.

So, H_0 should be rejected at any reasonable significance level.

Paired data

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Paired data arise when two different observations are made on the **same set** of n individuals in a sample.

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Assumptions

- n independently selected pairs of observations: $\mathbb{E}(X_i) = \mu_1, \mathbb{E}(Y_i) = \mu_2$.

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- n independently selected pairs of observations: $\mathbb{E}(X_i) = \mu_1, \mathbb{E}(Y_i) = \mu_2$.
- Differences betweeen pairs $D_i = X_i - Y_i$ are normally distributed with mean μ_D and variance σ_D^2

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- n independently selected pairs of observations: $\mathbb{E}(X_i) = \mu_1, \mathbb{E}(Y_i) = \mu_2$.
- Differences betweeen pairs $D_i = X_i - Y_i$ are normally distributed with mean μ_D and variance σ_D^2
  - If n is small, conduct t-test
  - If n is large, conduct z-test

Paired t-test

Paired t-test

Null hypothesis:

Paired t-test

Null hypothesis: $H_0 : \mu_D = \mu_0$.

Paired $\boldsymbol{t}$-test

Null hypothesis: $H_0 : \mu_D = \mu_0$.
Test statistic:

$$t = \frac{\overline{d} - \mu_0}{s_D / \sqrt{n}} \tag{10}$$

Paired t-test

Null hypothesis: $H_0 : \mu_D = \mu_0$.
Test statistic:

$$t = \frac{\overline{d} - \mu_0}{s_D/\sqrt{n}} \tag{10}$$

where $\overline{d}$ is the sample difference, μ_0 is the null difference and s_D is the sample standard deviation

Alternative Hypothesis	Rejection Region for level α test
$H_1 : \mu_D > \mu_0$	$t \geq t_{\alpha,n-1}$ (upper-tailed)
$H_1 : \mu_D < \mu_0$	$t \leq t_{\alpha,n-1}$ (lower-tailed)
$H_1 : \mu_D \neq \mu_0$	$t \leq t_{\alpha/2,n-1}$ or $t \geq t_{(1-\alpha/2),n-1}$ (both tails)

Paired t-test example

Example 2: Intervention for ergonomic improvements

Musculoskeletal neck-and-shoulder disorders are all too common among office staff who perform repetitive tasks using visual display units. A study was conducted to determine whether more varied work conditions would have any impact on arm movement.[a]

Paired t-test example

Example 2: Intervention for ergonomic improvements

Musculoskeletal neck-and-shoulder disorders are all too common among office staff who perform repetitive tasks using visual display units. A study was conducted to determine whether more varied work conditions would have any impact on arm movement.[a]

The accompanying data was obtained from a sample of $n = 16$ subjects. Each observation is the amount of time, expressed as a proportion of total time observed, during which arm elevation was 30°. The two measurements from each subject were obtained 18 months apart.

[a] "Upper-Arm Elevation During Office Work" (*Ergonomics*, 1996: 1221-1230)

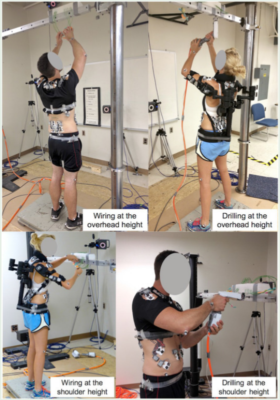

Illustration of upper-arm elevation. Source: https://www.sciencedirect.com/science/article/pii/S0003687018300590

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change?

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change? Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change? Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.

Step 1. $H_0 : \mu_D = 0$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

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Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

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Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Step 2. $H_1 : \mu_D \neq 0$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

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Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Step 2. $H_1 : \mu_D \neq 0$

Step 3. Compute the parameters:

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change? Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.

Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Step 2. $H_1 : \mu_D \neq 0$

Step 3. Compute the parameters:

$$n = 16$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change? Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.

Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Step 2. $H_1 : \mu_D \neq 0$

Step 3. Compute the parameters:

$$
\begin{aligned}
n &= 16 \\
\sum d_i &= 108
\end{aligned}
$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change? Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.

Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Step 2. $H_1 : \mu_D \neq 0$

Step 3. Compute the parameters:

$$\begin{aligned} n &= 16 \\ \sum d_i &= 108 \\ \sum d_i^2 &= 1746 \end{aligned}$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change? Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.

Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Step 2. $H_1 : \mu_D \neq 0$

Step 3. Compute the parameters:

$$
\begin{aligned}
n &= 16 \\
\sum d_i &= 108 \\
\sum d_i^2 &= 1746 \\
\overline{d} &= 6.75
\end{aligned}
$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

During this period, work conditions were changed, and subjects were allowed to engage in a wider variety of work tasks. Do the data suggest that true average time during which elevation is below 30° differs after the change from what it was before the change? Let μ_D denote the true average difference between elevation time before the change in work conditions and time after the change.

Step 1. $H_0 : \mu_D = 0$ (i.e. there is no difference between true average time before the change and true average time after the change)

Step 2. $H_1 : \mu_D \neq 0$

Step 3. Compute the parameters:

$$
\begin{aligned}
n &= 16 \\
\sum d_i &= 108 \\
\sum d_i^2 &= 1746 \\
\overline{d} &= 6.75 \\
s_D &= 8.234
\end{aligned}
$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Step 5. Compute the T-statistic:

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Step 5. Compute the T-statistic:

$$t = \frac{\overline{d} - 0}{s_D/\sqrt{n}}$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Step 5. Compute the T-statistic:

$$
\begin{aligned}
t &= \frac{\overline{d} - 0}{s_D/\sqrt{n}} \\
&= \frac{6.75}{8.234/\sqrt{16}}
\end{aligned}
$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Step 5. Compute the T-statistic:

$$\begin{aligned} t &= \frac{\overline{d} - 0}{s_D/\sqrt{n}} \\ &= \frac{6.75}{8.234/\sqrt{16}} = 3.28 \approx 3.3 \end{aligned}$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Step 5. Compute the T-statistic:

$$
\begin{aligned}
t &= \frac{\overline{d} - 0}{s_D/\sqrt{n}} \\
&= \frac{6.75}{8.234/\sqrt{16}} = 3.28 \approx 3.3
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Step 6. Find the p-value:

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

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$$
\begin{aligned}
t &= \frac{\overline{d} - 0}{s_D/\sqrt{n}} \\
&= \frac{6.75}{8.234/\sqrt{16}} = 3.28 \approx 3.3
\end{aligned}
$$

Step 6. Find the p-value:

$$p\text{-value} = 2 \cdot (1 - F_T(3.3, 15)) \quad (2 * \texttt{t.sf}(3.3, 15))$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Step 5. Compute the T-statistic:

$$\begin{aligned} t &= \frac{\overline{d} - 0}{s_D/\sqrt{n}} \\ &= \frac{6.75}{8.234/\sqrt{16}} = 3.28 \approx 3.3 \end{aligned}$$

Step 6. Find the p-value:

$$\begin{aligned} p\text{-value} &= 2 \cdot (1 - F_T(3.3, 15)) \quad (2 * \texttt{t.sf}(3.3, 15)) \\ &= 2(0.0024) \end{aligned}$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

Step 5. Compute the T-statistic:

$$
\begin{aligned}
t &= \frac{\bar{d} - 0}{s_D/\sqrt{n}} \\
&= \frac{6.75}{8.234/\sqrt{16}} = 3.28 \approx 3.3
\end{aligned}
$$

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$$
\begin{aligned}
p\text{-value} &= 2 \cdot (1 - F_T(3.3, 15)) \quad (2 * \texttt{t.sf}(3.3, 15)) \\
&= 2(0.0024) \approx 0.005
\end{aligned}
$$

Paired t-test in practice

Example 2: Intervention for ergonomic improvements (cont.)

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Paired t-test in practice

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- When population variances are unknown, we use the *t*-distribution with degrees of freedom computed using a complicated formula or a simple shortcut.
- Paired data arise when two different observations are made on the same set of individuals in a sample.
- The paired *t*-test is used to test hypotheses about the mean difference between paired observations.

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- Two-sample z-test (known variances):

$$z = \frac{\overline{x_1} - \overline{x_2} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Key equations (cont.)

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- Paired data differences $D_i = X_i - Y_i$ are normally distributed with mean μ_D and variance σ_D^2.
- Paired t-test:

$$t = \frac{\overline{d} - \mu_0}{s_D/\sqrt{n}}$$

where $\overline{d} = \overline{x_1} - \overline{x_2}$ is the sample mean difference and s_D is the sample standard deviation of differences