CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture M3c: Lognormal and Exponential Distributions

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College of Engineering

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Outline

Recap of normal distribution

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- In Python, the `norm.cdf(x, mu, sigma)` and `norm.ppf(p, mu, sigma)` can be used to compute probabilities and inverse CDFs of the normal distribution, respectively.

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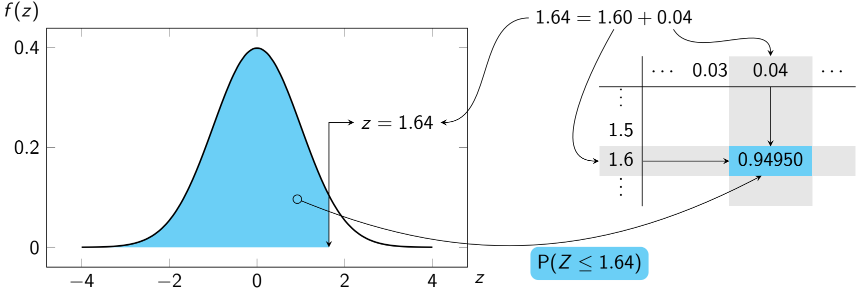

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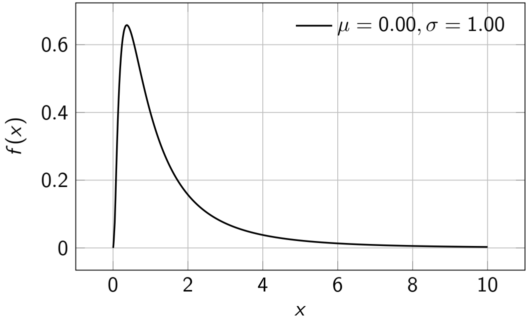

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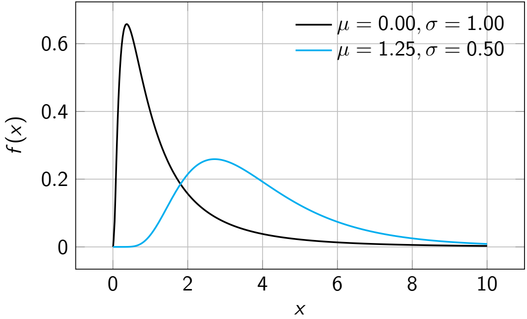

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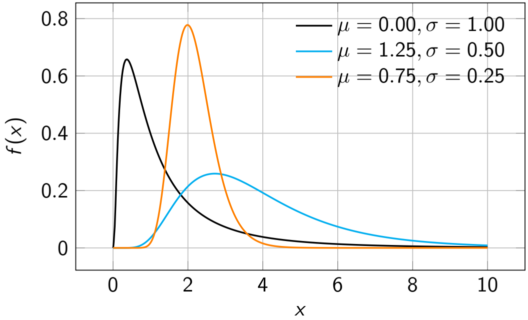

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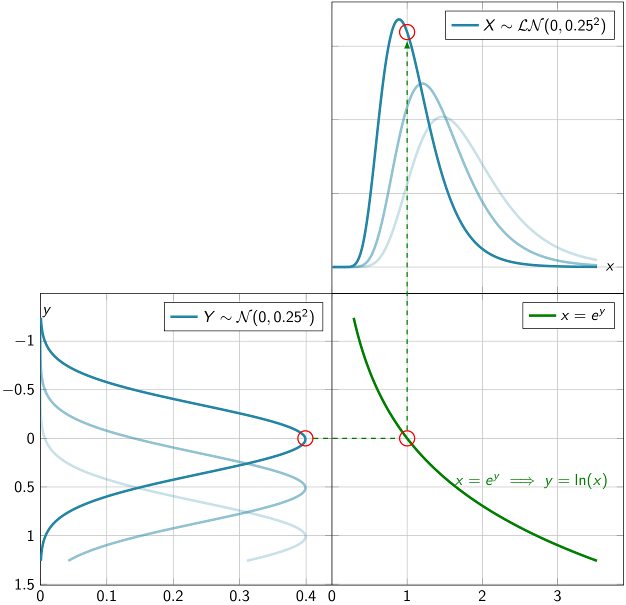

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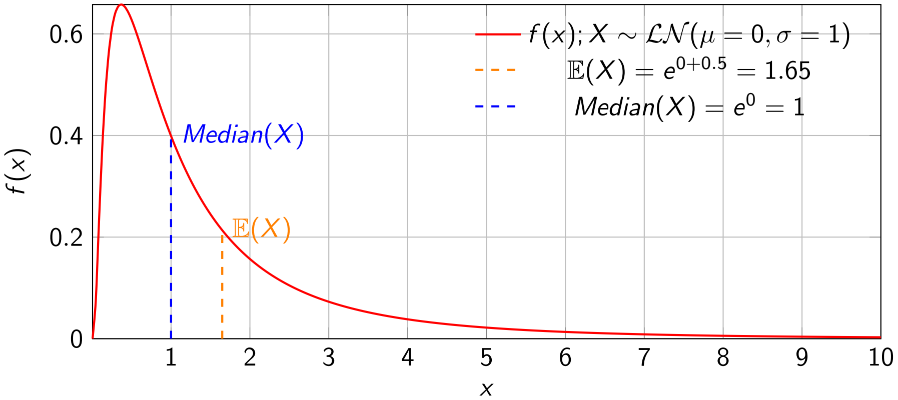

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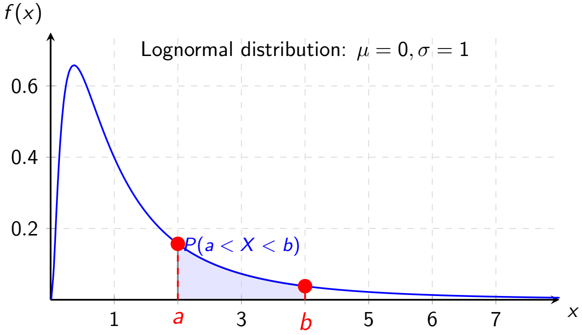

Figure: Lognormal distribution with $\mu = 0$, $\sigma = 1$, showing $P(a < X < b)$ where $a = 2$ and $b = 4$

Probability of a lognormal random variate (cont.)

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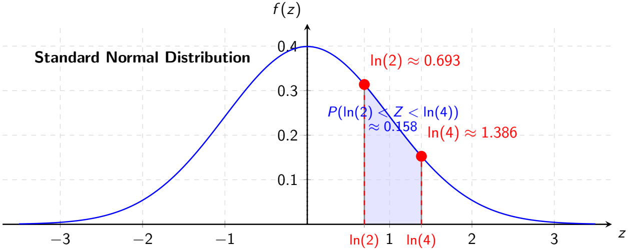

Example 2: Equipment breakdown

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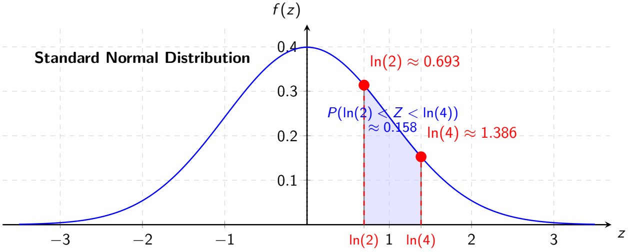

Figure: Standard normal distribution showing $P(\ln(2) < Z < \ln(4))$

Example 2: Probability of incubation period

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The incubation period of the COVID-19 infection is assumed to be lognormally distributed with a median of about 5 days and $\sigma^2 = 0.42$. What is the probability that a randomly selected person will show symptoms within 7 days of exposure (i.e., $P(X \leq 7)$)?

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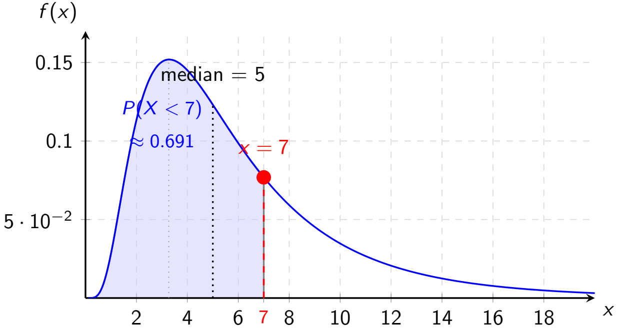

Figure: Lognormal distribution with $\mu = \ln(5)$, $\sigma^2 = 0.42$, showing $P(X \leq 7)$

Example 2: Probability of incubation period (cont.)

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Solution (using Python)

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from scipy import stats
import numpy as np
p = stats.lognorm.cdf(7, s=0.648, scale=5)
```

Example 2: Probability of incubation period (cont.)

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where:

- $\mathtt{s} = \sigma = 0.648 = \sqrt{0.42}$ (shape parameter/standard deviation of underlying normal)
- $\mathtt{scale} = \exp(\mu) = \exp(\ln(5)) = 5$ (scale parameter/median)

The result is: `p = 0.691`, i.e., about 69.1% of the people will show symptoms within 7 days of exposure.

Example 2: Probability of incubation period (cont.)

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where $\mu = \ln(5) = 1.609$ and $\sigma = \sqrt{0.42} = 0.648$. Thus:

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where $\mu = \ln(5) = 1.609$ and $\sigma = \sqrt{0.42} = 0.648$. Thus:

$$\begin{aligned} P(X \leq 7) &= \Phi\left(\frac{\ln 7 - 1.609}{0.648}\right) \\ &= \Phi(0.539) \\ &\approx 0.7054 \text{ (from standard normal table)} \end{aligned}$$

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The result is: $P(X \leq 7) \approx 0.7054$, i.e., about 70.54% of the people will show symptoms within 7 days of exposure.

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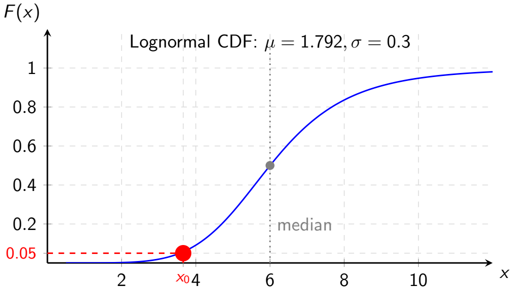

Figure: Lognormal CDF showing the 5th percentile at $x = x_0 = 3.66$

Example 3: Equipment breakdown (cont.)

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Solution (using Python)

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$P(X \leq x_0) = 0.05$

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Code:

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import numpy as np
x0 = stats.lognorm.ppf(q=0.05, s=0.30, scale=6)
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Example 3: Equipment breakdown (cont.)

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Note the following about the *arguments*`stats.lognorm.ppf` function:

- The first, `q`, is the cumulative probability or quantile (0.05)
- The second, `s`, is the shape parameter σ (0.30)
- The third, `scale`, is e^{μ} (median) (6)

This returns $x_0 = 3.66$ months.

Example 3: Equipment breakdown (cont.)

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Solution (using tables)

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Thus:

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Modeling probabilities of elapsed times

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Now consider the variable Y representing the **elapsed time** between successive arrivals.

- What is the probability the time between the third and fourth arrivals is less than y minutes, for instance?
- This is modeled by the **exponential distribution** with parameter λ.

PDF of exponential distribution

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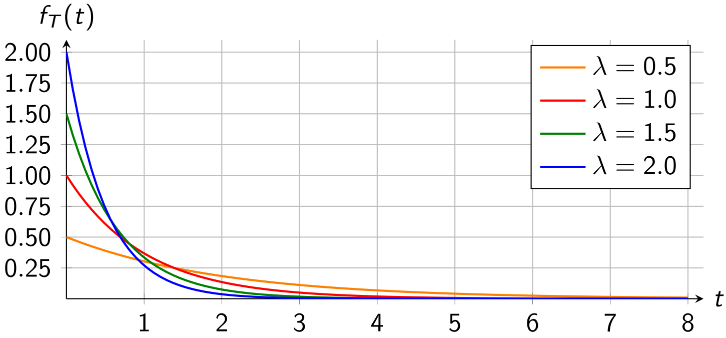

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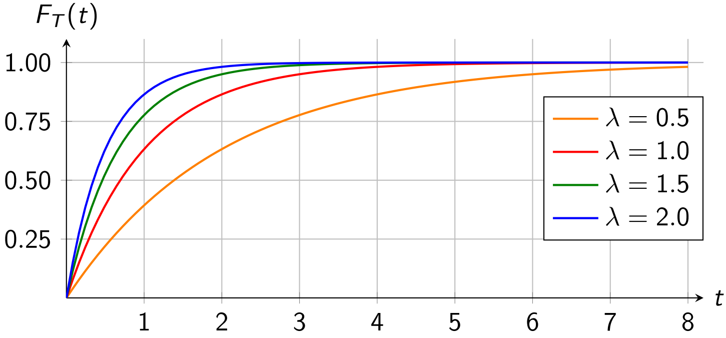

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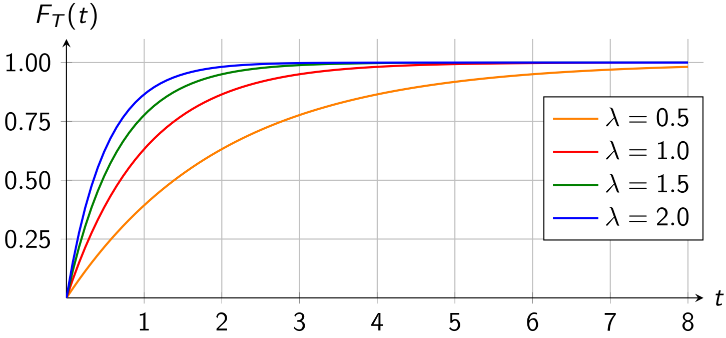

Note that $P(X \leq x) = 1 - e^{-\lambda x}$,

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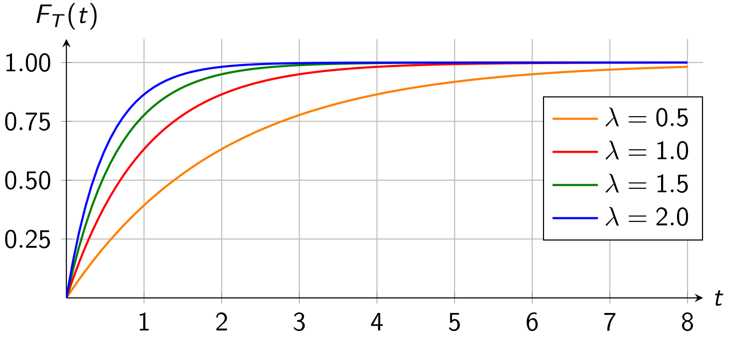

Note that $P(X \leq x) = 1 - e^{-\lambda x}$, while $P(X > x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$

Mean and variance of the exponential distribution

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Let $X \sim \text{Exponential}(\lambda)$.

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The mean of X is given by:

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Example 3: Waiting for a flight

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The delay time T of a flight is exponentially distributed wtih $\lambda = 2$ (delays per hour).

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The delay time T of a flight is exponentially distributed wtih $\lambda = 2$ (delays per hour). Answer the following questions:

(a) What is the mean delay (waiting) time, $\mathbb{E}(T)$?

(b) What is the variance of the delay time $\mathbb{V}(T)$?

(c) Find the probability that a flight will be delayed by no more than 10 minutes.

(d) Assuming you have been waiting for a flight for an hour, what is the probability that the flight will be delayed for an additional 30 minutes? (i.e. Find $P(T > 1.5 | T > 1)$).

Example 3: Waiting for a flight (cont.)

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Solution

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(a) The mean delay is given by

$$\mathbb{E}(T) = \frac{1}{\lambda} = \frac{1}{2} = \boxed{0.5\text{hr}}$$

Example 3: Waiting for a flight (cont.)

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(a) The mean delay is given by

$$\mathbb{E}(T) \quad = \quad \frac{1}{\lambda} = \frac{1}{2} = \boxed{0.5\text{hr}}$$

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Example 3: Waiting for a flight (cont.)

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Example 3: Waiting for a flight (cont.)

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(c) The probability the flight will be delayed by no more than 10 minutes ($\frac{1}{6}$ hr) is given by:

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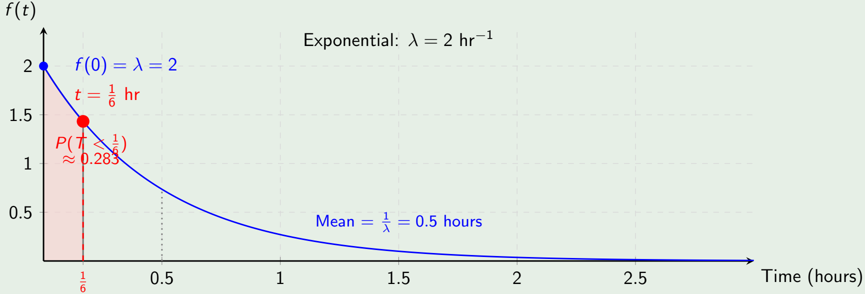

Example 3: Waiting for a flight (cont.)

Solution

(c) In python, we can use the `scipy.stats.expon.cdf` function to find $P(T \leq 1/6)$:

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```
from scipy import stats
import numpy as np
p = stats.expon.cdf(1/6, scale=1/2)
```

where:

- 1/6 is the value at which we want to evaluate the CDF
- `scale` $= 1/\lambda = 1/2$ (scale parameter/mean)

This returns `p = 0.283`, i.e., about 28.3% probability that the flight will be delayed by no more than 10 minutes.

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(d) The probability that the flight will be delayed by a further 0.5hr after 1hr of waiting is given by:

$$P(T > (0.5 + 1)|T > 1) = P(T > 1.5|T > 1)$$

Example 3: Waiting for a flight (cont.)

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(d) The probability that the flight will be delayed by a further 0.5hr after 1hr of waiting is given by:

$$\begin{aligned} P(T > (0.5+1) | T > 1) &= P(T > 1.5 | T > 1) \\ &= \frac{P((T > 1.5) \cap (T > 1))}{P(T > 1)} \end{aligned}$$

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In Python: `p = 1 - stats.expon.pdf(.5, scale=1/2)` also returns `p = 0.37`.

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That is, it does not matter from which time the waiting begins (i.e. conditioning); the probability of an elapsed time remains the same.

Recap

- **Lognormal distribution:** $X \sim \mathcal{LN}(\mu, \sigma^2)$
  CDF: $F_X(x) = P(X \leq x) = \Phi((\ln(x) - \mu)/\sigma)$

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