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CEE 260/MIE 273: Probability and Statistics in Civil Engineering
Lecture 6C: Analysis of Variance (ANOVA)

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UMass Amherst

College of Engineering

November 25, 2025

Outline

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Inferences concerning two population variances

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- Coined in 1934 by George W. Snedecor in honor of Sir Ronald A. Fisher who developed the ANOVA method, among other achievements.
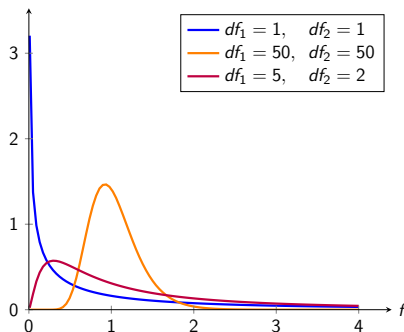
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- Effects of the presence of four different sugar solutions on bacterial growth
- Investigating if hardwood concentration in pulp has an effect on tensile strength of bags made from the pulp
- Deciding whether the color density of fabric specimens depends on the amount of dye used

Single-factor ANOVA

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Comparison of more than two populations or treatment means.

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$$H_0: \qquad \mu_1 = \mu_2 = \cdots = \mu_k$$

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Hypotheses:

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H_0 &: & \mu_1 = \mu_2 = \cdots = \mu_k \\
H_1 &: & \text{at least two of the other } \mu_i\text{'s are different}
\end{aligned}
$$

The ANOVA F-test

Assumption All k populations are normally distributed with means μ_i and equal variance σ^2.

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The ANOVA F-test

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$$H_0 : \qquad \mu_1 = \mu_2 = \cdots = \mu_k$$
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Test Statistic

$$f = \frac{\text{mean square among groups}}{\text{mean square error}}$$

Compare (a) Find critical value $F_{(1-\alpha, df_1, df_2)}$.

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Different sums of squares in ANOVA

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Total sum of squares (SST)

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \overline{x})^2 \tag{2}$$

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and $n = n_1 + n_2 + \cdots + n_k$

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Different sums of squares in ANOVA (cont.)

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Different sums of squares in ANOVA (cont.)

Sum of squares between groups (SSG)

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Different sums of squares in ANOVA (cont.)

Sum of squares between groups (SSG)

Measures the variance *among* the k sample means

$$SSG = \sum_{i=1}^{k} n_i(\overline{x_i} - \overline{x})^2 \tag{4}$$

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Measures the pooled variance *within* each of the k samples

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$$SSE = (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2 = \sum_{i=1}^{k}(n_i - 1)s_i^2 \tag{5}$$

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where s_i^2 is the sample variance for group i.

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Fundamental identities and d.o.f.

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Fundamental identity

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Fundamental identities and d.o.f.

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$$SST = SSG + SSE \qquad (6)$$

(You can prove this identity as an exercise.)

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Degrees of freedom (df):

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It also follows that:

$$df(SST) = df(SSG) + df(SSE) \tag{10}$$

Variances among and within groups

Ultimately, we want to determine how the variance is porportioned among the samples.

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Variances among and within groups

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Mean square among groups (MSG)

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$$MSG = \frac{1}{k-1} \times SSG = \frac{1}{k-1} \sum n_i (\overline{x_i} - \overline{x})^2 \qquad (11)$$

- Variance among the groups/treatments
- Also called **mean square for treatments** (MSTr)

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- Pooled variance within each of the groups.

Summary of ANOVA notation and sums

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$$SST = \sum (x_{ij} - \overline{x})^2 = \sum x_{ij}^2 - CM$$

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$$T_i = \sum_j x_{ij} \quad \text{(total of the observations for group } i\text{)}$$

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Test statistic

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Test statistic

The test procedure compares a measure of differences among the sample means ("between-samples" variation)

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Test statistic

The test procedure compares a measure of differences among the sample means ("between-samples" variation) to a measure of variation calculated from within each of the samples.

Test statistic for one-way ANOVA

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Test statistic

The test procedure compares a measure of differences among the sample means ("between-samples" variation) to a measure of variation calculated from within each of the samples.

Test statistic for one-way ANOVA

$$F = \frac{MSG}{MSE} \tag{13}$$

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- $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$

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- When H_0 is not acceptable, $\mathbb{E}(MSG) > \mathbb{E}(MSE) = \sigma^2 \implies f > 1$

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$$df_1 \quad = \quad k - 1 \tag{14}$$

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- When H_0 is acceptable, f follows the F-distribution with:

$$
\begin{aligned}
df_1 &= k - 1 \tag{14} \\
df_2 &= n - k \tag{15}
\end{aligned}
$$

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The ANOVA F-test

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[1]Use `f.ppf(1 - alpha, df1, df2)` in Python

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The ANOVA F-test

Assumption All k populations are normally distributed with means μ_i and equal variance σ^2.

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[1]Use `f.ppf(1 - alpha, df1, df2)` in Python

The ANOVA F-test

Assumption All k populations are normally distributed with means μ_i and equal variance σ^2. The samples are independent and random.

Hypotheses

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The ANOVA F-test

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Hypotheses

$$H_0 : \qquad \mu_1 = \mu_2 = \cdots = \mu_k$$
$$H_1 : \quad \text{at least one the other } \mu_i\text{'s differs from the others}$$

Test Statistic

$$f = \frac{MSG}{MSE}$$

Compare (a) Find critical value $F_{(1-\alpha, df_1, df_2)}$[1] (also called f^*).

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Compare (a) Find critical value $F_{(1-\alpha, df_1, df_2)}$[1] (also called f^*).
If $f \leq F_{(1-\alpha, df_1, df_2)}$, **fail to reject H_0**

---

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If $f \leq F_{(1-\alpha, df_1, df_2)}$, **fail to reject H_0**
If $f > F_{(1-\alpha, df_1, df_2)}$, **reject H_0** OR

---

[1]Use `f.ppf(1 - alpha, df1, df2)` in Python

The ANOVA F-test

Assumption All k populations are normally distributed with means μ_i and equal variance σ^2. The samples are independent and random.

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If $f \leq F_{(1-\alpha, df_1, df_2)}$, **fail to reject H_0**
If $f > F_{(1-\alpha, df_1, df_2)}$, **reject H_0** OR
(b) Find p-value.

---

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The ANOVA F-test

Assumption   All k populations are normally distributed with means μ_i and equal variance σ^2. The samples are independent and random.

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  If $f \leq F_{(1-\alpha, df_1, df_2)}$, **fail to reject H_0**
  If $f > F_{(1-\alpha, df_1, df_2)}$, **reject H_0** OR
  (b) Find p-value.
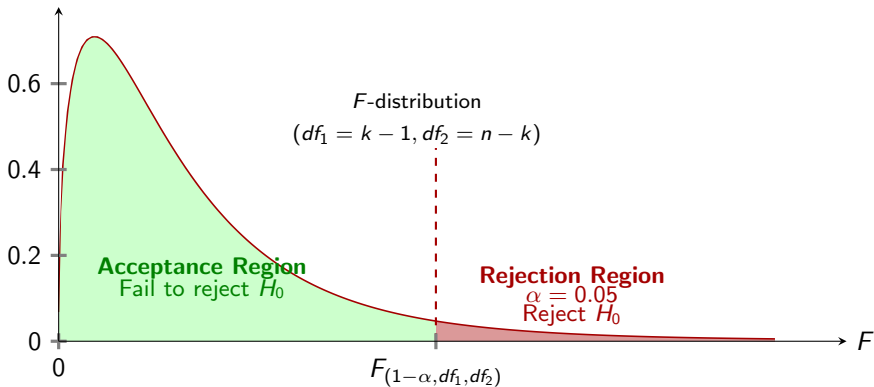 If p-value $\geq \alpha$, **fail to reject H_0**

---

[1]Use `f.ppf(1 - alpha, df1, df2)` in Python

The ANOVA F-test
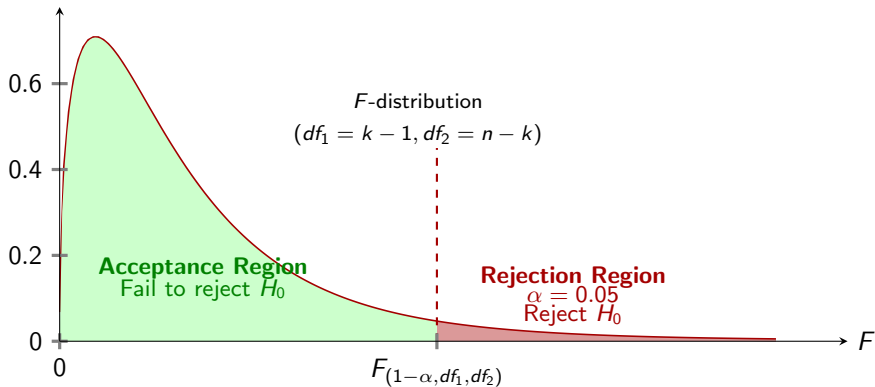
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Hypotheses

$$H_0 : \qquad \mu_1 = \mu_2 = \cdots = \mu_k$$
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Compare (a) Find critical value $F_{(1-\alpha, df_1, df_2)}$[1] (also called f^*).
If $f \leq F_{(1-\alpha, df_1, df_2)}$, **fail to reject H_0**
If $f > F_{(1-\alpha, df_1, df_2)}$, **reject H_0** OR
(b) Find p-value.
If p-value $\geq \alpha$, **fail to reject H_0**
If p-value $< \alpha$, **reject H_0**

---

[1]Use `f.ppf(1 - alpha, df1, df2)` in Python

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ANOVA F-test visualization

ANOVA F-test visualization

ANOVA F-test visualization

F-distribution
$(df_1 = k - 1, df_2 = n - k)$

Acceptance Region
Fail to reject H_0

Rejection Region
$\alpha = 0.05$
Reject H_0

0

$F_{(1-\alpha, df_1, df_2)}$

F

Decision Rule

ANOVA F-test visualization

F-distribution
$(df_1 = k - 1, df_2 = n - k)$

Acceptance Region
Fail to reject H_0

Rejection Region
$\alpha = 0.05$
Reject H_0

0

$F_{(1-\alpha, df_1, df_2)}$

F
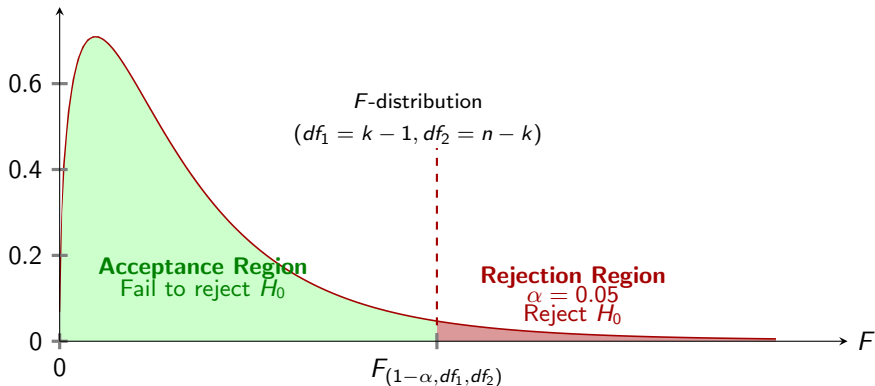
Decision Rule

- If $f \leq F_{(1-\alpha, df_1, df_2)}$: Fail to reject H_0 (means are equal)

ANOVA F-test visualization

F-distribution
$(df_1 = k - 1, df_2 = n - k)$

Acceptance Region
Fail to reject H_0

Rejection Region
$\alpha = 0.05$
Reject H_0
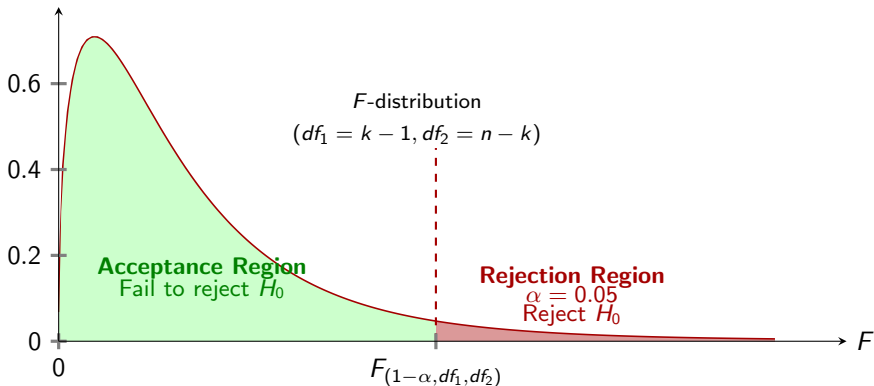
$F_{(1-\alpha, df_1, df_2)}$

Decision Rule

- If $f \leq F_{(1-\alpha, df_1, df_2)}$: Fail to reject H_0 (means are equal)
- If $f > F_{(1-\alpha, df_1, df_2)}$: Reject H_0 (at least one mean differs)

ANOVA table

When conducting one-way ANOVA, computations are usually summarized in an ANOVA table

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When conducting one-way ANOVA, computations are usually summarized in an ANOVA table

Source of variation	d.o.f.	Sum of Squares	Mean Square	f

ANOVA table

When conducting one-way ANOVA, computations are usually summarized in an ANOVA table

Source of variation	d.o.f.	Sum of Squares	Mean Square	f
Groups	$k-1$	SSG	$MSG = SSG/(k-1)$	MSG/MSE

ANOVA table

When conducting one-way ANOVA, computations are usually summarized in an ANOVA table

Source of variation	d.o.f.	Sum of Squares	Mean Square	f
Groups	$k-1$	SSG	$MSG = SSG/(k-1)$	MSG/MSE
Error	$n-k$	SSE	$MSE = SSE/(n-k)$	

ANOVA table

When conducting one-way ANOVA, computations are usually summarized in an ANOVA table

Source of variation	d.o.f.	Sum of Squares	Mean Square	f
Groups	$k-1$	SSG	$MSG = SSG/(k-1)$	MSG/MSE
Error	$n-k$	SSE	$MSE = SSE/(n-k)$	
Total	$n-1$	SST		

Example 1: Elastic moduli of alloys

Using the ANOVA table

Example 1: Elastic moduli of alloys

An experiment was performed to measure the elastic modulus (GPa) of an alloy produced using three different casting processes. Let μ_1, μ_2, μ_3 denote the true average elastic moduli for the 3 different processes. Using the ANOVA table given below, test the null hypothesis that all three means are equal (using the p-value approach).

Using the ANOVA table

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Source of variation	d.o.f.	Sum of Squares	Mean Square	*f*
Treatments	2	7.93	3.965	**?**

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Using the ANOVA table

Example 1: Elastic moduli of alloys

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Source of variation	d.o.f.	Sum of Squares	Mean Square	*f*
Treatments	2	7.93	3.965	**?**
Error	19	6.00	.3158	

Using the ANOVA table

Example 1: Elastic moduli of alloys

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Source of variation	d.o.f.	Sum of Squares	Mean Square	*f*
Treatments	2	7.93	3.965	**?**
Error	19	6.00	.3158	
Total	21	13.93		

Using the ANOVA table

Example 1: Elastic moduli of alloys (cont.)

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Using the ANOVA table

Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute f:

Using the ANOVA table

Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute f:

$$f$$

Using the ANOVA table

Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute f:

$$f = \frac{MSG}{MSE}$$

Using the ANOVA table

Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute f:

$$f = \frac{MSG}{MSE} = \frac{3.965}{.3158} = 12.56$$

Using the ANOVA table

Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute f:

$$f = \frac{MSG}{MSE} = \frac{3.965}{.3158} = 12.56$$

Step 2. Second, we find the p-value[a]:

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Using the ANOVA table

Example 1: Elastic moduli of alloys (cont.)

Step 1. First, we compute f:

$$f = \frac{MSG}{MSE} = \frac{3.965}{.3158} = 12.56$$

Step 2. Second, we find the p-value[a]:

$$p\text{-value} = 1 - F(12.56, 2, 19) = 0.0003$$

Step 3. We conclude: since the p-value is very small, we can **reject the null hypothesis** at any reasonable significance level.

---

[a]In MATLAB, use: `1 - f.cdf(12.56, 2, 19)` OR `f.sf(12.56, 2, 19)`

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One-way ANOVA in practice

One-way ANOVA in practice

Example 2: Recording tape quality

One-way ANOVA in practice

Example 2: Recording tape quality

In a effort to improve the quality of recording tapes, the effects of four kinds of coatings A, B,C, D on the reproducing quality of sound are compared. Independent samples are obtained foreach kind of coating.

One-way ANOVA in practice

Example 2: Recording tape quality

In a effort to improve the quality of recording tapes, the effects of four kinds of coatings A, B,C, D on the reproducing quality of sound are compared.
Independent samples are obtained foreach kind of coating.
The following values on distortion are measured:

Coating	Observations	Sample Sizes
A	10, 15, 8, 12, 15	5
B	14, 18, 21, 15	4
C	17, 16, 14, 15, 17, 15, 18	7
D	12, 15, 17, 15, 16, 15	6

With the help of such a sample we want to decide if the four different coatings result in different mean distortions, i.e. test if $\mu_A = \mu_B = \mu_C = \mu_D$ (use $\alpha = 0.05$).

One-way ANOVA in practice

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 1. Assemble the parameters:

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 1. Assemble the parameters:

$$k \quad = \quad 4$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 1. Assemble the parameters:

$$
\begin{aligned}
k &= 4 \\
n_1 &= 5, n_2 = 4, n_3 = 7, n_4 = 6
\end{aligned}
$$

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 1. Assemble the parameters:

$$
\begin{aligned}
k &= 4 \\
n_1 &= 5, n_2 = 4, n_3 = 7, n_4 = 6 \\
T_1 &= 60, T_2 = 68, T_3 = 112, T_4 = 90
\end{aligned}
$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 1. Assemble the parameters:

$$
\begin{aligned}
k &= 4 \\
n_1 &= 5, n_2 = 4, n_3 = 7, n_4 = 6 \\
T_1 &= 60, T_2 = 68, T_3 = 112, T_4 = 90 \\
n &= \sum n_i = 22
\end{aligned}
$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 1. Assemble the parameters:

$$
\begin{aligned}
k &= 4 \\
n_1 &= 5, n_2 = 4, n_3 = 7, n_4 = 6 \\
T_1 &= 60, T_2 = 68, T_3 = 112, T_4 = 90 \\
n &= \sum n_i = 22 \\
\sum x_{ij} &= \sum T_i = 330
\end{aligned}
$$

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

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Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

$$CM = \frac{1}{n} \left(\sum x_{ij} \right)^2 = \frac{330^2}{22} = 4950$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

$$
\begin{aligned}
CM &= \frac{1}{n}\left(\sum x_{ij}\right)^2 = \frac{330^2}{22} = 4950 \\
SST &= \sum x_{ij}^2 - CM = (10^2 + 15^2 + 8^2 + \cdots + 15^2 + 5^2) - CM
\end{aligned}
$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

$$
\begin{aligned}
CM &= \frac{1}{n}\left(\sum x_{ij}\right)^2 = \frac{330^2}{22} = 4950 \\
SST &= \sum x_{ij}^2 - CM = (10^2 + 15^2 + 8^2 + \cdots + 15^2 + 5^2) - CM \\
&= 5112 - 4950 = 162
\end{aligned}
$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

$$
\begin{aligned}
CM &= \frac{1}{n}\left(\sum x_{ij}\right)^2 = \frac{330^2}{22} = 4950 \\
SST &= \sum x_{ij}^2 - CM = (10^2 + 15^2 + 8^2 + \cdots + 15^2 + 5^2) - CM \\
&= 5112 - 4950 = 162 \\
SSG &= \sum \frac{T_i^2}{n_i} - CM
\end{aligned}
$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

$$
\begin{aligned}
CM &= \frac{1}{n}\left(\sum x_{ij}\right)^2 = \frac{330^2}{22} = 4950 \\
SST &= \sum x_{ij}^2 - CM = (10^2 + 15^2 + 8^2 + \cdots + 15^2 + 5^2) - CM \\
&= 5112 - 4950 = 162 \\
SSG &= \sum \frac{T_i^2}{n_i} - CM \\
&= \left(\frac{60^2}{5} + \frac{68^2}{4} + \frac{112^2}{7} + \frac{90^2}{6}\right) - CM = 68
\end{aligned}
$$

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 2. Compute the sums:

$$
\begin{aligned}
CM &= \frac{1}{n}\left(\sum x_{ij}\right)^2 = \frac{330^2}{22} = 4950 \\
SST &= \sum x_{ij}^2 - CM = (10^2 + 15^2 + 8^2 + \cdots + 15^2 + 5^2) - CM \\
&= 5112 - 4950 = 162 \\
SSG &= \sum \frac{T_i^2}{n_i} - CM \\
&= \left(\frac{60^2}{5} + \frac{68^2}{4} + \frac{112^2}{7} + \frac{90^2}{6}\right) - CM = 68 \\
SSE &= SST - SSG = 162 - 68 = 94
\end{aligned}
$$

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 3. Create the ANOVA table:

Source	d.o.f.	SS	MS	f
Coating	3	68	$MSG = 22.67$	$MSG/MSE = 4.343$
Error	18	94	$MSE = 5.22$	
Total	21	162		

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad \text{(from Python)}$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad \text{(from Python)}$$

Notes:
- Use `f.ppf(0.95, 3, 18)` to find the critical value.

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95, 3, 18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad \text{(from Python)}$$

Notes:
- Use `f.ppf(0.95, 3, 18)` to find the critical value.
- The F curve is NOT symmetric. So, `-f.ppf(0.05, 3, 18)` will NOT give the same answer.

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad \text{(from Python)}$$

Notes:
- Use `f.ppf(0.95, 3, 18)` to find the critical value.
- The F curve is NOT symmetric. So, `-f.ppf(0.05, 3, 18)` will NOT give the same answer.
- However, the F-curve has the property

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 4. Find critical value:

$$F_{0.95,3,18} = F^{-1}(0.95, 3, 18) = 3.1599 \quad \text{(from Python)}$$

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- However, the F-curve has the property

$$F_{(\alpha, df_1, df_2)} = \frac{1}{F_{(1-\alpha, df_2, df_1)}} \tag{16}$$

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

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$$F_{(\alpha, df_1, df_2)} = \frac{1}{F_{(1-\alpha, df_2, df_1)}} \tag{16}$$

- So, in this case, `f.ppf(0.05, 18, 3)` will give the same answer as `f.ppf(0.95, 3, 18)`.

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Example 2: Recording tape quality (cont.)

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One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 5. Compare:

$$f = 4.343 > F_{0.95,3,18} = 3.1599$$

One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 5. Compare:

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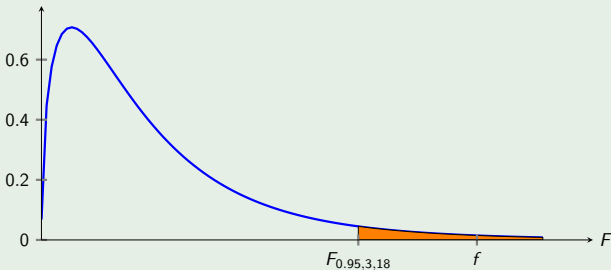
One-way ANOVA in practice

Example 2: Recording tape quality (cont.)

Step 5. Compare:

$$f = 4.343 > F_{0.95,3,18} = 3.1599$$
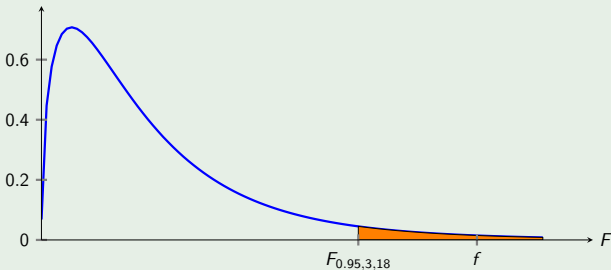
Thus, f lies in the critical region, and we **reject the null hypothesis** that the mean distortions from the four coatings are equal.

ANOVA in MATLAB/Python

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- The computations in ANOVA are tedious and prone to human error.
- Usually ANOVA is conducted using software platforms such as MATLAB or Python.

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Example 3: Investigating fuel efficiency of cars manufactured in different countries

The `carbig` dataset in MATLAB contains variables[a] for 400 cars from the 70s and 80s. The `carsmall` dataset is a subset of `carbig` (years '70, '76 and '82). We want to check if the true average fuel efficiency (MPG) of vehicles from different countries (USA/Europe/Japan) are equal.

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[a]For more on visualizing some of these variables, visit
https://www.mathworks.com/help/stats/examples/visualizing-multivariate-data.html

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 1. Examine the contents of the `6C-anova-cars.ipynb` script. Run the script.

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 1. Examine the contents of the `6C-anova-cars.ipynb` script. Run the script.

Step 2. Figure 1 is a boxplot of the MPG across the different origins. What can you say about the variance *within* each of the countries/regions, and *among* them?
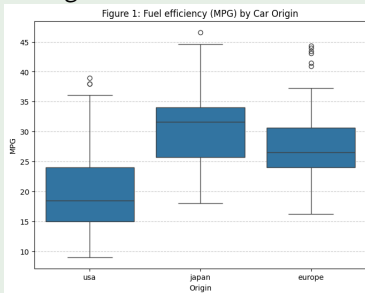
Figure 1: Fuel efficiency (MPG) by Car Origin

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 3. The function `scipy.stats.f_oneway` generates the ANOVA results (though not formatted as nicely as in MATLAB—see Appendix). More detail is provided by `statsmodels.stats.oneway.anova_oneway`.

Step 4. What is the p-value for this test?

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 3. The function `scipy.stats.f_oneway` generates the ANOVA results (though not formatted as nicely as in MATLAB—see Appendix). More detail is provided by `statsmodels.stats.oneway.anova_oneway`.

Step 4. What is the p-value for this test? Based on this, do we **reject/fail to reject** the null hypothesis at any reasonable significance level?

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

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Step 4. What is the p-value for this test? Based on this, do we **reject/fail to reject** the null hypothesis at any reasonable significance level?

Step 5. In Figure 3, we have a notched boxplot, which denotes the confidence interval of the median. What are your observations?

Multiple comparisons in ANOVA

Multiple comparisons in ANOVA

Multiple comparisons in ANOVA

- If after performing a one-way ANOVA, we reject the null hypothesis, naturally we might want to find out which populations significantly differ from each other
- Various methods have been developed for multiple pairwise comparisons (out of scope of this lecture):
  - One option: perform pairwise t-tests between group means using the Bonferroni correction for α
  - Use Tukey's range test, which performs a significance test based on the studentized range distribution

Multiple comparisons in ANOVA (cont.)

Example 3: Investigating fuel efficiency of cars (cont.)

Multiple comparisons in ANOVA (cont.)

Example 3: Investigating fuel efficiency of cars (cont.)

Step 5. Use the `pairwise_tukeyhsd` function from `statsmodels.stats.multicomp` to perform multiple comparisons.

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Multiple comparisons in ANOVA (cont.)

Example 3: Investigating fuel efficiency of cars (cont.)

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Multiple comparisons in ANOVA (cont.)
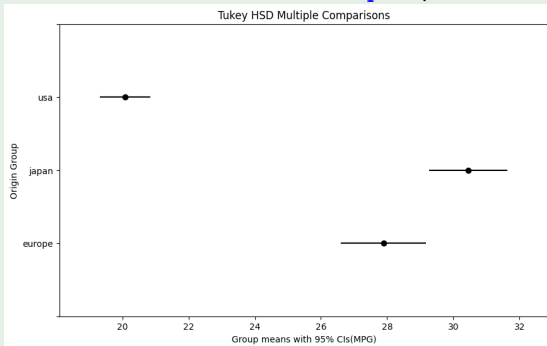
Example 3: Investigating fuel efficiency of cars (cont.)

Step 5. Use the `pairwise_tukeyhsd` function from
`statsmodels.stats.multicomp` to perform multiple comparisons.
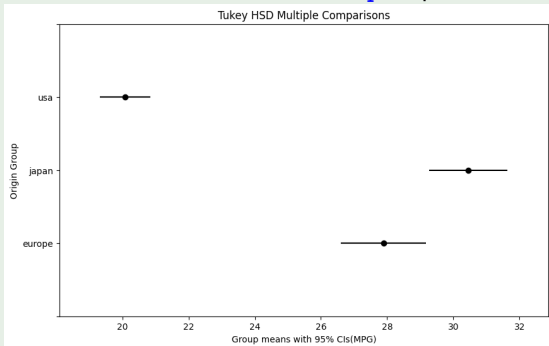
Which regions have significantly different average fuel efficiency?

Summary

- The analysis of variance (ANOVA) test is used to determine whether the population means across multiple samples are equal
- It uses the test statistic ($f = MSG/MSE$) which is F-distributed
- Reading: Section 7.5 (Open Intro Statistics)

Summary

- The analysis of variance (ANOVA) test is used to determine whether the population means across multiple samples are equal
- It uses the test statistic ($f = MSG/MSE$) which is F-distributed
- Reading: Section 7.5 (Open Intro Statistics)
- If interested, read Section 7.4 on "power" analysis (but we will consider this out of scope for the course)

ANOVA in MATLAB (cont.)

> Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

ANOVA in MATLAB (cont.)

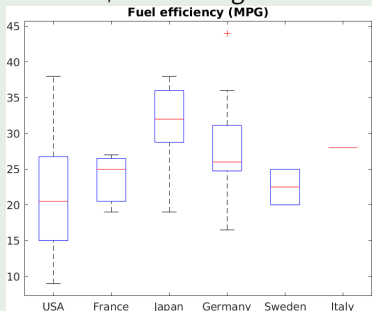### Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 1. Examine the contents of the example3.m script. Run the script.

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 1. Examine the contents of the `example3.m` script. Run the script.

Step 2. Figure 1 is a boxplot of the MPG across the different origins (6 countries). What can you say about the variance *within* each of the countries, and *among* them?

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 3. The function anova1 generates 2 figures: the first is the ANOVA table (Figure 2) and the second a notched boxplot (Figure 3)

ANOVA Table					
Source	SS	df	MS	F	Prob>F
Groups	1629.87	5	325.974	6.56	3.14631e-05
Error	4375.41	88	49.721		
Total	6005.28	93			

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

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ANOVA Table

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Step 4. The column `Prob>F` is the *p*-value.

ANOVA in MATLAB (cont.)

Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

Step 3. The function anova1 generates 2 figures: the first is the ANOVA table (Figure 2) and the second a notched boxplot (Figure 3)

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Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

ANOVA in MATLAB (cont.)

### Example 3: Investigating fuel efficiency of cars manufactured in different countries (cont.)

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ANOVA in MATLAB (cont.)

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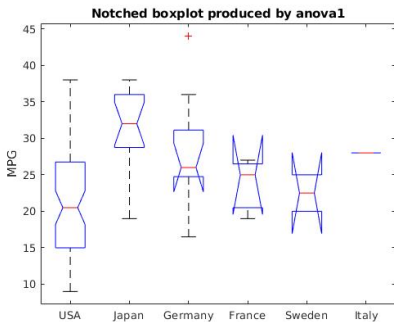
Multiple comparisons in ANOVA

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Multiple comparisons in ANOVA (cont.)

Multiple comparisons in ANOVA (cont.)

Example 3: Investigating fuel efficiency of cars (cont.)

Multiple comparisons in ANOVA (cont.)

Example 3: Investigating fuel efficiency of cars (cont.)

Step 5. Uncomment lines 13 and 14 and rerun the example3.m script.

Multiple comparisons in ANOVA (cont.)

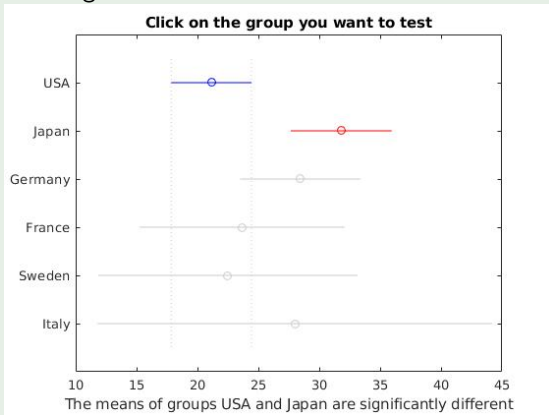## Example 3: Investigating fuel efficiency of cars (cont.)

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Multiple comparisons in ANOVA (cont.)

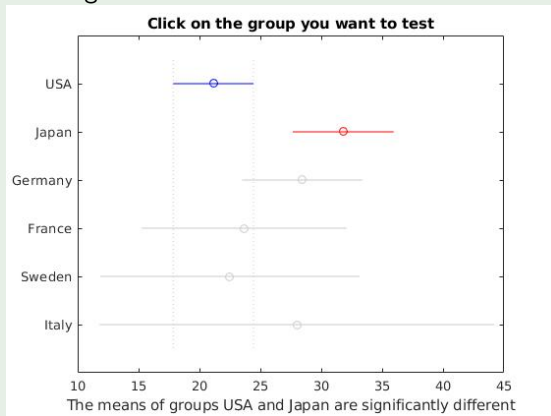## Example 3: Investigating fuel efficiency of cars (cont.)

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Click on the group you want to test

The means of groups USA and Japan are significantly different

Multiple comparisons in ANOVA (cont.)

Example 3: Investigating fuel efficiency of cars (cont.)

Step 5. Uncomment lines 13 and 14 and rerun the example3.m script. Interact with Figure 5.

Click on the group you want to test

The means of groups USA and Japan are significantly different

Which countries have significantly different average fuel efficiency?