CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 3f: Joint Distributions

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Outline

Today's objectives

Joint distributions:

- Understand the concept of jointly distributed random variables
- Differentiate between marginal and conditional distributions
- Manipulate joint distributions to compute probabilities

Joint distributions

Given two random variables X and Y:

Discrete case

The joint PMF is:

$$p_{X,Y}(x_i, y_j) = P(X = x_i, Y = y_j) \tag{1}$$

The CDF is:

$$F_{X,Y}(x, y) = \sum_{x_i \le x} \sum_{y_j \le y} p_{X,Y}(x_i, y_j) \tag{2}$$

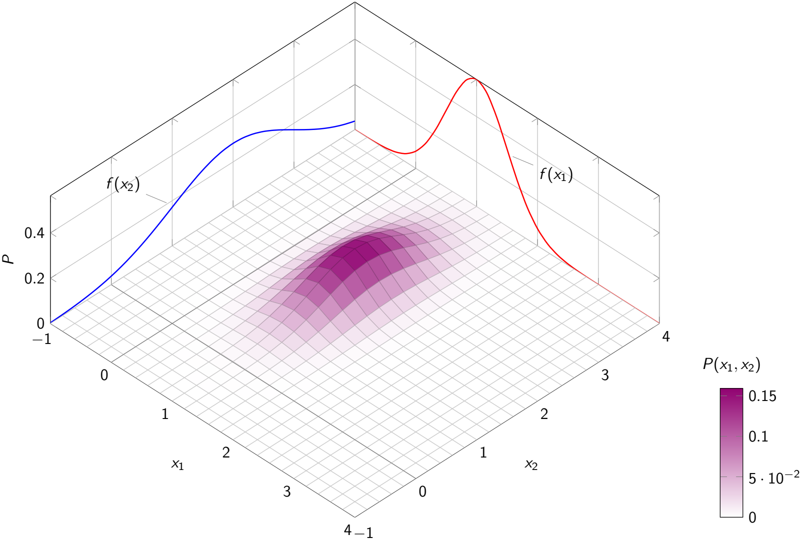
Continuous case

The joint probability is given by:

$$P(a < X \le b, c < Y \le d) = \int_a^b \int_c^d f_{X,Y}(x, y) dy dx \tag{3}$$

Joint and marginal distributions

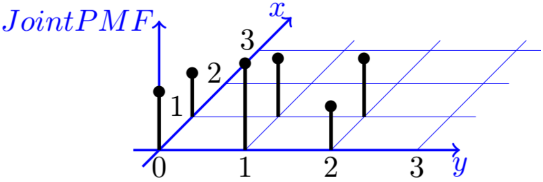
Marginal distributions: $f(x_1)$ and $f(x_2)$
Joint distribution: $f(x_1, x_2)$.



Example 1: Joint PMF of two random variables

Given two random variables X and Y with joint PMF indicated in the table below:

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	?
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	?
$p_Y(y)$	?	?	?	



(a) Find $P(X=0, Y \leq 1)$

(b) Find the marginal PMFs of X and Y

(c) Find $P(Y=1|X=0)$

(d) Are X and Y independent?

Example 1: Joint PMF of two random variables (cont.)

Solution

(a) To find the probability $P(X = 0, Y \leq 1)$, simply add up the cells *jointly* satisfying the conditions.

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	?
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	?
$p_Y(y)$	?	?	?	

$$\begin{aligned} P(X = 0, Y \leq 1) &= \frac{1}{6} + \frac{1}{4} \\ &= \frac{4 + 6}{24} = \frac{10}{24} = \frac{5}{12} \end{aligned}$$

Example 1: Joint PMF of two random variables (cont.)

Solution

(b) To find the marginal PMFs of X and Y, i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases} \qquad p_Y(y) = \begin{cases} \frac{7}{24}, & y = 0 \\ \frac{5}{12}, & y = 1 \\ \frac{7}{24}, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$
\begin{aligned}
P(Y=1|X=0) &= \frac{P(Y=1\cap X=0)}{P(X=0)} = \frac{P(Y=1,X=0)}{P(X=0)} \\
&= \frac{p_{XY}(0,1)}{p_X(0)} = \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}
\end{aligned}
$$

Example 1: Joint PMF of two random variables (cont.)

	$Y=0$	$Y=1$	$Y=2$	$p_X(x)$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{6}=\frac{11}{24}$
$p_Y(y)$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$	$\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$	

Solution

(d) Two r.v.'s are independent if $P(Y=y_j|X=x_i)=P(X=x_i)$ or $P(X=x_i|Y=y_j)=P(Y=y_j)$ for all i and j. Here,

$$P(Y=1|X=0)=\frac{6}{13}\neq P(Y=1)=\frac{5}{12}$$

The independence condition fails. Hence X and Y are not independent.

Conditional distributions of continuous random variables

Recall the definition of conditional probability (multiplication rule):

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{4}$$

$$P(AB) = P(A|B)P(B) = P(B|A)P(A) \tag{5}$$

Similarly, for two continuous r.v.'s, the conditional PDF of X given Y is:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{6}$$

Joint PDF and CDF of two variables

The joint PDF is given by:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x) \tag{7}$$

While the joint CDF is given by:

$$F_{X,Y}(a,b) = P(X \le a, Y \le b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{X,Y}(x,y)dydx \tag{8}$$

Marginal distributions of continuous random variables

Recall the theorem of total probability:

$$P(A) = \sum_{i=1}^{n} P(A|E_i)P(E_i) \tag{9}$$

Similarly, the marginal PDFs from a joint distribution of two continuous r.v.'s X and Y is given as:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)dy = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy & (10) \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dx = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx & (11) \end{aligned}$$

Example 2: Water levels

The daily water levels of two reservoirs A and B are denoted by two r.v.'s X and Y having the following joint PDF:

$$f(x,y) = \frac{6}{5}\left(x + y^2\right), \quad 0 < x < 1; 0 < y < 1$$

(a) Determine the marginal density function of the daily water level for reservoir A.

(b) If reservoir A is half full on a given day, what is the probability that the water level will be more than half full?

Example 2: Water levels (cont.)

$$f(x, y) = \frac{6}{5}\left(x + y^2\right), \quad 0 < x < 1; 0 < y < 1$$

Solution

(a) We obtain the marginal density function $f_X(x)$ by integrating the joint PDF over y:

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{6}{5}\left(x + y^2\right) dy \\ &= \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_0^1 \\ &= \frac{2}{5}(3x + 1) \quad (0 < x < 1) \end{aligned}$$

This is the **marginal distribution**

Example 2: Water levels (cont.)

$$f(x,y) = \frac{6}{5}\left(x + y^2\right), \quad 0 < x < 1; 0 < y < 1$$

Solution

(b) If reservoir A is half full on a given day, what is the probability that the water level will be more than half full? We first find the conditional distribution:

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{6}{5}\left(x+y^2\right)}{\frac{2}{5}\left(3x+1\right)^2} = 3\frac{x+y^2}{3x+1} \\
\text{Thus: } P(Y > 0.5|X = 0.5) &= \int_{0.5}^{1} f_{Y|X}(y|x=0.5)dy \\
&= 3\int_{0.5}^{1} \frac{0.5+y^2}{1.5+1}dy \\
&= \left(\frac{3}{2.5}\right)\left[0.5y + \frac{y^3}{3}\right]_{0.5}^{1} = \boxed{0.65}
\end{aligned}$$

Reading

- Exponential distribution: Section 4.7 (Navidi)
- Also read up on the Uniform Distribution in Section 4.8 (Navidi)
- Joint distributions: Section 2.6 (Navidi)