

CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture M7a: Correlation and Variance Analyses in Linear Regression

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December 4, 2025

Outline

Today's objectives

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- Learn how to compute and interpret the correlation coefficient
- Understand and apply linear regression
- Analyze regression fitness metrics (in particular, R^2)

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 - E.g. the height of the son of an taller-than-average father was greater than average but not by as much as his father's
 - And the height of the son a shorter-than-average father was lower but not by as much as his father's

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- $x = \text{age of a child}$; $y = \text{size of child's vocabulary}$
- $x = \text{size of an engine}$; $y = \text{fuel efficiency for a car equipped with engine}$
- $x = \text{applied tensile force}$; $y = \text{deformation of a metal strip}$

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In this module, we will cover the following key topics:

- Correlation and variance analyses
- Simple Linear Regression and Least Squares Estimation
- Inference for Linear Regression

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This can also be rewritten as:

$$\text{Cov}(X, Y) = \begin{cases} \sum_x \sum_y (x - \mu_x)(y - \mu_y)f(x, y) & (X, Y) \text{ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y)f(x, y)dxdy & (X, Y) \text{ continuous} \end{cases} \quad (3)$$

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Thus, the correlation coefficient ranges from -1 (perfectly linear negative relationship) to $+1$ (perfectly linear positive relationship).

identify_relationships_lin_neg

identify_relationships_lin_pos

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$$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) \quad (6)$$

$$S_{xx} = \sum (x_i - \bar{x})^2 \quad (7)$$

$$S_{yy} = \sum (y_i - \bar{y})^2 \quad (8)$$

Properties of the sample correlation coefficient $\hat{\rho}$

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- ① Value does not depend on which of the two variables is labeled x or y
- ② Independent of the units in which x and y are measured
- ③ $-1 \leq \hat{\rho} \leq 1$
- ④ $\hat{\rho} = 1$ if and only if all data pairs lie on a straight line with positive slope and
 $\hat{\rho} = -1$ iff¹ all pairs lie on a straight line with negative slope

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$$\sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{1}{n} \left(\sum x_i \right)^2 \quad (12)$$

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Computing the correlation coefficient from summary data

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Find the sample correlation coefficient.

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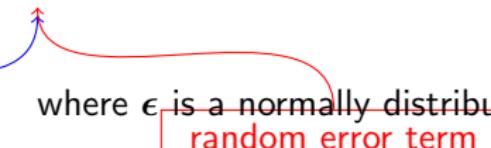
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β_0 (intercept) and β_1 (slope) are the **regression coefficients**

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Given n independent observations $(x_1, y_1) \dots (x_n, y_n)$, the random error term ϵ allows (x_i, y_i) to fall:

- above the line: $\epsilon > 0$
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Image source: https://sigmazone.com/labrea_scatter_plots/

The observed errors in model predictions are known as **residuals**.

Alternative notation

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Define

$\mu_{Y|x} = \mathbb{E}(Y|X = x)$ = expected (or mean) value of Y when $X = x$

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Thus, $\mu_{Y|x}$ is the mean of all y values for which $X = x$ and $\sigma_{Y|x}^2$ describes the variability of y values when $X = x$.

Example: Age and vocabulary size of children

Let:

x = age of a child

y = vocabulary size

Then $\mu_{Y|5}$ is the average vocabulary size for all 5-year-old children in the population.

And $\sigma_{Y|5}^2$ indicates the amount of variability in vocabulary size for 5-year-olds.

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$$\mu_{Y|x} = \mathbb{E}(Y|X=x)$$

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Analyzing a regression equation

Example 2: Flow rate

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The flow rate y (m^3/min) in a device used for air quality measurement depends on the pressure drop x (in. of water) across the device's filter. Suppose that for x values between 5 and 20, the variables are related by the regression model:

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- For a pressure drop of 10 in., what is the probability that the observed flow rate will exceed 0.835?

Analyzing a regression equation

Example 2: Flow rate (cont.)

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- (a) The **expected change** in flow rate associated with a 1-inch increase in pressure drop is the slope:

Analyzing a regression equation

Example 2: Flow rate (cont.)

From the model equation: $\beta_0 = -0.12$ and $\beta_1 = 0.095$

- (a) The **expected change** in flow rate associated with a 1-inch increase in pressure drop is the slope: $0.095 \text{ m}^3/\text{min}$.

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- (a) The **expected change** in flow rate associated with a 1-inch increase in pressure drop is the slope: $0.095 \text{ m}^3/\text{min}$.
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$$\Delta^2 = \sum_n [y_i - (\beta_0 + \beta_1 x_i)]^2 \quad (15)$$

Then minimizing Δ^2 yields the estimates of the regression coefficients:

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Least squares regression in MATLAB

Example 3: Relationship between population and number of accidents

Least squares regression in MATLAB

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Using the accidents dataset in MATLAB, perform a least-squares regression of accidents in a state *on* the population of the state^a:

```
load accidents  
x = hwydata(:,14); (Population of state)  
y = hwydata(:,4); (Accidents per state)
```

- (a) What are the slope and intercept estimates, $\hat{\beta}_0$ and $\hat{\beta}_1$?
- (b) How can you evaluate the strength of the relationship?

^aSee the file `ex3_119_least_squares_regression.m`

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- Residual = Observed – Expected

Residuals illustrated

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What patterns do you observe?

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The degrees of freedom $df = n - 2$ because 2 parameters must first be estimated before computing $\hat{\sigma}^2$: β_0 and β_1

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- The ratio SSE/SST is the proportion of total variation unexplained by the simple linear regression model

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$$R^2 = 1 - \frac{SSE}{SST} \quad (22)$$

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- As R^2 decreases \rightarrow a weaker linear fit
- $\hat{\rho}^2$ approximates R^2 for large n

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Coefficient of determination $R^2 = 1 - \frac{SSE}{SST} = \frac{SSR}{SST}$ (30)

Note

The variance estimate $\hat{\sigma}^2$ is also defined as the *conditional variance*, $\mathbb{V}(Y|X=x)$