

CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 3f: Joint Distributions

Prof. Oke

UMassAmherst

College of Engineering

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Outline

- ① Preamble
- ② Joint distributions
- ③ Discrete random variables
- ④ Continuous random variables



Today's objectives

Joint distributions:

- Understand the concept of jointly distributed random variables
- Differentiate between marginal and conditional distributions
- Manipulate joint distributions to compute probabilities

Joint distributions

Given two random variables X and Y :

Discrete case

The joint PMF is:

$$p_{X,Y}(x_i, y_j) = P(X = x_i, Y = y_j) \quad (1)$$

The CDF is:

$$F_{X,Y}(x, y) = \sum_{x_i \leq x} \sum_{y_j \leq y} p_{X,Y}(x_i, y_j) \quad (2)$$

Continuous case

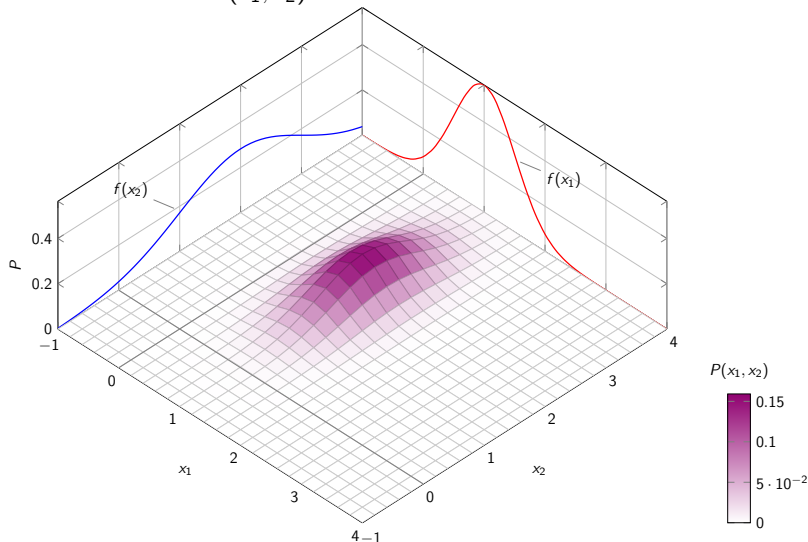
The joint probability is given by:

$$P(a < X \leq b, c < Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y) dy dx \quad (3)$$

Joint and marginal distributions

Marginal distributions: $f(x_1)$ and $f(x_2)$

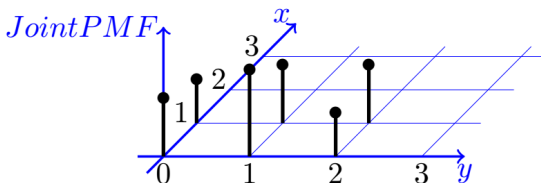
Joint distribution: $f(x_1, x_2)$.



Example 1: Joint PMF of two random variables

Given two random variables X and Y with joint PMF indicated in the table below:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$?
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$?
$p_Y(y)$?	?	?	



- Find $P(X = 0, Y \leq 1)$
- Find the marginal PMFs of X and Y
- Find $P(Y = 1|X = 0)$
- Are X and Y independent?

Example 1: Joint PMF of two random variables (cont.)

Solution

- (a) To find the probability $P(X = 0, Y \leq 1)$, simply add up the cells *jointly* satisfying the conditions.

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$?
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$?
$p_Y(y)$?	?	?	

$$\begin{aligned} P(X = 0, Y \leq 1) &= \frac{1}{6} + \frac{1}{4} \\ &= \frac{4 + 6}{24} = \frac{10}{24} = \frac{5}{12} \end{aligned}$$

Example 1: Joint PMF of two random variables (cont.)

Solution

- (b) To find the marginal PMFs of X and Y , i.e. $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$, respectively, sum up the rows and columns in the table:

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Thus, we obtain:

$$p_X(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases} \quad p_Y(y) = \begin{cases} \frac{7}{24}, & y = 0 \\ \frac{5}{12}, & y = 1 \\ \frac{7}{24}, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Solution

(c) Here we use the conditional probability formula:

$$\begin{aligned}
 P(Y = 1|X = 0) &= \frac{P(Y = 1 \cap X = 0)}{P(X = 0)} = \frac{P(Y = 1, X = 0)}{P(X = 0)} \\
 &= \frac{p_{XY}(0, 1)}{p_X(0)} = \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}
 \end{aligned}$$

Example 1: Joint PMF of two random variables (cont.)

	$Y = 0$	$Y = 1$	$Y = 2$	$p_X(x)$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$
$p_Y(y)$	$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$	$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$	$\frac{1}{8} + \frac{1}{6} = \frac{7}{24}$	

Solution

- (d) Two r.v.'s are independent if $P(Y = y_j | X = x_i) = P(Y = y_j)$ or $P(X = x_i | Y = y_j) = P(X = x_i)$ for all i and j . Here,

$$P(Y = 1 | X = 0) = \frac{6}{13} \neq P(Y = 1) = \frac{5}{12}$$

The independence condition fails. Hence X and Y are not independent.

Conditional distributions of continuous random variables

Recall the definition of conditional probability (multiplication rule):

$$P(A|B) = \frac{P(AB)}{P(B)} \quad (4)$$

$$P(AB) = P(A|B)P(B) = P(B|A)P(A) \quad (5)$$

Similarly, for two continuous r.v.'s, the conditional PDF of X given Y is:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad (6)$$

Joint PDF and CDF of two variables

The joint PDF is given by:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x) \quad (7)$$

While the joint CDF is given by:

$$F_{X,Y}(a,b) = P(X \leq a, Y \leq b) = \int_{-\infty}^a \int_{-\infty}^b f_{X,Y}(x,y) dy dx \quad (8)$$

Marginal distributions of continuous random variables

Recall the theorem of total probability:

$$P(A) = \sum_{i=1}^n P(A|E_i)P(E_i) \quad (9)$$

Similarly, the marginal PDFs from a joint distribution of two continuous r.v.'s X and Y is given as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)dy = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \quad (10)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dx = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \quad (11)$$

Example 2: Water levels

The daily water levels of two reservoirs A and B are denoted by two r.v.'s X and Y having the following joint PDF:

$$f(x, y) = \frac{6}{5} (x + y^2), \quad 0 < x < 1; 0 < y < 1$$

- (a) Determine the marginal density function of the daily water level for reservoir A.
- (b) If reservoir A is half full on a given day, what is the probability that the water level will be more than half full?

Example 2: Water levels (cont.)

$$f(x, y) = \frac{6}{5} (x + y^2), \quad 0 < x < 1; 0 < y < 1$$

Solution

- (a) We obtain the marginal density function $f_X(x)$ by integrating the joint PDF over y :

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{6}{5} (x + y^2) dy \\ &= \frac{6}{5} \left[xy + \frac{y^3}{3} \right]_0^1 \\ &= \frac{2}{5} (3x + 1) \quad (0 < x < 1) \end{aligned}$$

This is the **marginal distribution**

Example 2: Water levels (cont.)

$$f(x, y) = \frac{6}{5} (x + y^2), \quad 0 < x < 1; 0 < y < 1$$

Solution

- (b) If reservoir A is half full on a given day, what is the probability that the water level will be more than half full? We first find the conditional distribution:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{\frac{6}{5} (x + y^2)}{\frac{2}{5} (3x + 1)^2} = 3 \frac{x + y^2}{3x + 1}$$

$$\begin{aligned} \text{Thus: } P(Y > 0.5 | X = 0.5) &= \int_{0.5}^1 f_{Y|X}(y|x = 0.5) dy \\ &= 3 \int_{0.5}^1 \frac{0.5 + y^2}{1.5 + 1} dy \\ &= \left(\frac{3}{2.5} \right) \left[0.5y + \frac{y^3}{3} \right]_{0.5}^1 = \boxed{0.65} \end{aligned}$$

Reading

- Exponential distribution: Section 4.7 (Navidi)
- Also read up on the Uniform Distribution in Section 4.8 (Navidi)
- Joint distributions: Section 2.6 (Navidi)