CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 5B: Inference for Difference of Two Proportions

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November 6, 2025

Outline

1. Test for normality
2. CI for $p_1 - p_2$
3. Hypothesis testing
4. Pooled proportion
5. Summary

Today's objectives

- Normality test for difference of two proportions ($p_1 - p_2$)

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- Compute CIs for $p_1 - p_2$
- Conduct hypothesis tests for $p_1 - p_2$
- Using the pooled proportion $\hat{p}_{pooled}$

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- However, there are cases whereby we want to compare proportions from *two groups/populations*
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- Thus, the parameter of interest is $p_1 - p_2$, and we define the following sample statistics:

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  - $\hat{p}_1$: sample proportion for group 1
  - $\hat{p}_2$: sample proportion for group 2
  - $\hat{p}_1 - \hat{p}_2$: difference two sample proportions
  - n_1: sample size of group 1
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If these conditions hold, then we can use the normal distribution to find appropriate critical values in order to compute CIs and perform hypothesis tests.

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Also:

- Φ^{-1} is the inverse CDF function of the standard normal distribution
- α is defined as the **level of significance**
- $1 - \alpha$ is the **level of confidence**

Relationship between confidence and significance levels

For example, if 95% is the desired confidence level, then $1 - \alpha = .95$ and $\alpha = 0.05$

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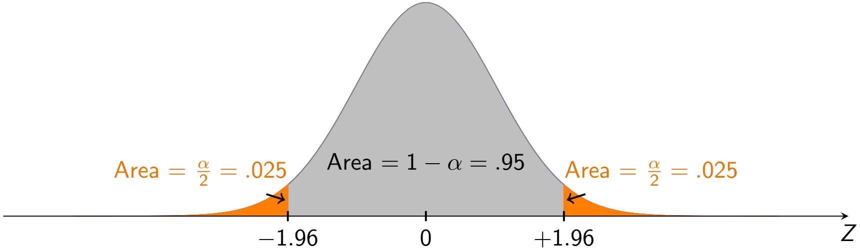
Below is the standard normal distribution showing the critical Z-scores corresponding to the desired confidence level of 95%:



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Thus, the CI is given by:

$$\langle p_t - p_c \rangle_{.95} \quad = \quad 0.13 \pm (1.96 \times 0.095) = 0.13 \pm 0.186$$

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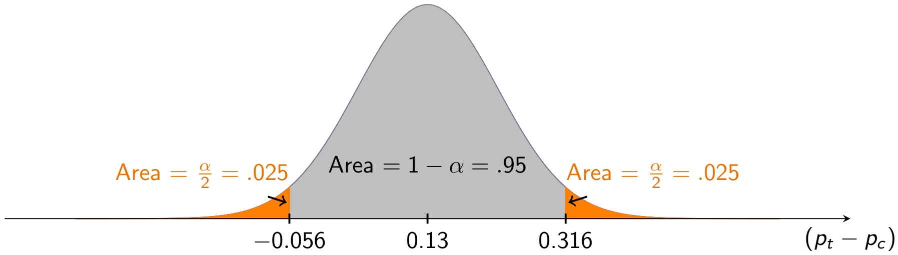
Now, we show the corresponding sampling distribution of $p_t - p_c$ with the computed CIs (i.e. the standard normal distribution converted into the actual scale of the point estimate $\hat{p}_1 - \hat{p}_2$)



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Interpretation

Thus, we are 95% confident (*OR* 95% of the time), the difference in survival rate between those who are treated by the blood thinner and those who are not lies between -5.6 and 31.6 percentage points.

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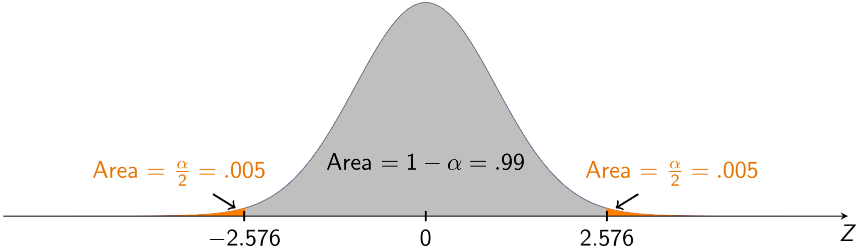

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Thus, the CI is given by:

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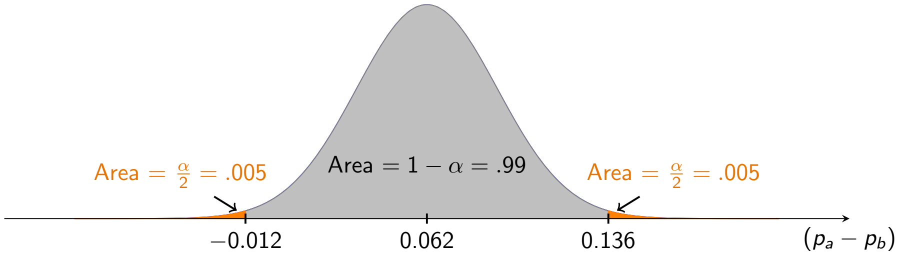
Now, we show the corresponding sampling distribution of $p_a - p_b$ with the computed CIs (i.e. the standard normal distribution converted into the actual scale of the point estimate $\hat{p}_1 - \hat{p}_2$)



Example 2: Political Polling Study (cont.)

Interpretation

We are 99% confident that the difference in support for the infrastructure referendum between County A and County B lies between -1.2 and 13.6 percentage points.

Example 2: Political Polling Study (cont.)

Interpretation

We are 99% confident that the difference in support for the infrastructure referendum between County A and County B lies between -1.2 and 13.6 percentage points. Since the interval contains zero, we cannot conclude at the 99% confidence level that there is a significant difference in support between the two counties.

Hypothesis testing using critical value

1. State the hypotheses H_0 and H_1
2. Compute the point estimate $\hat{p}_1 - \hat{p}_2$
3. Find the standard error SE
4. Find the test statistic $z = \frac{(\hat{p}_1 - \hat{p}_2) - \Delta_0}{SE}$, where Δ_0 is the null value of the difference
5. Find the critical value(s) z^*
6. Compare the test statistic to the critical value
7. Clearly state the outcome from your hypothesis test
8. Write a final concluding statement in response to the question

Hypothesis testing using p-value

1. State the hypotheses H_0 and H_1
2. Compute the point estimate $\hat{p}_1 - \hat{p}_2$
3. Find the standard error SE
4. Find the p-value
5. Compare the p-value to the level of significance α
6. Clearly state the outcome from your hypothesis test
7. Write a final concluding statement in response to the question

Pooled proportion

Pooled proportion

If the hypothesis test is to check whether $p_1 = p_2$ or that $p_1 - p_2 = 0$ (null hypothesis), then:

$$H_0 : p_1 - p_2 = \Delta_0 = 0 \quad (10)$$

$$H_1 : p_1 - p_2 \neq 0 \quad (11)$$

In this case, we use the **pooled proportion** $\hat{p}_{pooled}$ to verify the success-failure condition and to estimate the standard error

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$$\hat{p}_{pooled} = \frac{\text{number of successes}}{\text{number of successes} + \text{number of failures}} \quad (12)$$

$$= \frac{\text{number of successes}}{\text{total number of cases}} \quad (13)$$

$$= \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2} \quad (14)$$

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Thus:

$$SE \approx \sqrt{\frac{\hat{p}_{pooled}(1 - \hat{p}_{pooled})}{n_1} + \frac{\hat{p}_{pooled}(1 - \hat{p}_{pooled})}{n_2}}$$

Example 3: Concrete quality control study

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	Yes	No
Supplier A (a)	500	44,425
Supplier B (b)	505	44,405

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	Failed quality test?	
	Yes	No
Supplier A (a)	500	44,425
Supplier B (b)	505	44,405

Conduct a hypothesis test to determine whether there is significant evidence to suggest that the two suppliers have different failure rates. Use a significance level $\alpha = 0.05$

Hypotheses

Thus, the null hypothesis H_0 is: $p_a - p_b = 0$ (i.e. the null difference $\Delta_0 = 0$).
And the alternative hypothesis H_1 is: $p_a - p_b \neq 0$.

Example 3 (cont.)

In this case the pooled proportion is given by

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$$\begin{aligned}
\hat{p}_{pooled} &= \frac{\#\text{ samples that failed quality test in the entire study}}{\#\text{ samples in the entire study}} \\
&= \frac{500 + 505}{500 + 44,425 + 505 + 44,405} \\
&= 0.0112
\end{aligned}$$

Thus, the SE is given by:

$$\begin{aligned}
SE &\approx \sqrt{\frac{\hat{p}_{pooled}(1 - \hat{p}_{pooled})}{n_a} + \frac{\hat{p}_{pooled}(1 - \hat{p}_{pooled})}{n_b}} \\
&= \sqrt{\frac{0.0112(1 - 0.0112)}{44,925} + \frac{0.0112(1 - 0.0112)}{44,910}} = 0.0007
\end{aligned}$$

Example 3 (cont.): test statistic

Now we find the point estimate $\hat{p}_a - \hat{p}_b$:

$$\hat{p}_a - \hat{p}_b = \frac{500}{44,925} - \frac{505}{44,910} = -0.00012 \tag{15}$$

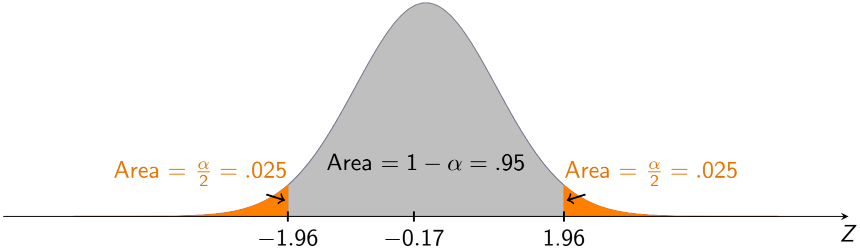
Then, the test statistic is given by:

$$\begin{aligned} z &= \frac{(\hat{p}_a - \hat{p}_b) - \Delta_0}{SE} \\ &= \frac{-0.00012 - 0}{.0007} = -0.17 \end{aligned}$$

Example 3 (cont.): critical value

The critical value z^* in this case is given by:

$$z^* \quad = \quad \Phi^{-1}(1 - \alpha/2) = \Phi^{-1}(1 - .025) = 1.96$$



Since $-0.17 > -1.96$ and $-0.17 < 1.96$ (i.e. the test statistic is contained within the interval $(-1.96, 1.96)$), then we **fail to reject** the null hypothesis H_0. **Thus, we conclude that there is no significant evidence to suggest that the two suppliers have different failure rates for concrete quality**.

Example 3 (cont.)

Next, we will conduct the hypothesis test using p-values (to be continued)

Example 4: Concrete quality control (p-value approach)

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Using the same data from Example 3, we now conduct the hypothesis test using the p-value approach.

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Using the same data from Example 3, we now conduct the hypothesis test using the p-value approach.

	Failed quality test?	
	Yes	No
Supplier A (a)	500	44,425
Supplier B (b)	505	44,405

Hypotheses

H_0: $p_a - p_b = 0$ (no difference in failure rates)
H_1: $p_a - p_b \neq 0$ (different failure rates)
Significance level: $\alpha = 0.05$

Example 4 (cont.): Computing the test statistic

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\hat{p}_{pooled} &= 0.0112 \\
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Example 4 (cont.): Computing the test statistic

From Example 3, we already computed:

$$\begin{aligned} \hat{p}_{pooled} &= 0.0112 \\ SE &= 0.0007 \\ \hat{p}_a - \hat{p}_b &= -0.00012 \end{aligned}$$

Example 4 (cont.): Computing the test statistic

From Example 3, we already computed:

$$
\begin{aligned}
\hat{p}_{pooled} &= 0.0112 \\
SE &= 0.0007 \\
\hat{p}_a - \hat{p}_b &= -0.00012 \\
z &= \frac{(\hat{p}_a - \hat{p}_b) - \Delta_0}{SE} = \frac{-0.00012 - 0}{0.0007} = -0.17
\end{aligned}
$$

Example 4 (cont.): Computing the p-value

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For a two-tailed test with test statistic $z = -0.17$:

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For a two-tailed test with test statistic $z = -0.17$:

$$\begin{aligned} p\text{-value} &= 2 \times P(Z \leq -0.17) \\ &= 2 \times \Phi(-0.17) \end{aligned}$$

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For a two-tailed test with test statistic $z = -0.17$:

$$\begin{aligned} p\text{-value} &= 2 \times P(Z \leq -0.17) \\ &= 2 \times \Phi(-0.17) \\ &= 2 \times 0.4325 \end{aligned}$$

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For a two-tailed test with test statistic $z = -0.17$:

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For a two-tailed test with test statistic $z = -0.17$:

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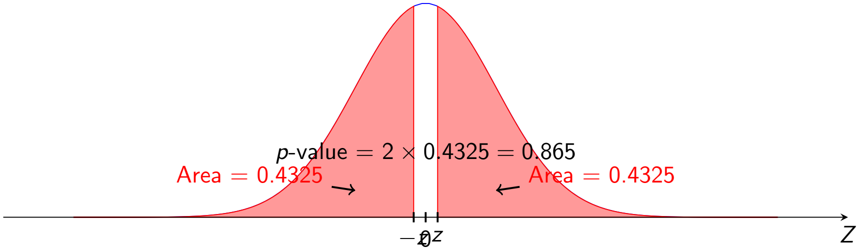
We multiply by 2 because this is a two-tailed test (alternative hypothesis is $p_a - p_b \neq 0$).

Example 4 (cont.): Visualization of p-value

The p-value represents the total area in both tails beyond $z = \pm 0.17$:



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- If p-value $\geq \alpha$, fail to reject H_0

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- p-value $= 0.865$
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- Since $0.865 > 0.05$, we **fail to reject** H_0

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Conclusion

There is no significant evidence to suggest that the two cement suppliers have different failure rates for concrete quality. The observed difference in failure rates could easily be due to random chance.

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There is no significant evidence to suggest that the two cement suppliers have different failure rates for concrete quality. The observed difference in failure rates could easily be due to random chance.

Note: This conclusion matches our result from Example 3 using the critical value approach!

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Point Estimate:

$$\hat{p}_1 - \hat{p}_2$$

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Critical Value:

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- For 95% CI: $z^* = 1.96$
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- For 95% CI: $z^* = 1.96$
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Confidence Interval:

$$(\hat{p}_1 - \hat{p}_2) \pm z^* \times SE$$

Hypothesis Testing for Difference of Two Proportions

Hypotheses:

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Test Statistic:

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - \Delta_0}{SE}$$

Decision Rules for Hypothesis Testing

Critical Value Approach:

- Two-tailed test: Reject H_0 if $|z| > z^*$ where $z^* = \Phi^{-1}(1 - \alpha/2)$
- Right-tailed test: Reject H_0 if $z > z^*$ where $z^* = \Phi^{-1}(1 - \alpha)$
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Key Difference:

- For CI: Use individual sample proportions $\hat{p}_1$ and $\hat{p}_2$ in SE
- For hypothesis testing (when H_0: $p_1 = p_2$): Use pooled proportion in SE

Python Commands for Critical Values

Import the necessary library:

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z_star = norm.ppf(1 - 0.05/2)  # 1.96
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  ```

- General formula: `norm.ppf(1 - alpha/2)`

Python Commands for p-values

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```
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```

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- Left-tailed test:

  `p_value = norm.cdf(z)`

Example: For $z = -0.17$ (two-tailed):

```
p_value = 2 * (1 - norm.cdf(abs(-0.17)))
# Result: 0.865
```