CEE 260/MIE 273: Probability and Statistics in Civil Engineering

# Lecture M4c: Hypothesis Testing and p-values

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# Outline

# Today's objectives

- Learn how to conduct a hypothesis test on the mean of a population
- Know when to use a lower-tailed, upper-tailed or two-tailed test
- Understand Type I errors and their relationship to $p$-values
- Learn how to use $p$-values to conduct a hypothesis test

# Hypothesis testing

- Hypothesis testing provides a framework for evaluating parameter(s) of a population with respect to a desired or known outcome.
- Given that the in most cases, we can only estimate these parameters, hypothesis testing allows us to determine if the estimate supports a **research hypothesis**.
- The results of this testing is useful for **decision-making**.

# Formulating a hypothesis test

A hypothesis is a statement regarding a parameter.
In a test, there are usually two competing hypotheses:

- $H_0$: the **null** hypothesis
- $H_1$: the **alternative** hypothesis ($H_A$ is also used to denote this)

The null hypothesis is usually framed as an equality, i.e.:

$$H_0 : p = p_0 \tag{1}$$

where $p_0$ is the specified standard.
The alternative is given by

$$H_1 : p \neq p_0 \tag{2}$$

# Outcomes of a hypothesis test

The null hypothesis is presumed unless there is sufficient evidence to discard it. The alternative hypothesis, however, is what we hope to support.

> *In experimental design, we frame the null hypothesis in such a way as to reject it.*

Thus there are **two outcomes** of a hypothesis test:

- **Reject $\boldsymbol{H_0}$**: because of sufficient sample evidence in support of $H_1$
- **Fail to reject $\boldsymbol{H_0}$**: because of insufficient evidence in support of $H_1$

**No truth test for the null hypothesis**

The failure to reject $H_0$ does not mean that $H_0$ is true.

# Further explanation of hypothesis test outcomes

## Example 1: Outcome of a trial

In a jury trial, the hypotheses are:

- $H_0$: defendant is innocent
- $H_1$: defendant is guilty (not innocent)

The null hypothesis $H_0$ is **rejected** if there evidence beyond reasonable doubt that the defendant is guilty.

However, **failure to reject** $H_0$ does not imply the defendant is innocent, only that there is **insufficient evidence to prove otherwise**.

# Hypothesis testing in practice

## Example 2: Chip manufacturing

A company manufacturing RAM chips claims the defective rate of the population is 5%. Using a 500-chip sample from production, formulate a hypothesis test to evaluate the validity of the company's claim.
Let $p$ denote the *true* defective probability.
We structure our hypothesis test as follows:

$$H_0 : p = 0.05$$
$$H_1 : p > 0.05$$

Note: this is a one-sided hypothesis test (testing in one direction only)

# Hypothesis testing in practice

## Example 2: Chip manufacturing (cont.)

In order to test the hypotheses, we must choose a **test statistic**.
Here, we let $X$ denote the number of defective chips in the sample of 1000.
Then in order to determine whether or not to reject $H_0$, we must decide on a **critical value**.
We note that this is a Bernoulli process. Thus, if $p = 0.05$, then the expected number of defective chips is

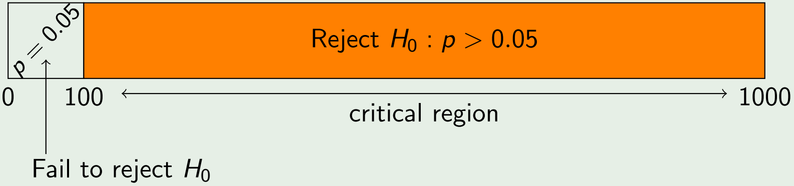
$$\overline{X} = np = 1000 \times 0.05 = 50$$

Say the critical value were $p^* = .1$, then $p \geq .1$ could then be considered as strong evidence that $p > 0.05$.

# Hypothesis testing in practice

## Example 2: Chip manufacturing (cont.)

Thus, we would reject $H_0$.



# Errors in hypothesis testing

Since we are working with finite samples, errors are bound to occur in decision-making.

The decision matrix is:

| | **$H_0$ is true** | **$H_1$ is true** |
|---|---|---|
| **Fail to reject $H_0$** | Correct decision | Type II error |
| **Reject $H_0$** | Type I error | Correct decision |

# Type I error

## Definitions

- The incorrect rejection of $H_0$ is a Type I error.
- Also known as a *false positive*
- The probability of a Type I error is the **level of significance**, $\alpha$

## Examples of Type I error

- Convicting a defendant of a crime when they are innocent (Example 1)
- Diagnosing a patient with a disease when in fact they do not have it (i.e. the null hypothesis is that the disease is NOT present)

# Level of significance

A Type I error is less likely as $\alpha$ reduces.
We revisit Example 2.

## Example 2: Chip manufacturing (cont.)

Find the level of significance $\alpha$ when the critical value of $p$ is .1.
In other words, what is the probability of incorrectly rejecting $H_0$ when it is true?

$$
\begin{aligned}
\alpha &= P(\text{Type I error}) \qquad (3)\\
&= P(p > 0.1) \qquad (4)\\
&= 1 - \Phi\left(\frac{p^* - p_0}{SE_{p_0}}\right) = 1 - \Phi\left(\frac{.1 - .05}{\sqrt{.05(.95)/1000}}\right) \qquad (5)\\
&= \boxed{0} \qquad (6)
\end{aligned}
$$

# Type II errors

## Definitions

- Failure to reject $H_0$ when in fact $H_1$ is true is a Type II error.
- Also known as a *false negative*
- The probability of a Type II error is denoted $\boldsymbol{\beta}$

## Note

We cannot compute $\beta$ except the alternative hypothesis $H_1$ is specified.
Much of our focus will be on dealing with the level of significance, $\alpha$.

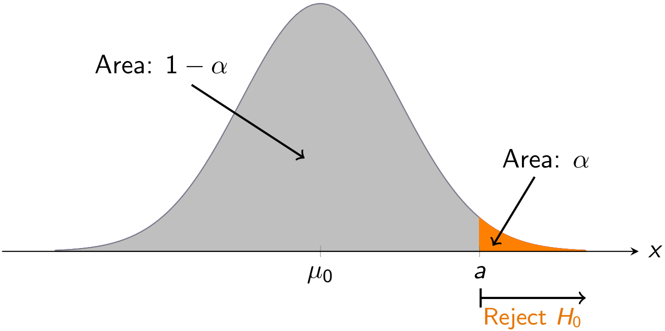
# Summary of hypothesis testing approach

1. *Define* the **null** ($H_0$) and **alternative** ($H_1$) hypotheses
2. *Determine* the appropriate **test statistic** (and distribution)
3. *Estimate* the test statistic from the sample data
4. *Specify* or *identify* the **level of significance** ($\alpha$)
5. *Define* the **region of rejection/critical region** of the null hypothesis by choosing the **critical value**.
6. *Decide.* If the test statistic is in the critical region, reject $H_0$. If not, do not reject $H_0$ (fail to reject it)

# One-sided tests

**Case A: upper tail**

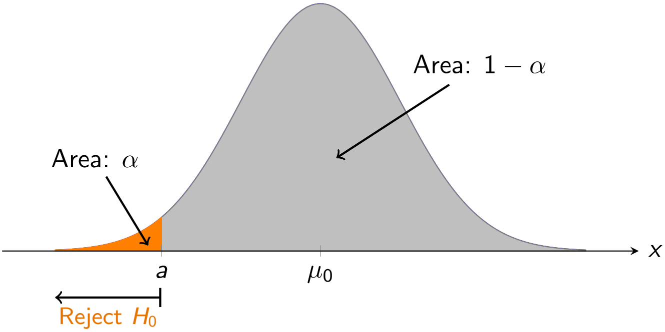
- $H_0 : \mu = \mu_0$
- $H_1 : \mu > \mu_0$



# One-sided tests (cont.)

**Case B: lower tail**

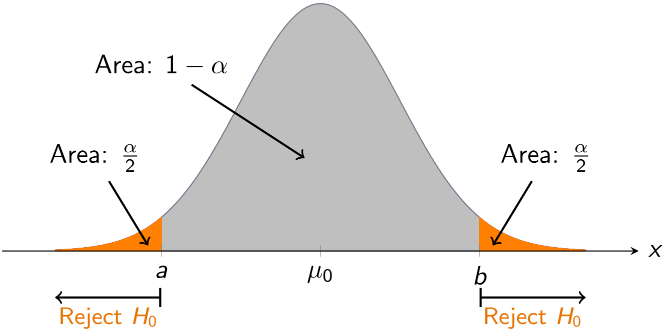
- $H_0 : \mu = \mu_0$
- $H_1 : \mu < \mu_0$



# Two-sided tests

**Case C: both tails**

- $H_0 : \mu = \mu_0$
- $H_1 : \mu \neq \mu_0$



# Distribution of the test statistic

In this lecture, the test statistic is the **sample proportion**.
We will assume the normal distribution is the success-failure condition holds.

The sample proportion is **normally** distributed and its variance is :

$$\mathbb{V}(p) = \frac{p(1-p)}{n} \tag{7}$$

And thus, the standard error is:

$$SE_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} \tag{8}$$

Thus, to compute the probability (area under curve) of the test statistic, we use the z-score:

$$z^* = \frac{p - p_0}{SE_p} \tag{9}$$

which is normally distributed.

# What is a *p*-value?

**Definition**

The $p$-value is the smallest level of significance at which $H_0$ would be rejected when a specified test procedure is used on a given dataset.
Equivalently,this is the minimum probability of a Type I error.

# Motivating the usage of $p$-values

## Example 3: Nicotine content

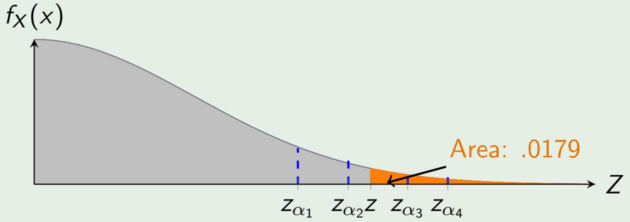
Based on data from a sample of cigarettes, the $Z$ statistic is $z = 2.10$. You want to verify if the true nicotine content (measured in proportion of tobacco weight) is $p = .015$ ($H_0$) versus the alternative hypothesis that is greater: $H_1 : p > .015$).
This is an **upper-tailed** hypothesis test.
What are your conclusions from testing at the following significance levels:

- $\alpha_1 = 0.05$
- $\alpha_1 = 0.025$
- $\alpha_1 = 0.01$
- $\alpha_1 = 0.005$

# Motivating the usage of $p$-values

## Example 3: Nicotine content (cont.)

The $p$-value $1 - \Phi(2.10)$ (area to the right of $z$) $\therefore p = 1 - 0.9821 = 0.0179$.




Your conclusions are as follows:

| Level of significance $\alpha$ | Rejection Region | Conclusion |
|---|---|---|
| $\alpha_1 = 0.05$ | $z \geq 1.645$ | Reject $H_0$ |
| $\alpha_2 = 0.025$ | $z \geq 1.96$ | Reject $H_0$ |
| $\alpha_3 = 0.01$ | $z \geq 2.33$ | Fail to reject $H_0$ |
| $\alpha_4 = 0.005$ | $z \geq 2.58$ | Fail to reject $H_0$ |

# Usefulness of $p$-value

- Provides more information about the strength of a test
- Indicates the smallest level at which the data is significant
- Can be compared with $\alpha$ irrespective of which type of test was used

**Alternative definition**

The $p$-value is the probability of obtaining a test statistic value at least as contradictory to $H_0$ as the value that actually resulted. The smaller the $p$-value, the more contradictory are the data to $H_0$.

# Hypothesis testing with the $p$-value

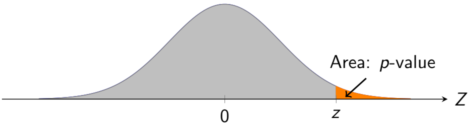
Step 1. Formulate your hypotheses

Step 2. Determine the $p$-value from the test statistic

Step 3. Conclude the test based on a chosen level of significance:

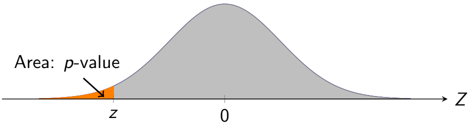
1. $p$-value $\leq \alpha \implies$ reject $H_0$ at level $\alpha$.
2. $p$-value $> \alpha \implies$ do not reject $H_0$ at level $\alpha$.

# *p*-value for *z* tests



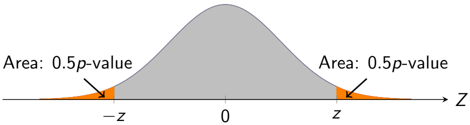

**$p$-value: area in upper tail**

$$p = 1 - \Phi(z) \tag{10}$$




**$p$-value: area in lower tail**

$$p = \Phi(z) \tag{11}$$




**$p$-value: sum of area in both tails**

$$p = 2(1 - \Phi(|z|)) \tag{12}$$

# Hypothesis testing using $p$-value approach

## Example 4: Getting enough sleep (OS 5.21)

400 students were randomly sampled from a large university, and 289 said they did not get enough sleep. Conduct a hypothesis test to check whether this represents a statistically significant difference from 50%, and use a significance level of 0.01

Step 1. Parameter of interest: $p$ (proportion of students not getting enough sleep)

Step 2. Null hypothesis: $H_0 : p = 289/400 = .723$.

Step 3. Alternative hypothesis: $H_1 : p \neq .723$.

Step 4. Formula for test statistic value: $z = \frac{p - p_0}{SE_p}$

# Hypothesis testing using $p$-value approach

## Example 4: Getting enough sleep (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{.723 - .5}{\sqrt{.5(.5)/400}} = 8.92$$

Step 6. Determine $p$-value (two-tailed test):

$$p\text{-value} = 2(1 - \Phi(8.92)) = 0.0$$

Step 7. Conclude:
Using a significance level of 0.01, we reject $H_0$ since $0.0204 > 0.01$. Thus, at the 1% significance level, there is sufficient evidence to conclude that true proportion differs from the target value of 0.5.

# Recap of this lecture

- Definition of hypothesis testing
  - Null hypothesis (default/expected outcome) $H_0$
  - Alternate hypothesis (what we want to test/support; research hypothesis) $H_1$ or $H_A$
  - One-tailed or two-tailed
- Types of errors:
  - Type I: false positive
  - Type II: false negative
- Test statistic:
  - Sample proportion with independent observations and large enough sample size (normal distribution); $Z$-statistic:

$$z^* = \frac{p - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \tag{13}$$

- The $p$-value is the minimum probability of a Type I error.
  - Upper-tailed test: $p - \text{value} = 1 - \Phi(z)$; MATLAB: `normcdf(z, 'upper')`
  - Lower-tailed test: $p - \text{value} = \Phi(z)$; MATLAB: `normcdf(z)`
  - Two-tailed test: $p - \text{value} = 2(1 - \Phi(|z|))$; MATLAB: `2 * normcdf(abs(z))`