CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture M5c: Goodness of Fit Testing; Chi-square

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Outline

Today's objectives

- Get introduced to the χ^2 distribution

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Reading: Section 6.3, Open Intro Statistics

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Group	A	B	C	D	Total
Observed counts	$\hat{n}_A$	$\hat{n}_B$	$\hat{n}_C$	$\hat{n}_D$	
Expected counts	n_A	n_B	n_C	n_D	

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and so on, then it turns out that χ^2 follows a special distribution called the χ^2 distribution

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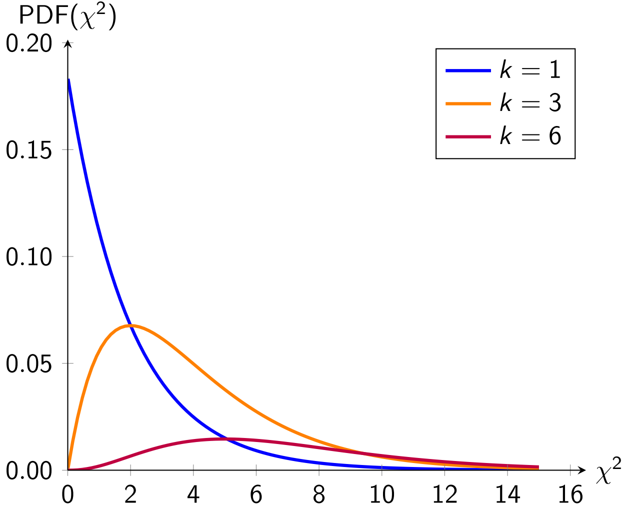

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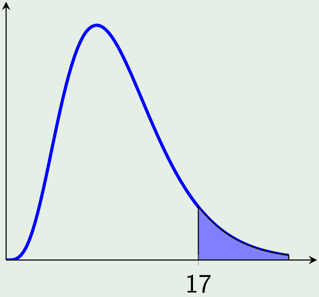

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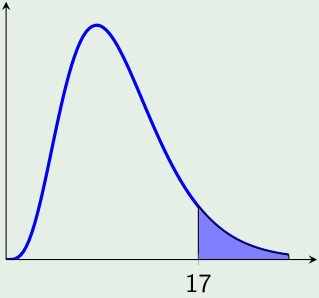

(a) 0.05

(b) 0.02

(c) between 0.02 and 0.05

(d) between 0.05 and 0.1

(e) between 0.01 and 0.02

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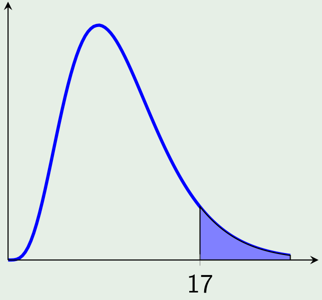

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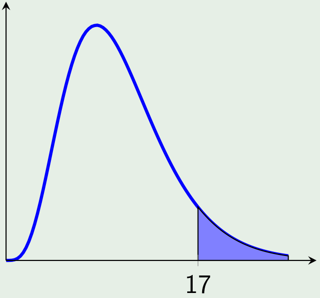

In Python, you can find the area using `1 − chi2.cdf(17, 9)` $= 0.0487$.

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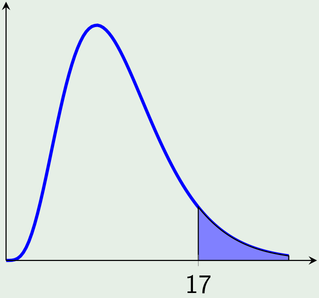

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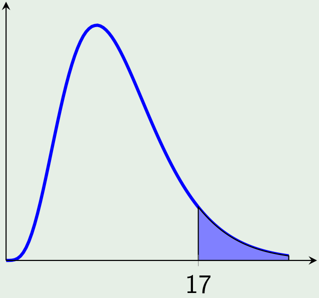

In Python, you can find the area using `1 − chi2.cdf(17, 9)` $= 0.0487$.
Answer: (c) between 0.02 and 0.05

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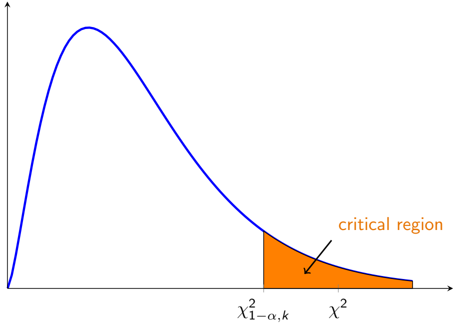

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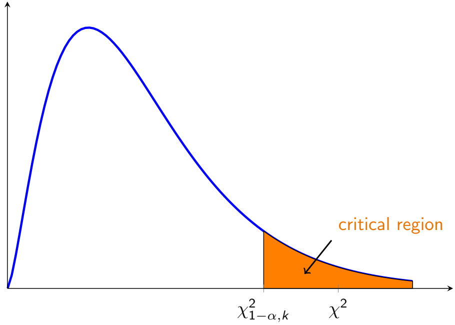

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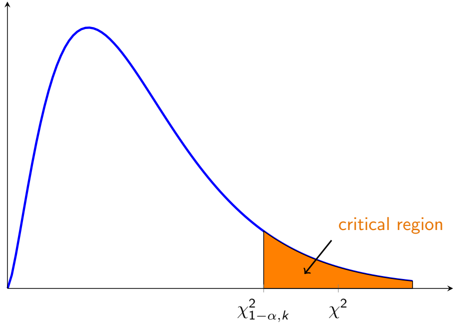

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- If $\chi^2 > \chi^2_{1-\alpha,k}$, **reject null hypothesis**
- If $\chi^2 \leq \chi^2_{1-\alpha,k}$, **fail to reject null hypothesis**

p-values for chi-square tests

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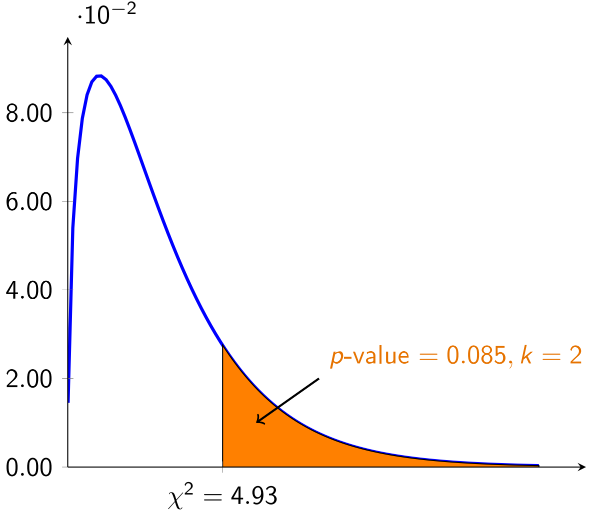
As our focus is on upper-tailed tests, the p-value is given as the area to the right of the test statistic under the chi-square curve (`chi2.sf(test, df)`)



Working with the chi-square distribution

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What conclusion would be appropriate for an upper-tailed chi-square test given the following:

Working with the chi-square distribution

Example 2: Hypothesis testing

What conclusion would be appropriate for an upper-tailed chi-square test given the following:

$$\begin{aligned} \alpha &= 0.05 \\ \chi^2 &= 12.25 \quad \text{(Test statistic)} \\ k &= 4 \quad (\chi^2 \text{ parameter: degrees of freedom}) \end{aligned}$$

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Example 2: Hypothesis testing (cont.)

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The critical value is given by:

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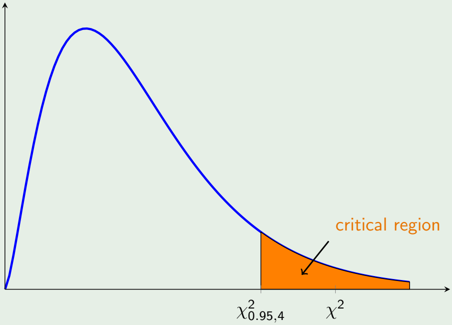

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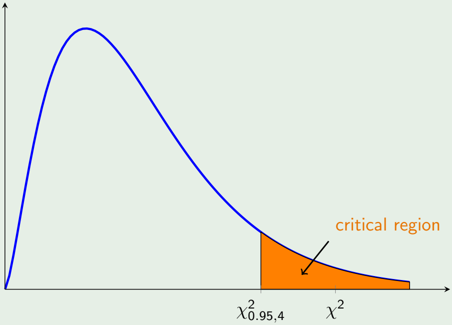

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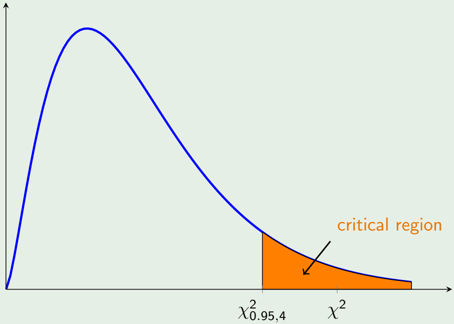

The test statistic is in the critical region. We therefore **reject the null hypothesis**.

Working with the chi-square distribution (cont.)

Example 3: p-value

Working with the chi-square distribution (cont.)

Example 3: *p*-value

Give as much information as you can about the *p*-value for an upper-tailed chi-square test in this situation:

Working with the chi-square distribution (cont.)

Example 3: *p*-value

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$$
\begin{aligned}
k &= 4 \\
\chi^2 &= 7.5
\end{aligned}
$$

Working with the chi-square distribution (cont.)

Example 3: *p*-value (cont.)

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Example 3: p-value (cont.)

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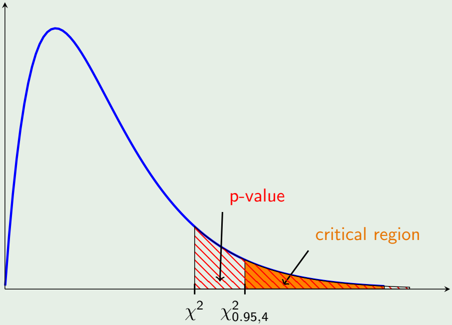

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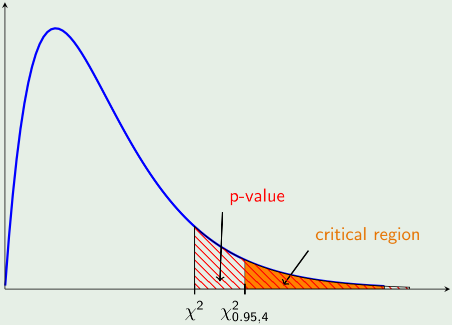

Since the test statistic, $\chi^2 = 7.5$

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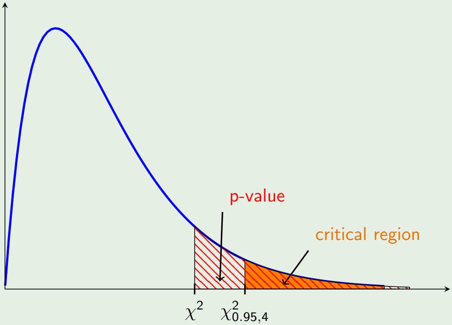

Since the test statistic, $\chi^2 = 7.5 < \chi^2_{0.95,4} = 9.4877$, then the test statistic is in the region of non-rejection.

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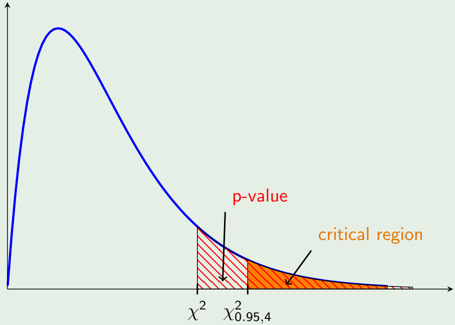

Since the test statistic, $\chi^2 = 7.5 < \chi^2_{0.95,4} = 9.4877$, then the test statistic is in the region of non-rejection.

We can therefore say that the *p*-value > 0.05.

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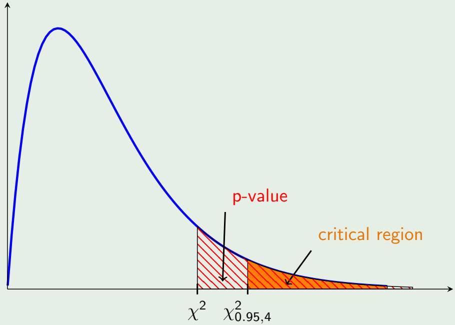

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We can therefore say that the *p*-value > 0.05.
But how much greater?

Working with the chi-square distribution (cont.)

Example 3: *p*-value (cont.)

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Example 3: *p*-value (cont.)

We can increase α further and try $\alpha = 0.10$. Thus:

Working with the chi-square distribution (cont.)

Example 3: *p*-value (cont.)

We can increase α further and try $\alpha = 0.10$. Thus:

$$\chi^2_{0.90,4} = \texttt{chi2inv}(.9, 4) = 7.7794$$

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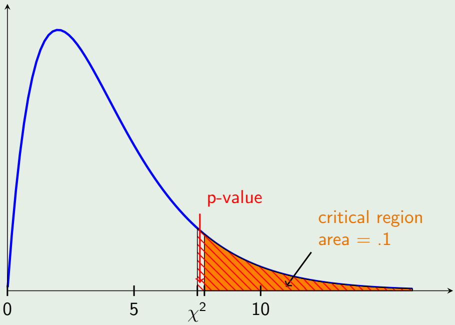

We see that the critical value is still greater than χ^2, but the difference is much smaller.

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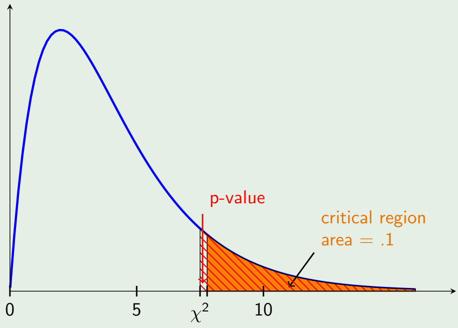

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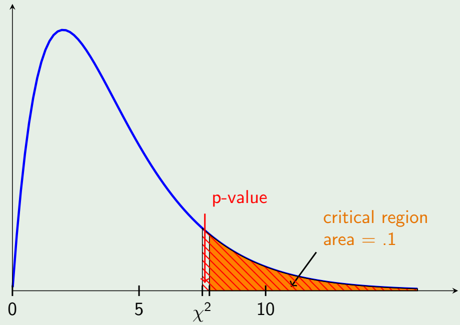

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$$\chi^2 = 7.5 < \chi^2_{0.90,4} = 7.7794$$

Thus, we can say that the *p*-value is quite close to 0.10.

Working with the chi-square distribution (cont.)

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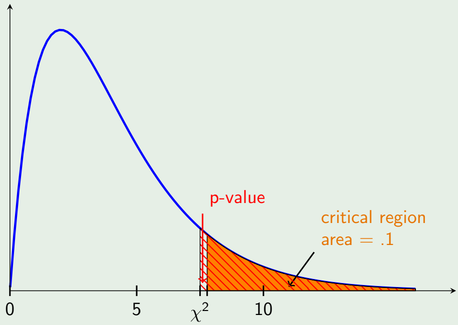

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Example 3: *p*-value (cont.)

We can also see that $\chi^2_{.8,4} = \texttt{chi2inv}(.8, 4) = 5.9886$.

Working with the chi-square distribution (cont.)

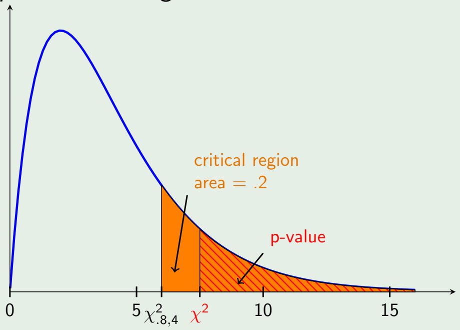
Example 3: *p*-value (cont.)

We can also see that $\chi^2_{.8,4} = \texttt{chi2inv}(.8, 4) = 5.9886$. This is evidence that the *p*-value is not greater than 0.20.

Working with the chi-square distribution (cont.)

Example 3: *p*-value (cont.)

We can also see that $\chi^2_{.8,4} = \texttt{chi2inv}(.8, 4) = 5.9886$. This is evidence that the *p*-value is not greater than 0.20.



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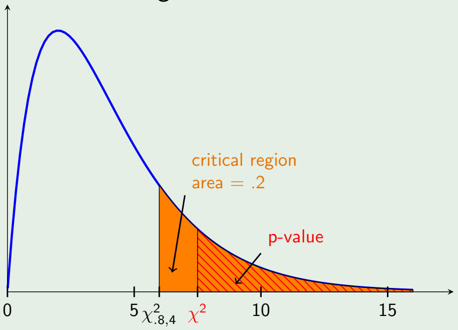

From MATLAB, we find that the *p*-value is given by `chi2cdf(7.5,4,'upper')`:

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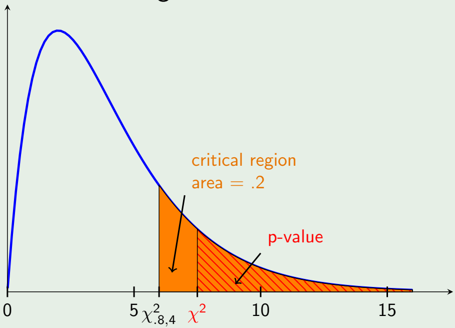

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$$F(\chi^2, k) = F(7.5, 4) = \boxed{0.1117}$$

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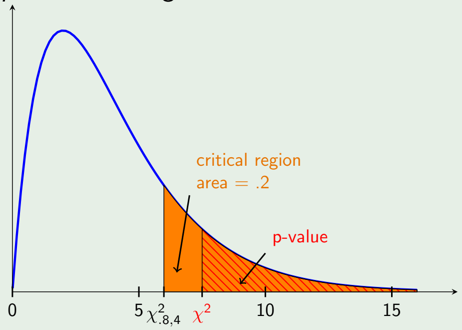

From MATLAB, we find that the *p*-value is given by `chi2cdf(7.5,4,'upper')`:

$$F(\chi^2, k) = F(7.5, 4) = \boxed{0.1117}$$

Note that F here represents the CDF of the χ^2 distribution (not the F distribution).

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- Chi-square tests can be used to examine whether data from a sample fit expected values from a theoretical model/distribution

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  where o_i is the number of observations in each group or interval i, and e_i is the expected/theoretical frequency for each group/interval.
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Given n observations with expected probabilities p_{i0}, the statistic:

$$\chi^2 = \sum \frac{(\text{observed} - \text{expected})^2}{\text{expected}} = \sum_{i=1}^{k} \frac{(o_i - np_{i0})^2}{np_{i0}} \tag{4}$$

is approximately **chi-square** distributed with $k = k - 1$, **provided that** $np_i \geq 5$ **for all** i.

Steps to perform chi-square goodness-of-fit tests

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Step 6. Perform the hypothesis test and conclude

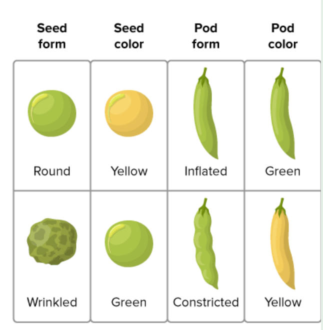
Goodness-of-fit testing

Example 4: Peas in a pod

The distribution of the dominant alleles (forms of a gene) in peas (Y = yellow color, R = round shape) is binomial with $p = \frac{9}{16}$. An experiment was performed to test the hypothesis that all the observed frequencies are equal to the theoretical ones (null hypothesis) versus the alternative that at least one is not. In this experiment, n four-seed pods were examined. In a randomly selected pod, possible X values were 0, 1, 2, 3 and 4.

The following data are given (**one-way table**):

Cell i	**1**	**2**	**3**	**4**	**5**
YR peas/pods	**0**	**1**	**2**	**3**	**4**
Observed	16	45	100	82	26
Expected					
$(o_i - e_i)^2/e_i$					



Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

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Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

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Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

$$e_i = n \times \binom{n}{i-1} p^{i-1}(1-p)^{4-(i-1)}$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

$$\begin{aligned} e_i &= n \times \binom{n}{i-1} p^{i-1}(1-p)^{4-(i-1)} \\ e_1 &= 269 \times \binom{4}{0} \left(\frac{9}{16}\right)^0 \left(\frac{7}{16}\right)^4 = \end{aligned}$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

$$
\begin{aligned}
e_i &= n \times \binom{n}{i-1} p^{i-1}(1-p)^{4-(i-1)} \\
e_1 &= 269 \times \binom{4}{0} \left(\frac{9}{16}\right)^0 \left(\frac{7}{16}\right)^4 = 9.86
\end{aligned}
$$

Goodness-of-fit testing

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Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

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e_i &= n \times \binom{n}{i-1} p^{i-1}(1-p)^{4-(i-1)} \\
e_1 &= 269 \times \binom{4}{0}\left(\frac{9}{16}\right)^0\left(\frac{7}{16}\right)^4 = 9.86 \\
e_2 &= 269 \times \binom{4}{1}\left(\frac{9}{16}\right)^1\left(\frac{7}{16}\right)^3 =
\end{aligned}
$$

Goodness-of-fit testing

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Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

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Goodness-of-fit testing

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Goodness-of-fit testing

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Goodness-of-fit testing

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Goodness-of-fit testing

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Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

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Goodness-of-fit testing

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Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

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Given $n = 16 + 45 + 100 + 82 + 26 = 269$, we compute the theoretical frequencies:

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e_5 &= 269 \times \binom{4}{4} \left(\frac{9}{16}\right)^4 \left(\frac{7}{16}\right)^0 = 26.93
\end{aligned}$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

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Our updated table is now:

Goodness-of-fit testing

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Observed	16	45	100	82	26
Expected	9.86	50.38	97.75	83.78	26.93
$(o_i - e_i)^2/e_i$					

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Example 4: Peas in a pod (cont.)

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We can then compute the normalized squared deviations

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Example 4: Peas in a pod (cont.)

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$(n_1 - e_1)^2/e_1 \quad =$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$(n_1 - e_1)^2/e_1 \quad = \quad (16 - 9.86)^2/9.86$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$(n_1 - e_1)^2/e_1 = (16 - 9.86)^2/9.86 = 3.823$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &=
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$\begin{aligned}(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\ (n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68\end{aligned}$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637 \\
(n_3 - e_3)^2/e_3 &=
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637 \\
(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637 \\
(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75 = 0.052
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637 \\
(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75 = 0.052 \\
(n_4 - e_4)^2/e_4 &=
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637 \\
(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75 = 0.052 \\
(n_4 - e_4)^2/e_4 &= (82 - 83.78)^2/83.78
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637 \\
(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75 = 0.052 \\
(n_4 - e_4)^2/e_4 &= (82 - 83.78)^2/83.78 = 0.038
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

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\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
(n_2 - e_2)^2/e_2 &= (45 - 50.68)^2/50.68 = 0.637 \\
(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75 = 0.052 \\
(n_4 - e_4)^2/e_4 &= (82 - 83.78)^2/83.78 = 0.038 \\
(n_5 - e_5)^2/e_5 &=
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
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(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
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(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75 = 0.052 \\
(n_4 - e_4)^2/e_4 &= (82 - 83.78)^2/83.78 = 0.038 \\
(n_5 - e_5)^2/e_5 &= (26 - 26.93)^2/26.93
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

$$
\begin{aligned}
(n_1 - e_1)^2/e_1 &= (16 - 9.86)^2/9.86 = 3.823 \\
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(n_3 - e_3)^2/e_3 &= (100 - 97.75)^2/97.75 = 0.052 \\
(n_4 - e_4)^2/e_4 &= (82 - 83.78)^2/83.78 = 0.038 \\
(n_5 - e_5)^2/e_5 &= (26 - 26.93)^2/26.93 = 0.032
\end{aligned}
$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Goodness-of-fit testing

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Now, we have all we need to compute the chi-square test statistic:

Goodness-of-fit testing

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$$\chi^2 = \sum_{i=1}^{G} (o_i - e_i)^2 / e_i$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Now, we have all we need to compute the chi-square test statistic:

$$\begin{aligned}\chi^2 &= \sum_{i=1}^{G} (o_i - e_i)^2/e_i \\ &= 3.823 + 0.637 + 0.052 + 0.038 + 0.032\end{aligned}$$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Now, we have all we need to compute the chi-square test statistic:

$$\begin{aligned}\chi^2 &= \sum_{i=1}^{G}(o_i - e_i)^2/e_i \\ &= 3.823 + 0.637 + 0.052 + 0.038 + 0.032 \\ &= 4.582\end{aligned}$$

Goodness-of-fit testing

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Now, we have all we need to compute the chi-square test statistic:

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The degrees of freedom, $k = G - 1 = 5 - 1 = 4$.

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Now, we have all we need to compute the chi-square test statistic:

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\chi^2 &= \sum_{i=1}^{G} (o_i - e_i)^2 / e_i \\
&= 3.823 + 0.637 + 0.052 + 0.038 + 0.032 \\
&= 4.582
\end{aligned}
$$

The degrees of freedom, $k = G - 1 = 5 - 1 = 4$.
Thus, the critical value is given by $\chi^2_{0.99,4} =$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Now, we have all we need to compute the chi-square test statistic:

$$\begin{aligned}\chi^2 &= \sum_{i=1}^{G}(o_i - e_i)^2/e_i \\ &= 3.823 + 0.637 + 0.052 + 0.038 + 0.032 \\ &= 4.582\end{aligned}$$

The degrees of freedom, $k = G - 1 = 5 - 1 = 4$.
Thus, the critical value is given by $\chi^2_{0.99,4} = 13.2767$.

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Since $\chi^2 = 4.582$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Since $\chi^2 = 4.582 < \chi^2_{0.99,4} =$

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Since $\chi^2 = 4.582 < \chi^2_{0.99,4} = 13.2767$, we **fail to reject** the null hypothesis.

Goodness-of-fit testing

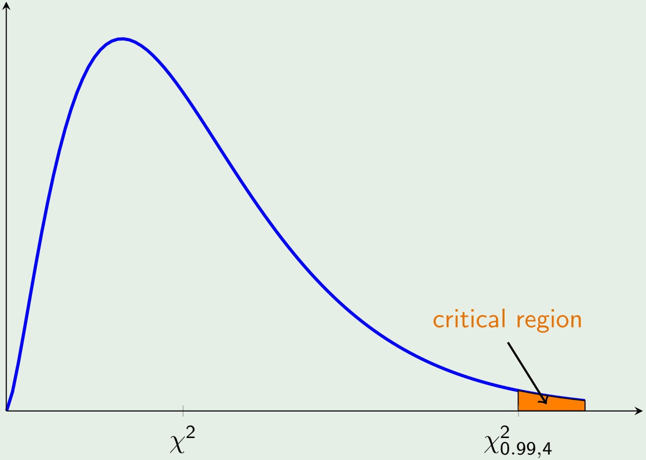
Example 4: Peas in a pod (cont.)

Since $\chi^2 = 4.582 < \chi^2_{0.99,4} = 13.2767$, we **fail to reject** the null hypothesis. This indicates that the binomial model is a good fit for the data.

Goodness-of-fit testing

Example 4: Peas in a pod (cont.)

Since $\chi^2 = 4.582 < \chi^2_{0.99,4} = 13.2767$, we **fail to reject** the null hypothesis. This indicates that the binomial model is a good fit for the data.



Example 5

Example 5

We will revisit Scenario 2 as an in-class activity:

- Each of you will be assigned to a group
- In your group, count the following:
    - number of CEE majors ($\hat{n}_A$)
    - number of ME majors ($\hat{n}_B$)
    - number of IE majors ($\hat{n}_C$)
    - number of Other majors ($\hat{n}_D$)
- Goal: test whether the distribution of majors in your sample (group) is representative of the expected (true) counts (n_A, n_B, n_C, n_D) based on the proportions obtained from the class-wide survey you just completed.

Summary

- The χ^2 distribution has a single parameter k (degrees of freedom)
- It is the underlying the distribution for chi-square tests, which are used to evaluate whether observed data fit a given distribution (or that proportions/frequencies observed from a sample follow a theoretical model/assumption).
- Critical value: $\chi^2_{1-\alpha,k} \equiv$ `chi2.ppf(1—alpha, k)`
- p-value: `chi2.cdf(test, k)`
- When given a one-way table, the test statistic χ^2 is computed as

$$\chi^2 = \sum_{i=1}^{G} \frac{(o_i - e_i)^2}{e_i}, \quad k = G - 1 \tag{7}$$

  where G is the number of groups
- Reject null hypothesis: p-value $< \alpha$ OR $\chi^2 > \chi^2_{1-\alpha,k}$
- Fail to reject null hypothesis: p-value $\geq \alpha$ OR $\chi^2 \leq \chi^2_{1-\alpha,k}$
- Reading: Open Intro Stats, Section 6.3