Problem Set 6

Prof. Oke

CEE 260/MIE 273: Probability & Statistics in Civil Engineering *10.29.2024*

Due Tuesday, November 12, 2024 at 11:59 PM as PDF uploaded on Moodle. *I strongly encourage you to write/type your responses directly on this document and upload it.* ***Show as much work as possible in order to get FULL credit.*** *There are 7 problems with a total of 67 points available.* ***Important:*** *If you use MATLAB/Python for any probability computations, briefly write/include the statements you used to arrive at your answers. If instead you use probability tables, note this in the respective solution, as well.*

Problem 1 *(8 points)*

Respond "T" (*True*) or "F" (*False*) to the following statements. Use the boxes provided. Each response is worth 1 point.

(i) T — A sample of size $n = 100$ has a proportion parameter estimate $\hat{p} = 0.3$. If the sample observations are independent, then the sample satisfies the success-failure condition.

(ii) T — For a given confidence level, the margin of error decreases as the sample size increases.

(iii) F — If the 95% confidence interval for a proportion p is $(0.669, 0.731)$, then the margin of error is 0.062.

(iv) T — In computing a confidence interval, if the critical Z-score is 2.58, then the α corresponding to the confidence level of interest is 0.01.

(v) F — In a hypothesis test, the significance level α can also be understood as the probability of making a Type II error.

(vi) T — In hypothesis testing, failure to reject the null hypothesis H_0 does not mean the null hypothesis is necessarily true.

(vii) F — In a hypothesis test, if the p-value for the test statistic is 0.084 and $\alpha = 0.05$, then we would reject the null hypothesis.

(viii) T — In a hypothesis test, if the p-value for the test statistic is 3.2×10^{-5} and $\alpha = 0.01$, then we would reject the null hypothesis.

Problem 2 *Confidence interval of a proportion (10 points)*

A poll conducted in 2013 found that 52% of U.S. adult X (formerly Twitter) users get at least some news on X. The standard error for this estimate was 2.4%, and a normal distribution may be used to model the sample proportion $\hat{p}$. Now answer the following step-by-step questions in order to construct a 95% confidence interval for the fraction of U.S. adult X users who get some news on X.

[1] **(a)** State the value of $\hat{p}$.

$$\hat{p} = 0.52$$

(b) Find the corresponding critical Z-score for a 95% confidence level. Show how you obtained
[1] your answer (wehther its via Matlab/Python or another source.)

$$z = \texttt{norm.ppf(1 - (1 - 0.95) / 2)}$$
$$z \approx 1.96$$

[2] **(c)** Compute the margin of error.

$$\begin{aligned} \text{Margin of Error}(ME) &= z \times \text{SE} \\ &= 1.96 \times 0.024 \\ &= 0.04704 \end{aligned}$$

[2] **(d)** Write the 95% confidence interval of p.

The Confidence Interval is given by:

$$\begin{aligned} < p >_{0.95} &= \hat{p} \pm \text{ME} \\ &= 0.52 \pm 0.047 \\ &= (0.473, 0.567) \end{aligned}$$

[2] **(e)** Briefly interpret the interval in the context of this question.

We are 95% confident that the true proportion of U.S. adult X users who get at least some news on X lies between 47.3% and 56.7%.

(f) Would you expect the 99% confidence interval to be narrower (smaller) or wider (larger)?
[2] Provide a reason for your answer.

The 99% confidence interval would be wider, as increasing the confidence level requires a larger margin of error.

Problem 3 *CI of proportion and sample size (10 points)*

An article reports that when each football helmet in a random sample of 37 suspension-type helmets was subjected to a certain impact test, 24 showed damage. Let p denote the proportion of all helmets of this type that would show damage when tested in the prescribed manner.

(a) Calculate a 99% CI (confidence interval) for p. *[6]*

$$\begin{aligned}
\hat{p} &= \frac{24}{37} \approx 0.6486 \\
z &= 2.576 \\
SE &= \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\
&= \sqrt{\frac{0.6486 \times 0.3514}{37}} \approx 0.07848 \\
\text{ME} &= z \times SE \\
&= 2.576 \times 0.07848 \approx 0.2022 \\
\therefore < p >_{0.99} &= \hat{p} \pm \text{ME} \\
&= 0.6486 \pm 0.2022 \\
&= (0.4464, 0.8508)
\end{aligned}$$

(b) What sample size would be required for the width of a 99% CI to be 0.1? *[4]*

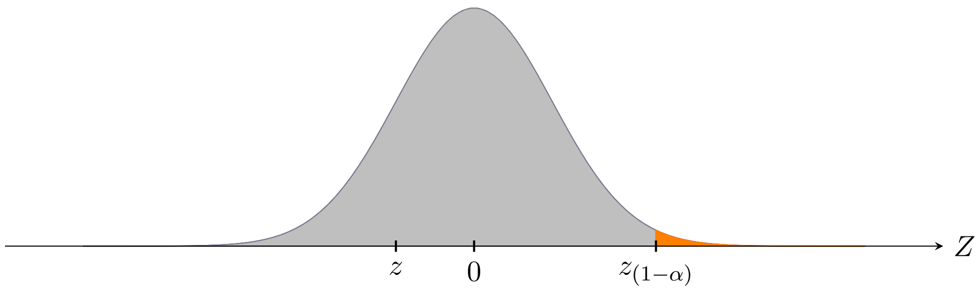
$$\begin{aligned}
\text{Width} = 2 \times \text{ME} &= 0.1 \\
\therefore \text{ME} &= 0.05 \\
z &= 2.576 \\
\hat{p} &\approx 0.6486
\end{aligned}$$

$$\begin{aligned}
\text{ME} &= z \times SE \\
0.05 &= 2.576 \times \sqrt{\frac{0.6486 \times 0.3514}{n}} \\
n &\approx 605
\end{aligned}$$

Problem 4 *Hypothesis testing (5 points)*

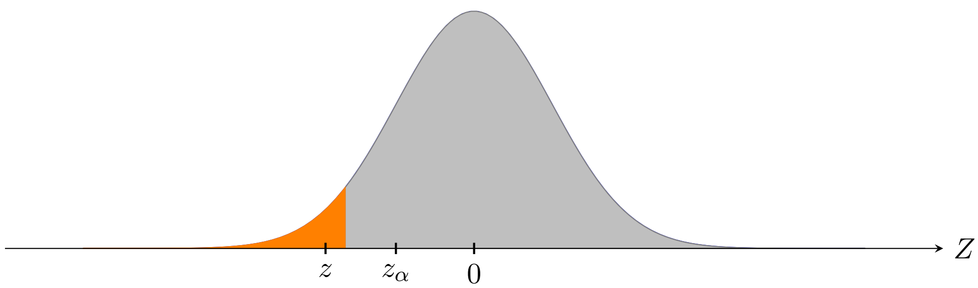
In the following hypothesis tests, decide whether to (i) "Reject H_0" or (ii) "Fail to reject H_0" by comparing the Z-scores (z) to the critical values ($z_{\frac{\alpha}{2}}$, $z_{(1-\frac{\alpha}{2})}$, etc.) at the boundaries of the critical regions (in orange). Circle the correct decision in each case.

(a) $H_0 : p = p_0; H_1 : p > p_0$ Fail to reject H_0



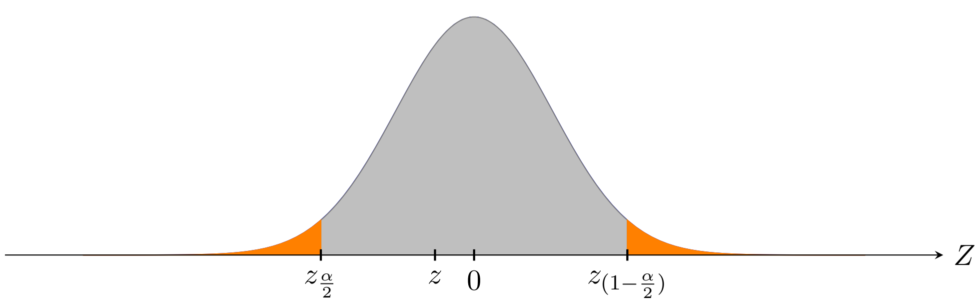
i. Reject H_0 **ii.** Fail to reject H_0

(b) $H_0 : p = p_0; H_1 : p > p_0$ Reject H_0



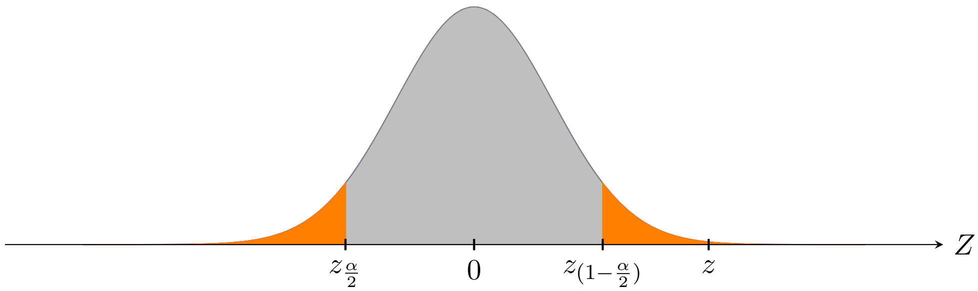
i. Reject H_0 **ii.** Fail to reject H_0

(c) $H_0 : p = p_0; H_1 : p \neq p_0$ Fail to reject H_0



i. Reject H_0 **ii.** Fail to reject H_0

(d) $H_0 : p = p_0; H_1 : p \neq p_0$ Reject H_0



i. Reject H_0 **ii.** Fail to reject H_0

Problem 5 *Two-tailed hypothesis test using critical values (10 points)*

400 students were randomly sampled from a large university, and 289 said they did not get enough sleep. Conduct a hypothesis test to check whether this represents a statistically significant difference from 50%, and use a significance level of 0.05. In this problem, you are required to use the critical value appreoach (so no need to compute p-values). Your response will be graded on the following steps:

- State the hypotheses (there are two) *[2]*
- Compute test statistic *[2]*
- Compute the critical values *[2]*
- Compare test statistic to critical values *[2]*
- State outcome of test and write concluding statement *[2]*

1. State the hypotheses.

$$H_0 : p = 0.5$$
$$H_1 : p \neq 0.5$$

2. Compute the test statistic.

$$\hat{p} = \frac{289}{400} = 0.7225$$
$$SE = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.5 \times 0.5}{400}} = 0.025$$
$$z = \frac{\hat{p} - p_0}{SE} = \frac{0.7225 - 0.5}{0.025} = 8.9$$

3. Compute the critical values.

Since this is a two-tailed test with a significance level of $\alpha = 0.05$, we split α into two tails (0.025 in each tail).
We need the z-scores that correspond to the upper and lower 2.5% of the normal distribution. These z-scores are found using the inverse cumulative distribution function, which gives:

$$\begin{aligned} z_{\alpha/2} &= \texttt{norm.ppf(0.975)} \\ &\approx 1.96 \end{aligned}$$

Thus, the critical values for this test are:

$$z = \pm 1.96$$

4. Compare test statistic to critical values.

$$\text{The calculated } z\text{-score is } 8.9\text{, which is greater than the critical value of } 1.96.$$
$$\text{Since } 8.9 > 1.96, \text{ we fall within the rejection region.}$$

5. State outcome of test and write concluding statement.

Outcome: We reject H_0.
Conclusion: At the 5% significance level, there is sufficient evidence to conclude that the proportion of students who report not getting enough sleep is significantly different from 50%.

Problem 6 *Two-tailed hypothesis test using p-value (12 points)*

It is believed that nearsightedness affects about 8% of all children. In a random sample of 194 children, 21 are nearsighted. Conduct a hypothesis test ($\alpha = .05$) for the following question: do these data provide evidence that the 8% value is inaccurate? Your response will be graded on the following steps:

- State the hypotheses (there are two) *[2]*
- Find the standard error *[2]*
- Find the test statistic *[2]*
- Find the p-value *[2]*
- Compare the appropriate values *[1]*
- Clearly state the outcome from your hypothesis test *[1]*
- Write a final concluding statement in response to the question *[2]*

1. State the hypotheses.

$$H_0 : p = 0.08$$
$$H_1 : p \neq 0.08$$

2. Find the standard error.

$$SE = \sqrt{\frac{p_0(1 - p_0)}{n}} = \sqrt{\frac{0.08 \times 0.92}{194}} \approx 0.0195$$

3. Find the test statistic.

$$\hat{p} = \frac{21}{194} \approx 0.1082$$
$$z = \frac{\hat{p} - p_0}{SE} = \frac{0.1082 - 0.08}{0.0195} \approx \pm 1.45$$

4. Find the p-value.

$$\begin{aligned} \text{p-value} &= 2 \times (1 - \phi(z)) \\ &= 2 \times (1 - \phi(1.45)) \\ &= 2 \times (1 - \texttt{norm.cdf(1.45)}) \\ &= 2 \times 0.0735 \\ \text{p-value} &= 0.147 \end{aligned}$$

5. Compare the p-value to α.

$$\alpha = 0.05$$
$$\therefore \text{p-value} > \alpha,$$
$$\text{Since } 0.147 > 0.05.$$

6. State the outcome of the test.

   Outcome: We fail to reject H_0.

7. Write a concluding statement.

   Conclusion: At the 5% significance level, there is insufficient evidence to conclude that the true proportion of nearsighted children differs from 8%.

Problem 7 *Difference of two proportions hypothesis testing using p-values (12 points)*

A quadcopter company is considering a new supplier for rotor blades. The new supplier would be more expensive, but they claim their higher-quality blades are more reliable, with 3% more blades passing inspection than their competitor. The company's quality control engineer collects a sample of blades, examining 1000 blades from each supplier, and she finds that 899 blades pass inspection from the current supplier and 958 pass inspection from the prospective supplier. Using these data, set up and evaluate the hypotheses with a significance level $\alpha = .01$. Your response will be graded on the following steps:

- State the hypotheses (there are two) *[2]*
- Find the standard error *[2]*
- Find the test statistic *[2]*
- Find the p-value *[2]*
- Compare the appropriate values *[1]*
- Clearly state the outcome from your hypothesis test *[1]*
- Write a final concluding statement in response to the question *[2]*

1. State the hypotheses.

Option 1 (One-tailed test):

$$H_0 : p_2 - p_1 = 0.03$$
$$H_1 : p_2 - p_1 > 0.03$$

Option 2 (two-tailed test):

$$H_0 : p_2 - p_1 = 0.03$$
$$H_1 : p_2 - p_1 \neq 0.03$$

Note: Both tests are valid interpretations. The upper-tail test aligns better with testing for a higher pass rate, but the two-sided test is also acceptable. A solution for each option is provided below.

Solve with option 1 (One-tailed test)

1. Find the standard error.

$$\hat{p}_{pooled} = \frac{899 + 958}{2000} = 0.9285$$
$$SE = \sqrt{\hat{p}_{pooled}(1 - \hat{p}_{pooled})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$
$$= \sqrt{0.9285 \times (1 - 0.9285) \times \left(\frac{1}{1000} + \frac{1}{1000}\right)}$$
$$\approx 0.0115$$

2. Find the test statistic.

$$
\begin{aligned}
z &= \frac{(\hat{p}_2 - \hat{p}_1) - \Delta_0}{SE} \\
\hat{p}_1 &= \frac{899}{1000} = 0.899 \\
\hat{p}_2 &= \frac{958}{1000} = 0.958 \\
\Delta_0 &= 0.03
\end{aligned}
$$

$$
\begin{aligned}
\therefore z &= \frac{(0.958 - 0.899) - 0.03}{0.0115} \\
&= \frac{0.059 - 0.03}{0.0115} \\
&= \frac{0.029}{0.0115} \\
&\approx 2.52
\end{aligned}
$$

3. Find the p-value.

$$
\begin{aligned}
\text{p-value} &= 1 - \phi(z) \\
&= 1 - \texttt{norm.cdf(2.52)} \\
\therefore \text{p-value} &= 0.0059
\end{aligned}
$$

4. Compare the p-value to α.

$$
\begin{aligned}
\alpha &= 0.01 \\
\therefore \text{p-value} &< \alpha, \\
0.0059 &< 0.01.
\end{aligned}
$$

5. State the outcome of the test.

   Outcome: We reject H_0.

6. Write a concluding statement.

   There is sufficient evidence to support the new supplier's claim that their blades are more reliable, with 3% more blades passing inspection than their competitor, at the 0.01 significance level.

Solve with option 2 (two-tailed test)

1. State the hypotheses.

$$
\begin{aligned}
H_0 &: p_2 - p_1 = 0.03 \\
H_1 &: p_2 - p_1 \neq 0.03
\end{aligned}
$$

2. Find the standard error.

$$
\begin{aligned}
\hat{p}_{\text{pooled}} &= \frac{899 + 958}{2000} = 0.9285 \\
SE &= \sqrt{\hat{p}_{\text{pooled}}(1 - \hat{p}_{\text{pooled}}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)} \\
&= \sqrt{0.9285 \times (1 - 0.9285) \times \left(\frac{1}{1000} + \frac{1}{1000} \right)} \\
&\approx 0.0115
\end{aligned}
$$

3. Find the test statistic.

$$
\begin{aligned}
z &= \frac{(\hat{p}_2 - \hat{p}_1) - \Delta_0}{SE} \\
\hat{p}_1 &= \frac{899}{1000} = 0.899 \\
\hat{p}_2 &= \frac{958}{1000} = 0.958 \\
\Delta_0 &= 0.03 \\
z &= \frac{(0.958 - 0.899) - 0.03}{0.0115} \\
&= \frac{0.059 - 0.03}{0.0115} \\
&= \frac{0.029}{0.0115} \\
&\approx 2.52
\end{aligned}
$$

4. Find the p-value.

$$\begin{aligned}\text{p-value} &= 2 \times \big(1 - \phi(2.52)\big) \\ &= 2 \times (1 - \texttt{norm.cdf}(2.52)) \\ &\approx 2 \times 0.0059 \\ &\approx 0.0118\end{aligned}$$

5. Compare the p-value to α.

$$\begin{aligned}&\alpha = 0.01 \\ &\therefore \text{p-value} > \alpha, \\ &\quad 0.0118 > 0.01.\end{aligned}$$

6. State the outcome of the test.

   Outcome: We fail to reject H_0.

7. Write a concluding statement.

   There is insufficient evidence to conclude that the new supplier's blades are more reliable, with a 3% higher pass rate, at the 0.01 significance level.