Introduction to Probability and Statistics Twelve Edition

Chapter 6

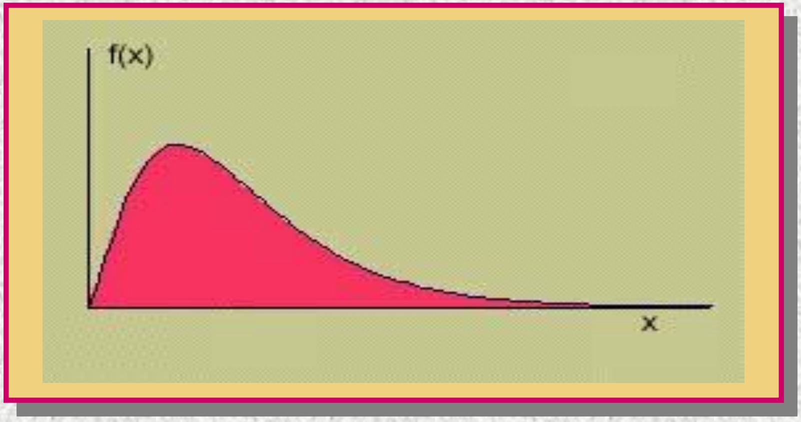
The Normal Probability Distribution

Continuous Random Variables

- Continuous random variables can assume the infinitely many values corresponding to points on a line interval.
- **Examples:**
  - Heights, weights
  - Blood pressure
  - length of life of a particular product

Continuous Random Variables

- For each continuous random variable x, a nonnegative function $f(x)$ describes the probability distribution of x.



- $f(x)$ is called the **probability distribution** or **probability density function** for x.

Properties of Continuous Probability Distributions

1) f is a nonnegative function such that

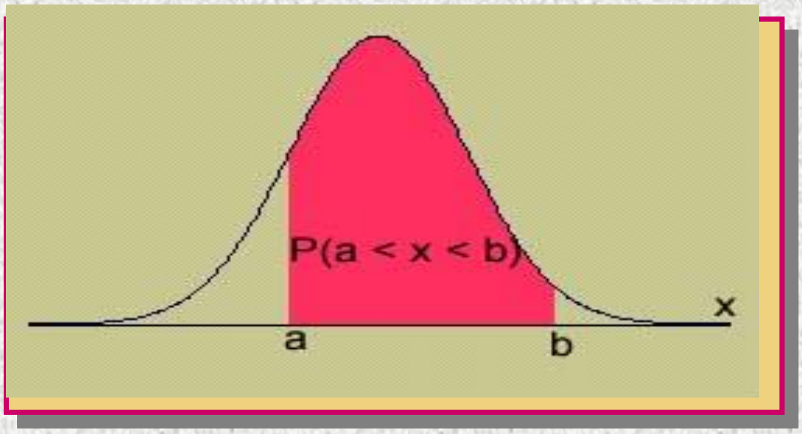
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

That is the area under the curve $f(x)$ and above the x-axis is equal to 1.

Properties of Continuous Probability Distributions

2. $P(a \le x \le b) = \int_a^b f(x)dx$

= area under the curve between a and b.



Continuous Probability Distributions

- As a result, there is no probability attached to any single value of x. That is,

$$P(x = a) = 0, \text{ for any real value } a$$

- Also,

$$\begin{aligned} (a \le x \le b) &= P(a < x \le b) \\ &= P(a \le x < b) \\ &= P(a < x < b) \end{aligned}$$

Continuous Random Variable

- The expected value or mean of a continuous random variable x is

$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

- The variance of x is

$$\sigma^2 = E[(x-\mu)^2]$$
$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

- The standard deviation of x is

The Normal Distribution

- A random variable x is said to be normally distributed with mean μ and standard deviation σ if its probability distribution is

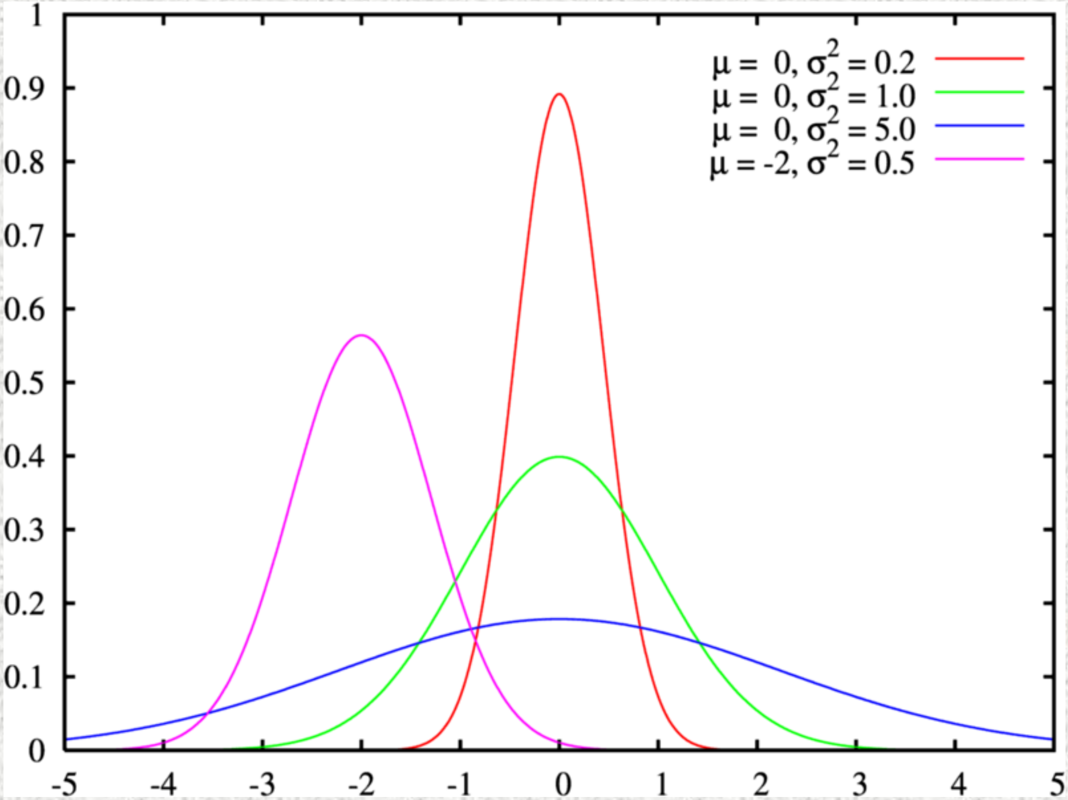
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for} -\infty < x < \infty$$

$$\text{where } e \approx 2.7183, \quad \pi \approx 3.1416$$

$$-\infty < \mu < \infty \text{ and } \sigma > 0$$

Note that the total area under the curve $f(x)$ and above the x–axis is equal to 1.

The Normal Distribution



The shape and location of the normal curve changes as the mean and standard deviation change.

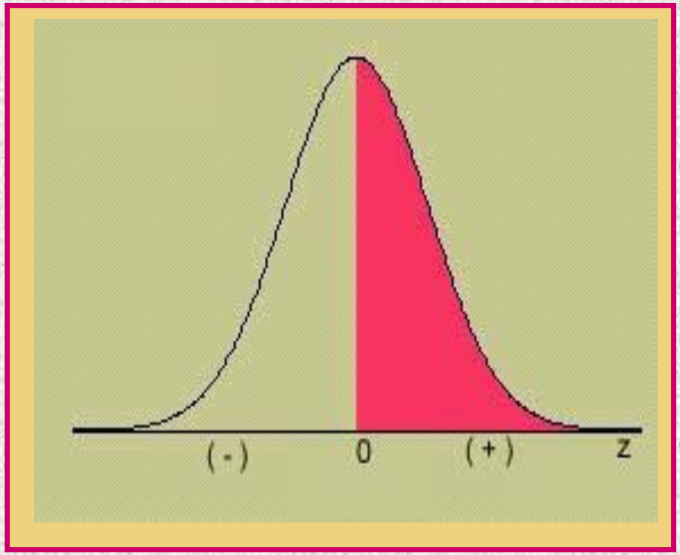
Properties of a Normal Distribution

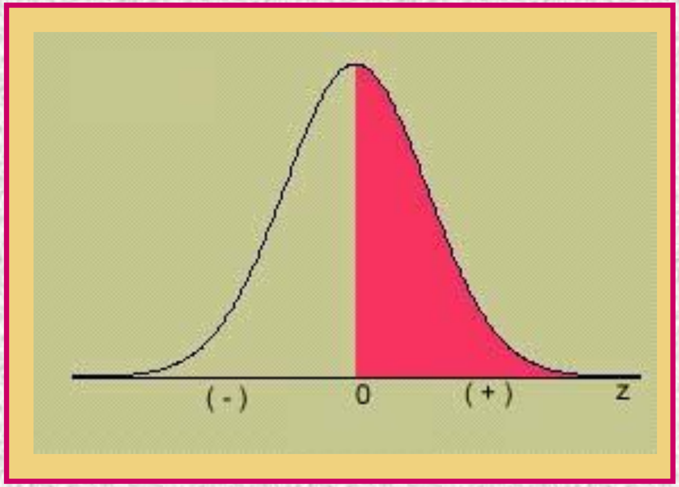
- The normal distribution is completely determined by its mean μ and standard deviation σ.
- The normal distribution is symmetrical about the mean μ.
- The mean is at the middle of the distribution and divide the area into halves. Thus, mean = median.
- The mode is the value of x such that $f(x)$ becomes maximum
  - Thus, mean = median = mode.

The Standard Normal (*z*) Distribution

A normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$ is called the ***standard normal distribution***.

The standard normal random variable is denoted by *z*.





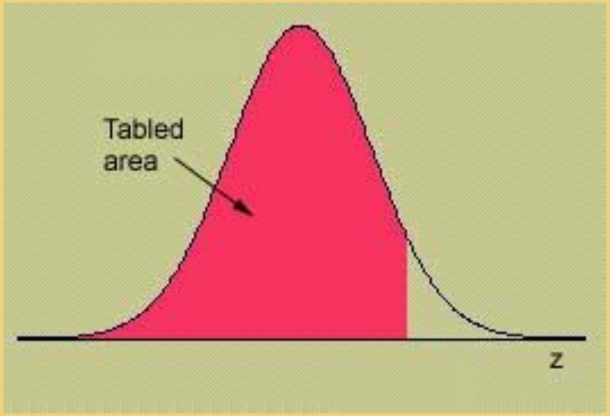
The Standard Normal (z) Distribution

- Mean = 0; Standard deviation = 1
- Symmetric about $z = 0$
- Values of z to the left of center are negative
- Values of z to the right of center are positive
- Total area under the curve is 1.

Using Table 3

The four digit probability in a particular row and column of Table 3 gives the area under the z curve to the left that particular value of z.

z	.00	.01	.02	.03	.04	.05	.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051
0.9	0.8159	0.8186	0.8212	0.8328	0.8264	0.8289	0.8315
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278



Area for $z = 1.36$

Example 1

Use Table 3 to calculate these probabilities:

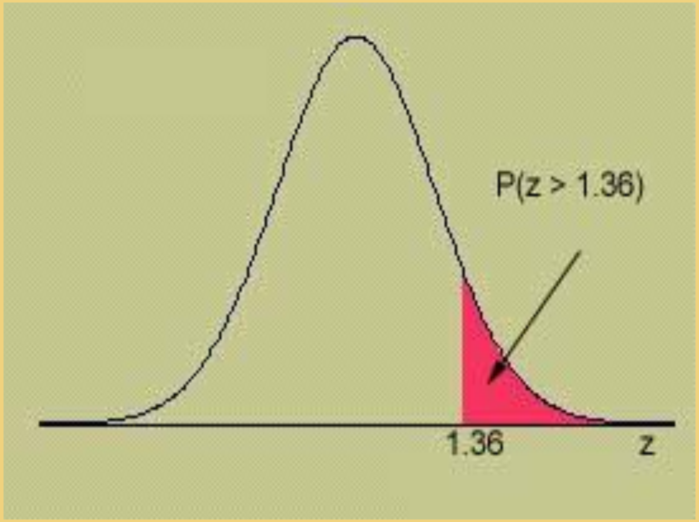
$$P(z \leq 1.36) = 0.9131$$

$$P(z > 1.36)$$

$$= 1 - P(z \leq 1.36)$$

$$= 1 - 0.9131$$

$$= 0.0869$$



Example 2

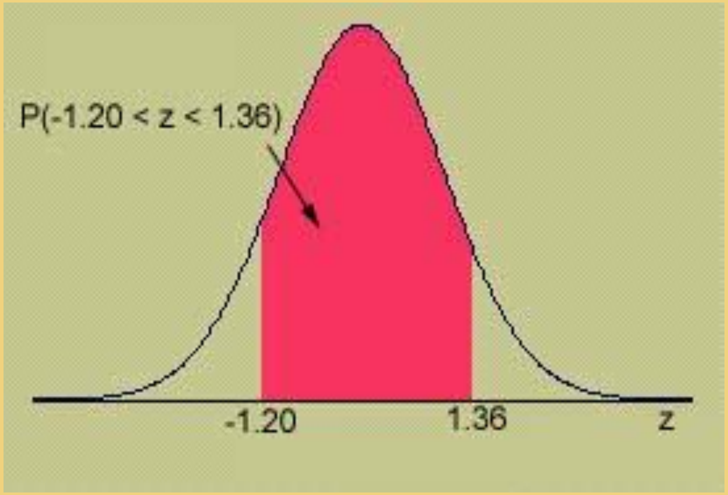
Use Table 3 to calculate the probability:

$P(-1.20 < z < 1.36)$

$= P(z < 1.36) - P(z \leq -1.2)$

$= 0.9131 - 0.1151$

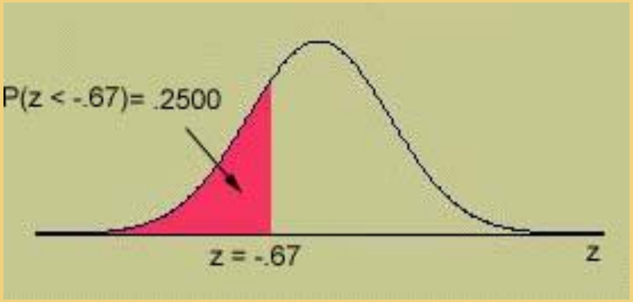
$= 0.7980$



Working Backwards

Example 3: Find the value of z that has area 0.25 to its left.

1. Look for the four digit area closest to 0.2500 in Table 3.
2. What row and column does this value correspond to?
3. $z \approx -0.67$



4. What percentile does this value represent?

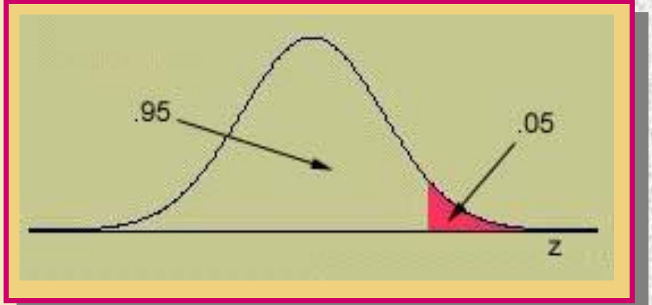
	.00	.01	.02	.03	.04	.05	.06	.07	.08
	...841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635
	...119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810

25th percentile, or 1st quartile (Q_1)

Working Backwards

Example 4: Find the value of z that has area .05 to its right.

1. The area to its left will be 1 - 0.05 = 0.95
2. Look for the four digit area closest to 0.9500 in Table 3.
3. Since the value 0.9500 is halfway between 0.9495 and 0.9505, we choose z halfway between 1.64 and 1.65.
4. $z \approx 1.645$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Class Activity

1. Consider a standard normal random variable with $\mu = 0$ and standard deviation $\sigma = 1$. Use Table 3 to find the following probabilities:
   a) $P(z < 2)$
   b) $P(z \leq 2)$
   $\Big\} = 0.9772$
   c) $P(z > 1.16)$ $= 1 - (z \leq 1.16) = 1 - 0.8770 = 0.123$
   d) $P(-2.33 < z < 2.33)$
   e) $P(z < 1.88)$

Class Activity

2. Find a z_0
   a) that has the area 0.9505 to its left. $z = 1.65$
   b) that has the area 0.72 to its right. $z = -0.58$
   c) such that $P(-z_0 < z < z_0) = 0.80$.

Finding Probabilities for the General Normal Random Variable

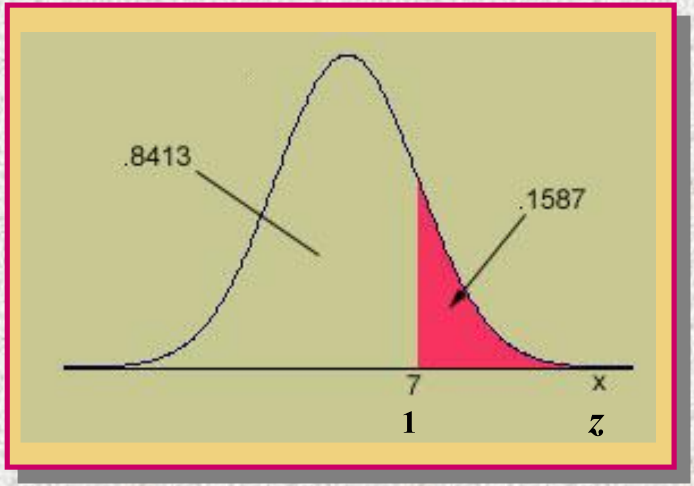
- To find an area for a normal random variable x with mean μ and standard deviation σ, *standardize or rescale* each value of x by expressing it as a z-score,
- Find the appropriate area using Table 3.

$$\boxed{z = \frac{x - \mu}{\sigma}}$$

Finding Probabilities for the General Normal Random Variable

Example 5: If x has a normal distribution with $\mu = 5$ and $\sigma = 2$, find $P(x > 7)$.

$$P(x > 7) = P(z > \frac{7-5}{2})$$
$$= P(z > 1)$$
$$= 1 - 0.8413$$
$$= 0.1587$$



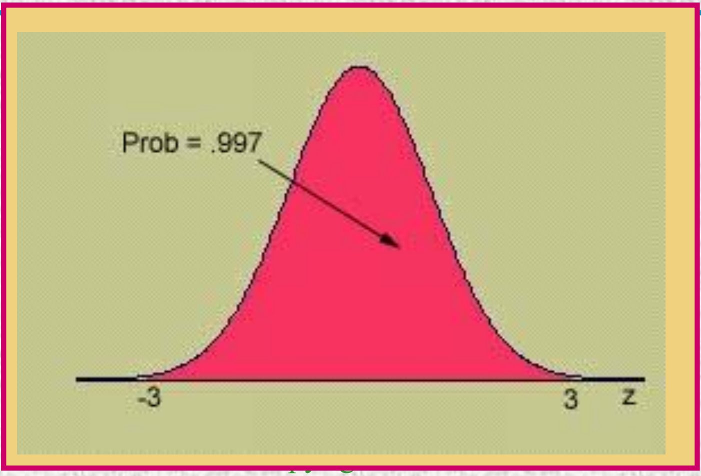
Finding Probabilities for the General Normal Random Variable

Example 6: If x has a normal distribution with $\mu = 5$ and $\sigma = 2$, find $P(\mu - 3\sigma < x < \mu + 3\sigma)$.

Using Table 3

- ✓ To find an area to the left of a z-value, find the area directly from the table.
- ✓ To find an area to the right of a z-value, find the area in Table 3 and subtract from 1.
- ✓ To find the area between two values of z, find the two areas in Table 3, and subtract one from the other.

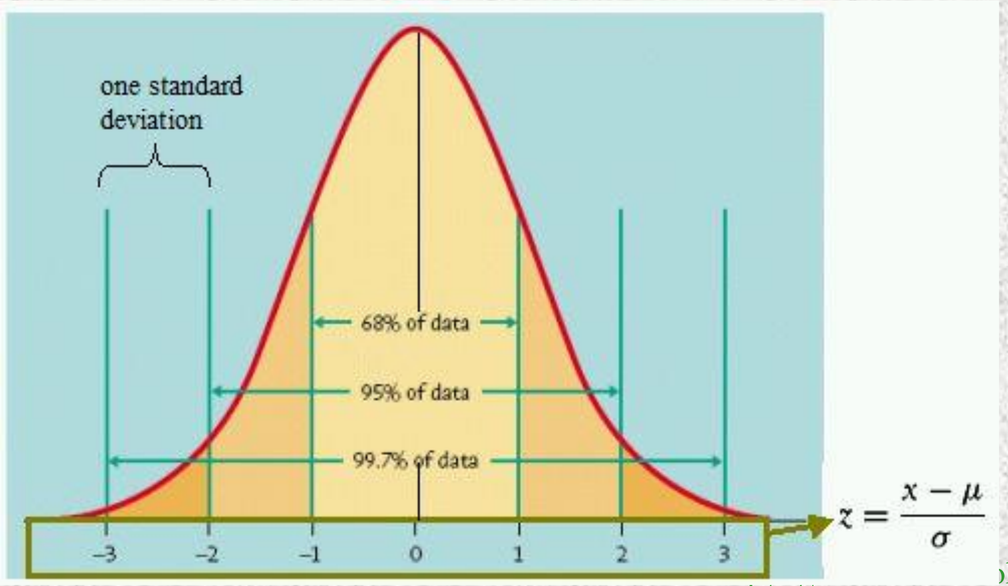
$P(-3 \leq z \leq 3)$
$= 0.9987 - 0.0013$
$= 0.9974$



Using Table 3

$P(-3 \leq z \leq 3) = 0.9974$

Approximately 99.7% of the measurements lie within 3 standard deviations of the mean.

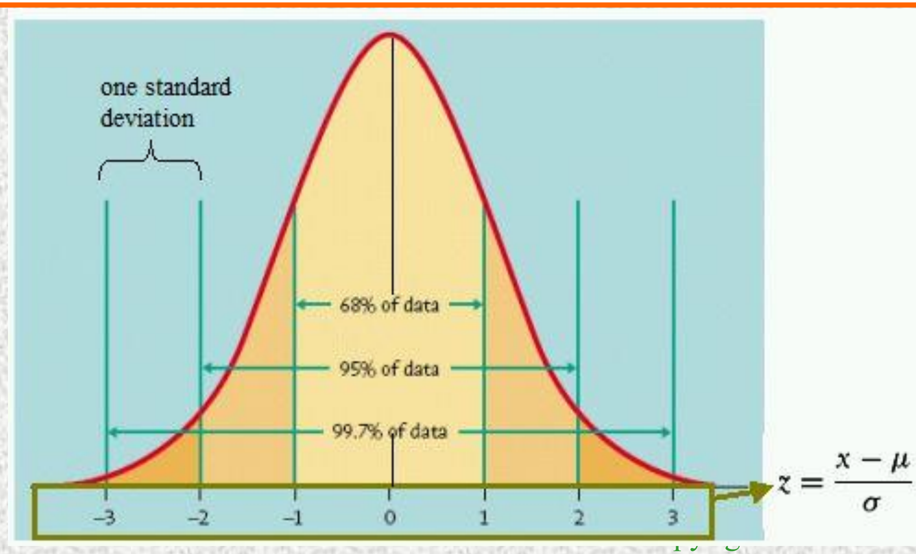


Using Table 3

$P(-2 \leq z \leq 2) = 0.9544, \quad P(-1 \leq z \leq 1) = 0.6826$

Approximately 95% of the measurements lie within 2 standard deviations of the mean.

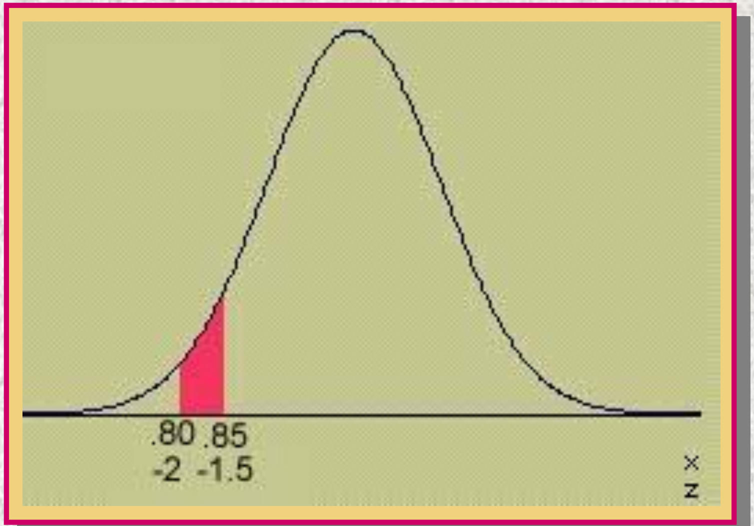
Approximately 68% of the measurements lie within 1 standard deviation of the mean.



Example 7

The weights of packages of ground beef are normally distributed with mean 1 pound and standard deviation 0.10. What is the probability that a randomly selected package weighs between 0.80 and 0.85 pounds?

$$P(0.80 \le x \le 0.85)$$
$$= P(-2 < z < -1.5)$$
$$= 0.0668 - 0.0228$$
$$= 0.0440$$



Example 8

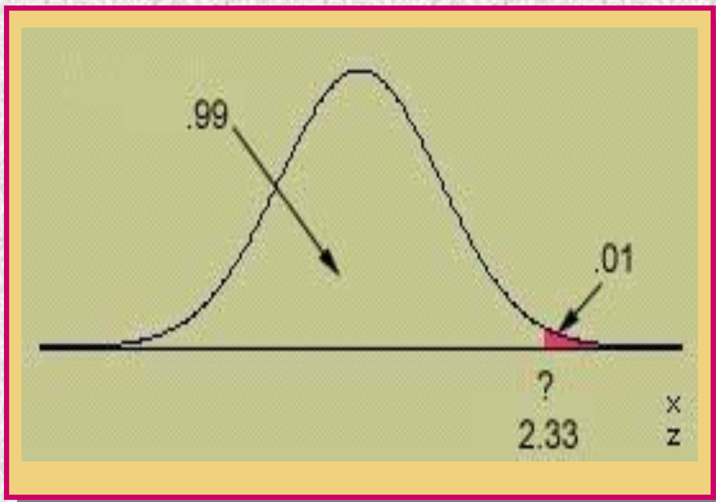
What is the weight of a package such that only 1% of all packages exceed this weight?

$$P(x > ?) = 0.01$$

$$P(z > \frac{? - 1}{.1}) = 0.01$$

$$\text{From Table 3, } \frac{? - 1}{0.1} = 2.33$$

$$? = 2.33(0.1) + 1 = 1.233$$



Class Activity

3. A normal random variable x has mean μ =1.2 and standard deviation σ = 0.15.
   a) Find the probability associated with each of the following intervals.
      i. $1.00 < x < 1.10$
      ii. $x > 1.38$
   b) Find the value c such that $P(x < c) = 0.23$.
   c) Find the 65th percentile of x.

Class Activity / Homework 7

4. **Bacteria in Drinking Water.** Suppose the numbers of a particular type of bacteria in samples of 1 ml of drinking water tend to be approximately normal distributed, with a mean of 85 and a standard deviation of 9.

   a) What is the probability that a given 1-ml sample with contain more than 100 bacteria?

   b) Find the 20^{th} percentile of number of bacteria.

Some important *z*-values

Some important *z*-values have tail areas as follows:

Tail Area:	0.005	0.01	0.025	0.05	0.10
z-Value:	2.58	2.33	1.96	1.645	1.28