Prof. Dr. Carmen Gräßle & Jannis Marquardt, M.Sc.
Institute for Partial Differential Equations

Assignment 5

Upload your solution until
Mon, 13. May 2024, 3:00 pm.

Assignment 5.1 *Difference quotients warmup* **(3 + 3 + 3 = 9 points)**

(a) Create a file `difference_quotients.py` in which you implement three functions `forward_diff_quot(f,x0,h)`, `backward_diff_quot(f,x0,h)` and `central_diff_quot(f,x0,h)`. As the names suggest, these functions shall return the approximations of the first derivative of a function f at the position x_0 with accuracy h using the forward, backward and central difference quotient.

(b) Use these functions in order to approximate the first derivative of $f(x) = \sin(x) \cdot \ln(x)$ at $x_0 = \frac{1}{2}$. Furthermore, let your program show a plot which depicts $f(x)$ as well as $T_{f,x_0}(x)$ in the interval $(0, 1]$. The function $T_{f,x_0}(x)$ denotes the tangent of $f(x)$ at x_0. Choose the function from (a) for the difference quotient from which you expect the best results. Make a suitable choice for $h > 0$ as well.

(c) Let $f(x)$ and x_0 as in (b). Compare the three methods to approximate $f'(x_0)$. Therefore, compute the value $f'(x_0)$ by hand first. Then use the methods from (a) to compute approximations for different values of $h = 10^{-5}, \ldots, 10^{-1}$. Depict the errors between the true value and the approximations of the derivatives graphically by creating a plot with h on the x-axis and plot all three errors-curves into it.
(You may compare the difference between `plot`, `semilogx`, `semilogy` and `loglog`. Which one depicts the error best?)

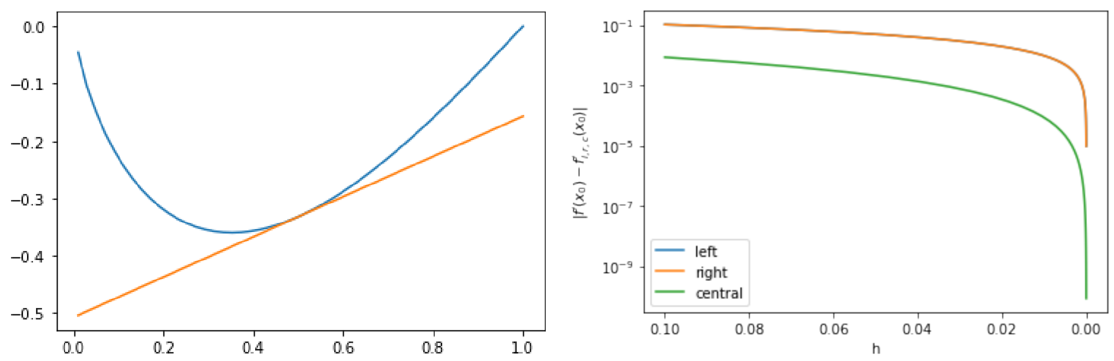

Figure 1: Reference plots for Assignment 5.1

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Assignment 5.2 *Poisson's equation* **(7 + 2 + 2 = 11 points)**

Consider the partial differential equation

$$\begin{cases} -\Delta u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = g(x) & \text{for } x \in \partial\Omega, \end{cases}$$

in the one dimensional case where $\Omega = (a, b)$. Create a file `poisson.py` in which you compute a solution of this problem numerically. In order to do so, proceed as follows:

(a) Discretise the domain into $N+1$ equally spaced points $a = x_0 < \ldots < x_N = b$. Denote u_i as the approximation of $u(x_i)$ and f_i as the approximations of $f(x_i)$ for $0 \leq i \leq N$. Substitute the second derivative in Poisson's equation by the central difference scheme for the second derivative to obtain a system of linear equations. This leads to a tridiagonal matrix equation of the form $AU = F$, where U is the vector of unknowns $U = (u_0, \ldots, u_N)^T \in \mathbb{R}^{N+1}$ and $F = (f_0, \ldots, f_N)^T \in \mathbb{R}^{N+1}$ the vector representing f.

(b) Before we can solve this system, we have to integrate the boundary conditions into the system. Alter the first and the last line of your system of equations, sucht that they represent the condition $u_0 = g(x_0)$ as well as $u_N = g(x_N)$. (You have to alter both A and F)

(c) Solve the system of equations for $a = 0$, $b = 1$, $g(x) = 1 - x$ and $f(x) = -\mu \cdot \sin(\pi x)$. Try out different stepsizes and choices of $\mu > 0$. Depict the resulting discretised functions in a common plot.

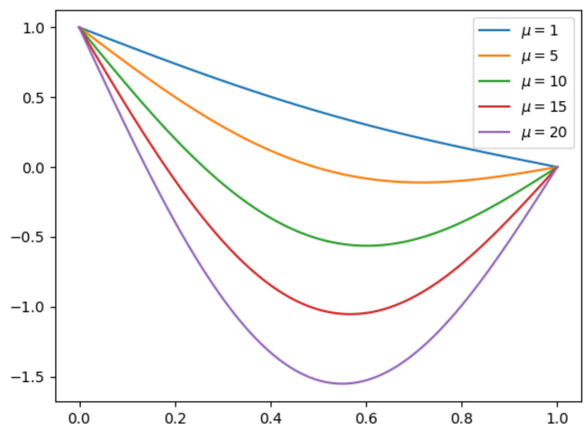

Figure 2: Reference plot for Assignment 5.2 with $N = 100$