Technische Universität Braunschweig

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Assignment 9

Upload your solution until
Mon, 17. June 2024, 3:00 pm.

Assignment 9.1 *Lorenz System* **(10 points)**

In this exercise, we want to have a look at the chaotic system

$$\begin{aligned}\frac{d}{dt}x_1(t) &= \sigma(x_2(t) - x_1(t)),\\ \frac{d}{dt}x_2(t) &= x_1(t)(\rho - x_3(t)) - x_2(t),\\ \frac{d}{dt}x_3(t) &= x_1(t)x_2(t) - \beta x_3(t)\end{aligned}$$

for positive times $t > 0$ and parameters σ, ρ and β. This system is called Lorenz system and is named after the mathematician and meteorologists Edward Lorenz. It was developed to predict flows in earth's atmosphere.
The Lorenz system is a prominent example for a system, which shows completely different behaviours for slight changes in the initial conditions ($\to$ chaos).

Create a file `lorenz.py` in which you repeat the following steps for the Crank-Nicolson method, the Euler-Heun method and the improved Euler method. Compare the three methods in the end.

- Compute a numerical solution of the system for the parameters $\sigma = 10, \rho = 28$ and $\beta = \frac{8}{3}$.
- Solve the system for each of the initial values

$$x(0) = (1,1,1)^T \quad x(0) = (1,1,1.01)^T \quad x(0) = (1,1,1.02)^T$$

  and plot the solution trajectories (i.e. the polygonal chain which approximates $x(t)$ for each initial value) in a common plot. Use different colors.
- Decide on your own for a suitable number of steps N and stepsize $h > 0$. Your plot should look similar to Figure 1.

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Assignment 9.2 *The Earth, the Moon and the numerical methods* **(10 points)**

The differential equation

$$\begin{aligned}
x'' &= x + 2y' - (1-\mu)\frac{x+\mu}{((x+\mu)^2+y^2)^{\frac{3}{2}}} - \mu\frac{x-(1-\mu)}{((x-1+\mu)^2+y^2)^{\frac{3}{2}}}\\
y'' &= y - 2x' - (1-\mu)\frac{y}{((x+\mu)^2+y^2)^{\frac{3}{2}}} - \mu\frac{y}{((x-1+\mu)^2+y^2)^{\frac{3}{2}}}\\
x(0) &= 1.2\\
y(0) &= 0\\
x'(0) &= 0\\
y'(0) &= -1.049357509830350
\end{aligned}$$

describes the periodic trajectory of a satellite in a two dimensional scaled model of the Earth-Moon system. The constant $\mu = \frac{1}{82.45}$ denotes the relative mass of the Moon. For a full circulation, the satellite requires $T = 6.19$ seconds. The goal of this assignment is the numerical implementation of the system with the

(i) Explicit Euler method,

(ii) Implicit Euler method,

(iii) Crank-Nicolson method,

(iv) Improved Euler method

for the constant stepsize $h = 0.001$. Follow these steps while doing so:

- Familiarise yourself with the code provided in the `earth_moon.py` file from StudIp. It gives you a coordinate system in which the Earth, the Moon, and some reference points (to verify that your code outputs what it is supposed to) are already plotted.
- First, solve the differential equation system above using a built-in Python solver (e.g., `scipy.integrate.odeint`). Plot the solution in the given coordinate system.
- Now calculate numerical solutions using the methods (i) - (iv). Plot the solution trajectories in the same coordinate system as before. You may find a reference solution Figure 2

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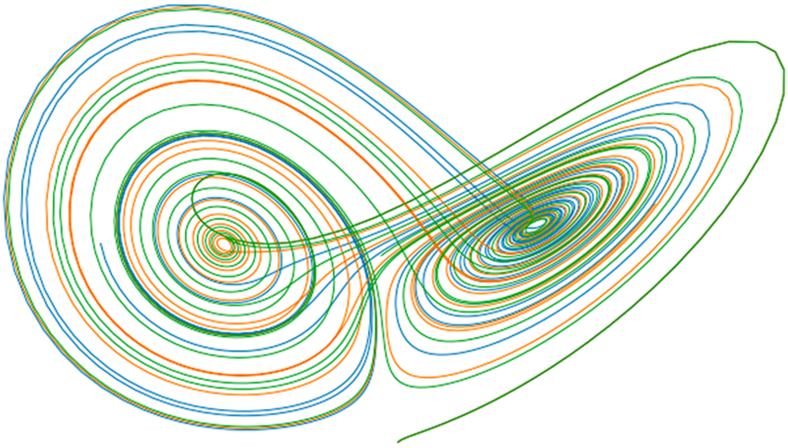

Figure 1: Solution of the Lorenz system for one method and different initial conditions

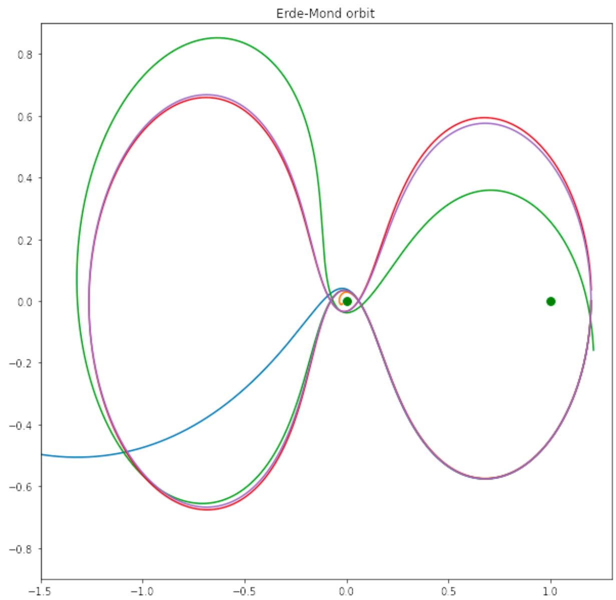

Figure 2: Plot of the Earth-Moon system.