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Institute for Partial Differential Equations

Assignment 6

Upload your solution until
Mon, 27. May 2024, 3:00 pm.

Assignment 6.1 *Newton's method* **(6 + 5 + 3 + 6 = 20 points)**

Create a file `newton.py` and implement the Newton's method to solve the equation $F(\mathbf{x}) = \mathbf{0}$ with $\mathbf{x} \in \mathbb{R}^d$ and $F : \mathbb{R}^d \to \mathbb{R}^n$. Proceed as follows:

(a) Write a function `jacobian(func,x0)`, which returns the approximation (using difference quotients) of the Jacobian matrix of a function `func` at `x0`.

(b) Write a function `newton(func,x0)`, which returns a matrix $X \in \mathbb{R}^{d\times m}$, where the columns of X are the vectors which occur after the repeated application of the Newton iteration rule, i.e. $X = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots)$. The Newton iterations should stop after 50 iterations or earlier if the relative error bound

$$\frac{\|\mathbf{x}_{i+1} - \mathbf{x}_i\|_2}{\|\mathbf{x}_i\|_2} \leq 10^{-6}$$

is reached. Keep in mind, that the computation of the inverse of a matrix is numerically unstable und should therefore be avoided.

(c) Use the Newton's method to solve $F(\mathbf{x}) = \mathbf{0}$ for an initial guess $\mathbf{x}_0 = \mathbf{0}$ and

$$F(\mathbf{x}) = F(x_1, x_2) = \begin{pmatrix} \exp(-\exp(-(x_1 + x_2))) - x_2(1 + x_1^2) \\ x_1 \cos(x_2) + x_2 \sin(x_1) - \frac{1}{2} \end{pmatrix}.$$

Plot the path which Newton's method takes in order to reach a zero of F on a suitable clipping of $\mathbb{R}^2$. Connect the single waypoints (vectors) by straight lines (cf. Figure 1).

(d) Use Newton's method to find an extremal point of

$$G(\mathbf{x}) = G(x, y) = e^{-\frac{1}{3}x^3 + x - y^2}.$$

Classify the extremal point using the second derivative (Hessian matrix) of $G(\mathbf{x})$. Do not compute any of the derivative of F by hand! Notice, that we haven't computed all difference schemes which you need for the computation of the Hessian matrix.

Continue on next page ⇝

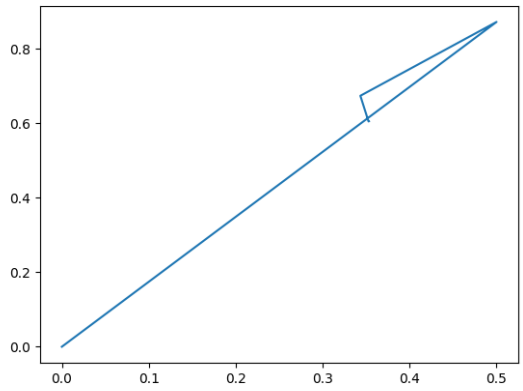

Figure 1: Path which Newton's method takes in Assignment 6.1 (c)

Organisational matters

The deadline of this assignment is the 27. May 2024 due to the excursion week from 19. May until 26. May. Please have in mind, that we will spend the first 45 minutes of the exercise class on 27. May with the mock exam (refer to the Stud.Ip announcement for further information).