Model Predictive Control: Mini-Project

In this project, you will develop an MPC controller to fly a rocket prototype.

The project is worth 40% of your final grade and is due on January 31st

It is strongly recommended that you complete the project before the end of term, so as to not interfere with the exam period.

Report and handing-in instructions

- Group sign up and report hand-in is via Moodle.
- You can do the project in groups of one, two or three.
- Include everyone's name and SCIPER on the title page of your project report.
- When you have completed the project, hand in one report (pdf) per group and your Matlab code (zip).
- Report:
  - Your report should contain headings according to the **Deliverables** listed below in the project description.
  - **You will be graded on the Deliverables**, and not on the Todos.
  - The report should be written in English.
  - Explain what you're doing and why for each deliverable, but don't be excessive. The entire report should be less than 20 pages.
- Code:
  - Include a directory for each deliverable containing all the m-files to run the deliverable.
  - Create a file in each directory `Deliverable_xxx.m` which can be run to produce all the the figures for the deliverable.
  - Compress all the code / directories into a single zip file for submission.

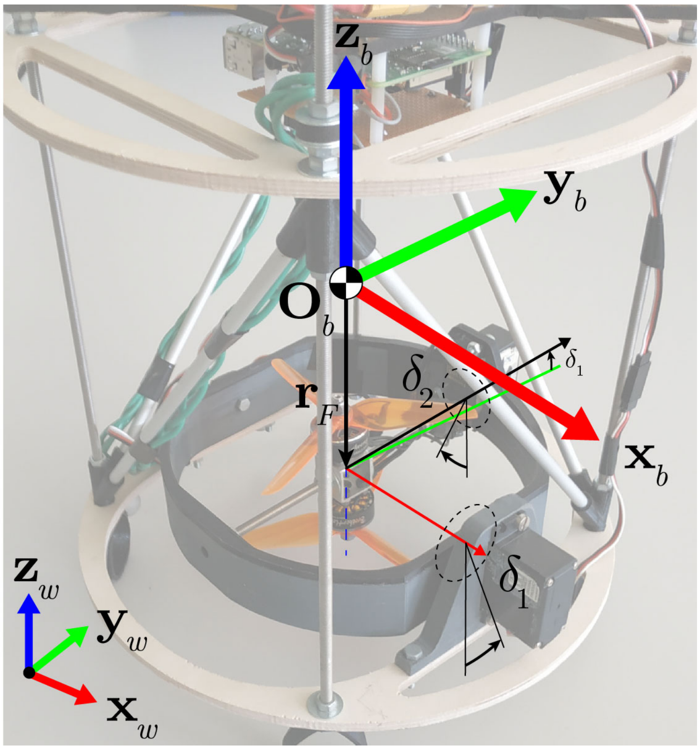
Before you start

- Make sure you have installed YALMIP, MPT3, Gurobi and Casadi according to the course exercise setup instructions on Moodle.
- Download and unpack the file `rocket_project.zip` from Moodle.
- Run `rocket = Rocket(1/20);` If this executes correctly, then your setup should be ready to go.

Part 1 | System Dynamics

Building a model of the system dynamics from physical principles is a crucial step in the development of an MPC controller and is a significant part of the task in practice. However, as this process is out of the scope of this course, you are not required to model the rocket by yourself and instead, we provide you with a nonlinear model.

On the way towards thrust vector control for combustion engine rockets, we study a small-scale prototype where the rocket engine is replaced by high-performance drone racing propellers as depicted in the picture below. The propellers counter-rotate such that their torques cancel each other out in stationary flight. The propeller pair is mounted on a gimbal and can be tilted by two servos.



System Definition In order to derive a nonlinear model consider two reference frames. The first one is the body frame (subscript b) with the origin O_b attached to the center of mass of the rocket (see picture). The second one is the world frame (subscript w) which is a fixed, inertial frame. We are going to derive a 12-state description of the system with the state vector

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\omega}^T & \boldsymbol{\varphi}^T & \mathbf{v}^T & \mathbf{p}^T \end{bmatrix}^T, \quad [\mathbf{x}] = \begin{bmatrix} \text{rad/s} & \text{rad} & \text{m/s} & \text{m} \end{bmatrix}^T$$

where $\boldsymbol{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$ are the angular velocities about the body axes. The Euler angles $\boldsymbol{\varphi} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^T$ represent the attitude of the body frame with respect to the world frame. The rotation from world to body frame is obtained by three consecutive rotations about the body axes: 1. α about $\mathbf{x}_b$, 2. β about $\mathbf{y}_b$, 3. γ about $\mathbf{z}_b$. The velocity and position, $\mathbf{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T$ and $\mathbf{p} = \begin{bmatrix} x & y & z \end{bmatrix}^T$, are expressed in the world frame, i.e., $\mathbf{v} = \dot{\mathbf{p}}$.

The input vector of the model is

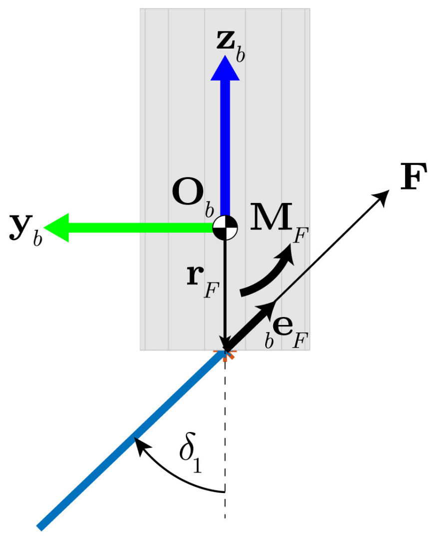
$$\mathbf{u} = \begin{bmatrix} \delta_1 & \delta_2 & P_{avg} & P_{diff} \end{bmatrix}^T, \quad [\mathbf{u}] = \begin{bmatrix} \text{rad} & \text{rad} & \% & \% \end{bmatrix}^T$$

where δ_1 and δ_2 are the deflection angles of servo 1 (about $\mathbf{x}_b$) and servo 2 (about rotated $\mathbf{y}_b$), respectively, up to $\pm 15°$ ($= 0.26\,\mathrm{rad}$).

Although the physical controls are the power settings of motor 1 and 2, we use a convenient abstraction that corresponds to the resulting behavior of the motor pair. $P_{avg} = (P_1 + P_2)/2$ is the average throttle and $P_{diff} = P_2 - P_1$ is the throttle difference between the motors. To hold the throttle of each individual motor within $[0, 100\%]$ while P_{diff} might be up to $\pm 20\%$, we have to limit the valid range for P_{avg} to $[20\%, 80\%]$.

Forces and Moments Simplified, the combined propellers produce a thrust force of magnitude $F(P_{avg})$ and a differential moment of magnitude $M_\Delta(P_{diff})$. They apply along/about the motor axis that is tilted by servo 1 and 2 with deflection angles δ_1, δ_2:

$$_b\mathbf{e}_F(\delta_1, \delta_2) = \begin{bmatrix} \sin\delta_2 \\ -\sin\delta_1\cos\delta_2 \\ \cos\delta_1\cos\delta_2 \end{bmatrix} \tag{1}$$



When the axis is tilted, the thrust vector introduces another moment about the center of mass, $_b\mathbf{M}_F = \mathbf{r}_F \times {_b\mathbf{F}}$, so that the resulting force and moment acting on the body are, expressed in body frame:

$$_b\mathbf{F} = F \cdot {_b\mathbf{e}_F} \tag{2}$$

$$_b\mathbf{M} = M_\Delta \cdot {_b\mathbf{e}_F} + {_b\mathbf{M}_F} \tag{3}$$

where $\times$ is the cross product.

Linear and Angular Dynamics The acceleration of the center of mass in the inertial (world) frame is given by

$$\dot{\mathbf{v}} = \mathbf{T}_{wb} \cdot {}_b\mathbf{F}/m - \mathbf{g} \tag{4}$$

where $\mathbf{T}_{wb}(\boldsymbol{\varphi})$ is the direction cosine matrix that transforms a vector from body to world frame (i.e., ${}_w\mathbf{F} = \mathbf{T}_{wb} \cdot {}_b\mathbf{F}$), m is the body mass (`rocket.mass`), and $\mathbf{g} = \begin{bmatrix} 0 & 0 & g \end{bmatrix}^T$ is the gravitational acceleration.

The angular dynamics in the body frame are given by

$$\dot{\boldsymbol{\omega}} = \mathbf{J}^{-1} \left(-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + {}_b\mathbf{M} \right), \tag{5}$$

where $\mathbf{J}$ is the inertia matrix of the vehicle .

Attitude Kinematics The rate of change of the Euler angles is a function of the attitude $\boldsymbol{\varphi} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^T$ expressing the rotating body frame and the angular velocity in the body frame according to the kinematic differential equation

$$\dot{\boldsymbol{\varphi}} = \begin{bmatrix} 1 & \sin(\alpha) * \tan(\beta) & -\cos(\alpha) * \tan(\beta) \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha)/\cos(\beta) & \cos(\alpha)/\cos(\beta) \end{bmatrix} \boldsymbol{\omega}. \tag{6}$$

When we put all of this together, we get the dynamic equations for the rocket

$$\dot{x} = f(x, u)$$

which have been implemented in the function `f` in the Matlab class `Rocket.m`, which is in the directory `src`.

Todo 1.1 | Study the functions `f` and `getForceAndMomentFromThrust` in the Matlab `Rocket` class to confirm that it implements the dynamics of the system as described above.

To evaluate the functions, you can call them independently:

```
Ts = 1/20;
rocket = Rocket(Ts);
u = [d1, d2, Pavg, Pdiff]'; % (Assign appropriately)
[b_F, b_M] = rocket.getForceAndMomentFromThrust(u)

x = [w,  phi,  v,  p]'; % (Assign appropriately)
x_dot = rocket.f(x, u)
```

Todo 1.2 | Simulate the rocket with various step inputs to confirm that the dynamics respond as you expect.

To simulate the nonlinear model for two seconds starting from $\mathbf{x}_0$ with a constant input, you can use:

```
rocket = Rocket(Ts);
Tf = 2.0;   % Time to simulate for

x0 = [deg2rad([2 -2 0, -2 2 0]),  0 0 0,  0 0 0]'; % (w, phi, v, p) Initial state
u  = [deg2rad([2 0]),  60,   0  ]';                % (d1 d2 Pavg Pdiff) Constant input
```

```
[T, X, U] = rocket.simulate_f(x0, Tf, u);        % Simulate nonlinear dynamics f

rocket.anim_rate = 1.0;
rocket.vis(T, X, U);         % Trajectory visualization at 1.0x real-time
```

A few things to try to see if the rocket is behaving as you think it should. Find input **u** that will cause the rocket to:

- Ascend/descend vertically without tipping over.
- Rotate about its body x/y/z axis.
- Fly along the x/y/z axis.
- Hover in space.

Part 2 | Linearization

In the first part of the project, we are going to control a linearized version of the rocket.

Todo 2.1 | Use the following code to generate a trimmed[1] and linearized version of the rocket:

```
rocket = Rocket(Ts);

[xs, us] = rocket.trim() % Compute steady-state for which 0 = f(xs,us)
sys = rocket.linearize(xs, us)     % Linearize the nonlinear model about trim point
```

Go through the functions `trim` and `linearize` to see how they work.

Note that we have named all the states in the linearized model. Type `sys` and you will see the ordering of the states.

> **An aside on trimming and linearization**
>
> We have computed a trim point (x_s, u_s), which is a steady-state state and input pair $0 = f(x_s, u_s)$. We have then linearized our system around this point, $\dot{x} \approxeq A(x - x_s) + B(u - u_s)$. However, when we design our controllers, we do so for the linear system $\dot{x} = Ax + Bu$. This means that if our MPC controller applies an input $u^\star$, that the true input applied to the system is $u = u^\star + u_s$, and therefore we will need to take the trim value into account when determining the constraints for our MPC controller.
>
> Note that the `simulate` function will add the trim to your MPC controller's input when you provide it with a linear model.

Study the resulting **A**, **B**, **C** and **D** matrices until you recognize that the linearized system about the trim point can be broken into four independent/non-interacting systems.

Deliverable 2.1 | Explain from an intuitive physical / mechanical perspective, why this separation into independent subsystems occurs.

Todo 2.2 | Compute the four independent systems above using the following command

```
[sys_x, sys_y, sys_z, sys_roll] = rocket.decompose(sys, xs, us)
```

Four models are produced:

`sys_x`	Thrust vector angle δ_2 to position x. The system has four states: ω_y, β, v_x, x.
`sys_y`	Thrust vector angle δ_1 to position y. The system has four states: ω_x, α, v_y, y.
`sys_z`	Average throttle P_{avg} to height z. The system has two states: v_z and z.
`sys_roll`	Differential throttle P_{diff} to roll angle γ. The system has two states: ω_z and γ.

Note that these are all **continuous time** models.

[1] Trimming a system $\dot{x} = f(x, u)$ means to find a state and input pair $\bar{x}, \bar{u}$ such that $f(\bar{x}, \bar{u}) = 0$.

Part 3 |

Design MPC Controllers for Each Sub-System

For each of the dimensions x, y, z and $roll$, your goal is to design a recursively feasible, stabilizing MPC controller that can track step references.

Throughout this section, we use a sampling period of $T_s = 1/20$ seconds. The continuous time models produced in the previous section must be discretized using `sys_d = c2d(sys, Ts)`. This is done for you when using the provided `MPC_Control_*` template files.

Constraints

Because our linearization is approximate, we must place constraints on the maximum angle that the rocket can take so that our approximation is valid:

$$|\alpha| \le 5^\circ = 0.0873 \text{ rad}$$
$$|\beta| \le 5^\circ = 0.0873 \text{ rad}$$

In addition to the mechanical input constraints specified in Part 1, another requirement comes up from an engineering perspective. Experiments with the prototype have shown that the rocket descends too quickly when less than 50% average throttle is given. For safety reasons, we want to limit the downward acceleration that can occur to the rocket, and we therefore require a minimum average throttle of 50% at all times:

$$50\% \le P_{avg} \le 80\%$$

Design MPC Regulators

Todo 3.1 | Design four MPC controllers for x, y, z and $roll$ with the following properties:

- Recursive satisfaction of the input and angle constraints.
- Stabilization of the system to the origin (i.e., all states equal to zero).
- Settling time of around eight seconds when starting stationary at five meters from the origin (for x, y and z) or stationary at 45° for roll.

To help you design the controllers, we've created four files: • `MPC_Control_x.m` • `MPC_Control_y.m` • `MPC_Control_z.m` • `MPC_Control_roll.m` which you'll find in the `templates` directory. Copy these into the `Deliverable_3_1` directory and then make your changes. Your job is to fill in the function `setup_controller` in each file.

You can then evaluate your control via:

```
Ts       = 1/20; % Sample time
rocket   = Rocket(Ts);
[xs, us] = rocket.trim();
sys      = rocket.linearize(xs, us);
[sys_x, sys_y, sys_z, sys_roll] = rocket.decompose(sys, xs, us);

% Design MPC controller
```

```
H = ?; % Horizon length in seconds
mpc_x = MPC_Control_x(sys_x, Ts, H);

% Get control input
ux = mpc_x.get_u(x)
```

You can use the `simulate` function to simulate a continuous-time linear system that you have obtained from the `Rocket` class. To have the control law evaluated during closed-loop simulation, hand it over as a function handle, and give a zero reference for regulation. The function `plotvis_sub` plots the states and inputs of the sub-system and visualizes its trajectory in 3D by setting all other sub states and inputs to the trim point $(\mathbf{x}_s, \mathbf{u}_s)$.

```
[T, X_sub, U_sub] = rocket.simulate(sys_x, x0, Tf, @mpc_x.get_u, 0);
ph = rocket.plotvis_sub(T, X_sub, U_sub, sys_x, xs, us);
```

Where `x0` here is the state of the particular subsystem being simulated.

Note: If you get warnings that the physical limits of your inputs are violated during simulation, then your controller is outputing infeasible inputs and it's incorrect.

Deliverable 3.1 |

- Explanation of design procedure that ensures recursive constraint satisfaction.
- Explanation of choice of tuning parameters. (e.g., $\mathbf{Q}$, $\mathbf{R}$, H, terminal components).
- Plot of terminal invariant set for each of the dimensions, and explanation of how they were designed and tuning parameters used.
  Hint: If X_f is higher than two dimensions, you can plot its projections with:

  ```
  Xf.projection(1:2).plot();
  Xf.projection(2:3).plot();
  Xf.projection(3:4).plot();
  ```

- Plot for each dimension starting stationary at five meters from the origin (for x, y and z) or stationary at 45° for roll.
- Matlab code for the four controllers, and code to produce the plots in the previous step.

Design MPC Tracking Controllers

Todo 3.2 | Extend your controllers so that they can track constant references

For x, y and z the reference is a position, and for $roll$ it is an angle in radians.

You may drop the requirement of invariance here (i.e., no terminal set is required).[2]

To implement your controllers, modify your four controllers from the previous section,
• `MPC_Control_x.m` • `MPC_Control_y.m` • `MPC_Control_z.m` • `MPC_Control_roll.m` and fill in the function `setup_steady_state_target` in each one.

[2] It is possible to use a terminal set for tracking here by noticing that there are no constraints on the position of the rocket and that all steady states are zero for all non-position states. This means that a shifted version of the terminal invariant set for regulation is still invariant. i.e., if $\mathcal{X}_f = \{x \,|\, Gx \leq g\}$ is invariant, then so is $\mathcal{X}_f + x_s = \{x \,|\, G(x - x_s) \leq g\} = \{x \,|\, Gx \leq g + Gx_s\}$.

Make sure you're editing your code in the directory for Deliverable 3.2. Don't overwrite Deliverable 3.1

You can now get your control input via:

```
ux = mpc_x.get_u(x, x_position_reference)
```

For showing the reference in the plots, you can enter it to `plotvis_sub`:

```
[T, X_sub, U_sub] = rocket.simulate(sys_x, x0, Tf, @mpc_x.get_u, x_ref);
ph = rocket.plotvis_sub(T, X_sub, U_sub, sys_x, xs, us, x_ref);
```

Deliverable 3.2 |

- Explanation of your design procedure and choice of tuning parameters.
- Plot for each dimension of the system starting at the origin and tracking a reference to -5 meters from the origin (for x, y and z) and to 45° for roll.
- Matlab code for the four controllers, and code to produce the plots in the previous step.

Part 4 | **Simulation with Nonlinear Rocket**

In this section, you will use your controllers to have the nonlinear rocket track a given path.

Todo 4.1 | Simulate the full nonlinear system with your four controllers from the origin as initial state.

You can simulate your system to track the given reference path with the command:

```
% Merge four sub-system controllers into one full-system controller
mpc = rocket.merge_lin_controllers(xs, us, mpc_x, mpc_y, mpc_z, mpc_roll);

% Setup reference function
Tf = 30;
ref = @(t_, x_) rocket.MPC_ref(t_, Tf);

x0 = zeros(12,1);
[T, X, U, Ref] = rocket.simulate_f(x0, Tf, mpc, ref);

% Plot pose
rocket.anim_rate = 1;  % Increase this to make the animation faster
                       % anim_rate = 4 is about right for printing in the report
ph = rocket.plotvis(T, X, U, Ref);
ph.fig.Name = 'Nonlin. sim..........'    % Set a figure title
```

Deliverable 4.1 | A plot of your controllers successfully tracking the MPC reference path and reference roll angle. If your tracking performance is not good, explain how you adapt your tuning to improve it.

Part 5 | **Offset-Free Tracking**

In this section, we assume that the mass of the rocket changes, and we want to extend your z-controller to compensate.

The dynamics of the system in the $z-$direction is now

$$\mathbf{x}^+ = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}d$$

where d is a constant, unknown disturbance. Your goal is to update your controller to reject this disturbance and track setpoint references with no offset.

Todo 5.1 | For the z dimension only, design an offset-free tracking controller.

- Implement the `setup_estimator` function in the `MPC_Control_z.m` file
- Update the functions `setup_controller` and `setup_steady_state_target` in `MPC_Control_z.m` to provide offset-free tracking.

In this section, you will need to use an observer to estimate the offset and the state of the system and therefore, we can no longer ensure constraint satisfaction. So for this section, you can drop the terminal set.

You can simulate your system with a mass mismatch by manipulating the `rocket.mass` property **after** having created the controller, i.e., just before simulating.

```
Ts = 1/20;
rocket = Rocket(Ts);
[xs, us] = rocket.trim();
sys = rocket.linearize(xs, us);
[sys_x, sys_y, sys_z, sys_roll] = rocket.decompose(sys, xs, us);

mpc_x = MPC_Control_x(sys_x, Ts, H);
...

rocket.mass = 1.783; % Manipulate mass for simulation
[T, X, U, Ref] = rocket.simulate_f(x0, Tf, mpc, ref);
```

Once you have written the `setup_estimator` function, you can test it in simulation by replacing the function `rocket.simulate_f` with `rocket.simulate_f_est_z`. In the z direction, the controller will now act based on the state estimates from the observer. You can obtain the estimates of the z states from the corresponding columns of the `Z_hat` output. The last row is the disturbance estimate d.

```
[T, X, U, Ref, Z_hat] = rocket.simulate_f_est_z(x0, Tf, mpc, ref, mpc_z, sys_z);
```

Deliverable 5.1 |

- Explanation of your design procedure and choice of tuning parameters.
- Plot showing the impact of changing the mass on your original controller from Part 4, and then another plot showing that your controller now achieves offset-free tracking.
- Matlab code for your controllers, and code to produce the plots in the previous step.

Part 6 | Nonlinear MPC

Todo 6.1 | Develop a nonlinear MPC controller for the rocket using CASADI. Your controller should take the full state as input, and provide four input commands (i.e., we do not decompose the rocket into four sub-systems here).

Use the template code `NMPC_Control.m`. Complete the "YOUR CODE HERE" block to setup your controller. The rest of the code is just a wrapper to solve the optimization problem efficiently.

You can then simulate and plot the result of your controller with the following code:

```
Ts = 1/10;   % Note that we choose a larger Ts here to speed up the simulation
rocket = Rocket(Ts);

H = ???
nmpc = NMPC_Control(rocket, H);

[T, X, U, Ref] = rocket.simulate_f(x0, Tf, nmpc, ref);
```

Hint: The computation time here can get very slow. If this is the case for your setup, you may want to try a shorter horizon than you had for the linear case. Note that we've reduced the sample period to 0.1s in this part to improve computation speed.

Hint: Because it's likely that you'll want a shorter horizon here than ideal in order to keep the computation time under control, you may a terminal cost very helpful. A common approximate terminal cost is to linearize your system and compute a terminal cost based on this (don't forget to discretize).

Hint: As in our NMPC exercise, you can evaluate the dynamics of the rocket using CASADI variables `x` and `u` via the call `rocket.f(x,u)`

Deliverable 6.1 |

- Explanation of your design procedure and choice of tuning parameters.
- Discuss the pros and cons of your nonlinear controller vs the linear ones you developed earlier.
- Plots showing the performance of your controller.
- Matlab code for your controllers, and code to produce the plots in the previous step.