Pose Solution in Geometric Algebra $\mathcal{G}_{3,0,1}$

J. Russell Carpenter*
NASA Goddard Space Flight Center, Greenbelt, MD, 20771

Notation Conventions: The typeface for a variable distinguishes what kind of object it represents, as follows: $\mathcal{G}_{3,0,1}$ objects are upper-case bold, ordinary column vectors are lower-case bold, matrices are upper-case italic.

I. Introduction

Geometric algebra is both an approach to generalizing algebra over the reals to algebras over arbitrary combinations of real, imaginary, and dual numbers, and an approach to definitively associating such numbers with geometrical objects, such as points, lines, planes, and volumes, and their transformations. The former has its roots in the work of Clifford, and the latter in the works of Hamilton and Grassmann[1]. Geometric algebra may also be viewed as a form of compact notation for multilinear operations that are often performed using tensor algebras[2].

This work describes the pose solution of a generalized version of Wahba's Problem, using Lengyel's conventions[3] for expressing the geometric algebra $\mathcal{G}_{3,0,1}$, and the null-space method of Perwass[4], as well as a two-step method inspired by a classic paper by Walker, Shao, and Volz that used dual quaternions[5].

II. Preliminaries Concerning $\mathcal{G}_{3,0,1}$

Geometric algebras are essentially Clifford algebras in that they consist of a scalar element, denoted by $\mathbf{e}_0$, along with additive basis elements $\mathbf{e}_i$, which multiply with themselves according to the signature of the algebra to produce either $+1\mathbf{e}_0$, $-1\mathbf{e}_0$, or $0\mathbf{e}_0$, thus supporting aggregations of real, imaginary, and dual numbers respectively. The notation $\mathcal{G}_{3,0,1}$ denotes an algebra with three basis elements, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, that multiply with themselves to produce $+1\mathbf{e}_0$, zero elements that multiply to produce $-1\mathbf{e}_0$, and one basis element, $\mathbf{e}_4$, that multiples with itself to produce $0\mathbf{e}_0$. When two basis elements multiply with each other, e.g. $\mathbf{e}_1\mathbf{e}_2$ then they create a new basis element, $\mathbf{e}_{12}$, with the property that $\mathbf{e}_{12} = -\mathbf{e}_{21}$. The number of subscripts of a basis element is its grade; scalars and vectors are thus grade-zero and grade-one objects; the terms bivector and trivector refer to grade-two and grade-three objects; and an object of arbitrary or mixed grades is a multivector. The product of all the basis elements, $\Pi_i \mathbf{e}_i$, behaves as a sort of scalar, and is called the pseudoscalar. Thus, in $\mathcal{G}_{3,0,1}$, a general multivector, $\boldsymbol{A}$, is written as

$$\begin{aligned}
\boldsymbol{A} = {} & \alpha_0\mathbf{e}_0 \quad \text{(scalar)} \\
& + \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 + \alpha_4\mathbf{e}_4 \quad \text{(vector)} \\
& + \alpha_{23}\mathbf{e}_{23} + \alpha_{31}\mathbf{e}_{31} + \alpha_{12}\mathbf{e}_{12} + \alpha_{43}\mathbf{e}_{43} + \alpha_{42}\mathbf{e}_{42} + \alpha_{41}\mathbf{e}_{41} \quad \text{(bivector)} \\
& + \alpha_{321}\mathbf{e}_{321} + \alpha_{412}\mathbf{e}_{412} + \alpha_{431}\mathbf{e}_{431} + \alpha_{423}\mathbf{e}_{423} \quad \text{(trivector)} \\
& + \alpha_{1234}\mathbf{e}_{1234} \quad \text{(pseudoscalar)}
\end{aligned} \tag{1}$$

It is often convenient to view a multivector in terms of a set of dual basis vectors, in which the identities of the various grades appear reversed:

$$\begin{aligned}
\boldsymbol{A} = {} & \alpha_0\boldsymbol{\varepsilon}_{1234} \quad \text{(pseudoscalar)} \\
& + \alpha_1\boldsymbol{\varepsilon}_{423} + \alpha_2\boldsymbol{\varepsilon}_{431} + \alpha_3\boldsymbol{\varepsilon}_{412} + \alpha_4\boldsymbol{\varepsilon}_{321} \quad \text{(trivector)} \\
& + \alpha_{23}\boldsymbol{\varepsilon}_{41} + \alpha_{31}\boldsymbol{\varepsilon}_{42} + \alpha_{12}\boldsymbol{\varepsilon}_{43} + \alpha_{43}\boldsymbol{\varepsilon}_{12} + \alpha_{42}\boldsymbol{\varepsilon}_{31} + \alpha_{41}\boldsymbol{\varepsilon}_{23} \quad \text{(bivector)} \\
& + \alpha_{321}\boldsymbol{\varepsilon}_4 + \alpha_{412}\boldsymbol{\varepsilon}_3 + \alpha_{431}\boldsymbol{\varepsilon}_2 + \alpha_{423}\boldsymbol{\varepsilon}_1 \quad \text{(vector)} \\
& + \alpha_{1234}\boldsymbol{\varepsilon}_0 \quad \text{(scalar)}
\end{aligned} \tag{2}$$

When working with the dual basis vectors, it can also be convenient to utilize a dual to the product operation, such that under this dual product, dual basis vectors multiply with themselves according to the same rules as basis elements. Thus, under the dual product in $\mathcal{G}_{3,0,1}$, three dual basis elements, $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3$ multiply with themselves to produce $+1\boldsymbol{\varepsilon}_0$, zero elements that multiply to produce $-1\boldsymbol{\varepsilon}_0$, and one element, $\boldsymbol{\varepsilon}_4$, that multiplies with itself to produce $0\boldsymbol{\varepsilon}_0$.

*Deputy Project Manager/Technical, Space Science Mission Operations Project, 8800 Greenbelt Rd/Code 444, Fellow, AIAA, AAS.

The algebra $\mathcal{G}_{3,0,1}$ is of particular interest because by associating $\mathbf{e}_4$ (or equivalently, $\boldsymbol{\varepsilon}_{321}$), with the direction of infinity, one obtains homogeneous projective coordinates, where the hyperplane located along $\mathbf{e}_4$ (or $\boldsymbol{\varepsilon}_{321}$) with the coordinate of one is ordinary three-space. In this context, one may associate vectors with points, bivectors with lines, and trivectors with planes:

$$\boldsymbol{P} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4 \quad \text{(point; } p_w = 1 \text{ is a point in ordinary three-space)} \tag{3}$$

$$\begin{aligned}\boldsymbol{L} &= m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12} + v_z\mathbf{e}_{43} + v_y\mathbf{e}_{42} + v_x\mathbf{e}_{41} \quad \text{(line, with direction } \boldsymbol{v} \text{ and moment } \boldsymbol{m}\text{)} \\ &= m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12} + v_z\boldsymbol{\varepsilon}_{12} + v_y\boldsymbol{\varepsilon}_{31} + v_x\boldsymbol{\varepsilon}_{23}\end{aligned} \tag{4}$$

$$\begin{aligned}\boldsymbol{F} &= f_w\mathbf{e}_{321} + f_z\mathbf{e}_{412} + f_y\mathbf{e}_{431} + f_x\mathbf{e}_{423} \quad \text{(plane, with normal vector } [f_x, f_y, f_z] \text{ and origin distance } f_w\text{)} \\ &= f_w\boldsymbol{\varepsilon}_4 + f_z\boldsymbol{\varepsilon}_3 + f_y\boldsymbol{\varepsilon}_2 + f_x\boldsymbol{\varepsilon}_1\end{aligned} \tag{5}$$

Or, in the dual basis, one may prefer to view vectors as planes, etc.

Multivectors in $\mathcal{G}_{3,0,1}$ may also encapsulate dual quaternions $\boldsymbol{q} = \boldsymbol{r} + \epsilon\boldsymbol{s}$:

$$\begin{aligned}\boldsymbol{Q} &= s_w\mathbf{e}_0 + s_x\mathbf{e}_{23} + s_y\mathbf{e}_{31} + s_z\mathbf{e}_{12} + r_z\mathbf{e}_{43} + r_y\mathbf{e}_{42} + r_x\mathbf{e}_{41} + r_w\mathbf{e}_{1234} \\ &= s_w\boldsymbol{\varepsilon}_{1234} + s_x\boldsymbol{\varepsilon}_{41} + s_y\boldsymbol{\varepsilon}_{42} + s_z\boldsymbol{\varepsilon}_{43} + r_z\boldsymbol{\varepsilon}_{12} + r_y\boldsymbol{\varepsilon}_{31} + r_x\boldsymbol{\varepsilon}_{23} + r_w\boldsymbol{\varepsilon}_0\end{aligned} \tag{6}$$

III. Pose in $\mathcal{G}_{3,0,1}$

In $\mathcal{G}_{3,0,1}$, an object called a motor, which is a generalization of a dual quaternion, when unitized performs the screw transformation of an object $\boldsymbol{M}^i$, which could be a point, line, plane, or another motor, into the object $\boldsymbol{N}^i$, via

$$\boldsymbol{N}^i = \boldsymbol{Q}\boldsymbol{M}^i\underset{\sim}{\boldsymbol{Q}} \tag{7}$$

where juxtaposition indicates the dual geometric product, $\boldsymbol{Q}$ is the motor, and $\underset{\sim}{\boldsymbol{Q}}$ denotes an operation that generalizes the conjugate of a quaternion, such that $\underset{\sim}{\boldsymbol{Q}}\boldsymbol{Q} = \boldsymbol{Q}\underset{\sim}{\boldsymbol{Q}} = 1\boldsymbol{\varepsilon}_0$. Thus, it also holds that

$$\boldsymbol{N}^i\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{M}^i \tag{8}$$

Any multivector may be written as the sum of terms involving the basis element involving $\mathbf{e}_4$, and those that do not. Denoting the former with the subscript $\circ$ and the latter with the subscript $\bullet$, Eq. (7) becomes

$$\boldsymbol{N}^i_\circ + \boldsymbol{N}^i_\bullet = (\boldsymbol{Q}_\circ + \boldsymbol{Q}_\bullet)(\boldsymbol{M}^i_\circ + \boldsymbol{M}^i_\bullet)(\underset{\sim}{\boldsymbol{Q}}_\circ + \underset{\sim}{\boldsymbol{Q}}_\bullet) \tag{9}$$

Since all dual products involving $\boldsymbol{\varepsilon}_4$ with itself are zero, Eq. (9) reduces to

$$\boldsymbol{N}^i_\circ + \boldsymbol{N}^i_\bullet = \boldsymbol{Q}_\circ\boldsymbol{M}^i_\circ\underset{\sim}{\boldsymbol{Q}}_\circ + \boldsymbol{Q}_\circ\boldsymbol{M}^i_\circ\underset{\sim}{\boldsymbol{Q}}_\bullet + \boldsymbol{Q}_\bullet\boldsymbol{M}^i_\circ\underset{\sim}{\boldsymbol{Q}}_\circ + \boldsymbol{Q}_\circ\boldsymbol{M}^i_\bullet\underset{\sim}{\boldsymbol{Q}}_\circ \tag{10}$$

As with a dual quaternion, where the real part performs pure rotations and the dual part performs the remainder of the screw operation, for a motor, $\boldsymbol{Q}_\circ$ performs the pure rotation. For other geometric objects, the $\circ$ part corresponds to orientation information: for a plane, $\circ$ gives the normal vector; for a line, $\circ$ gives the direction vector; and for a point, $\circ$ gives the homogenous coordinate, i.e. the component at infinity. Thus, $\boldsymbol{Q}_\circ\boldsymbol{M}^i_\circ\underset{\sim}{\boldsymbol{Q}}_\circ$ performs pure rotations on the orientations of $\mathcal{G}_{3,0,1}$ objects via

$$\boldsymbol{N}^i_\circ = \boldsymbol{Q}_\circ\boldsymbol{M}^i_\circ\underset{\sim}{\boldsymbol{Q}}_\circ \tag{11}$$

from which it follows that

$$\boldsymbol{N}^i_\circ\boldsymbol{Q}_\circ = \boldsymbol{Q}_\circ\boldsymbol{M}^i_\circ \tag{12}$$

One may also rewrite Eq. (8) as

$$\boldsymbol{N}^i\boldsymbol{Q}_\circ - \boldsymbol{Q}_\circ\boldsymbol{M}^i = \boldsymbol{Q}_\bullet\boldsymbol{M}^i - \boldsymbol{N}^i\boldsymbol{Q}_\bullet \tag{13}$$

The pose problem in $\mathcal{G}_{3,0,1}$ thus becomes the following. Given a set of multivector objects $\boldsymbol{M}^i$ and $\boldsymbol{N}^j$, corresponding to points, lines, planes, and/or motors, where $\boldsymbol{M}^i$ ar e members of a known model object set, and $\boldsymbol{N}^j$ are members of an observed object set, find the unique motor $\boldsymbol{Q}$ that represents the screw transformation of $\boldsymbol{M}^i$ into $\boldsymbol{N}^i$. With the above decomposition of the motor equation, and if knows the correspondences that uniquely associate each $\boldsymbol{M}^i$ with each $\boldsymbol{N}^j$, finding the pose reduces to two steps:

1) Solving a generalized version of Wahba's Problem to find the unique (best-fit) $\boldsymbol{Q}_\circ$ that satisfies Eq. (12).

2) "Completing the screw" by using $\boldsymbol{Q}_\circ$ in Eq. (13) to find a unique (best-fit) $\boldsymbol{Q}_\bullet$.

To proceed, one may use results from the Appendix to show that Eqs. (12) and (13) become the following linear algebra relations:

$$\left[\Psi(\boldsymbol{n}_\circ^i) - \Xi(\boldsymbol{m}_\circ^i)\right] \boldsymbol{q}_\circ = 0 \tag{14}$$

$$\left[\Psi(\boldsymbol{n}^i) - \Xi(\boldsymbol{m}^i)\right] \boldsymbol{q}_\circ = \left[\Xi(\boldsymbol{m}^i) - \Psi(\boldsymbol{n}^i)\right] \boldsymbol{q}_\bullet \tag{15}$$

Also, it is convenient to define $\hat{\boldsymbol{q}}_\circ = H_\circ^\mathsf{T} \boldsymbol{q}_\circ$ and $\hat{\boldsymbol{q}}_\bullet = H_\bullet^\mathsf{T} \boldsymbol{q}_\bullet$, which select the four non-zero elements of $\boldsymbol{q}_\circ$ and $\boldsymbol{q}_\bullet$, respectively.

1. Solving the Generalized Wahba Problem

Since Eq. (14) must hold for each i, one may stack the i relations into a system of equations, leading to the result that

$$\sum_i \left(\left[\Psi(\boldsymbol{n}_\circ^i) - \Xi(\boldsymbol{m}_\circ^i)\right]\right)^\mathsf{T} \left(\left[\Psi(\boldsymbol{n}_\circ^i) - \Xi(\boldsymbol{m}_\circ^i)\right]\right) \boldsymbol{q}_\circ = 0 \tag{16}$$

$$C_\circ \boldsymbol{q}_\circ = 0 \tag{17}$$

from which it is clear that $\boldsymbol{q}_\circ$ must form a column of the null space of $C_\circ$. If $C_\circ$ is full-rank, then $\boldsymbol{q}_\circ$ will form its unique null space. Since any scaling of a null-space column remains in the null space, one may choose $\boldsymbol{q}_\circ$ to have unit norm, hence satisfying the constraint that the norm of a quaternion specifying a rotation should be unity. For the case of noisy (but unbiased) observations, choosing $\boldsymbol{q}_\circ$ to be the right singular vector of $C_\circ$ corresponding to its minimum singular value will guarantee a best fit to the observations.

Recognizing that $\boldsymbol{q}_\circ$ has only four non-zero values, corresponding to the four elements of the real rotation quaternion, one gains efficiency and accuracy by finding the null space of the 4×4 matrix $H_\circ^\mathsf{T} C_\circ H_\circ$, rather than considering the full 16×16 matrix $C_\circ$.

2. Completing the Screw

With the solution for $\boldsymbol{q}_\circ$ from above, a least-squares solution for $\boldsymbol{Q}_\bullet$ in Eq. (13) is possible. However, such a solution typically will not satisfy the constraint that $\hat{\boldsymbol{q}}_\circ^\mathsf{T} \hat{\boldsymbol{q}}_\bullet = 0$. Appending this constraint to the usual constrained least-squares normal equations, and economizing using $H_\circ$ and $H_\bullet$, gives

$$\begin{bmatrix} H_\bullet^\mathsf{T} C H_\bullet & \hat{\boldsymbol{q}}_\circ \\ \hat{\boldsymbol{q}}_\circ^\mathsf{T} & 0 \end{bmatrix} \begin{pmatrix} \hat{\boldsymbol{q}}_\bullet \\ \lambda \end{pmatrix} = \begin{pmatrix} -H_\bullet^\mathsf{T} C H_\circ \hat{\boldsymbol{q}}_\circ \\ 0 \end{pmatrix} \tag{18}$$

where λ is the Lagrange multiplier used to append the constraint, and

$$C = \sum_i \left[\Psi(\boldsymbol{n}^i) - \Xi(\boldsymbol{m}^i)\right]^\mathsf{T} \left[\Psi(\boldsymbol{n}^i) - \Xi(\boldsymbol{m}^i)\right] \tag{19}$$

which one may readily solve for $\hat{\boldsymbol{q}}_\bullet$.

IV. Examples

A. Stellar Attitude Determination

In stellar attitude determination, one has two sets of corresponding unit vectors that represent directions to stars. One set are directions from a star catalog expressed in a reference coordinate frame, while the other are observed directions expressed in some frame that rotates with the observer. The distances to the stars from the observer are so large that one may assume the observer is at the origin of the reference frame. In the present context, the reference and observed star directions are therefore zero-moment lines containing the origin and the observed star,

$$\boldsymbol{M}^i = v_z^{\text{ref}} \boldsymbol{\varepsilon}_{12} + v_y^{\text{ref}} \boldsymbol{\varepsilon}_{31} + v_x^{\text{ref}} \boldsymbol{\varepsilon}_{23} \tag{20}$$

$$\boldsymbol{N}^i = v_z^{\text{obs}} \boldsymbol{\varepsilon}_{12} + v_y^{\text{obs}} \boldsymbol{\varepsilon}_{31} + v_x^{\text{obs}} \boldsymbol{\varepsilon}_{23} \tag{21}$$

$$\tag{22}$$

and solving for the optimal pose in $\mathcal{G}_{3,0,1}$ reduces to solving for the optimal attitude quaternion, $\hat{\boldsymbol{q}}_\circ^*$. Two such (non-collinear) directions are clearly sufficient to solve for $\hat{\boldsymbol{q}}_\circ^*$, since each of the two unit vectors has two independent degrees of freedom, allowing for the three rotational degrees of freedom and one constraint ($\hat{\boldsymbol{q}}_\circ^\mathsf{T}\hat{\boldsymbol{q}}_\circ = 1$) that $\hat{\boldsymbol{q}}_\circ^*$ requires.

B. Pose from Heterogeneous Object Measurements

Three-dimensional pose sensors such as lidars permit an observer to generate a collection of points on the surface of (a) neighboring object(s), referenced to a frame centered on the observer. The observer may extract lines and planes from subsets of the observed points. Two-dimensional pose sensors such as cameras provide projections of such objects onto the camera's image plane. Solving for the pose in $\mathcal{G}_{3,0,1}$ that relates these observations to corresponding rigid bodies extracted from a model of the observed object(s) provides a unified and seamless method for forming the optimal solution from such a heterogenous mixture of observations. Not all combinations of observations necessarily permit a robust solution however. In general, for a solution the observations must contain three rotational degrees of freedom, and three translational degrees of freedom; the two constraints $\boldsymbol{q}_\circ^\mathsf{T}\boldsymbol{q}_\circ = 1$ and $\boldsymbol{q}_\bullet^\mathsf{T}\boldsymbol{q}_\circ = 0$ allow for solution of the remaining two elements of the motor. Consider the various objects individually and in combination:

Points (as defined herein) by their nature have no orientation, which is why $\boldsymbol{P}_\circ = 1$ for any point $\boldsymbol{P}$. Thus points provide no information toward solving for $\boldsymbol{q}_\circ$. Completing the screw requires a solution for $\boldsymbol{q}_\circ$, so a pose solution is not possible from points alone.

Planes are defined by their normal vector, constrained to unit length, and a scalar distance from the origin. Thus, two (non-parallel) planes are sufficient to solve for $\boldsymbol{q}_\circ$, since their normal vectors provide the same information as directions (zero-moment lines). But with only two origin distances, there is not enough information to obtain a full pose. The addition of a third (non-parallel) plane provies the necessary additional origin distance for a pose solution, along with redundant information from its normal vector.

Lines have direction and moment, but these must satisfy the constraints that directions have unit length, and direction and moment for any given line must be perpendicular. Thus any particular line has only four independent degrees of freedom, two of which contribute to rotational information and two of which contribute to translational information. As the discussion of stellar attitude determination above describes, any two non-collinear zero-moment lines are sufficient to solve for $\boldsymbol{q}_\circ$. The addition of the moment information from two unique lines provides the additional information for a pose solution.

Combinations From the above considerations, the following minimal combinations permit a pose solution:
- One point and two planes.
- One point and two directions (zero-moment lines).

C. Motors as Observations

Considering Eq. (7), if $\boldsymbol{M}^i$ is itself a motor, it is not hard to show that $\boldsymbol{N}^i$ is a motor sharing the same rotation angle and scalar displacement as $\boldsymbol{M}^i$, but referenced to a line that is rotated and displaced by the action of $\boldsymbol{Q}$ relative to the original line associated with $\boldsymbol{M}^i$. So for example, suppose the observer consists of two pose sensors, rigidly attached to each other with unknown, or poorly known relative pose between them. Suppose each sensor outputs a motor relating a frame fixed to a single rigid body that both sensors simultaneously observe, to a frame fixed to each pose sensor. Choosing one of these pose solutions as $\boldsymbol{M}$ and the other as $\boldsymbol{N}$ one could attempt to solve for the relative pose $\boldsymbol{Q}$ relating the two sensors to each other, thereby providing a relative calibration for future observations. However, motors are subject to similar constraints as lines, with these constraints applying to four-element components of the motor rather than three-element components of the line. Thus, at least two targets must be simultaneously observed, or the same target be observed at two different locations and orientations, and the data must include at least one point observed by each sensor in addition to the motor observations.

Appendix

The geometric dual product $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ used in this work may be expressed as a linear algebra matrix-vector product as

$$\boldsymbol{c} = \Psi(\boldsymbol{a})\boldsymbol{b} \tag{23}$$

where multivectors given by, e.g.

$$\begin{aligned}\boldsymbol{A} = {} & s\mathbf{e}_0 + p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4 \\ & + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12} + v_z\mathbf{e}_{43} + v_y\mathbf{e}_{42} + v_x\mathbf{e}_{41} \\ & + f_w\mathbf{e}_{321} + f_z\mathbf{e}_{412} + f_y\mathbf{e}_{431} + f_x\mathbf{e}_{423} + \sigma\mathbf{e}_{1234}\end{aligned} \tag{24}$$

cast into a linear algebra column vectors as

$$\boldsymbol{a} = [s, p_x, p_y, p_z, p_w, m_x, m_y, m_z, v_z, v_y, v_x, f_w, f_z, f_y, f_x, \sigma]^\top \tag{25}$$

$$= [s, \boldsymbol{p}_{xyz}^\top, p_w, \boldsymbol{m}_{xyz}^\top, \boldsymbol{v}_{zyx}^\top, f_w, \boldsymbol{f}_{zyx}^\top, \sigma]^\top \tag{26}$$

$$= [s, \boldsymbol{p}_{xyzw}^\top, \boldsymbol{m}_{xyz}^\top, \boldsymbol{v}_{zyx}^\top, \boldsymbol{f}_{wzyx}^\top, \sigma]^\top \tag{27}$$

and

$$\Psi(\boldsymbol{a}) =$$
$$\begin{pmatrix}
\sigma & -f_x & -f_y & -f_z & -f_w & -v_x & -v_y & -v_z & -m_z & -m_y & -m_x & p_w & p_z & p_y & p_x & s \\
-f_x & \sigma & -v_z & v_y & m_x & -p_w & f_z & -f_y & p_y & -p_z & f_w & v_x & m_y & -m_z & s & p_x \\
-f_y & v_z & \sigma & -v_x & m_y & -f_z & -p_w & f_x & -p_x & f_w & p_z & v_y & -m_x & s & m_z & p_y \\
-f_z & -v_y & v_x & \sigma & m_z & f_y & -f_x & -p_w & f_w & p_x & -p_y & v_z & s & m_x & -m_y & p_z \\
0 & 0 & 0 & 0 & \sigma & 0 & 0 & 0 & -f_z & -f_y & -f_x & 0 & -v_z & -v_y & -v_x & p_w \\
v_x & p_w & -f_z & f_y & -p_x & \sigma & -v_z & v_y & m_y & -m_z & s & f_x & -p_y & p_z & -f_w & m_x \\
v_y & f_z & p_w & -f_x & -p_y & v_z & \sigma & -v_x & -m_x & s & m_z & f_y & p_x & -f_w & -p_z & m_y \\
v_z & -f_y & f_x & p_w & -p_z & -v_y & v_x & \sigma & s & m_x & -m_y & f_z & -f_w & -p_x & p_y & m_z \\
0 & 0 & 0 & 0 & -f_z & 0 & 0 & 0 & \sigma & v_x & -v_y & 0 & -p_w & -f_x & f_y & v_z \\
0 & 0 & 0 & 0 & -f_y & 0 & 0 & 0 & -v_x & \sigma & v_z & 0 & f_x & -p_w & -f_z & v_y \\
0 & 0 & 0 & 0 & -f_x & 0 & 0 & 0 & v_y & -v_z & \sigma & 0 & -f_y & f_z & -p_w & v_x \\
-p_w & -v_x & -v_y & -v_z & s & f_x & f_y & f_z & -p_z & -p_y & -p_x & \sigma & -m_z & -m_y & -m_x & f_w \\
0 & 0 & 0 & 0 & v_z & 0 & 0 & 0 & p_w & f_x & -f_y & 0 & \sigma & v_x & -v_y & f_z \\
0 & 0 & 0 & 0 & v_y & 0 & 0 & 0 & -f_x & p_w & f_z & 0 & -v_x & \sigma & v_z & f_y \\
0 & 0 & 0 & 0 & v_x & 0 & 0 & 0 & f_y & -f_z & p_w & 0 & v_y & -v_z & \sigma & f_x \\
0 & 0 & 0 & 0 & -p_w & 0 & 0 & 0 & -v_z & -v_y & -v_x & 0 & f_z & f_y & f_x & \sigma
\end{pmatrix} \tag{28}$$

As with quaternions, it is sometimes useful to consider two different versions of the product between geometric algebra objects, in particular to get around their general lack of commutativity. Thus, the geometric product $\boldsymbol{D} = \boldsymbol{B}\boldsymbol{A}$ may be expressed as a linear algebra matrix-vector product as

$$\boldsymbol{d} = \Xi(\boldsymbol{a})\boldsymbol{b} \tag{29}$$

where

$$\Xi(\boldsymbol{a}) =$$

$$\begin{pmatrix}
\sigma & f_x & f_y & f_z & f_w & -v_x & -v_y & -v_z & -m_z & -m_y & -m_x & -p_w & -p_z & -p_y & -p_x & s \\
f_x & \sigma & v_z & -v_y & -m_x & p_w & f_z & -f_y & -p_y & p_z & f_w & v_x & m_y & -m_z & -s & p_x \\
f_y & -v_z & \sigma & v_x & -m_y & -f_z & p_w & f_x & p_x & f_w & -p_z & v_y & -m_x & -s & m_z & p_y \\
f_z & v_y & -v_x & \sigma & -m_z & f_y & -f_x & p_w & f_w & -p_x & p_y & v_z & -s & m_x & -m_y & p_z \\
0 & 0 & 0 & 0 & \sigma & 0 & 0 & 0 & -f_z & -f_y & -f_x & 0 & -v_z & -v_y & -v_x & p_w \\
v_x & -p_w & -f_z & f_y & p_x & \sigma & v_z & -v_y & -m_y & m_z & s & -f_x & -p_y & p_z & f_w & m_x \\
v_y & f_z & -p_w & -f_x & p_y & -v_z & \sigma & v_x & m_x & s & -m_z & -f_y & p_x & f_w & -p_z & m_y \\
v_z & -f_y & f_x & -p_w & p_z & v_y & -v_x & \sigma & s & -m_x & m_y & -f_z & f_w & -p_x & p_y & m_z \\
0 & 0 & 0 & 0 & -f_z & 0 & 0 & 0 & \sigma & -v_x & v_y & 0 & -p_w & f_x & -f_y & v_z \\
0 & 0 & 0 & 0 & -f_y & 0 & 0 & 0 & v_x & \sigma & -v_z & 0 & -f_x & -p_w & f_z & v_y \\
0 & 0 & 0 & 0 & -f_x & 0 & 0 & 0 & -v_y & v_z & \sigma & 0 & f_y & -f_z & -p_w & v_x \\
p_w & -v_x & -v_y & -v_z & -s & -f_x & -f_y & -f_z & -p_z & -p_y & -p_x & \sigma & m_z & m_y & m_x & f_w \\
0 & 0 & 0 & 0 & v_z & 0 & 0 & 0 & p_w & -f_x & f_y & 0 & \sigma & -v_x & v_y & f_z \\
0 & 0 & 0 & 0 & v_y & 0 & 0 & 0 & f_x & p_w & -f_z & 0 & v_x & \sigma & -v_z & f_y \\
0 & 0 & 0 & 0 & v_x & 0 & 0 & 0 & -f_y & f_z & p_w & 0 & -v_y & v_z & \sigma & f_x \\
0 & 0 & 0 & 0 & -p_w & 0 & 0 & 0 & -v_z & -v_y & -v_x & 0 & f_z & f_y & f_x & \sigma
\end{pmatrix} \tag{30}$$

By denoting the usual skew-symmetric cross-product matrix as $K(\boldsymbol{x})$, and introducing the convention that reflecting the symbol for a matrix left-to-right indicates a similar reflection of its columns, $\Psi(\boldsymbol{a})$ and $\Xi(\boldsymbol{a})$ become

$$\Psi(\boldsymbol{a}) = \begin{pmatrix}
\sigma & -\boldsymbol{f}^{\mathsf{T}}_{xyz} & -f_w & -\boldsymbol{v}^{\mathsf{T}}_{xyz} & -\boldsymbol{m}^{\mathsf{T}}_{zyx} & p_w & \boldsymbol{p}^{\mathsf{T}}_{zyx} & s \\
-\boldsymbol{f}_{xyz} & \sigma I + K(\boldsymbol{v}_{xyz}) & \boldsymbol{m}_{xyz} & -p_w I - K(\boldsymbol{f}_{xyz}) & f_w \reflectbox{I} + \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{v}_{xyz} & s\reflectbox{I} + \reflectbox{K}(\boldsymbol{m}_{xyz}) & \boldsymbol{p}_{xyz} \\
0 & O_{1\times3} & \sigma & O_{1\times3} & -\boldsymbol{f}^{\mathsf{T}}_{zyx} & 0 & -\boldsymbol{v}^{\mathsf{T}}_{zyx} & p_w \\
\boldsymbol{v}_{xyz} & p_w I + K(\boldsymbol{f}_{xyz}) & -\boldsymbol{p}_{xyz} & \sigma I + K(\boldsymbol{v}_{xyz}) & s\reflectbox{I} + \reflectbox{K}(\boldsymbol{m}_{xyz}) & \boldsymbol{f}_{xyz} & -f_w \reflectbox{I} - \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{m}_{xyz} \\
O_{3\times1} & O_{3\times3} & -\boldsymbol{f}_{zyx} & O_{3\times3} & \sigma I - K(\boldsymbol{v}_{zyx}) & O_{3\times1} & -p_w I + K(\boldsymbol{f}_{zyx}) & \boldsymbol{v}_{zyx} \\
-p_w & -\boldsymbol{v}^{\mathsf{T}}_{xyz} & s & \boldsymbol{f}^{\mathsf{T}}_{xyz} & -\boldsymbol{p}^{\mathsf{T}}_{zyx} & \sigma & -\boldsymbol{m}^{\mathsf{T}}_{zyx} & f_w \\
O_{3\times1} & O_{3\times3} & \boldsymbol{v}_{zyx} & O_{3\times3} & p_w I - K(\boldsymbol{f}_{zyx}) & O_{3\times1} & \sigma I - K(\boldsymbol{v}_{zyx}) & \boldsymbol{f}_{zyx} \\
0 & O_{1\times3} & -p_w & O_{1\times3} & -\boldsymbol{v}^{\mathsf{T}}_{zyx} & 0 & \boldsymbol{f}^{\mathsf{T}}_{zyx} & \sigma
\end{pmatrix} \tag{31}$$

and

$$\Xi(\boldsymbol{a}) = \begin{pmatrix}
\sigma & \boldsymbol{f}^{\mathsf{T}}_{xyz} & f_w & -\boldsymbol{v}^{\mathsf{T}}_{xyz} & -\boldsymbol{m}^{\mathsf{T}}_{zyx} & -p_w & -\boldsymbol{p}^{\mathsf{T}}_{zyx} & s \\
\boldsymbol{f}_{xyz} & \sigma I - K(\boldsymbol{v}_{xyz}) & -\boldsymbol{m}_{xyz} & p_w I - K(\boldsymbol{f}_{xyz}) & f_w \reflectbox{I} - \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{v}_{xyz} & -s\reflectbox{I} + \reflectbox{K}(\boldsymbol{m}_{xyz}) & \boldsymbol{p}_{xyz} \\
0 & O_{1\times3} & \sigma & O_{1\times3} & -\boldsymbol{f}^{\mathsf{T}}_{zyx} & 0 & -\boldsymbol{v}^{\mathsf{T}}_{zyx} & p_w \\
\boldsymbol{v}_{xyz} & -p_w I + K(\boldsymbol{f}_{xyz}) & \boldsymbol{p}_{xyz} & \sigma I - K(\boldsymbol{v}_{xyz}) & s\reflectbox{I} - \reflectbox{K}(\boldsymbol{m}_{xyz}) & -\boldsymbol{f}_{xyz} & f_w \reflectbox{I} - \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{m}_{xyz} \\
O_{3\times1} & O_{3\times3} & -\boldsymbol{f}_{zyx} & O_{3\times3} & \sigma I + K(\boldsymbol{v}_{zyx}) & O_{3\times1} & -p_w I - K(\boldsymbol{f}_{zyx}) & \boldsymbol{v}_{zyx} \\
p_w & -\boldsymbol{v}^{\mathsf{T}}_{xyz} & -s & -\boldsymbol{f}^{\mathsf{T}}_{xyz} & -\boldsymbol{p}^{\mathsf{T}}_{zyx} & \sigma & \boldsymbol{m}^{\mathsf{T}}_{zyx} & f_w \\
O_{3\times1} & O_{3\times3} & \boldsymbol{v}_{zyx} & O_{3\times3} & p_w I + K(\boldsymbol{f}_{zyx}) & O_{3\times1} & \sigma I + K(\boldsymbol{v}_{zyx}) & \boldsymbol{f}_{zyx} \\
0 & O_{1\times3} & -p_w & O_{1\times3} & -\boldsymbol{v}^{\mathsf{T}}_{zyx} & 0 & \boldsymbol{f}^{\mathsf{T}}_{zyx} & \sigma
\end{pmatrix} \tag{32}$$

Some useful properties associated with these relations are the following:

$$\Psi(\boldsymbol{a})\boldsymbol{b} = \Xi(\boldsymbol{b})\boldsymbol{a} \tag{33}$$

$$\Psi(\Psi(\boldsymbol{a})\boldsymbol{b}) = \Psi(\boldsymbol{a})\Psi(\boldsymbol{b}) \tag{34}$$

$$\Xi(\Xi(\boldsymbol{a})\boldsymbol{b}) = \Xi(\boldsymbol{a})\Xi(\boldsymbol{b}) \tag{35}$$

$$\Psi(\Xi(\boldsymbol{a})\boldsymbol{b}) = \Psi(\boldsymbol{b})\Psi(\boldsymbol{a}) \tag{36}$$

$$\Xi(\Psi(\boldsymbol{a})\boldsymbol{b}) = \Xi(\boldsymbol{b})\Xi(\boldsymbol{a}) \tag{37}$$

$$\Psi(\boldsymbol{a}_\circ) = \Psi(\underset{\sim}{\boldsymbol{a}}_\circ)^\mathsf{T} \tag{38}$$

$$\Xi(\boldsymbol{a}_\circ) = \Xi(\underset{\sim}{\boldsymbol{a}}_\circ)^\mathsf{T} \tag{39}$$

$$\Psi(\boldsymbol{a}_\circ)\Xi(\underset{\sim}{\boldsymbol{b}}_\circ) = \Xi(\boldsymbol{b}_\circ)^\mathsf{T}\Psi(\boldsymbol{a}_\circ) \tag{40}$$

$$\Psi(\boldsymbol{a}_\bullet)\Xi(\underset{\sim}{\boldsymbol{b}}_\circ) = \Xi(\boldsymbol{b}_\circ)^\mathsf{T}\Psi(\boldsymbol{a}_\bullet) \tag{41}$$

$$\Psi(\boldsymbol{a}_\circ)\Xi(\underset{\sim}{\boldsymbol{b}}_\bullet) = \Psi(\underset{\sim}{\boldsymbol{a}}_\circ)^\mathsf{T}\Xi(\underset{\sim}{\boldsymbol{b}}_\bullet) \tag{42}$$

Note that the $\circ$, $\bullet$, and $\underset{\sim}{\cdot}$ operations merely select and/or change the signs of particular elements of the objects they modify, as follows.

$$\underset{\sim}{\boldsymbol{a}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -I_{10} & 0 \\ 0 & 0 & I_5 \end{bmatrix} \boldsymbol{a} = \operatorname{diag}(1, -I_{10}, I_5)\boldsymbol{a} = U\boldsymbol{a} \tag{43}$$

$$\boldsymbol{a}_\circ = \operatorname{diag}(O_4, 1, O_3, I_3, 0, I_4)\boldsymbol{a} = W\boldsymbol{a} \tag{44}$$

$$\boldsymbol{a}_\bullet = \operatorname{diag}(I_4, 0, I_3, O_3, 1, O_4)\boldsymbol{a} = B\boldsymbol{a} \tag{45}$$

When the multivectors are motors corresponding to dual quaternions $\boldsymbol{q} = \boldsymbol{r} + \epsilon\boldsymbol{s}$, then $\Psi(\boldsymbol{q})$ and $\Xi(\boldsymbol{q})$ become

$$\Psi(\boldsymbol{q}) = \begin{pmatrix}
r_w & O_{1\times3} & 0 & -\boldsymbol{r}_{xyz}^\mathsf{T} & -\boldsymbol{s}_{zyx}^\mathsf{T} & 0 & O_{1\times3} & s_w \\
O_{3\times1} & r_w I + K(\boldsymbol{r}_{xyz}) & \boldsymbol{s}_{xyz} & O_{3\times3} & O_{3\times3} & \boldsymbol{r}_{xyz} & s_w \rotatebox[origin=c]{180}{I} + \rotatebox[origin=c]{180}{K}(\boldsymbol{s}_{xyz}) & O_{3\times1} \\
0 & O_{1\times3} & r_w & O_{1\times3} & O_{1\times3} & 0 & -\boldsymbol{r}_{zyx}^\mathsf{T} & 0 \\
\boldsymbol{r}_{xyz} & O_{3\times3} & O_{3\times1} & r_w I + K(\boldsymbol{r}_{xyz}) & s_w \rotatebox[origin=c]{180}{I} + \rotatebox[origin=c]{180}{K}(\boldsymbol{s}_{xyz}) & O_{3\times1} & O_{3\times3} & \boldsymbol{s}_{xyz} \\
O_{3\times1} & O_{3\times3} & O_{3\times1} & O_{3\times3} & r_w I - K(\boldsymbol{r}_{zyx}) & O_{3\times1} & O_{3\times3} & \boldsymbol{r}_{zyx} \\
0 & -\boldsymbol{r}_{xyz}^\mathsf{T} & s_w & O_{1\times3} & O_{1\times3} & r_w & -\boldsymbol{s}_{zyx}^\mathsf{T} & 0 \\
O_{3\times1} & O_{3\times3} & \boldsymbol{r}_{zyx} & O_{3\times3} & O_{3\times3} & O_{3\times1} & r_w I - K(\boldsymbol{r}_{zyx}) & O_{3\times1} \\
0 & O_{1\times3} & 0 & O_{1\times3} & -\boldsymbol{r}_{zyx}^\mathsf{T} & 0 & O_{1\times3} & r_w
\end{pmatrix} \tag{46}$$

and

$$\Xi(\boldsymbol{q}) = \begin{pmatrix}
r_w & O_{1\times3} & 0 & -\boldsymbol{r}_{xyz}^\mathsf{T} & -\boldsymbol{s}_{zyx}^\mathsf{T} & 0 & O_{1\times3} & s_w \\
O_{3\times1} & r_w I - K(\boldsymbol{r}_{xyz}) & \boldsymbol{s}_{xyz} & O_{3\times3} & O_{3\times3} & \boldsymbol{r}_{xyz} & -s_w \rotatebox[origin=c]{180}{I} + \rotatebox[origin=c]{180}{K}(\boldsymbol{s}_{xyz}) & O_{3\times1} \\
0 & O_{1\times3} & r_w & O_{1\times3} & O_{1\times3} & 0 & -\boldsymbol{r}_{zyx}^\mathsf{T} & 0 \\
\boldsymbol{r}_{xyz} & O_{3\times3} & O_{3\times1} & r_w I - K(\boldsymbol{r}_{xyz}) & s_w \rotatebox[origin=c]{180}{I} - \rotatebox[origin=c]{180}{K}(\boldsymbol{s}_{xyz}) & O_{3\times1} & O_{3\times3} & \boldsymbol{s}_{xyz} \\
O_{3\times1} & O_{3\times3} & O_{3\times1} & O_{3\times3} & r_w I + K(\boldsymbol{r}_{zyx}) & O_{3\times1} & O_{3\times3} & \boldsymbol{r}_{zyx} \\
0 & -\boldsymbol{r}_{xyz}^\mathsf{T} & -s_w & O_{1\times3} & O_{1\times3} & r_w & \boldsymbol{s}_{zyx}^\mathsf{T} & 0 \\
O_{3\times1} & O_{3\times3} & \boldsymbol{r}_{zyx} & O_{3\times3} & O_{3\times3} & O_{3\times1} & r_w I + K(\boldsymbol{r}_{zyx}) & O_{3\times1} \\
0 & O_{1\times3} & 0 & O_{1\times3} & -\boldsymbol{r}_{zyx}^\mathsf{T} & 0 & O_{1\times3} & r_w
\end{pmatrix} \tag{47}$$

Then one may notice that $A(\boldsymbol{q}) = \Psi(\boldsymbol{q})\Xi(\underset{\sim}{\boldsymbol{q}})$ corresponds to a homogenous transformation of the multivector:

$$A(\boldsymbol{q}) = \begin{pmatrix}
1 & & & & \\
& \begin{bmatrix} R(\boldsymbol{r}) & 2\boldsymbol{t}(\boldsymbol{r}, \boldsymbol{s}) \\ O_{1\times3} & 1 \end{bmatrix} & & \Large{O} & \\
& & \begin{bmatrix} R(\boldsymbol{r}) & 2\rotatebox[origin=c]{180}{t}(\boldsymbol{r}, \boldsymbol{s}) \\ O_{3\times3} & \rotatebox[origin=c]{180}{R}(\boldsymbol{r}) \end{bmatrix} & & \\
& \Large{O} & & \begin{bmatrix} 1 & 2\boldsymbol{u}_{zyx}^\mathsf{T}(\boldsymbol{r}, \boldsymbol{s}) \\ O_{3\times3} & \rotatebox[origin=c]{180}{R}(\boldsymbol{r}) \end{bmatrix} & \\
& & & & 1
\end{pmatrix} \tag{48}$$

where $R(\boldsymbol{r})$ is the usual 3×3 coordinate system rotation matrix given by

$$R(\boldsymbol{r}) = (r_w^2 - \boldsymbol{r}_{xyz}^{\mathsf{T}} \boldsymbol{r}_{xyz}) I + 2 \boldsymbol{r}_{xyz} \boldsymbol{r}_{xyz}^{\mathsf{T}} + 2 r_w K(\boldsymbol{r}_{xyz}) \tag{49}$$

and $\boldsymbol{t}(\boldsymbol{r}, \boldsymbol{s})$ is the translation vector given by

$$\boldsymbol{t}(\boldsymbol{r}, \boldsymbol{s}) = r_w \boldsymbol{s}_{xyz} - s_w \boldsymbol{r}_{xyz} + \boldsymbol{r}_{xyz} \times \boldsymbol{s}_{xyz} \tag{50}$$

and where $\boldsymbol{u}(\boldsymbol{r}, \boldsymbol{s}) = -R(\boldsymbol{r})^{\mathsf{T}} \boldsymbol{t}(\boldsymbol{r}, \boldsymbol{s})$ and $T(\boldsymbol{r}, \boldsymbol{s}) = R(\boldsymbol{r}) K(-\boldsymbol{u}(\boldsymbol{r}, \boldsymbol{s}))$.

Note that one may derive many of the results of this Appendix by starting from a multilinear structure coefficient tensor that encodes the geometric product relations[2].

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