Multipoint evaluation of polynomials

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1 Objective

The objective is to evaluate the given polynomialof degree $3^d - 1$ at d different points.
Let the given polynomial be:

$$A(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} \tag{1}$$

We wish to evaluate $A(x)$ at n points $x_0, x_1, ..., x_{n-1}$, such that:

$$x_k = \frac{2k\pi}{n}; 0 \leq k \leq n-1 \tag{2}$$

2 Three way Decomposition

In order to evaluate the given polynomial at a particular point we can decompose given polynomial $A(x)$ into the following three polynomials:
$A_1(x) = a_0 + a_3 x + a_6 x^2 + ...$
$A_2(x) = a_1 + a_4 x + a_7 x^2 + ...$
$A_3(x) = a_2 + a_5 x + a_8 x^2 + ...$
And $A(x)$ is given by:

$$A(x) = A_1(x^3) + xA_2(x^3) + x^2 A_3(x^3) \tag{3}$$

Evaluating $A(x)$ at α^k we get:
$A(\alpha^k) = A_1(\alpha^{3k}) + \alpha^k A_2(\alpha^{3k}) + \alpha^{2k} A_3(\alpha^{3k})$

At $\alpha^{k+\frac{n}{3}}$ we have:
$A(\alpha^{k+\frac{n}{3}}) = A_1(\alpha^{3k+n}) + \alpha^{k+\frac{n}{3}} A_2(\alpha^{3k+n}) + \alpha^{2(k+\frac{n}{3})} A_3(\alpha^{3k+n})$

Now if we take α to be the n^{th} root of 1 then $\alpha^n = 1$ then the above equation becomes:
$A(\alpha^{k+\frac{n}{3}}) = A_1(\alpha^{3k}) + \alpha^{k+\frac{n}{3}} A_2(\alpha^{3k}) + \alpha^{2(k+\frac{n}{3})} A_3(\alpha^{3k})$

Similarly,
$A(\alpha^{k+\frac{2n}{3}}) = A_1(\alpha^{3k}) + \alpha^{2(k+\frac{2n}{3})} A_2(\alpha^{3k}) + \alpha^{k+\frac{n}{3}} A_3(\alpha^{3k})$

Thus we observe that the evaluation of a polynomial of degree n, at 3 points has reduced to the evaluation of a polynomial of degree $\frac{n}{3}$ at one point namely α.

Further, $A_1(\alpha), A_2(\alpha)$ and $A_3(\alpha)$ can also be recursively calculated from $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, ...$ and so on.

3 Complexity Analysis

Thus we notice that we can devise a divide and conquer strategy to evaluate the polynomial. Since at each step we evaluate the polynomial by further dividing it into three smaller polynomials, we arrive at the recurrence relation:

$$T(n) = \begin{cases} a, & n = 1 \\ 3T(\frac{n}{3}) + bn + c, & n > 1, n = 3^d, d \geq 0 \end{cases}$$

For example, let us analyse the evaluation of a polynomial of degree 9, at 9 points.It is first decomposed into 3 polynomials, each requiring evaluation at 3 points. These 3 polynomials are further divided into 3 polynomials each, as shown below schematically:

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<tr><td colspan="3" align="center">3</td><td colspan="3" align="center">3</td><td colspan="3" align="center">3</td></tr>
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The number of such decompositions is proportional to $\lg_3 n$
The complexity can be thought of in terms of area of the diagram. The length is proprotional to n and the depth is proportional to $\lg_3 n$.
Thus we arrive at the Asymptotic bound: $T(n) \in O(n \lg_3 n)$