Seismic analysis of asymmetric multistorey buildings including foundation interaction and P-Δ effects

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This paper presents a method of seismic analysis of three-dimensional asymmetric multistorey buildings founded on flexible foundations. The building–foundation system considered in this study is a linear elastic N-storey asymmetric building with a rigid footing, resting on the surface of a linear elastic soil half-space. The method of analysis also includes the P-Δ effects, in which the additional overturning moment and torsional moment at each storey due to P-Δ effects have been replaced by fictitious lateral forces and torques. The whole system has $3N+5$ displacement degrees-of-freedom. The necessary governing equations have been developed considering the three motions of each floor and the five motions of the whole building. Recognizing that the superstructure alone admits classical normal modes, the governing equations of the floors are first uncoupled in terms of footing displacements using the mode superposition method. Substitution of structural deformations, in combination with the dynamic soil–structure interaction force–displacement relationships into the governing equations of the whole system results in five integro-differential equations for footing displacements, which are then solved by numerical step-by-step time history analysis. The floor displacement responses, storey shears, storey torque, etc., are obtained by back substitution of footing accelerations into the relevant governing equations. As a demonstration of the method of analysis and in order to obtain the soil–structure interaction effects, P-Δ effects, and the eccentricity effects, a 10-storey asymmetric building on soft soil was subjected to El Centro 1940 earthquake excitations. The results show that the soft soil conditions increase the lateral deflections, but reduce the twists, storey shears and torques. Increasing eccentricity increases the floor twists and storey torques, however, it does not modify the lateral deflections at the centre of mass, and the total storey shears. The significance of P-Δ effects along with soil–structure interactions on the response of this building has also been discussed.

Keywords: seismic analysis, asymmetric, multistorey, buildings, mode superposition, soil–structure interaction, P-Δ effects, eccentricity

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The multistorey buildings become asymmetric when the floor masses and/or the load resisting elements are not symmetrically distributed with respect to the floor plan of the building. Asymmetric buildings exhibit coupled lateral–torsional movements when subjected to earthquake ground excitations, even when the ground excitation is uniform over the footing and contains no rotational components. Most of the studies[1–5] on the seismic response of torsionally coupled asymmetric buildings assume that the structure is supported by a rigid foundation, which is really an approximation to the real conditions. There are occasions, such as multistorey buildings founded on soft soil, when it becomes necessary to consider the effects of deformability of the foundation (soil–structure interaction effects). When the soil–structure interaction effects are included in the modelling of the building, the whole system may be referred to as a building–foundation system. Another factor that should be considered in the analysis of a building–foundation system subjected to earthquake excitation is the P-Δ effect, P referring to the vertical loads and Δ to the floor displacements. The governing equations of motion for a multistorey building–foundation system including the P-Δ effects, as well as the methods of solving these equations are relatively complex. The effects of soil–structure interaction on the seismic response of multistorey buildings have been studied[6–10]. These studies, however, were limited to two-dimensional (planar) multistorey frames, thus the conclusion reached may only be applied to symmetric buildings. Sivakumaran[11] presented a method of seismic analysis of monosymmetric multistorey building–foundation systems including the P-Δ effects. Recently, Sivakumaran *et al.*[12] presented a method for the seismic analysis of asymmetric building–foundation systems, however, it did not include the P-Δ effects. The objective of this paper is to formulate a method of analysis for the seismic response of three-dimensional asymmetric multistorey buildings including the soil–structure interaction effects and the P-Δ effects. The method of analysis developed herein may be used to answer the following questions: (i) Do local soft soil conditions adversely affect the response of asymmetric multistorey buildings? (ii) How important are the P-Δ effects on the response of such buildings? (iii) Would the P-Δ effects be more important when there is soil–structure interaction and vice versa?

Notation

EI	flexural rigidity
e	static eccentricity between centre of mass and centre of stiffness
G	shear modulus of elasticity of foundation soil
g	gravitational acceleration
h^*	height of floor from ground level
I	mass moment of inertia
$[K]$	stiffness submatrices
$[K_G]$	geometric stiffness submatrices
$[M]$	mass submatrix
$M(t)$	rocking soil–structure interaction forces
m	mass of floor or footing
N	number of storeys
P	vertical loads
$P(t)$	horizontal soil–structure interaction forces
q	normal co-ordinates
r	radius of gyration
t	time
U	horizontal displacements
V_s	shear wave velocity
α	coefficients
β	coefficients
γ	mode shapes
Δ	floor displacements
θ	twist of floor
λ	function given by equation (A2.6)
μ	function given by equation (A2.6)
ξ	damping ratio
ρ	density of foundation soil
τ	time
ϕ	rocking rotation
ω	natural frequency

Subscripts

D	damped natural frequency
G	associated with geometric effects (P-Δ effects)
g	associated with free-field ground motions
i	associated with floor level i
K	associated with Kth mode
N	associated with floor N, top floor
t	indicates total (i.e sum of properties associated with all floors and footing)
U	associated with translation
x	associated with x-direction
y	associated with y-direction
0	associated with ground floor level (i.e. footing)
θ	associated with twist (about z-axis)
ϕ	associated with rocking

Superscripts

x	associated with x-direction
y	associated with y-direction
θ	associated with twist (about z-axis)

Equations of motion and the method of solution

Figure 1 shows the building–foundation system used in the present study. The system consists of a N-storey three-dimensional asymmetric superstructure founded on rigid footing, which in turn, is supported by a flexible soil mass foundation. Structural asymmetry may arise due to asymmetric distribution of the masses or asymmetric distribution of the load resisting elements. In this study the floor is assumed to be rectangular in plan and the floor mass is assumed to be evenly distributed. Thus the centre of mass (CM) is at the geometric centre of the floor plan, and the CM of all the floors lies on one vertical axis, which is generally the case in many multistorey buildings unless the building has setback. The floor system is assumed to be rigid compared to the lateral load resisting system, which may be assumed to consist of massless, axially inextensible columns and walls. The lateral load resisting elements are assumed to be arranged parallel to the floor edges thus, they are either along the X-direction or the Y-direction as shown in *Figure 1*. The stiffnesses of the load resisting elements are such that the system is asymmetric, thus the centre of stiffness (CS) does not coincide with CM. The location of CS may vary from storey to storey, and in this study the CS associated with storey i is assumed to lie at eccentricities e_{ix} and e_{iy} from CM.

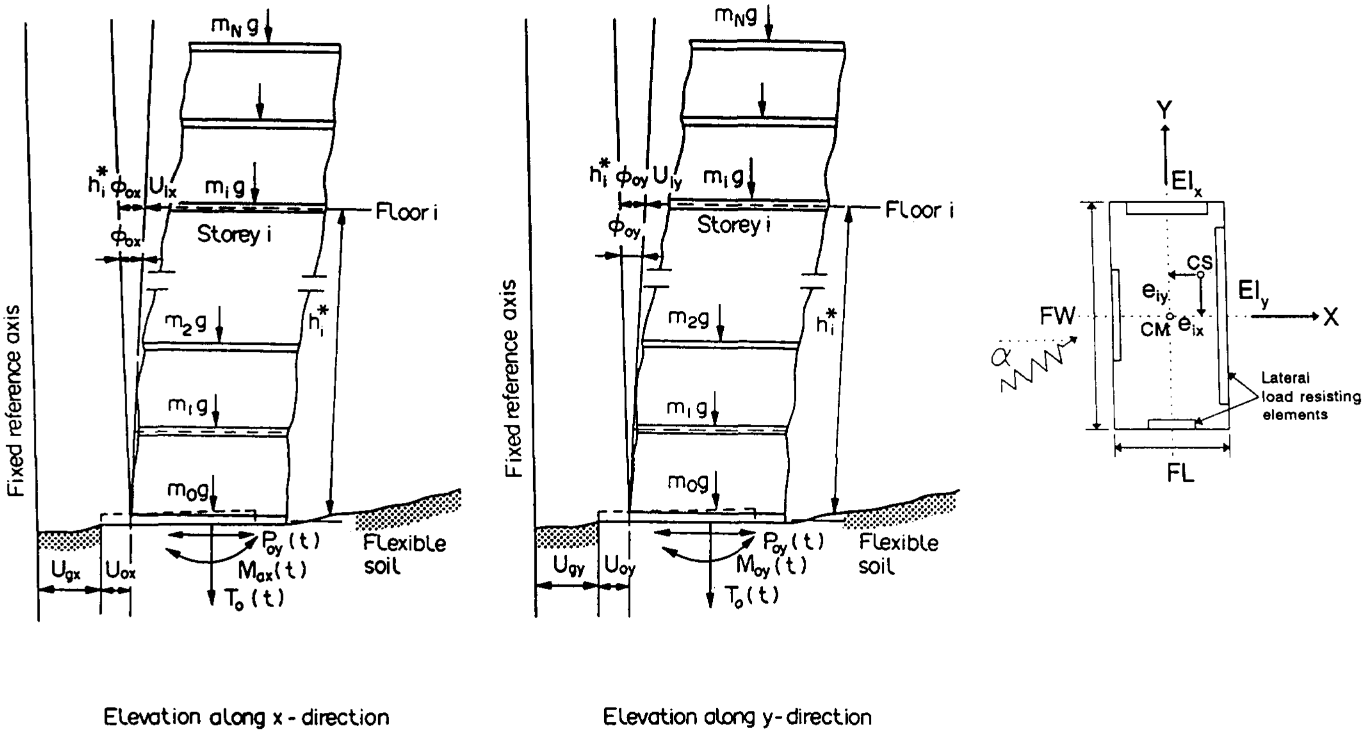

Figure 1 Three-dimensional asymmetric multistorey building–foundation system

The building–foundation system described herein is subjected to horizontal free-field ground motions, assumed to be acting at an angle α from the X-axis (the X-component acceleration is $\ddot{U}_{gx}$ and the Y-component acceleration is $\ddot{U}_{gy}$). Each floor, say floor i, has three displacement degrees of freedom, namely, two horizontal displacements U_{ix} and U_{iy}, and twist θ_i. In addition, due to the deformability of foundation the system has five more displacement degrees-of-freedom, namely, two horizontal translations of the footing (U_{0x} and U_{0y}), two rocking rotations of the footing (ϕ_{0x} and ϕ_{0y}), and the twist of the footing θ_0. Defining the horizontal translations and the twists, with respect to the vertical axis passing through CM prior to ground motions, the absolute floor displacements and the absolute twist of the centre of mass corresponding to floor i may be given as: along X-direction $= U_{gx} + U_{0x} + h_i^*\phi_{0x} + U_{ix}$, along Y-direction $= U_{gy} + U_{0y} + h_i^*\phi_{0y} + U_{iy}$, and twist $= \theta_0 + \theta_i$, where, h_i^* is the height of floor i from the ground level. Considering the equilibrium of each floor (about the CM), the governing equations of the N-floors of the asymmetric superstructure may be developed as

$$\begin{bmatrix}[M] & [0] & [0]\\ [0] & [M] & [0]\\ [0] & [0] & [M]\end{bmatrix}\begin{bmatrix}\{\ddot{U}_{ix}\}\\ \{r_i\ddot{\theta}_i\}\\ \{\ddot{U}_{iy}\}\end{bmatrix} + \begin{bmatrix}[K_{xx}-K_{Gx}] & [K_{x\theta}] & [0]\\ [K_{x\theta}]^{\mathrm{T}} & [K_{\theta\theta}-K_{g\theta}] & [K_{y\theta}]^{\mathrm{T}}\\ [0] & [K_{y\theta}] & [K_{yy}-K_{Gy}]\end{bmatrix}\begin{bmatrix}\{U_{ix}\}\\ \{r_i\theta_i\}\\ \{U_{iy}\}\end{bmatrix} = \begin{bmatrix}[M] & [0] & [0]\\ [0] & [M] & [0]\\ [0] & [0] & [M]\end{bmatrix}\begin{bmatrix}-\{\ddot{U}_{gx}+\ddot{U}_{0x}+h_i^*\ddot{\phi}_{0x}\}+g\phi_{0x}\\ -\{r_i\ddot{\theta}_0\}\\ -\{\ddot{U}_{gy}+\ddot{U}_{0y}+h_i^*\ddot{\phi}_{0y}\}+g\phi_{0y}\end{bmatrix} \quad (1)$$

Equation (1) is a $3N \times 3N$ coupled differential equation in terms of ground motions and the footing displacements. In the above equation, and in the equations that follow $\{a_i\}$ is an $N \times 1$ subvector of quantity a_i, where subscript i denotes that it is associated with floor i. r_i is the mass radius of gyration of the ith floor deck about the vertical axis through the centre of mass and g is the gravitational acceleration. $[M]$ is the diagonal mass submatrix of order $N \times N$; $[K]$ are the stiffness submatrices of order $N \times N$. $[K_G]$ are the geometric stiffness matrices of order $N \times N$, which account for the P-Δ effects. The coefficients of the submatrices are given in Appendix 1. It should be noted that the right-hand-side of equation (1) contains terms associated with g which arise due to the P-Δ effects. Note also that the second set of equations (equations corresponding to rotational displacement degrees-of-freedom) have been divided by the corresponding mass radius of gyration (r_i). For symmetric structures, that is when e_{ix} and e_{iy} are zero, the coupling matrices $[K_{x\theta}]$ and $[K_{y\theta}]$ become zero. Equation (1) then reduces to three sets of uncoupled equations, corresponding to the X-direction, Y-direction and the torsional motions of the building, respectively. Here, equation (1) has been shown for undamped systems only. However, in the analysis the system viscous damping may be assumed to be such a form that the building on rigid foundation admits decomposition into classical normal modes. Also, assuming that the building–foundation system possesses similar mode shapes as the classical normal modes of the building on rigid foundation, equation (1) can be uncoupled and written in terms of normal coordinates as

$$\ddot{q}_K + 2\zeta_K\omega_K\dot{q}_K + \omega_K^2 q_K = -F_K(t)\,; \quad K = 1 \text{ to } 3N \quad (2a)$$

where, ω_K and ζ_K are the undamped natural frequency and the damping ratio, respectively, corresponding to the Kth mode including P-Δ effects, and

$$F_K(t) = \alpha_{1K}[(\ddot{U}_{gx} + \ddot{U}_{0x}) - g\phi_{0x}] + \beta_{1K}\ddot{\phi}_{0x} + \alpha_{2K}r_0\ddot{\theta}_0 + \alpha_{3K}[(\ddot{U}_{gy} + \ddot{U}_{0y}) - g\phi_{0y}] + \beta_{3K}\ddot{\phi}_{0y}$$

Here, r_0 is the mass radius of gyration of the footing and α_{1K}, β_{1K}, α_{2K}, α_{3K} and β_{3K} are coefficients corresponding to the Kth mode, and are given by

$$\alpha_{1K} = \frac{\sum_{i=1}^{N} m_i\gamma_{iK}^x}{\sum_{i=1}^{N} m_i\{\gamma_{iK}^{x2}\}}; \qquad \beta_{1K} = \frac{\sum_{i=1}^{N} m_i h_i^*\gamma_{iK}^x}{\sum_{i=1}^{N} m_i\{\gamma_{iK}^{x2}\}};$$

$$\alpha_{2K} = \frac{\sum_{i=1}^{N} m_i\gamma_{iK}^\theta}{\sum_{i=1}^{N} m_i\{\gamma_{iK}^{\theta 2}\}}; \qquad \alpha_{3K} = \frac{\sum_{i=1}^{N} m_i\gamma_{iK}^y}{\sum_{i=1}^{N} m_i\{\gamma_{iK}^{y2}\}};$$

$$\beta_{3K} = \frac{\sum_{i=1}^{N} m_i h_i^*\gamma_{iK}^y}{\sum_{i=1}^{N} m_i\{\gamma_{iK}^{y2}\}}; \qquad K = 1 \text{ to } 3N \tag{2b}$$

where, m_i is the lumped mass at floor i and $\{\gamma_{ik}^x\}$, $\{\gamma_{ik}^\theta\}$, and $\{\gamma_{ik}^y\}$ are the subvectors of the mode shape corresponding to the Kth mode of vibration. The structural deformation of each floor may be obtained by the mode superposition relations as

$$\begin{Bmatrix} \{U_{ix}\} \\ \{r_i\theta_i\} \\ \{U_{iy}\} \end{Bmatrix} = \begin{bmatrix} [\gamma_{iK}^x] \\ [y_{iK}^\theta] \\ [\gamma_{iK}^y] \end{bmatrix}_{3N\times 3N} \{q_K\}_{3N\times 1} \tag{3}$$

where, $\{q_K\}$ is the array of $3N$ normal co-ordinates, which may be obtained in terms of ground motions and the footing displacements by solving equation (2) as

$$q_K = \frac{-1}{\omega_{DK}}\int_0^t F_K(\tau)e^{-\zeta_K\omega_K(t-\tau)}\sin\omega_{DK}(t-\tau)\mathrm{d}\tau \tag{4a}$$

where, $\omega_{DK} = \omega_K \cdot (1 - \zeta_K^2)^{1/2}$ is the damped natural frequency. Also note that

$$\ddot{q}_K(t) = \frac{\omega_K}{(1-\zeta_K^2)^{1/2}}\int_0^t F_K(\tau)e^{-\zeta_K\omega_K(t-\tau)} \cos[\omega_{DK}(t-\tau) - \Psi_K]\mathrm{d}\tau - F_K(t) \tag{4b}$$

where, $\Psi_K = \tan^{-1}(1-2\zeta_K^2)/(2\zeta_K(1-\zeta_K^2)^{1/2})$ is the phase angle.

Five more equations are needed in order to completely solve the problem. These equations are developed by considering the equilibrium of the whole system. They are given by

$$m_0(\ddot{U}_{gx} + \ddot{U}_{0x}) + \sum_{i=1}^{N} m_i(\ddot{U}_{gx} + \ddot{U}_{0x} + h_i^*\ddot{\phi}_{0x} + \ddot{U}_{ix}) + P_{0x}(t) = 0 \tag{5a}$$

$$m_0(\ddot{U}_{gy} + \ddot{U}_{0y}) + \sum_{i=1}^{N} m_i(\ddot{U}_{gy} + \ddot{U}_{0y} + h_i^*\ddot{\phi}_{0y} + \ddot{U}_{iy}) + P_{0y}(t) = 0 \tag{5b}$$

$$I_{tx}\ddot{\phi}_{0x} + \sum_{i=1}^{N} m_i h_i^*(\ddot{U}_{gx} + \ddot{U}_{0x} + h_i^*\ddot{\phi}_{0x} + \ddot{U}_{ix}) - \sum_{i=1}^{N} m_i g(h_i^*\phi_{0x} + U_{ix}) + M_{0x}(t) = 0 \tag{5c}$$

$$I_{ty}\ddot{\phi}_{0y} + \sum_{i=1}^{N} m_i h_i^*(\ddot{U}_{gy} + \ddot{U}_{0y} + h_i^*\ddot{\phi}_{0y} + \ddot{U}_{iy}) - \sum_{i=1}^{N} m_i g(h_i^*\phi_{0y} + U_{iy}) + M_{0y}(t) = 0 \tag{5d}$$

$$m_0 r_0^2\ddot{\theta}_0 + \sum_{i=1}^{N} m_i r_i^2(\ddot{\theta}_0 + \ddot{\theta}_i) + T_0(t) = 0 \tag{5e}$$

where, m_0, and r_0 are the mass and the radius of gyration of the footing, respectively. I_{tx} and I_{ty} are the total of the mass moment of inertia of the floors and the footing with respect to the X and Y axis, respectively. Again the terms associated with g are due to P-Δ effects. $P_{0x}(t)$, $P_{0y}(t)$, $M_{0x}(t)$, and $M_{0y}(t)$ are the horizontal soil–structure interaction forces and the rocking soil–structure interaction moments in the X- and Y- directions, respectively. $T_0(t)$ is the torsional soil–structure interaction moment. These soil–structure interactions can be related to footing displacements and have been presented in the next section. Substitution of floor deformations given by equation (3), and the corresponding floor accelerations, into equations (5) results in five integro-differential equations in terms of footing displacements U_{0x}, U_{0y}, ϕ_{0x}, ϕ_{0y} and θ_0, as given in Appendix 2.

Dynamic soil–structure interactions

As introduced in the preceding section, $P_{0x}(t)$, $P_{0y}(t)$, $M_{0x}(t)$, $M_{0y}(t)$, and $T_0(t)$ are the dynamic loads imposed by the structure on the foundation, and vice versa. These loads can be related to footing displacements U_{0x}, U_{0y}, ϕ_{0x}, ϕ_{0y} and θ_0, respectively. In general, the dynamic soil–structure interaction force–displacement relationships depend on the frequency of excitation, thus, they are usually developed in terms of frequency. These relationships may be used in a frequency domain analysis. Such an analysis requires firstly, a Fourier transformation of the governing equations of the building–foundation system into the frequency domain. Then the response of the system at various excitation frequencies may be determined. Response to an arbitrary ground motion is then determined by considering the range of frequencies over which the ground motion and building–foundation system have significant components, and then performing a Fourier synthesis of the frequency response to obtain the responses in the time domain. However, such a procedure would require a large computational effort, particularly for structures with many degrees-of-freedom such as three-dimensional asymmetric multistorey buildings. In consideration of this difficulty various frequency independent, but approximate, dynamic soil–structure interaction relationships have been proposed[13–15].

In this study the soil–structure interaction force–displacement relationships proposed by Veletsos and Verbic[13], and Veletsos and Nair[14] have been used. These relationships, in general, depend on the Poisson's ratio (ν) of the foundation material. For Poisson's ratio of $\nu = 0.45$, which may be considered as a representative value for soil, these relationships are as given below[13,14]

$$P_{0x}(t) = K_{Ux}\left[U_{0x}(t) + 0.6\left(\frac{r}{V_s}\right)\dot{U}_{0x}(t)\right] \tag{6a}$$

$$P_{0y}(t) = K_{Uy}\left[U_{0y}(t) + 0.6\left(\frac{r}{V_s}\right)\dot{U}_{0y}(t)\right] \tag{6b}$$

$$M_{0x}(t) = K_{\phi x}\left[0.55\phi_{0x}(t) + 0.36\left(\frac{r}{V_s}\right)\dot{\phi}_{0x}(t) + 0.023\left(\frac{r}{V_s}\right)^2\ddot{\phi}_{0x} + 0.563\left(\frac{V_s}{r}\right)\left\{\int_0^t \phi_{0x}(\tau)e^{-1.25\left(\frac{V_s}{r}\right)(t-\tau)}\,d\tau\right\}\right] \tag{6c}$$

$$M_{0y}(t) = K_{\phi y}\left[0.55\phi_{0y}(t) + 0.36\left(\frac{r}{V_s}\right)\dot{\phi}_{0y}(t) + 0.023\left(\frac{r}{V_s}\right)^2\ddot{\phi}_{0y} + 0.563\left(\frac{V_s}{r}\right)\left\{\int_0^t \phi_{0y}(\tau)e^{-1.25\left(\frac{V_s}{r}\right)(t-\tau)}\,d\tau\right\}\right] \tag{6d}$$

$$T_0(t) = K_\theta\left[0.575\theta_0(t) + 0.292\left(\frac{r}{V_s}\right)\dot{\theta}_0(t) + 0.619\left(\frac{V_s}{r}\right)\left\{\int_0^t \theta_0(\tau)e^{-1.456\left(\frac{V_s}{r}\right)(t-\tau)}\,d\tau\right\}\right] \tag{6e}$$

where the stiffness coefficients are given by

$$K_{Ux} = K_{Uy} = \frac{8Gr}{2-\nu}; \quad K_{\phi x} = K_{\phi y} = \frac{8Gr^3}{3(1-\nu)}; \quad K_\theta = \frac{16Gr^3}{3} \tag{6f}$$

and where V_s is the shear wave velocity of the foundation half-space, ν is the Poisson's ratio of foundation material, G is the shear modulus of elasticity of the foundation soil, which may be related as $G = V_s^2\rho$, where ρ is the mass density of the foundation material, and r is the equivalent radius of the rigid circular foundation. The original soil–structure interaction relationships as presented by Velotsos and Verbic[13] and by Velotsos and Nair[14] are for a rigid circular disc footing on elastic half-space. In this study, the time domain integro-differential relationships for a circular footing given by equations (6) have been used, with r taken as the radius of a circle having the same plan area of the rectangular rigid footing of the building under consideration. For soil having a different Poisson's ratio, the form of the dynamic force–displacement relationships are the same, however, the numerical constants in the relationships would be different.

Numerical step-by-step time history analysis

The coupled integro-differential equations (See Appendix 2) along with equations (6) are now the governing equations for the seismic behaviour of asymmetric multistorey building–foundation system. Note that the governing equations are in terms of the five footing displacement responses. These equations are solved in this study using a numerical step-by-step time history analysis. Without going through the mathematical details, in what follows, first the integrations and the differentials involved in these equations have been replaced by displacements only, using the trapezoidal rule and Newmark's constant-average-acceleration algorithms[11]. Assigning the terms associated with the current time step to the left-hand-side and the values associated with the previous time steps, and other known quantities to the right-hand-side, the governing equations have been reduced to the following system of linear equations.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \begin{Bmatrix} U_{0x}(t) \\ U_{0y}(t) \\ \phi_{0x}(t) \\ \phi_{0y}(t) \\ r_0\theta_0(t) \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{Bmatrix} \tag{7}$$

The expressions for the coefficients A_{ij} and B_i are given in Appendix 3. Note that in A_{ij} and B_i the terms containing the gravitational acceleration g represent the P-Δ effects. Sway effects may be neglected in the analysis by assigning $g = 0$. The shear wave velocity V_s indicates the soil–structure interaction. Rigid foundation may be simulated by assigning a high value for the shear wave velocity. Therefore, using the method of analysis proposed herein the soil–structure interaction effects and the P-Δ effects can be studied independently, if so desired. It may be noted that A_{ij} are constants at each time step, whereas, B_i requires summation of the quantities associated with time 0 to $t - \Delta t$. Thus, the calculation of B_i requires considerable computational effort, which has been greatly improved by the use of recurrence formulae[10–12]. The solution for equation (7) gives the footing displacements at each time step, from which footing accelerations at each time step may be established. The structural deformations, storey shears and storey torque may be obtained by back-substitution of footing accelerations into equations (4) and (3). Thus, in the method of analysis for the seismic response of the three-dimensional asymmetric multistorey building–foundation system presented in this paper, the governing equations of the whole system with $3N + 5$ degrees-of-freedom are reduced to five coupled linear equations. This reduction in degrees-of-freedom allows a considerable saving in computing time. It has been shown that the seismic response of a system having classical normal modes can be accurately obtained by considering only the first few modes[2]. It has also been shown that this phenomenon also exists in building–foundation systems, which do not possess classical normal modes[8,10,11]. Thus, additional saving in computing time can be achieved by considering only the first few modes of vibration in the summations involved in governing equations and the associated coefficients shown in Appendices 2 and 3.

Numerical example and discussions

In order to illustrate the application of the above developed method of seismic analysis and to obtain the soil–structure interaction effects, P-Δ effects, and the eccentricity effects on a three-dimensional asymmetric multistorey building, the configuration of the building as shown in *Figure 1* was considered. The building was 10 storeys high. All storeys were of the same height of 3.5 m. All floor dimensions, including the footing dimensions, were assumed to be identical with length of the building $FL = 10$ m, and the width of the building $FW = 20$ m. The radius of gyration for all the floors and the footing is thus 6.455 m. The masses at all the floor levels were the same ($m = 5$ kPa). The mass of the footing was taken to be $m_o = 10$ kPa. The floor mass and the footing mass were assumed to be uniformly distributed, thus, the centre of mass (CM) for all the floors and the footing lies at the geometric centre of the floor plan. The lateral load resisting system for the building consists of four unbraced steel frames joined at each floor level by a rigid diaphragm. The total storey flexural rigidities of these frames were taken to be: along the X-direction: $\Sigma EI_x = 4 \times 10^9$ N.m^2 from floor 1 to floor 5, and $\Sigma EI_x = 2 \times 10^9$ N.m^2 from floor 6 to floor 10; along the Y-direction: $\Sigma EI_y = 2 \times 10^9$ N.m^2 from floor 1 to floor 5, and $\Sigma EI_y = 1 \times 10^9$ N.m^2 from floor 6 to floor 10. Stiffness asymmetry and thus the building asymmetry was generated by dividing the above indicated flexural rigidities unequally among the two frames. If the flexural rigidities of the two frames in any given direction were the same, then the building becomes symmetric in that direction. In this study, the eccentricities of all the storeys were assumed to be the same, and the values of the eccentricities considered for the study ranged between 0.0 and 0.25 times the building dimensions. The natural periods and the mode shapes of the asymmetric superstructure on rigid foundation, including the torsional modes, were obtained first for the analysis proposed herein. Thus, these values were readily available. The range of periods of the symmetric superstructure (eccentricity = 0.0) on rigid foundation was between 0.63 s for the fundamental Y-direction translation period and 0.019 s for the 30th mode of vibration. The uncoupled fundamental torsional mode period for this case was 0.27 s, thus giving an uncoupled fundamental natural period ratio of 2.33. Eccentricity changes the periods. In general, increasing eccentricity increases the ranges of the period. The natural periods for a structure having $e_{ix} = 0.25$ FL and $e_{iy} = 0.25$ FW were between 0.65 s for the first mode of vibration which is primarily associated with the Y-direction translation and 0.018 s for the 30th mode of vibration. The P-Δ effects increase the natural periods of the structure. A maximum increase in the fundamental natural period of 2.5% was noted in this study. The damping ratio in the fundamental mode of vibration of the superstructure was taken as 5%.

It should be remembered that the dynamic soil–structure interaction relationships given by equations (6) were developed for a massless rigid circular disc. In this study, however, they have been applied to a rectangular footing with mass, in an approximate manner by the use of equivalent radius, where r was taken as the radius of a circle having the same plan area of the rectangular rigid footing of the building. The foundation soil was assumed to be clay ($\rho = 1280$ kg/m^3, $\nu = 0.45$), and in order to establish the effects of soil–structure interaction the shear wave velocity of the half-space material for the study was selected between 50 m/s (soft) and 5000 m/s (rigid). The responses of the asymmetric building–foundation system were established when it was subjected to the 1940 El Centro accelerogram having a peak acceleration of 0.348 g. The duration of the accelerogram considered for the analysis was 20 s. Since the study involved a three-dimensional asymmetric building–foundation system subjected to bidirectional earthquake excitation, here, the earthquake accelerogram was applied at 30° to the X-direction. Thus, in the analysis $\ddot{U}_{gx} = \ddot{U}_g \cos 30°$, and $\ddot{U}_{gy} = \ddot{U}_g \sin 30°$, where $\ddot{U}_g$ are the free-field ground accelerations. Even though the building on flexible foundation generally experiences ground motion which is significantly different from the free-field motion, here, the soft soil amplification of ground motions is not considered. The value of the time step Δt used in the numerical step-by-step time history analysis can influence the convergence and accuracy of the results. A time step $\Delta t = 0.01$ s was used in the results presented in this paper. It was indicated in the previous section that accurate results may be obtained by considering only the first few normal modes of vibration. However, the results shown in this paper are based on all 30 normal modes of vibration.

Figures 2 and *3* show the envelopes of the responses of the building–foundation system under consideration subjected to the 1940 El Centro accelerations. Envelopes indicate the maximum values of the responses that have occurred during the whole excitation. *Figure 2* shows the effects of shear wave velocity on the maximum responses of an asymmetric building–foundation system ($e_{ix} = 0.25$ FL, $e_{iy} = 0.25$ FW). *Figure 3* shows the effects of eccentricity on the responses of the multistorey building on flexible foundation (constant shear wave velocity $V_s = 50$ m/s). In these figures, the floor deflections, and twists indicate the absolute displacements of the centre of mass (CM) with respect to the ground, thus they also include the footing displacements. The displacements corresponding to floor level 0 show the footing displacements. Considerable footing displacements are induced, particularly when the foundation soil is very soft (i.e.; $V_s = 50$ m/s). The maximum values of the footing displacements observed in these analyses were $U_{0x} = 11.26$ mm, $U_{0y} = 6.38$ mm, $\phi_{0x} = 0.0051$ rad, $\phi_{0y} = 0.0028$ rad, and $\phi_0 = 0.000056$ rad. It should be noted that the foundation soil shear wave velocity (i.e., foundation flexibility) significantly influences the footing displacements, however, P-Δ effects do not appear to alter the amount of footing displacements. Larger eccentricities result in higher footing twists, however, the eccentricity does not appear to significantly influence the translational footing displacements and the rocking footing rotations.

The effects of deformability of the foundation can be studied from *Figure 2*, where, the results corresponding to rigid foundation ($V_s = 5000$ m/s) provide the basis for comparison. *Figures 2(a)*, and *2(c)* show the envelopes of the X-component and Y-component translational displacements of the centre of mass, respectively. The building experiences larger X-component displacements compared to Y-component displacements, which may be because a larger component of the ground acceleration was applied along the X-direction. From these figures, it is evident that the foundation flexibility results in substantially increased lateral deflections. For example, the roof X-component deflections of the building founded on soft soil with $V_s = 50$ m/s

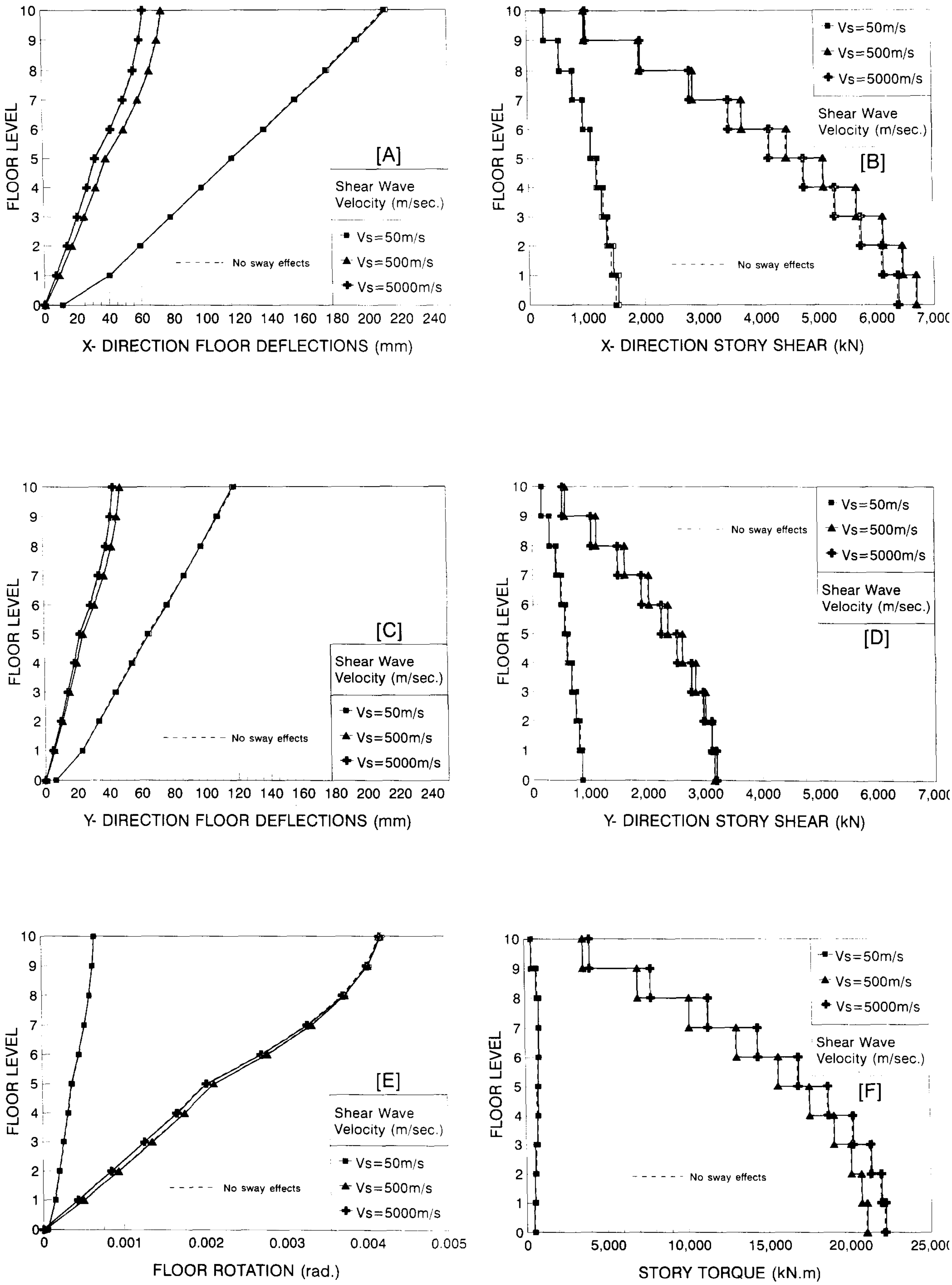

Figure 2 Soil-structure interaction and P-Δ effects on response envelopes of a 10-storey asymmetric building subjected to El Centro 1940 accelerogram (e_{ix} = 2.5 m, e_{iy} = 5 m)

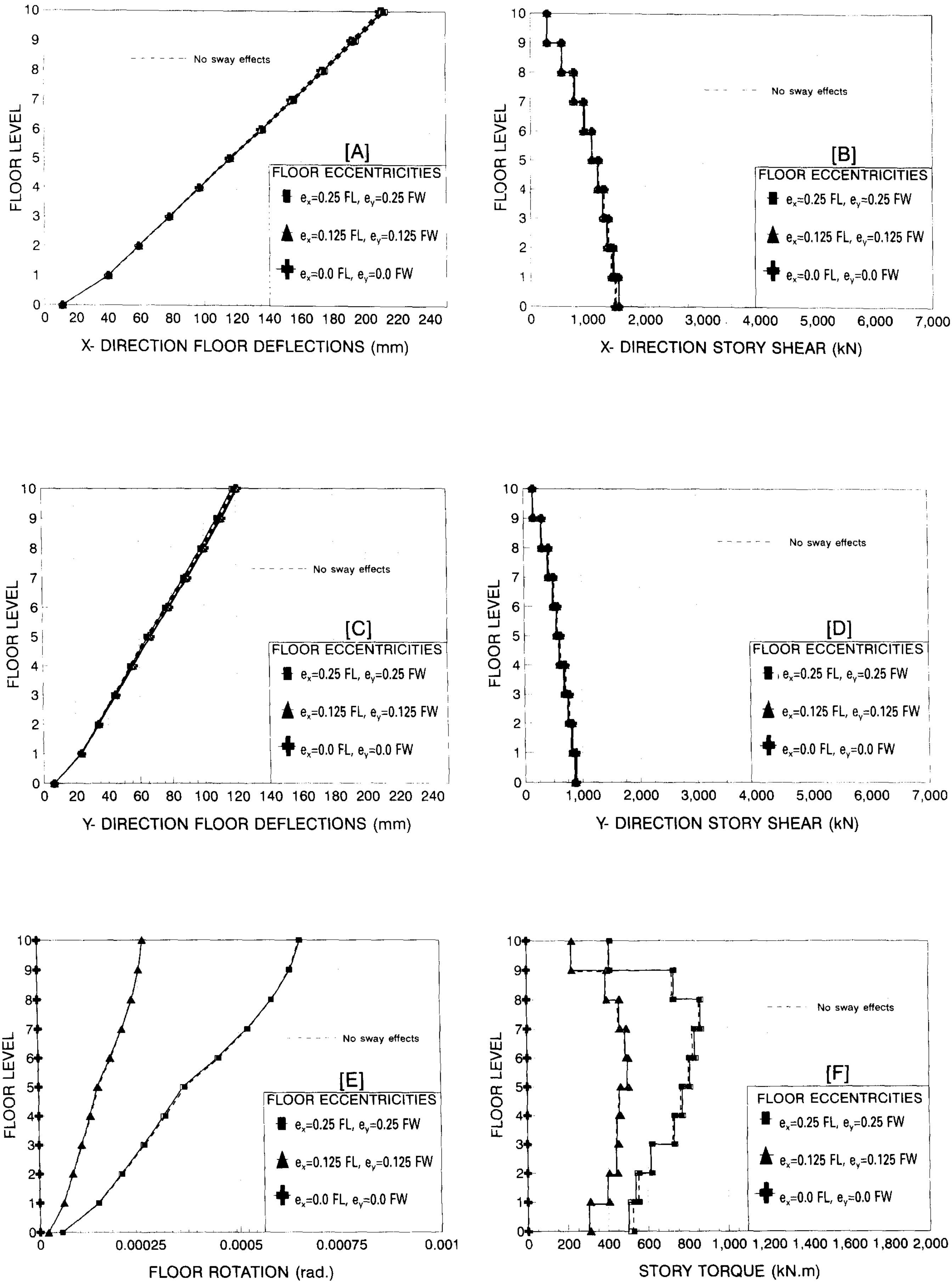

Figure 3 Eccentricity and P-Δ effects on response envelopes of 10-storey asymmetric building subjected to El Centro 1940 accelerogram (V_s = 50 m/s)

is increased by 340% relative to the building founded on rigid soil. *Figure 2(e)* shows the twists of the floors about the axis passing through the CM. The soft soil conditions reduce the amount of twists induced in an asymmetric building. *Figures 2(b), 2(d),* and *2(f)* show the X-component storey shear, Y-component storey shear, and the storey torque, respectively. From these figures it is clear that the soft soil conditions reduce the shear and torque exerted on the building. It may be noticed that when the building is founded on soft soil with $V_s = 50$ m/s, the storey shears have been reduced by 75%, and the storey torques have been reduced to almost 4% of that which would have been induced, if the building had been constructed on a rigid foundation. Irrespective of the soil conditions, whether rigid or soft soil, the P-Δ effects result in increased lateral displacements, twist, storey shear and storey torque. However, it appears that the P-Δ effects are negligible for the building–foundation system and the ground acceleration under consideration. From *Figure 2*, it may be stated that, the soft soil conditions do influence the responses of asymmetric buildings. Although, the lateral deflections were increased, the twists, storey shears and torques were decreased considerably indicating a beneficial effect due to soft soil conditions. P-Δ effects are not significant.

The effects of eccentricity may be studied from *Figure 3*, where, the results corresponding to symmetric buildings ($e_{ix} = 0.0$ and $e_{iy} = 0.0$) provide the basis for comparison. *Figures 3(a)*, and *3(c)* show the envelopes of the X-component, and Y-component translational displacements of the centre of mass, respectively. From these figures, it appears that the eccentricities do not change the maximum lateral deflections of the buildings. However, from *Figure 3(e)*, it is clear that the eccentricities do significantly increase the floor twists. Note, in this figure that the floor twists are zero when the building is symmetric, however, when eccentricities were increased the twists were increased considerably. As one would have expected, *Figures 3(b)* and *3(d)* show that the total storey shears taken at CM remain the same for increasing eccentricities. However, as shown in *Figure 3(f)*, significant storey torques are induced when the buildings are asymmetric. Again the P-Δ effects increase the responses marginally, and eccentricity does not alter the P-Δ effects. From *Figure 3*, it may be stated then that although the maximum lateral deflections and the lateral shears induced by the eccentricities are not significant, considerable twists and torques are induced, indicating that the deflections and the shears in the lateral load resisting frames further away from the centre of mass will be considerably increased due to increasing eccentricities.

Conclusions

This paper provides a method of analysis to determine the seismic responses of three-dimensional asymmetric building–foundation systems. The method of analysis also includes the P-Δ effects. In this method of analysis the computational efficiency has been greatly increased by: (i) the reduction of $3N+5$ governing equations to five coupled integro-differential equations; and (ii) the use of recurrence formulae to evaluate the convolution integrals. A numerical example was shown to illustrate the application of the method of analysis developed in this paper, and to obtain the soil–structure interaction effects, P-Δ effects and the effects of the eccentricity. The results indicate that the soft soil conditions influence the response of asymmetric buildings, particularly when the shear wave velocity of the foundation is lower. When the foundation soil is soft, the lateral deflections were increased, but the twists, storey shears and torques were decreased considerably indicating a beneficial effect due to soft soil conditions. P-Δ effects increase the lateral deflections, storey shear, and the storey torque, however, the amount of increase is negligible for the problem under consideration. Increasing asymmetry (i.e., eccentricity) does not alter the maximum lateral deflections of the centre of mass, and the total lateral shears induced in the building. However, increasing asymmetry induces larger twists and storey torques, resulting in higher deflections and shears, particularly at the lateral load resisting frames further away from the centre of mass.

Acknowledgments

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Appendix 1

Mass submatrix [M]

$$[M] = \begin{bmatrix} m_1 & & & & & & \\ & m_2 & & & & & \\ & & \cdot & & 0 & & \\ & & & m_i & & & \\ & 0 & & & \cdot & & \\ & & & & & m_{N-1} & \\ & & & & & & m_N \end{bmatrix} \tag{A1.1}$$

where, m_i is the lumped mass at floor i.

Stiffness submatrices

$$[K_{xx}] = \begin{bmatrix} K_{1x}+K_{2x} & -K_{2x} & & & & & \\ -K_{2x} & K_{2x}+K_{3x} & -K_{3x} & & 0 & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & -K_{ix} & K_{ix}+K_{(i+1)x} & -K_{(i+1)x} & \\ & & & & & \cdot & \\ & 0 & & & & \cdot & -K_{Nx} \\ & & & & & -K_{Nx} & K_{Nx} \end{bmatrix} \tag{A1.2}$$

$$[K_{yy}] = \begin{bmatrix} K_{1y}+K_{2y} & -K_{2y} & & & & & \\ -K_{2y} & K_{2y}+K_{3y} & -K_{3y} & & 0 & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & -K_{iy} & K_{iy}+K_{(i+1)y} & -K_{(i+1)y} & \\ & & & & & \cdot & \\ & 0 & & & & \cdot & -K_{Ny} \\ & & & & & -K_{Ny} & K_{Ny} \end{bmatrix} \tag{A1.3}$$

where, $K_{ix} = \Sigma k_{ixj}$ and $K_{iy} = \Sigma k_{iyj}$ are the total translational stiffnesses of the ith storey in the X- and Y- directions, respectively. k_{ixj} and k_{iyj} represent the translational stiffnesses of the jth load resisting element at storey i along the X- and Y- directions, respectively.

$$[K_{\theta\theta}] = \begin{bmatrix} \frac{K_{1\theta}+K_{2\theta}}{r_1^2} & \frac{-K_{2\theta}}{r_1 r_2} & & & & & \\ \frac{-K_{2\theta}}{r_2 r_1} & \frac{K_{2\theta}+K_{3\theta}}{r_2^2} & \frac{-K_{3\theta}}{r_2 r_3} & & 0 & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & \frac{-K_{i\theta}}{r_i r_{(i-1)}} & \frac{K_{i\theta}+K_{(i+1)\theta}}{r_i^2} & \frac{-K_{(i+1)\theta}}{r_i r_{(i+1)}} & \\ & & & & & \cdot & \\ & 0 & & & & \cdot & \frac{-K_{N\theta}}{r_{(N-1)} r_N} \\ & & & & & \frac{-K_{N\theta}}{r_N r_{(N-1)}} & \frac{K_{N\theta}}{r_N^2} \end{bmatrix} \tag{A1.4}$$

where, $K_{i\theta} = \Sigma k_{ixj} y_j^2 + \Sigma k_{iyj} x_j^2 + \Sigma k_{i\theta j}$, and where (x_j, y_j) is the coordinate of the jth element with respect to the X and Y axes. The rotational stiffness of the jth load resisting element at storey i may be defined as $k_{i\theta j}$.

$$[K_{x\theta}] = \begin{bmatrix} -\dfrac{e_{1y}K_{1x}+e_{2y}K_{2x}}{r_1} & \dfrac{e_{2y}K_{2x}}{r_2} & & & & & \\ \dfrac{e_{2y}K_{2x}}{r_1} & -\dfrac{e_{2y}K_{2x}+e_{3y}K_{3x}}{r_2} & \dfrac{e_{3y}K_{3x}}{r_3} & & 0 & & \\ & & \ddots & & & & \\ & & \dfrac{e_{iy}K_{ix}}{r_{(i-1)}} & -\dfrac{e_{iy}K_{ix}+e_{(i+1)y}K_{(i+1)x}}{r_i} & \dfrac{e_{(i+1)y}K_{(i+1)x}}{r_{(i+1)}} & & \\ & & & & \ddots & & \\ & 0 & & & & & \dfrac{e_{Ny}K_{Nx}}{r_N} \\ & & & & & \dfrac{e_{Ny}K_{Nx}}{r_{(N-1)}} & -\dfrac{e_{Ny}K_{Nx}}{r_N} \end{bmatrix} \quad \text{(A1.5)}$$

$$[K_{y\theta}] = \begin{bmatrix} \dfrac{e_{1x}K_{1y}+e_{2x}K_{2y}}{r_1} & -\dfrac{e_{2x}K_{2y}}{r_2} & & & & & \\ -\dfrac{e_{2x}K_{2y}}{r_1} & \dfrac{e_{2x}K_{2y}+e_{3x}K_{3y}}{r_2} & -\dfrac{e_{3x}K_{3y}}{r_3} & & 0 & & \\ & & \ddots & & & & \\ & & -\dfrac{e_{ix}K_{iy}}{r_{(i-1)}} & \dfrac{e_{ix}K_{iy}+e_{(i+1)x}K_{(i+1)y}}{r_i} & -\dfrac{e_{(i+1)x}K_{(i+1)y}}{r_{(i+1)}} & & \\ & & & & \ddots & & \\ & 0 & & & & & -\dfrac{e_{Nx}K_{Ny}}{r_N} \\ & & & & & -\dfrac{e_{Nx}K_{Ny}}{r_{(N-1)}} & \dfrac{e_{Nx}K_{Ny}}{r_N} \end{bmatrix} \quad \text{(A1.6)}$$

The storey eccentricities may be calculated from the stiffness and the location of the load resisting elements as:

$$e_{ix} = \frac{\Sigma\, k_{iyj}x_j}{K_{iy}} \qquad e_{iy} = \frac{\Sigma\, k_{ixj}y_j}{K_{ix}} \quad \text{(A1.7)}$$

Geometric stiffness submatrices

$$[K_{Gx}] = [K_{Gy}] = \begin{bmatrix} \dfrac{m_1^*}{h_1}g + \dfrac{m_2^*}{h_2}g & -\dfrac{m_2^*}{h_2}g & & & & & \\ -\dfrac{m_2^*}{h_2}g & \dfrac{m_2^*}{h_2}g + \dfrac{m_3^*}{h_3}g & -\dfrac{m_3^*}{h_3}g & & 0 & & \\ & & \ddots & & & & \\ & & -\dfrac{m_i^*}{h_i}g & \dfrac{m_i^*}{h_i}g + \dfrac{m_{(i+1)}^*}{h_{(i+1)}}g & -\dfrac{m_{(i+1)}^*}{h_{(i+1)}}g & & \\ & & & & \ddots & & \\ & 0 & & & & & -\dfrac{m_N^*}{h_N}g \\ & & & & & -\dfrac{m_N^*}{h_N}g & \dfrac{m_N^*}{h_N}g \end{bmatrix} \quad \text{(A1.8)}$$

where, g is the gravitational acceleration, and m_i^* is the storey mass given by $m_i^* = \Sigma_{i=j}^{N} m_j$

$$[K_{i\theta}] =$$

$$\begin{bmatrix}
\left(\frac{1}{r_1^2}\right)\left(\frac{D_1^2 m_1^*}{h_1} g + \frac{D_2^2 m_2^*}{h_2} g\right) & -\left(\frac{1}{r_1 r_2}\right)\frac{D_2^2 m_2^*}{h_2} g & & & & & \\
-\left(\frac{1}{r_2 r_1}\right)\frac{D_2^2 m_2^*}{h_2} g & \left(\frac{1}{r_2^2}\right)\left(\frac{D_2^2 m_2^*}{h_2} g + \frac{D_3^2 m_3^*}{h_3} g\right) & -\left(\frac{1}{r_2 r_3}\right)\frac{D_3^2 m_3^*}{h_3} g & & & & \\
 & & \cdot & & & & \\
 & & & \cdot & & & \\
 & & & -\left(\frac{1}{r_i r_{(i-1)}}\right)\frac{D_i^2 m_i^*}{h_i} g & \left(\frac{1}{r_i^2}\right)\left(\frac{D_i^2 m_i^*}{h_i} g + \frac{D_{(i+1)}^2 m_{(i+1)}^*}{h_{(i+1)}} g\right) & -\left(\frac{1}{r_i r_{(i+1)}}\right)\frac{D_{(i+1)}^2 m_{(i+1)}^*}{h_{(i+1)}} g & \\
 & & & & & \cdot & \\
 & & & & & \cdot & -\left(\frac{1}{r_{(N-1)} r_N}\right)\frac{D_N^2 m_N^*}{h_N} g \\
 & & & & & -\left(\frac{1}{r_N r_{(N-1)}}\right)\frac{D_N^2 m_N^*}{h_N} g & \left(\frac{1}{r_N^2}\right)\frac{D_N^2 m_N^*}{h_N} g
\end{bmatrix} \tag{A1.9}$$

Appendix 2

The integro-differential governing equations

$$\begin{aligned}
&m_0(\ddot{U}_{gx} + \ddot{U}_{0x}) + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gx} + \ddot{U}_{0x} - g\phi_{0x})\,\alpha_{1K1}\mu_K \mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0x}\beta_{1K1}\mu_K \mathrm{d}\tau \\
&+ \sum_{K=1}^{3N}\int_0^t r_0\ddot{\theta}_0\alpha_{2K1}\mu_K \mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y})\alpha_{3K1}\mu_K \mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0y}\beta_{3K1}\mu_K \mathrm{d}\tau \\
&- \left[\sum_{K=1}^{3N}\alpha_{1K1}(-g\phi_{0x}) + \sum_{K=1}^{3N}\alpha_{2K1}r_0\ddot{\theta}_0 + \sum_{K=1}^{3N}\alpha_{3K1}(\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y}) + \sum_{K=1}^{3N}\beta_{3K1}\ddot{\phi}_{0y}\right] + P_{0x}(t) = 0
\end{aligned} \tag{A2.1}$$

$$\begin{aligned}
&m_0(\ddot{U}_{gy} + \ddot{U}_{0y}) + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gx} + \ddot{U}_{0x} - g\phi_{0x})\,\alpha_{1K2}\mu_K \mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0x}\beta_{1K2}\mu_K \mathrm{d}\tau \\
&+ \sum_{K=1}^{3N}\int_0^t r_0\ddot{\theta}_0\alpha_{2K2}\mu_K \mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y})\alpha_{3K2}\mu_K \mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0y}\beta_{3K2}\mu_K \mathrm{d}\tau \\
&- \left[\sum_{K=1}^{3N}\alpha_{1K2}(\ddot{U}_{gx} + \ddot{U}_{0x} - g\phi_{0x})\right. \\
&\left. + \sum_{K=1}^{3N}\beta_{1K2}\ddot{\phi}_{0x} + \sum_{K=1}^{3N}\alpha_{2K2}r_0\ddot{\theta}_0 + \sum_{K=1}^{3N}\alpha_{3K2}(-g\phi_{0y})\right] + P_{0y}(t) = 0
\end{aligned} \tag{A2.2}$$

$$\begin{aligned}
&I_{tx}\ddot{\phi}_{0x} + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gx} + \ddot{U}_{0x} - g\phi_{0x})(\alpha_{1K3}\mu_K - g\alpha_{1K1}\lambda_K)\mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0x}(\beta_{1K3}\mu_K - g\beta_{1K1}\lambda_K)\mathrm{d}\tau \\
&+ \sum_{K=1}^{3N}\int_0^t r_0\ddot{\theta}_0(\alpha_{2K3}\mu_K - g\alpha_{2K1}\lambda_K)\mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y})(\alpha_{3K3}\mu_K - g\alpha_{3K1}\lambda_K)\mathrm{d}\tau \\
&+ \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0y}(\beta_{3K3}\mu_K - g\beta_{3K1}\lambda_K)\mathrm{d}\tau - \left[\sum_{K=1}^{3N}\alpha_{2K3}r_0\ddot{\theta}_0 + \sum_{K=1}^{3N}\alpha_{3K3}(\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y}) + \beta_{3K3}\ddot{\phi}_{0y}\right] + M_{0x}(t) = 0
\end{aligned} \tag{A2.3}$$

$$\begin{aligned}
&I_{ty}\ddot{\phi}_{0y} + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gx} + \ddot{U}_{0x} - g\phi_{0x})(\alpha_{1K4}\mu_K - g\alpha_{1K2}\lambda_K)\mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0x}(\beta_{1K4}\mu_K - g\beta_{1K2}\lambda_K)\mathrm{d}\tau \\
&+ \sum_{K=1}^{3N}\int_0^t r_0\ddot{\theta}_0(\alpha_{2K4}\mu_K - g\alpha_{2K2}\lambda_K)\mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y})(\alpha_{3K4}\mu_K - g\alpha_{3K2}\lambda_K)\mathrm{d}\tau
\end{aligned}$$

$$+\sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0y}(\beta_{3K4}\mu_K - g\beta_{3K2}\lambda_K)\mathrm{d}\tau - \left[\sum_{K=1}^{3N}\alpha_{1K4}(\ddot{U}_{gx}+\ddot{U}_{0x}-g\phi_{0x}) + \sum_{K=1}^{3N}\beta_{1K4}\ddot{\phi}_{0x} + \sum_{K=1}^{3N}\alpha_{2K4}r_0\ddot{\theta}_0\right] + M_{0y}(t) = 0 \qquad \text{(A2.4)}$$

$$\left[m_0r_0^2 + \sum_{i=1}^{N} m_ir_i^2\right]\ddot{\theta}_0 + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gx}+\ddot{U}_{0x}-g\phi_{0x})\,\alpha_{1K5}\mu_K\mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0x}\beta_{1K5}\mu_K\mathrm{d}\tau$$
$$+\sum_{K=1}^{3N}\int_0^t r_0\ddot{\theta}_0\alpha_{2K5}\mu_K\mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t (\ddot{U}_{gy}+\ddot{U}_{0y}-g\phi_{0y})\,\alpha_{3K5}\mu_K\mathrm{d}\tau + \sum_{K=1}^{3N}\int_0^t \ddot{\phi}_{0y}\beta_{3K5}\mu_K\mathrm{d}\tau$$
$$-\left[\sum_{K=1}^{3N}\alpha_{1K5}(\ddot{U}_{gx}+\ddot{U}_{0x}-g\phi_{0x}) + \sum_{K=1}^{3N}\beta_{1K5}\ddot{\phi}_{0x} + \sum_{K=1}^{3N}\alpha_{2K5}r_0\ddot{\theta}_0 + \sum_{K=1}^{3N}\alpha_{3K5}(\ddot{U}_{gy}+\ddot{U}_{0y}-g\phi_{0y}) + \sum_{K=1}^{3N}\beta_{3K5}\ddot{\phi}_{0y}\right] + T_0(t) = 0$$
$$\text{(A2.5)}$$

where

$$\mu_K(\tau) = \frac{\omega_K}{\sqrt{1-\zeta_K^2}}\,\mathrm{e}^{-\zeta_K\omega_K(t-\tau)}\cos[\omega_{DK}(t-\tau)-\psi_K]$$
$$\lambda_K(\tau) = -\frac{1}{\omega_{DK}}\,\mathrm{e}^{-\zeta_K\omega_K(t-\tau)}\sin[\omega_{DK}(t-\tau)]$$
$$\text{(A2.6)}$$

and the other coefficients in equations (A2.1)–(A2.5) are

$$\alpha_{1K1} = \alpha_{1K}\sum_{i=1}^{N} m_i\gamma_{iK}^x;\quad \alpha_{1K2} = \alpha_{1K}\sum_{i=1}^{N} m_i\gamma_{iK}^y;\quad \alpha_{1K3} = \alpha_{1K}\sum_{i=1}^{N} m_ih_i^*\gamma_{iK}^x;\quad \alpha_{1K4} = \alpha_{1K}\sum_{i=1}^{N} m_ih_i^*\gamma_{iK}^y;$$
$$\alpha_{1K5} = \alpha_{1K}\sum_{i=1}^{N} m_ir_i\gamma_{iK}^\theta;\quad K = 1 \text{ to } 3N \qquad \text{(A2.7)}$$

Expressions for β_{1K1}, α_{2K1}, α_{3K1}, β_{3K1}, etc. can be obtained from equation (A2.7) by replacing α_{1K} by β_{1K}, α_{2K}, α_{3K}, β_{3K}, respectively.

Appendix 3

Coefficients A_{ij} and B_j in equation (7)

$$A_{11} = K_{Ux}\left[1 + \frac{2}{\Delta t}0.6\left(\frac{r}{V_s}\right)\right] + \frac{4}{\Delta t^2}m_o + \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{1K1}\mu_K(0)$$
$$A_{12} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{3K1}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{3K1}$$
$$A_{13} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{1K1}\mu_K(0) - g\left\{\frac{\Delta t}{2}\alpha_{1K1}\mu_K(0) - \alpha_{1K1}\right\}$$
$$A_{14} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{3K1}\mu_K(0) - \frac{4}{\Delta t^2}\beta_{3K1} - g\left\{\frac{\Delta t}{2}\alpha_{3K1}\mu_K(0) - \alpha_{3K1}\right\}$$
$$A_{15} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{2K1}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{2K1}$$

$$A_{21} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{1K2}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{1K2}$$

$$A_{22} = K_{U_y}\left[1 + \frac{2}{\Delta t}0.6\left(\frac{r}{V_s}\right)\right] + \frac{4}{\Delta t^2}m_o + \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{3K2}\mu_K(0)$$

$$A_{23} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{1K2}\mu_K(0) - \frac{4}{\Delta t^2}\beta_{1K2} - g\left\{\frac{\Delta t}{2}\alpha_{1K2}\mu_K(0) - \alpha_{1K2}\right\}$$

$$A_{24} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{3K2}\mu_K(0) - g\left\{\frac{\Delta t}{2}\alpha_{3K2}\mu_K(0) - \alpha_{3K2}\right\}$$

$$A_{25} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{2K2}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{2K2}$$

$$A_{31} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{1K3}\mu_K(0)$$

$$A_{32} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{3K3}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{3K3}$$

$$A_{33} = K_{\phi x}\left[0.55 + \frac{2}{\Delta t}0.36\left(\frac{r}{V_s}\right) + \frac{4}{\Delta t^2}0.023\left(\frac{r}{V_s}\right)^2 + \frac{\Delta t}{2}0.563\left(\frac{V_s}{r}\right)\right] + \frac{4}{\Delta t^2}I_{tx} + \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{1K3}\mu_K(0) - g\left\{\frac{\Delta t}{2}\alpha_{1K3}\mu_K(0)\right\}$$

$$A_{34} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{3K3}\mu_K(0) - \frac{4}{\Delta t^2}\beta_{3K3} - g\left\{\frac{\Delta t}{2}\alpha_{3K3}\mu_K(0) - \alpha_{3K3}\right\}$$

$$A_{35} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{2K3}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{2K3}$$

$$A_{41} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{1K4}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{1K4}$$

$$A_{42} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{3K4}\mu_K(0)$$

$$A_{43} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{1K4}\mu_K(0) - \frac{4}{\Delta t^2}\beta_{1K4} - g\left\{\frac{\Delta t}{2}\alpha_{1K4}\mu_K(0) - \alpha_{1K4}\right\}$$

$$A_{44} = K_{\phi y}\left[0.55 + \frac{2}{\Delta t}0.36\left(\frac{r}{V_s}\right) + \frac{4}{\Delta t^2}0.023\left(\frac{r}{V_s}\right)^2 + \frac{\Delta t}{2}0.563\left(\frac{V_s}{r}\right)\right] + \frac{4}{\Delta t^2}I_{ty} + \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{3K4}\mu_K(0) - g\left\{\frac{\Delta t}{2}\alpha_{3K4}\mu_K(0)\right\}$$

$$A_{45} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{2K4}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{2K4}$$

$$A_{51} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{1K5}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{1K5}$$

$$A_{52} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{3K5}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{3K5}$$

$$A_{53} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{1K5}\mu_K(0) - \frac{4}{\Delta t^2}\beta_{1K5} - g\left\{\frac{\Delta t}{2}\alpha_{1K5}\mu_K(0) - \alpha_{1K5}\right\}$$

$$A_{54} = \sum_{K=1}^{3N}\frac{2}{\Delta t}\beta_{3K5}\mu_K(0) - \frac{4}{\Delta t^2}\beta_{3K5} - g\left\{\frac{\Delta t}{2}\alpha_{3K5}\mu_K(0) - \alpha_{3K5}\right\}$$

$$A_{55} = \left(\frac{K_\theta}{r_o}\right)\left[0.575 + \frac{2}{\Delta t}0.292\left(\frac{r}{V_s}\right) + \frac{\Delta t}{2}0.619\left(\frac{V_s}{r}\right)\right] + \frac{4}{\Delta t^2}\left(\frac{1}{r_o}\right)\left[m_o r_o^2 + \sum_{i=1}^{N} m_i r_i^2\right] + \sum_{K=1}^{3N}\frac{2}{\Delta t}\alpha_{2K5}\mu_K(0) - \frac{4}{\Delta t^2}\alpha_{2K5}$$

$$B_1 = 0.6K_{U_x}\left(\frac{r}{V_s}\right)\hat{Y}_2(U_{0x}) - m_0\ddot{U}_{gx}(t_n) + m_0\hat{Y}_3(U_{0x})$$

$$+\sum_{K=1}^{3N}\alpha_{1K1}\left[-\frac{\Delta t}{2}\ddot{U}_{gx}(t_n)\mu_K(0)+\frac{\Delta t}{2}\hat{Y}_3(U_{0x})\mu_K(0)-\hat{Y}_1\{(\ddot{U}_{gx}+\ddot{U}_{0x}-g\phi_{0x})\mu_K\}\right]$$

$$+\sum_{K=1}^{3N}\beta_{1K1}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0x})\mu_K(0)-\hat{Y}_1(\ddot{\phi}_{0x}\mu_K)\right]$$

$$+r_0\sum_{K=1}^{3N}\alpha_{2K1}\left[\frac{\Delta t}{2}\hat{Y}_3(\theta_0)\mu_K(0)-\hat{Y}_1(\ddot{\theta}_0\mu_K)-\hat{Y}_3(\theta_0)\right]$$

$$+\sum_{K=1}^{3N}\alpha_{3K1}\left[-\frac{\Delta t}{2}\ddot{U}_{gy}(t_n)\mu_K(0)+\frac{\Delta t}{2}\hat{Y}_3(U_{0y})\mu_K(0)-\hat{Y}_1\{(\ddot{U}_{gy}+\ddot{U}_{0y}-g\phi_{0y})\mu_K\}+\ddot{U}_{gy}(t_n)-\hat{Y}_3(U_{0y})\right]$$

$$+\sum_{K=1}^{3N}\beta_{3K1}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0y})\mu_K(0)-\hat{Y}_1(\ddot{\phi}_{0y}\mu_K)-\hat{Y}_3(\phi_{0y})\right]$$

$$B_2=0.6K_{Uy}\left(\frac{r}{V_s}\right)\hat{Y}_2(U_{0y})-m_0\ddot{U}_{gy}(t_n)+m_0\hat{Y}_3(U_{0y})$$

$$+\sum_{K=1}^{3N}\alpha_{1K2}\left[-\frac{\Delta t}{2}\ddot{U}_{gx}(t_n)\mu_K(0)+\frac{\Delta t}{2}\hat{Y}_3(U_{0x})\mu_K(0)-\hat{Y}_1\{(\ddot{U}_{gx}+\ddot{U}_{0x}-g\phi_{0x})\mu_K\}+\ddot{U}_{gx}(t_n)-\hat{Y}_3(U_{0x})\right]$$

$$+\sum_{K=1}^{3N}\beta_{1K2}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0x})\mu_K(0)-\hat{Y}_1(\ddot{\phi}_{0x}\mu_K)-\hat{Y}_3(\phi_{0x})\right]$$

$$+r_0\sum_{K=1}^{3N}\alpha_{2K2}\left[\frac{\Delta t}{2}\hat{Y}_3(\theta_0)\mu_K(0)-\hat{Y}_1(\ddot{\theta}_0\mu_K)-\hat{Y}_3(\theta_0)\right]$$

$$+\sum_{K=1}^{3N}\alpha_{3K2}\left[-\frac{\Delta t}{2}\ddot{U}_{gy}(t_n)\mu_K(0)+\frac{\Delta t}{2}\hat{Y}_3(U_{0y})\mu_K(0)-\hat{Y}_1\{(\ddot{U}_{gy}+\ddot{U}_{0y}-g\phi_{0y})\mu_K\}\right]$$

$$+\sum_{K=1}^{3N}\beta_{3K2}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0y})\mu_K(0)-\hat{Y}_1(\ddot{\phi}_{0y}\mu_K)\right]$$

$$B_3=K_{\phi x}\left[0.36\left(\frac{r}{V_s}\right)\hat{Y}_2(\phi_{0x})+0.023\left(\frac{r}{V_s}\right)^2\hat{Y}_3(\phi_{0x})-0.563\left(\frac{V_s}{r}\right)\hat{Y}_1(\phi_{0x}\eta_\phi)\right]+I_{tx}\hat{Y}_3(\phi_{0x})$$

$$+\sum_{K=1}^{3N}\alpha_{1K3}\left[-\frac{\Delta t}{2}\ddot{U}_{gx}(t_n)\mu_K(0)+\frac{\Delta t}{2}\hat{Y}_3(U_{0x})\mu_K(0)-\hat{Y}_1\left\{(\ddot{U}_{gx}+\ddot{U}_{0x}-g\phi_{0x})\left(\mu_K-g\frac{\alpha_{1K1}}{\alpha_{1K3}}\lambda_K\right)\right\}\right]$$

$$+\sum_{K=1}^{3N}\beta_{1K3}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0x})\mu_K(0)-\hat{Y}_1\left(\ddot{\phi}_{0x}\left(\mu_K-g\frac{\beta_{1K1}}{\beta_{1K3}}\lambda_K\right)\right)\right]$$

$$+r_0\sum_{K=1}^{3N}\alpha_{2K3}\left[\frac{\Delta t}{2}\hat{Y}_3(\theta_0)\mu_K(0)-\hat{Y}_1\left(\ddot{\theta}\mathrm{sub}0\left(\mu_K-g\frac{\alpha_{2K1}}{\alpha_{2K3}}\lambda_K\right)\right)-\hat{Y}_3(\theta_0)\right]$$

$$+\sum_{K=1}^{3N}\alpha_{3K3}\left[-\frac{\Delta t}{2}\ddot{U}_{gy}(t_n)\mu_K(0)+\frac{\Delta t}{2}\hat{Y}_3(U_{0y})\mu_K(0)-\hat{Y}_1\left\{(\ddot{U}_{gy}+\ddot{U}_{0y}-g\phi_{0y})\left(\mu_K-g\frac{\alpha_{3K1}}{\alpha_{3K3}}\lambda_K\right)\right\}+\ddot{U}_{gy}(t_n)-\hat{Y}_3(U_{0y})\right]$$

$$+\sum_{K=1}^{3N}\beta_{3K3}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0y})\mu_K(0)-\hat{Y}_1\left(\ddot{\phi}_{0y}\left(\mu_K-g\frac{\beta_{3K1}}{\beta_{3K3}}\lambda_K\right)\right)-\hat{Y}_3(\phi_{0y})\right]$$

$$B_4=K_{\phi y}\left[0.36\left(\frac{r}{V_s}\right)\hat{Y}_2(\phi_{0y})+0.023\left(\frac{r}{V_s}\right)^2\hat{Y}_3(\phi_{0y})-0.563\left(\frac{V_s}{r}\right)\hat{Y}_1(\phi_{0y}\eta_\phi)\right]+I_{ty}\hat{Y}_3(\phi_{0y})$$

$$+\sum_{K=1}^{3N}\alpha_{1K4}\left[-\frac{\Delta t}{2}\ddot{U}_{gx}(t_n)\mu_K(0)+\frac{\Delta t}{2}\hat{Y}_3(U_{0x})\mu_K(0)-\hat{Y}_1\left\{(\ddot{U}_{gx}+\ddot{U}_{0x}-g\phi_{0x})\left(\mu_K-g\frac{\alpha_{1K2}}{\alpha_{1K4}}\lambda_K\right)\right\}+\ddot{U}_{gx}(t_n)-\hat{Y}_3(U_{0x})\right]$$

$$+\sum_{K=1}^{3N}\beta_{1K4}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0x})\mu_K(0)-\hat{Y}_1\left(\ddot{\phi}_{0x}\left(\mu_K-g\frac{\beta_{1K2}}{\beta_{1K4}}\lambda_K\right)\right)-\hat{Y}_3(\phi_{0x})\right]$$

$$+ r_0\sum_{K=1}^{3N}\alpha_{2K4}\left[\frac{\Delta t}{2}\hat{Y}_3(\theta_0)\mu_K(0) - \hat{Y}_1\left(\ddot{\theta}_0\left(\mu_K - g\frac{\alpha_{2K2}}{\alpha_{2K4}}\lambda_K\right)\right) - \hat{Y}_3(\theta_0)\right]$$

$$+ \sum_{K=1}^{3N}\alpha_{3K4}\left[-\frac{\Delta t}{2}\ddot{U}_{gy}(t_n)\mu_K(0) + \frac{\Delta t}{2}\hat{Y}_3(U_{0y})\mu_K(0) - \hat{Y}_1\left\{(\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y})\left(\mu_K - g\frac{\alpha_{3K2}}{\alpha_{3K4}}\lambda_K\right)\right\}\right]$$

$$+ \sum_{K=1}^{3N}\beta_{3K4}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0y})\mu_K(0) - \hat{Y}_1\left(\ddot{\phi}_{0y}\left(\mu_K - g\frac{\beta_{3K2}}{\beta_{3K4}}\lambda_K\right)\right)\right]$$

$$B_5 = K_\theta\left[0.292\left(\frac{r}{V_s}\right)\hat{Y}_2(\theta_0) - 0.619\left(\frac{V_s}{r}\right)\hat{Y}_1(\theta_0\eta_\theta)\right] + \left[m_0r_0^2 + \sum_{i=1}^{N}m_ir_i^2\right]\hat{Y}_3(\theta_0)$$

$$+ \sum_{K=1}^{3N}\alpha_{1K5}\left[-\frac{\Delta t}{2}\ddot{U}_{gx}(t_n)\mu_K(0) + \frac{\Delta t}{2}\hat{Y}_3(U_{0x})\mu_K(0) - \hat{Y}_1\{(\ddot{U}_{gx} + \ddot{U}_{0x} - g\phi_{0x})\mu_K\} + \ddot{U}_{gx}(t_n) - \hat{Y}_3(U_{0x})\right]$$

$$+ \sum_{K=1}^{3N}\beta_{1K5}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0x})\mu_K(0) - \hat{Y}_1(\ddot{\phi}_{0x}\mu_K) - \hat{Y}_3(\phi_{0x})\right]$$

$$+ r_0\sum_{K=1}^{3N}\alpha_{2K5}\left[\frac{\Delta t}{2}\hat{Y}_3(\theta_0)\mu_K(0) - \hat{Y}_1(\ddot{\theta}\mathrm{sub}0\mu_K) - \hat{Y}_3(\theta_0)\right]$$

$$+ \sum_{K=1}^{3N}\alpha_{3K5}\left[-\frac{\Delta t}{2}\ddot{U}_{gy}(t_n)\mu_K(0) + \frac{\Delta t}{2}\hat{Y}_3(U_{0y})\mu_K(0) - \hat{Y}_1\{(\ddot{U}_{gy} + \ddot{U}_{0y} - g\phi_{0y})\mu_K\} + \ddot{U}_{gy}(t_n) - \hat{Y}_3(U_{0y})\right]$$

$$+ \sum_{K=1}^{3N}\beta_{3K5}\left[\frac{\Delta t}{2}\hat{Y}_3(\phi_{0y})\mu_K(0) - \hat{Y}_1(\ddot{\phi}_{0y}\mu_K) - \hat{Y}_3(\phi_{0y})\right]$$

where, μ_K are as defined at the end of equations (6), and

$$\eta_\phi = e^{-1.25\left(\frac{V_s}{r}\right)(t_n - t_i)} \quad \text{and} \quad \eta_\theta = e^{-1.456\left(\frac{V_s}{r}\right)(t_n - t_i)}$$

$\hat{Y}_1$, $\hat{Y}_2$, and $\hat{Y}_3$ are operators given by

$$\hat{Y}_1(\hat{x}) = \frac{\Delta t}{2}\hat{x}(t_0) + \Delta t\sum_{j=1}^{n-1}\hat{x}(t_j)\,; \quad \hat{Y}_2(\hat{x}) = \frac{2}{\Delta t}\hat{x}(t_{n-1}) + \dot{\hat{x}}(t_{n-1}) \quad \hat{Y}_3(\hat{x}) = \frac{4}{\Delta t^2}\hat{x}(t_{n-1}) + \frac{4}{\Delta t}\dot{\hat{x}}(t_{n-1}) + \ddot{\hat{x}}(t_{n-1})$$

where $\hat{x}$ is the variable.

Numerical scheme for operator $\hat{Y}_1$

Operator $\hat{Y}_1$ requires considerable computational effort. Direct operation at every time step is not numerically efficient. However, note that there are basically three different functions namely, μ_K, η_ϕ and η_θ which are involved along with the footing displacements U_{0x}, U_{0y}, ϕ_{0x}, ϕ_{0y} and θ_0. Since, these functions involve cosine and exponential functions it is possible to develop recurrence formulae[10,11], which greatly reduce the computational effort.