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Performance of base isolation systems for asymmetric building subject to random excitation

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The stochastic response of a one-storey model of a base isolated asymmetric building to random ground excitation is computed using the state space formulation. Three types of base isolation devices are considered: the laminated rubber bearing (LRB); the New Zealand system (NZ); and the resilient-friction base isolator (R-FBI). For the NZ and R-FBI systems, for which force-deformation behaviour is nonlinear, an equivalent linearization technique is applied to obtain an approximate response. The effect of linearization on the response is investigated by comparing the responses of the linearized solution with those obtained from simulation analysis. The RMS responses of the system are obtained under different parametric variations in order to investigate the effectiveness of different base isolation devices for stationary random ground motion. The study shows that the stochastic response of the asymmetric building isolated by an R-FBI system is less sensitive to the parametric variations.

Keywords: base isolation devices, asymmetric buildings

Probabilistic analysis of base-isolated structures has attracted considerable attention in recent years. The response of rigid block sliding on a randomly moving foundation has been studied by Ahmadi[1], Constantinou and Tadjbakhsh[2], Crandall *et al.*[3] and Su and Ahmadi[4]. The optimum design of a base isolated 2D system based on probabilistic excitation was also considered by Constantinou and Tadjbakhsh[5,6]. The stochastic response of a shear beam type structure isolated by different base isolation devices has been studied by Su and Ahmadi[7,8]. Chen and Ahmadi[9] have studied the stochastic response of the secondary system in a shear beam type structure to nonstationary earthquake excitation. These studies are confined to the translational response analysis of 2D idealized models of the buildings and intended to investigate the parametric behaviour of a base isolated building for different types of isolation devices such as: the laminated rubber bearing (LRB), and the laminated rubber bearing with lead core (NZ) system, the Electricite de France (EDF) system[10], the resilient-friction base isolator (R-FBI) system[11], and the sliding resilient-friction (SR-F) system[12]. Several authors have carried out a comparative study of the performance of different base isolators[7,8,12-15].

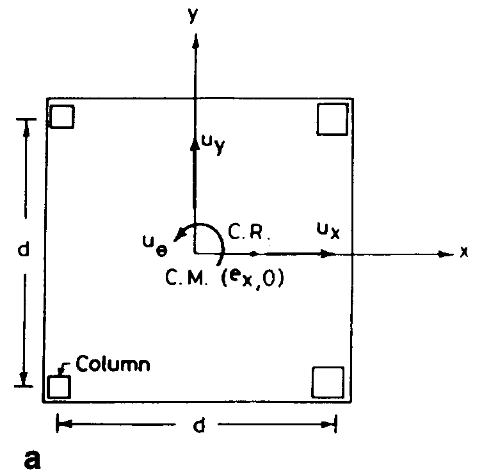
While these provide useful information on the design and optimization of base isolation systems, they are not strictly valid for asymmetric buildings having significant torsional coupling. Very few analytical studies have been reported on the seismic base isolation of 3D building models. Lee[16] had shown that the base isolation reduces the structural torque drastically, even if the structural eccentricity is large. Pan and Kelly[17] studied the effect of eccentricity on the elastic response of a rigid mass supported on a base isolator. Nagarajaiah *et al.*[18,19] studied the nonlinear response of a 3D base isolated building to the El Centro earthquake motion. Recently, Jangid and Datta[20-23] conducted extensive parametric studies on the response behaviour of a torsionally coupled base isolated system under random ground motion using simulation analysis.

In this paper, the stochastic response of a one-storey model of a torsionally coupled building isolated by LRB, NZ and R-FBI isolator systems to random ground motion is considered. The RMS responses are obtained using a state space formulation for a set of important parametric variations of the asymmetric building. For the NZ and R-FBI isolation systems, an equivalent linearization technique is adopted to obtain the approximate stochastic response. The main objectives of the study are: firstly, to investigate the effect of linearization on the response by comparing the linearized responses with those obtained by simulation analysis; and secondly, to investigate the response of each type of isolator to parametric variations.

Structural model

Figure 1 shows the structural model considered, which is an idealized one-storey building model mounted on base isolators. The rigid deck mass is supported by weightless discrete resisting elements (i.e. column or wall). The lateral dimension of the deck is d in the x-direction and b in the y-direction. The stiffness distribution of the resisting elements is symmetric about the x-axis but not about the y-axis, as a result, the structure displays a torsional effect when excited in the lateral direction-y. The torsional mass of the deck is varied to provide various torsionally coupled structural models. The base isolator consists of an array of isolators arranged between the base mass and the foundation. These isolators are fixed to the foundation at the bottom and to the base mass at the top. The following assumptions apply for the model under study.

(1) The superstructure (i.e. columns or walls) remains in the elastic range during the earthquake excitation
(2) Although, a torsional moment develops at each isolator, the contribution of this torsional moment to the total torque exerted at the base mass is insignificant and hence is not included
(3) The base isolator carries the vertical load without undergoing any vertical deformation
(4) An additional viscous damping is provided by the bearings
(5) The system is excited by the two components of random ground acceleration (i.e. in x- and y- directions) specified by the stationary power spectral density function



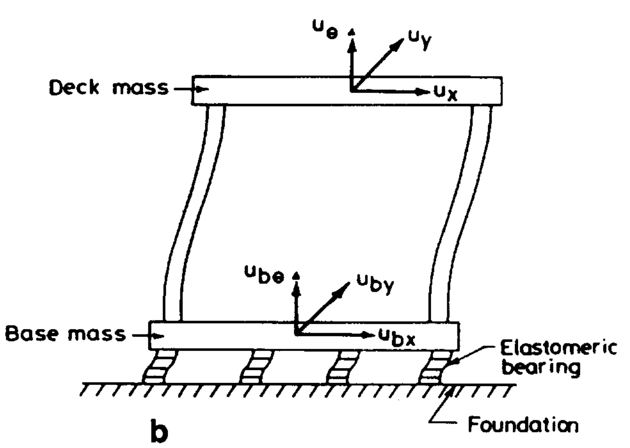


Figure Simple torsionally coupled base isolated system. (a) plan of deck; (b) elevation

Two lateral, u_x and u_y, and one rotational, u_θ degrees-of-freedom are considered at the centre of the deck mass. The base isolator permits the base motions relative to the ground in two lateral directions i.e. u_{bx} and u_{by} and a rotation $u_{b\theta}$ about the vertical axis as shown in *Figure 1(b)*.

Let k_{xi} and k_{yi} represent the lateral stiffnesses of the ith resisting element in the x- and y-directions, respectively. Then

$$K_x = \sum_i k_{xi} \quad \text{and} \quad K_y = \sum_i k_{yi} \tag{1}$$

are the lateral stiffnesses of the fixed base structure in the x- and y-directions, respectively. Let x_i and y_i denote the x- and y-co-ordinates of the ith resisting element with respect to the centre of the deck mass, respectively. Then the total torsional stiffness of the structure, defined about the centre of mass (CM) of the deck, is given by

$$K_\theta = \sum_i (k_{xi}\, y_i^2 + k_{yi}\, x_i^2) \tag{2}$$

in which the torsional stiffness of each individual resisting element is negligible and is not included. The eccentricity between the CM of the deck mass and the static centre of resistance (CR) of the resisting elements is given by

$$e_x = \frac{1}{K_y} \sum_i k_{yi}\, x_i \tag{3}$$

The three uncoupled frequency parameters of the superstructure are defined as

$$w_x = \left(\frac{K_x}{m}\right)^{1/2}; \quad w_y = \left(\frac{K_y}{m}\right)^{1/2}; \quad w_\theta = \left(\frac{K_\theta}{mr^2}\right)^{1/2} \tag{4}$$

in which, m is the mass of the superstructure (including additional lumped masses); and r is the radius of gyration of the deck mass about the vertical axis through the CM. The frequencies w_x and w_θ may be interpreted as the natural frequencies of the fixed base system if it were torsionally uncoupled i.e. a system with $e_x = 0$, but, m the mass of superstructure, K_x, K_y and K_θ are the same as in the coupled system. The torsional mass of the deck and base mass are varied to provide various uncoupled torsional to lateral frequency ratios. This can be achieved by varying the position of the lumped mass with respect to the CM.

Earthquake excitation

Earthquake ground motions are inherently random and multidimensional. To describe such ground motions, a multivariate random process model is generally used. In the present study, the ground motions are assumed to be stationary processes. The ground accelerations in the x- and y-directions are modelled as filtered white noise with a prescribed power spectral density function, such as the Kanai–Tajimi spectrum

$$S(w) = S_0 \frac{1 + 4\,\xi_g^2 \left(\dfrac{w}{w_g}\right)^2}{\left[1 - \left(\dfrac{w}{w_g}\right)^2\right]^2 + 4\,\xi_g^2 \left(\dfrac{w}{w_g}\right)^2}, \quad w > 0 \tag{5}$$

in which w_g and ξ_g are filter parameters, and S_0 is the spectral density of the input white-noise.

As there is generally no reason to use different spectral and intensity functions for the two orthogonal directions, it may be reasonable to assume that the parameters ξ_g and w_g are the same for both directions. Since the stochastic response analysis using the state space formulation requires the specification of the input as a white-noise process, the ground accelerations u_{gx} and u_{gy} in the x- and y-directions, respectively, from Reference 23 are expressed as

$$\ddot{u}_{gx} = -[2\xi_{gx}\, w_{gx}\, \dot{u}_{fx} + w_{gx}^2\, u_{fx}] \tag{6a}$$

$$\ddot{u}_{gy} = -[2\xi_{gy}\, w_{gy}\, \dot{u}_{fy} + w_{gy}^2\, u_{fy}] \tag{6b}$$

$$\ddot{u}_{fx} + 2\xi_{gx}\, w_{gx}\, \dot{u}_{fx} + w_{gx}^2\, u_{fx} = f_x(t) \tag{7a}$$

$$\ddot{u}_{fy} + 2\xi_{gy}\, w_{gy}\, \dot{u}_{fy} + w_{gy}^2\, u_{fy} = f_y(t) \tag{7b}$$

in which, u_{fx} and u_{fy} are the filter response, ξ_{gx} and ξ_{gy} are the damping of the filter, w_{gx} and w_{gy} are the ground filter frequencies in the x- and y-directions, respectively; $f_x(t)$ and $f_y(t)$ are the zero mean Gaussian white-noise processes with the following statistics

$$E[f_x(t) f_x(t+\tau)] = 2\pi\, S_{0x}\, \delta(\tau) \tag{8a}$$

$$E[f_y(t) f_y(t+\tau)] = 2\pi\, S_{0y}\, \delta(\tau) \tag{8b}$$

where, E is the expectation operator; $\delta(\cdot)$ is the Dirac delta function; and S_{0x} and S_{0y} are the constant power spectrum of white-noise processes, $f_x(t)$ and $f_y(t)$, respectively. The power spectral density function of the ground acceleration u_{gx} and u_{gy} as defined by equations (6(a)–7(b)) corresponds to the form given by equation (5).
Defining

$$\{\ddot{u}_g\} = \{\ddot{u}_{gx}, \ddot{u}_{gy}\}^{\mathrm{T}} \tag{9}$$

$$\{\ddot{u}_f\} = \{\ddot{u}_{fx}, \ddot{u}_{gy}\}^{\mathrm{T}} \tag{10}$$

$$\{f(t)\} = \{f_x(t),\ f_y(t)\}^{\mathrm{T}} \tag{11}$$

Equations (6) and (7) can be rewritten as

$$\{\ddot{u}_g\} = -[F_1]\{u_f\} - [F_2]\{u_f\} \tag{12}$$

$$\{\ddot{u}_f\} + [F_1]\{\dot{u}_f\} + [F_2]\{u_f\} = \{f(t)\} \tag{13}$$

in which, the matrix, $[F_1] = \mathrm{diag}[2\xi_{gx} w_{gx}, 2\xi_{gy} w_{gy}]$ and $[F_2] = \mathrm{diag}[w_{gx}^2, w_{gy}^2]$.

Equations of motion

The equations of motion of the superstructure remain the same for all base isolation systems which are written as follows

$$[M](\ddot{u} + \ddot{u}_b) + [C]\{\dot{u}\} + [K]\{u\} = -[M]\,[T_g]\,\{\ddot{u}_g\} \tag{14}$$

in which, $[M]$, $[K]$ and $[C]$ are the lumped mass, stiffness and damping matrices of the superstructure corresponding to the degrees-of-freedom (DOF) at the deck; $\{u\} = [u_x, u_y, u_\theta]^{\mathrm{T}}$, the vector of the displacements at the deck relative to the base mass; $\{u_b\} = [u_{bx}, u_{by}, u_{b\theta}]^{\mathrm{T}}$ is the vector of the base displacements relative to the ground; $\{\ddot{u}_g\}$ is the ground acceleration vector (equation (12)); and $[T_g]$ is the earthquake influence coefficient matrix. The stiffness matrix of the superstructure, $[K]$ is written as

$$[K] = \sum_i [T_i]^{\mathrm{T}}[k_i][T_i] \tag{15}$$

$$[k_i] = \mathrm{diag}[k_{xi},\ k_{yi}] \tag{16}$$

$$[T_i] = \begin{bmatrix} 1 & 0 & -y_i \\ 0 & 1 & x_i \end{bmatrix} \tag{17}$$

in which, $[T_i]$ is the transformation matrix of the ith resisting element. k_{xi} and k_{yi} are the lateral stiffness of the ith resisting element in the x- and y-directions, respectively. The damping matrix of the superstructure, $[C]$ is not explicitly known. It is constructed from the assumed modal damping for the fixed base structure using its mode shapes and frequencies.

LRB system

For the laminated rubber bearings (LRB) system, the equations of motion of the base raft corresponding to three base degrees-of-freedom are given as

$$[M_b]\{\ddot{u}_b\} + [C_b]\{\dot{u}_b\} + [K_b]\{u_b\} - [C]\{\dot{u}\} - [K]\{u\} = -[M_b][T_g]\{\ddot{u}_g\} \tag{18}$$

in which, $[M_b]$ and $[C_b]$ are the lumped mass and damping matrices corresponding to the degrees-of-freedom at the base mass. The damping matrix, $[C_b]$ of the isolator system is constructed by assuming the damping constants of the bearings. $[K_b]$ is the stiffness matrix of the isolator given by

$$[K_b] = \sum_i [T_{bi}]^{\mathrm{T}}[k_{it}][T_{bi}] \tag{19}$$

$$[k_{it}] = \mathrm{diag}[k_{bi},\ k_{bi}] \tag{20}$$

$$[T_{bi}] = \begin{bmatrix} 1 & 0 & -y_{bi} \\ 0 & 1 & x_{bi} \end{bmatrix} \tag{21}$$

Equations (12)–(14) and (18) can be rewritten as a system of 16 first-order stochastic differential equations as

$$\frac{d}{dt}\{Y\} = [H]\,\{Y\} + \{F\} \tag{22}$$

in which

$$\{Y\} = \{u_f,\ \dot{u}_f,\ u,\ u_b,\ \dot{u},\ \dot{u}_b\}^{\mathrm{T}} \quad (16 \times 1) \tag{23}$$

$$\{F\} = \{0, f(t), 0, 0, 0, 0\}^{\mathrm{T}} \quad (16 \times 1) \tag{24}$$

$[H]$ is the system property matrix including the characteristics of the ground filter and is given in Reference 23 and reproduced in Appendix 2.

The augmented response vector $\{Y\}$ is a Markov process. The corresponding covariance matrix $[V]$ satisfies the following equation[24]

$$\frac{d}{dt}[V] = [H]\,[V]^{\mathrm{T}} + [V]\,[H]^{\mathrm{T}} + [P] \tag{25}$$

where the elements of matrices $[V]$ and $[P]$ are given by

$$V_{ij} = E[Y_i Y_j] \quad i, j = 1, 2, \ldots, 16 \tag{26}$$

$$P_{ij} = E[f_i f_j] \quad i, j = 1, 2, \ldots, 16 \tag{27}$$

and $[V]^{\mathrm{T}}$ = the transpose of $[V]$, and

$$P_{ij} = 0 \quad \text{except} \quad P_{33} = 2\pi S_{0x} \quad \text{and} \quad P_{44} = 2\pi S_{0y} \tag{28}$$

If the excitation is stationary (i.e. matrices $[F_1]$ and $[F_2]$ are independent of time), then the stationary response of the system can be obtained by solving the Liapunov matrix equation as

$$[H]\,[V]^{\mathrm{T}} + [V]\,[H]^{\mathrm{T}} + [P] = 0 \tag{29}$$

Utilizing the symmetry of the matrix $[V]$, equation (29) may be written as a system of 136 linear algebraic equations. Solution of these equations provides the stationary value of variance of the response.

NZ system

The NZ bearings are generally represented by bilinear characteristics. Several models have been proposed to represent the biaxial force–deformation of the NZ system (Yaska *et al.*[25], Park *et al.*[26]). For the present study, the model of Park *et al.*[26] is used for the force–deformation characteristics of the bearing. The restoring force, F_{xi} and F_{yi} of the ith elastomeric bearing (isotropic) in the x- and y-directions are given by the relation

$$\begin{Bmatrix} F_{xi} \\ F_{yi} \end{Bmatrix} = \alpha k_{0i} \begin{Bmatrix} U_{xi} \\ U_{yi} \end{Bmatrix} + (1-\alpha) k_{0i} \begin{Bmatrix} Z_{xi} \\ Z_{yi} \end{Bmatrix}$$

or

$$\{F_i\} = \alpha\, k_{0i}\,\{U_i\} + (1-\alpha)\, k_{0i}\,\{Z_i\} \tag{30}$$

in which U_{xi} and U_{yi} are the lateral displacements, and Z_{xi} and Z_{yi} are the hysteretic displacement components of the restoring forces in the x- and y-directions, respectively; α is the post- to preyielding stiffness ratio; $k_{0i} = Q_i/q$ is the initial stiffness; Q_i and q are the yield force and displacement of the bearings, respectively; Z_{xi} and Z_{yi} satisfy the following coupled nonlinear first-order differential equations

$$\{\dot{Z}_i\} = [G]\{\dot{U}_i\} \tag{31}$$

$$\frac{[G]}{\beta} = \begin{bmatrix} A - \mathrm{sgn}(\dot{U}_{xi})|Z_{xi}|Z_{xi} - \tau Z_{xi}^2 & -\mathrm{sgn}(\dot{U}_{yi})|Z_{yi}|Z_{xi} - \tau Z_{xi}Z_{yi} \\ -\mathrm{sgn}(\dot{U}_{xi})|Z_{xi}|Z_{yi} - \tau Z_{xi}Z_{yi} & A - \mathrm{sgn}(\dot{U}_{yi})|Z_{yi}|Z_{yi} - \tau Z_{yi}^2 \end{bmatrix} \tag{32}$$

in which, $A = A'/\beta$; $\tau = \tau'/\beta$ and β, τ' and A' are the parameters which control the shape and size of the hysteresis loop; and sgn denotes the signum function. The force–deformation behaviour of the elastomeric bearing can be modelled by correctly selecting the parameters Q, q, α, β, τ' and A'. The off-diagonal elements of the matrix $[G]$ in equation (32) provide the bilinear interaction between the restoring forces of the bearing. A solution without bidirectional interaction may be obtained by neglecting these off-diagonal terms.

The equations of motion of the base mass corresponding to three base degrees-of-freedom are written as

$$\begin{aligned}[M_b]\{\ddot{u}_b\} + [C_b]\{\dot{u}_b\} + \{F_b\} - [C]\{\dot{u}\} - [K]\{u\} \\ = -[M_b][T_g]\{\ddot{u}_g\}\end{aligned} \tag{33}$$

in which, $\{F_b\}$ is the restoring force of the isolator and is given by

$$\{F_b\} = \sum_i [T_{bi}]^{\mathrm{T}}\,\{F_i\} \tag{34}$$

in which $[T_{bi}]$ is the transformation matrix given by equation (21).

The nonlinear differential equation of the hysteretic restoring force given by equations (31) and (32) of the NZ system are linearized as

$$\begin{pmatrix} \dot{Z}_{xi} \\ \dot{Z}_{yi} \end{pmatrix} = \begin{bmatrix} p_{xxi} & p_{xyi} \\ p_{yxi} & p_{yyi} \end{bmatrix} \begin{pmatrix} Z_{xi} \\ Z_{yi} \end{pmatrix} + \begin{bmatrix} q_{xxi} & q_{xyi} \\ q_{yxi} & q_{yyi} \end{bmatrix} \begin{pmatrix} \dot{U}_{xi} \\ \dot{U}_{yi} \end{pmatrix}$$

or

$$\{\dot{Z}_i\} = [P_i]\,\{Z_i\} + [Q_i]\,\{\dot{U}_i\} \tag{35}$$

$$\{U_i\} = [T_{bi}]\,\{u_b\} \tag{36}$$

The elements of matrices $[P_i]$ and $[Q_i]$ are given in Appendix 1. These coefficients are obtained by minimizing the error in the mean square response assuming that the vectors $\{\dot{U}_i\}$ and $\{Z_i\}$ are jointly Gaussian distributed.

Following a procedure similar to that described for the LRB system, the augmented first-order system can be written for equations (12)–(14) and (33)–(36) similar to those of equation (22) in which, vectors $\{Y\}$ and $\{F\}$ are given by

$$\{Y\} = \{u_f, \dot{u}_f, u, u_b, \dot{u}, \dot{u}_b, \bar{Z}\} \tag{37}$$

$$\{F\} = \{0, f(t), 0, 0, 0, 0, 0\} \tag{38}$$

in which, $\{\bar{Z}\} = \{Z_1, \ldots, Z_i, \ldots, Z_n\}^{\mathrm{T}}$, n is the number of bearings. Note that the system property matrix $[H]$ becomes different from that for the system isolated with LRB because of equation (35). The explicit form of the matrix $[H]$ is given in Reference 23 and is included in Appendix 2.

It is to be noted that the system of differential equations for the NZ system is similar to that of equations (22)–(29) for the LRB system. However, the equations governing the response of the NZ system are nonlinear due to the dependence of the elements of the matrix $[H]$ (due to matrices $[P_i]$ and $[Q_i]$) upon the elements of the response covariance matrix $[V]$. Thus, an iterative procedure is adopted to obtain the accurate response.

R-FBI system

The R-FBI system uses the parallel action resiliency of rubber and the friction of Teflon-coated steel plates. During the sliding state of the structure, the governing equations of motion of the base raft in corresponding to three degrees-of-freedom are written as

$$[M_b]\{\ddot{u}_b\} + [C_b]\{\dot{u}_b\} + [K_b]\{u_b\} + \{F_b\} - [C]\{\dot{u}\} - [K]\{u\} = -[M_b][T_g]\{\ddot{u}_g\} \qquad (39)$$

in which, $[K_b]$ is the stiffness matrix due to the deformation of the rubber core of the R-FBI system and provides the restoring action; and $\{F_b\}$ is the force required to overcome friction between the Teflon-coated steel plates and is given by

$$\{F_b\} = \sum_i [T_{bi}]^T \{F_{si}\} \qquad (40)$$

$$\{F_{si}\} = \begin{bmatrix} \mu_i\, m_i\, g\, \mathrm{sgn}(\dot{U}_{xi}) \\ \mu_i\, m_i\, g\, \mathrm{sgn}(\dot{U}_{yi}) \end{bmatrix} \qquad (41)$$

in which, μ_i is the coefficient of friction between the Teflon-coated steel plates of the R-FBI system; m_i is the lumped mass of the structure on the ith R-FBI system; and g is the acceleration due to gravity; sgn denotes the signum function. The nonlinear equation for the frictional force given by equation (42) is linearized by the equivalent linearization technique[14] as

$$\{F_{si}\} = [c_{ei}]\, \{\dot{U}_i\} \qquad (42)$$

$$[c_{ei}] = \begin{bmatrix} \left(\frac{2}{\pi}\right)^{1/2} \frac{\mu_i m_i g}{\sigma \dot{u}_{xi}} & 0 \\ 0 & \left(\frac{2}{\pi}\right)^{1/2} \frac{\mu_i m_i g}{\sigma \dot{u}_{yi}} \end{bmatrix} \qquad (43)$$

in which, $\sigma \dot{u}_{xi}$ and $\sigma \dot{u}_{yi}$ are the RMS of the velocity of the ith R-FBI system in the x- and y-directions, respectively. Equation (43) is obtained by ignoring the bilinear interaction between the resistive force (due to friction) in the two principal directions. This is done because a mathematical model for the interaction cannot be derived for the type of force–deformation behaviour of the R-FBI as proposed by Mostaghel and Khodaverdian[11]. It is to be noted that the force–deformation behaviour for the sliding bearing as experimentally verified by Mokha *et al.*[27] is of the type proposed by Park *et al.*[26]. One of the major advantages of the model is that it can take into consideration the bidirectional interaction. However, the parameters of the empirical equations as given by Mokha *et al.*[27] cannot be adjusted to obtain the idealized force–deformation (rigid–linear) behaviour for the R-FBI system as given in Reference 11. Thus, such interactions cannot be easily derived for the R-FBI model of Mostaghel and Khodaverdian[11]. Further, linearization implicitly assumes that the system is mostly in the state of slip and therefore, is incapable of capturing the entire range of stick-slip conditions. The effect of this assumption on the response of the system is examined by comparing its result with those of a simulation study where the entire stick-slip condition is fully accounted for.

On combining equations (36), (40) and (42), the restoring force vector can be expressed as

$$\{F_{si}\} = [C_e]\, \{\dot{u}_b\} \qquad (44)$$

in which

$$[C_e] = \sum_i [T_{bi}]^T [C_{ei}][T_{bi}] \qquad (45)$$

The augmented governing equation can be written as a system of 16 first-order differential equations which are identical to equation (22) except the damping matrix $[C_b]$ of the isolator which should be replaced by $[C_b] + [C_e]$. Further, the elements of $[C_e]$ are dependent on the elements of the response matrix $[V]$, as a result, an iteration procedure is required to find the response. The explicit form of matrix $[H]$ is same as that of the LRB system with the changed damping matrix as mentioned above.

Eccentricities and frequencies of the isolator

Let k_{bi} represent the stiffness of the ith isolator (i.e. the initial stiffness for the NZ system and the stiffness of the rubber core in the case of the R-FBI system). Then the total stiffness of the isolator, K_b in the lateral direction is given by

$$K_b = \sum_i k_{bi} \qquad (46)$$

k_{bi} represents the initial stiffness of the ith bearing which is the same in both x- and y-directions. Let x_{bi} and y_{bi} denote the x- and y-co-ordinates of the ith bearing with respect to the CM of the base mass, respectively. Then, the torsional stiffness of the isolator system defined about the CM of the base mass is given by

$$K_{b\theta} = \sum_i k_{bi}\, (x_{bi}^2 + y_{bi}^2) \qquad (47)$$

in which the torsional stiffness of each individual bearing is assumed to be small and is neglected.

The initial isolator stiffness eccentricity, e_{bx} between the CR and CM of the base mass is given by

$$e_{bx} = \frac{1}{K_b} \sum_i k_{bi}\, x_i \qquad (48)$$

The two uncoupled base isolation frequencies are defined as

$$w_b = \left(\frac{K_b}{m + m_b}\right)^{1/2} \quad \text{and} \quad w_{b\theta} = \left(\frac{K_{b\theta}}{mr^2 + m_b\, r_b^2}\right)^{1/2} \qquad (51)$$

The values of w_b and $w_{b\theta}$ are used to characterize the stiffness characteristics of the isolation device.

Numerical study

The response of the torsionally coupled base isolated system to the stationary Kanai–Tajimi model of the El Centro, 1940 earthquake[28] is discussed. The response is investigated with respect to the following parameters: e_x/d, e_{bx}/d, w_θ/w_x and $w_{b\theta}/w_b$ which characterize the torsional coupling arising due to both superstructure and isolator eccentricities. With the assumed set of values for the parameters chosen, the inputs for the numerical study are selected as: the uncoupled lateral frequencies of the superstructure,

$w_x = w_y = 2\pi$ rad/s (or time period of 1.0 s); the mass ratio (m_b/m) is taken as 1.5; $b = d = 10$ m; and the modal damping for the superstructure is taken as 5% of the critical for all modes.

The LRB system is designed for a 2 s lateral time period[12,14] of the base isolated structure (superstructure treated as rigid body). Note that the linear behaviour of the LRB system is valid up to a certain range of deformation beyond which it depicts somewhat complex characteristics. The present study assumes linear behaviour of the LRB system throughout its deformation state and thus the question of bidirectional interaction, as in the case of the NZ system, does not exist. The parameters for the force–deformation behaviour of the NZ system (described by equations (30)–(32)) are taken as: $\beta = 0.5/q^2$; $\tau' = 0.5/q^2$; and $A' = 1$. The yield displacement, q of the NZ system is taken as 25 mm. This value of yield displacement provides (for total yield strength $= 0.08W$ and post- to preyielding stiffness ratio $= 0.2$; W = total weight of the building) a time period of the base isolated structure in the lateral direction as 2.5 s based on the postyield stiffness of the isolator (as recommended in UBC-1991). For a building isolated by an R-FBI system the lateral base isolation frequency is taken as 0.5π rad/s (which corresponds to a 4 s time period of the base isolation system). The coefficient of friction between the Teflon-coated steel plate of the R-FBI system is taken as 0.04. The viscous damping for all base isolation devices is taken as 10%. These values are taken from Reference 13 in which the sensitivity of the various base isolation systems has been investigated. The properties of these isolation systems are not alike and therefore, the relative efficiencies of the three systems are not the objective of the present study. The parametric investigations are primarily intended to study the effect of linearization of the nonlinear behaviour of the NZ and R-FBI systems, and to study the isolation characteristics of each type of isolation to parametric variations. Since only the NZ system considers the bidirectional interaction, it is important to show the effect of this interaction on the response. Therefore, the responses of the system with NZ bearings are obtained with and without the interaction effect and are compared. For the parametric study, the responses without interaction are plotted, as the results for the LRB and R-FBI systems are obtained without interaction effect.

As there is generally no reason to use different spectral and intensity functions for the two orthogonal directions, it may be reasonable to assume that the filter parameters are the same for both directions (i.e. $w_{gx} = w_{gy} = w_g$, $\xi_{gx} = \xi_{gy} = \xi_g$) and $S_{0x} = S_{0y} = S_0$). The parameters of the PSDF of the stationary ground acceleration are taken corresponding to the El Centro, California earthquake of May 1940 which lasted for 25 s[28]. The parameters are $S_0 = 0.07$ m^2/s^3, $\xi_g = 0.5$ and $w_g = 4\pi$ rad/s.

The response quantities of interest are the response ratio R and the stationary RMS base displacements. The response ratio R is defined as the ratio of the RMS response of the asymmetric base isolated system to the RMS response of the corresponding fixed base system.

Figures 2 and *3* show the variations of the response ratio R and the RMS base displacements against the superstructure eccentricity, e_x/d. In order to assess the effect of linearization, the responses obtained from the simulation analysis are also plotted in the same figure. It is observed that there is generally a good agreement between the linearization results and those obtained by the simulation analysis. This shows that the assumptions made in the linearization process, especially for the R-FBI system do not affect the response significantly. Further, the ratio R for the response u_x and the displacement u_{bx}, which are parallel to the direction of eccentricity, are not influenced by the eccentricity. The ratio R for the response u_y increases with increase in eccentricity. The same observations hold good for the variation of the ratio R for the response u_θ. This indicates that the torsional coupling increases the lateral response of the superstructure perpendicular to the direction of eccentricity. Thus, the effectiveness of base isolation for an asymmetric system is overestimated if the eccentricity is ignored. The RMS base displacement, u_{by} does not significantly vary with e_x/d. The RMS base torsion increases with an increase in eccentricity.

In *Figure 4*, the variations of the response ratio R for u_y and u_θ against w_θ/w_x are shown. The ratio R for the response u_y increases with increase in w_θ/w_x (between 0.5 and 1) showing a decrease in the effectiveness of base isolation for the response u_y. When $w_\theta/w_x > 1$, u_y decreases (in the range $1 < w_\theta/w_x \leq 1.5$). As w_θ/w_x increases (in the range of $0.5 \leq w_\theta/w_x \leq 1$), the ratio R for u_θ decreases significantly, showing the effectiveness of base isolation in reducing the torsional response. It is to be noted that the ratio R for the responses u_y and u_θ remains almost insensitive to the variation of the w_θ/w_x ratio beyond 1.5. This is expected since for a torsionally stiff system ($w_\theta/w_x \gg 1$) the effect of torsional coupling is considerably reduced. As a result, R for u_y and u_θ become insensitive to the variation of w_θ/w_x. *Figure 5* shows the variation of RMS base displacements, u_{by} and base torsion $u_{b\theta}$ against w_θ/w_x. The response u_{by} is not much influenced by the variation w_θ/w_x ratio for both $e_{bx}/d = 0$ and $e_{bx}/d = 0.15$. However, for the LRB and the NZ system when $e_{bx}/d = 0.15$ the base torsion increases in the range ($0.5 < w_\theta/w_x < 1$) and beyond that it remains almost constant. *Figures 4* and *5* also show the difference between the responses with and without interaction being considered in the analysis for the NZ system. The interaction effect reduces the superstructure response but increases the base displacements. However, the difference between the two responses is not very significant. Further, the trend in variation of the response with w_θ/w_x remains the same for both cases.

The variation of the response ratio R for response u_y and u_θ against the isolator eccentricity (e_{bx}/d) is plotted in *Figure 6*. The ratio R for the response u_y, in the case of the LRB and NZ systems, decreases with increase in e_{bx}/d. However, the ratio R for the torsional response u_θ increases with increase in e_{bx}/d showing that the isolator eccentricity increases the superstructure torsional response. Further, the ratio R for both responses is found to be insensitive to the variation of e_{bx}/d in the case of the R-FBI system.

Figure 7 shows the variation of the RMS base displacement, u_{by} and the base torsion $u_{b\theta}$ against isolator eccentricity. The response u_{by} increases mildly with increases in e_{bx}/d for the R-FBI system. For the LRB and NZ systems, it increases with increase in e_{bx}/d when $w_{b\theta}/w_b = 2$ but for $w_{b\theta}/w_b = 1$ it decreases (in the range $0 < e_{bx}/d < 0.15$) and then it increases with further increase of eccentricity ($e_{bx}/d > 0.15$). The base torsion for the LRB and the NZ system increases significantly with the increase in the isolator eccentricity. It also increases when the structure is isolated by an R-FBI system but it is less pronounced.

In *Figure 8*, the variation of the response ratio R for u_y and u_θ is plotted against the $w_{b\theta}/w_b$ ratio. The ratio R for

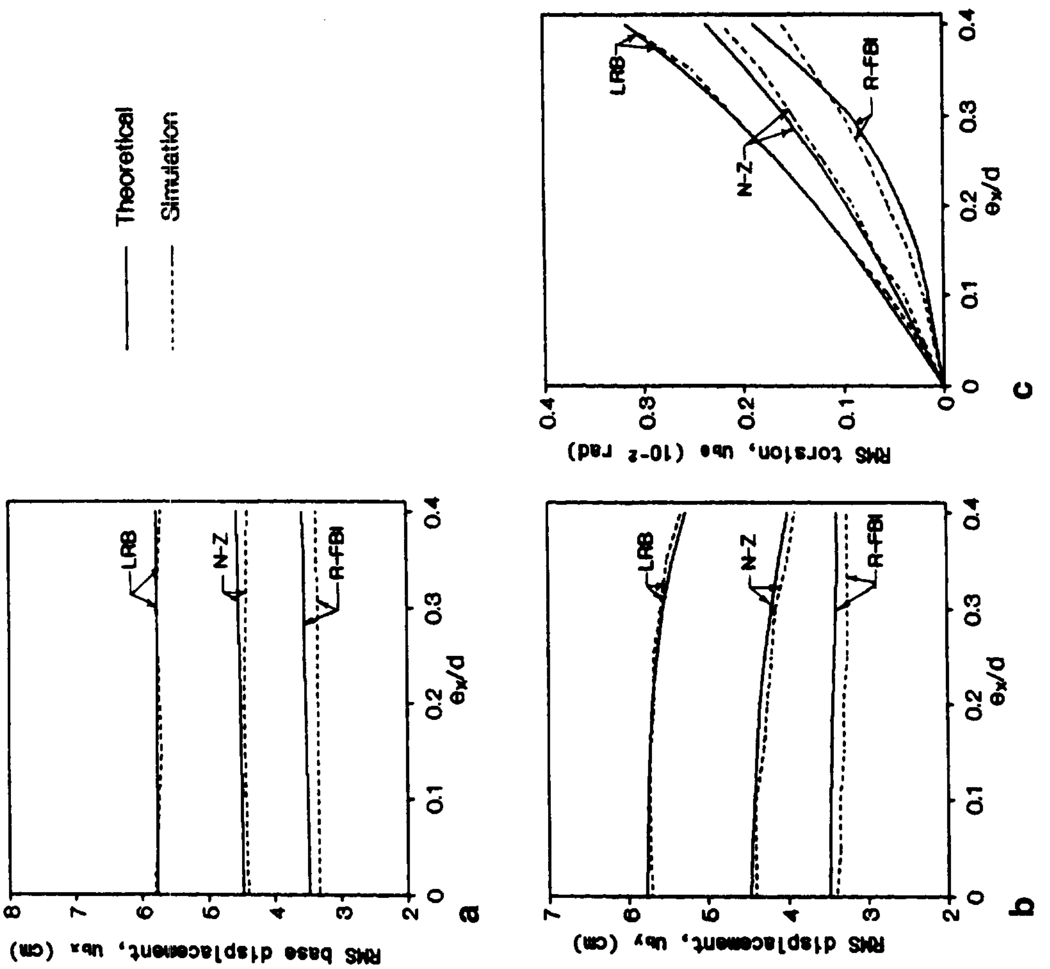


Figure 3 Variation of base displacements against e_x/d; $w_\theta/w_x = 1$ and $w_{b\theta}/w_b = 1$

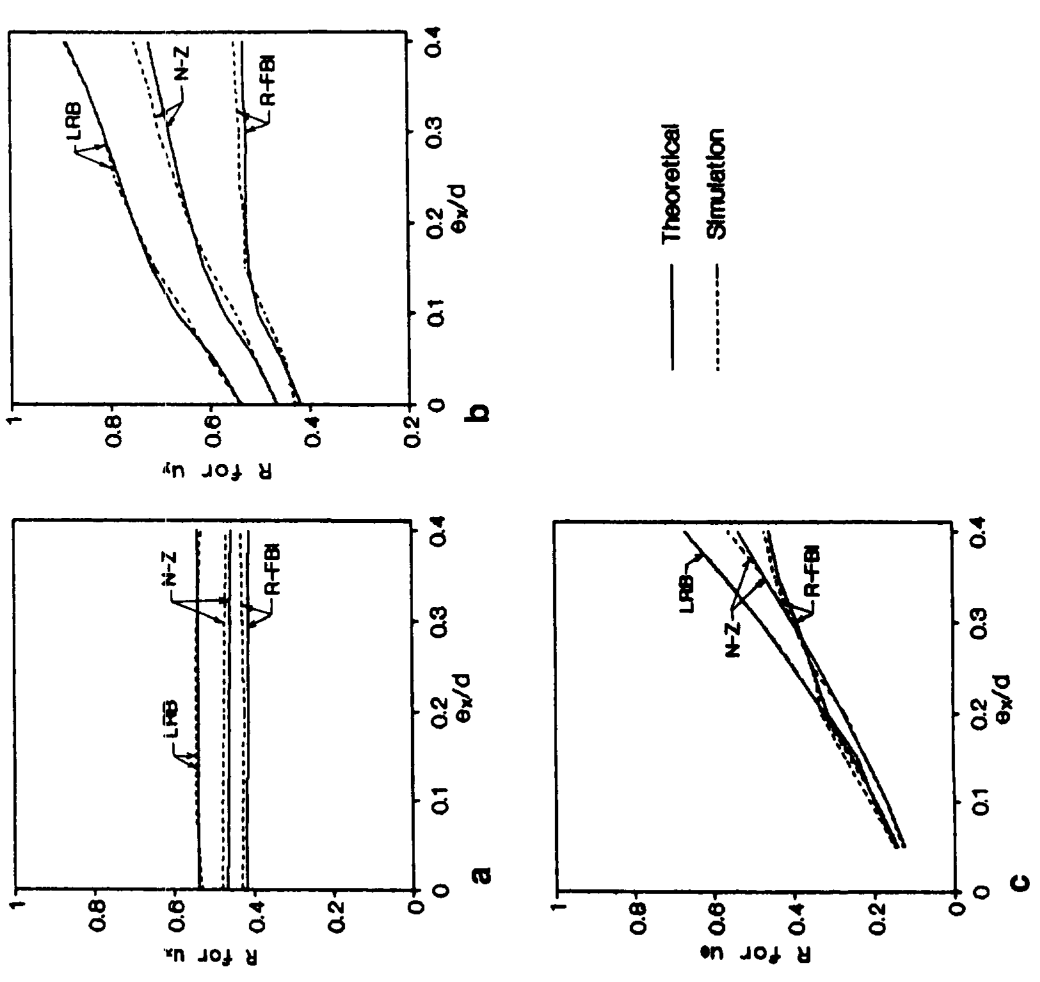


Figure 2 Variation of response ratio *R* against superstructure eccentricity; e_x/d; $w_\theta/w_x = 1$ and $w_{b\theta}/w_b = 1$

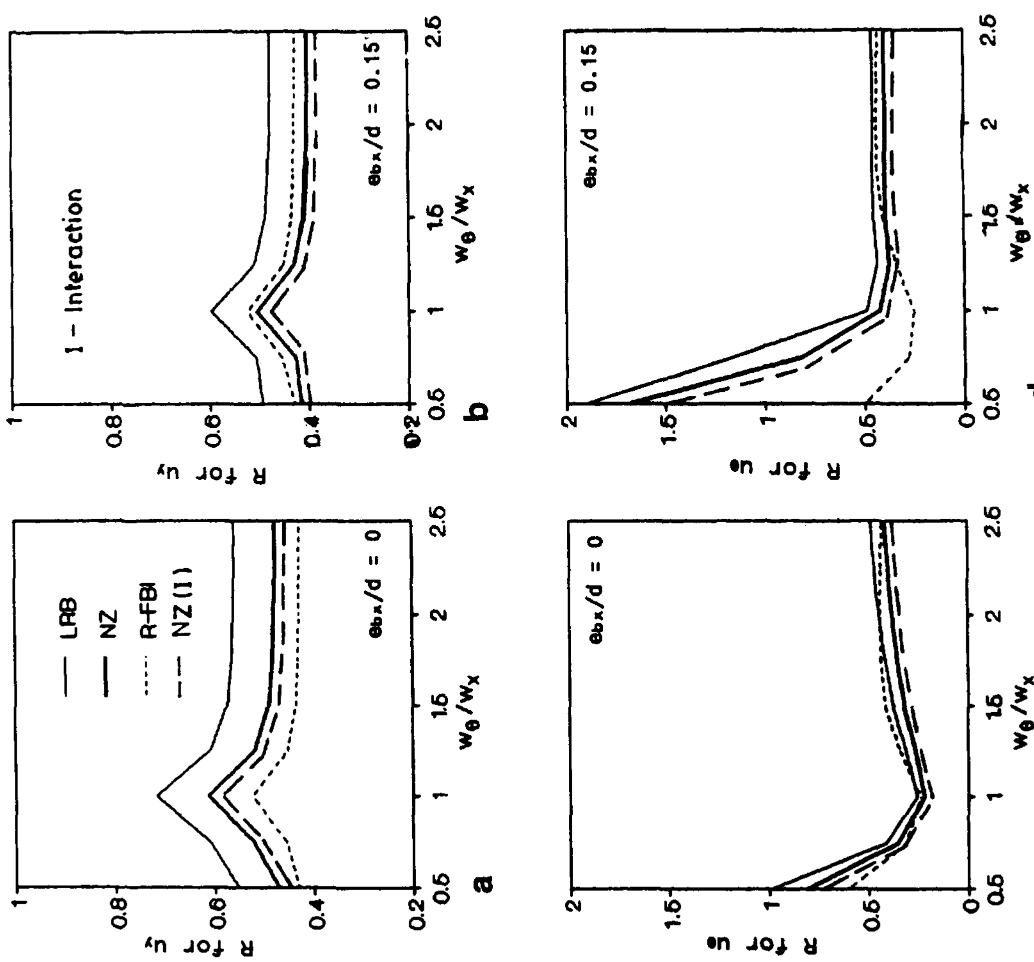


Figure 4 Variation of response ratio R against w_θ/w_x ratio for response u_y and u_θ; $e_x/d = 0.15$ and $w_{b\theta}/w_b = 1$

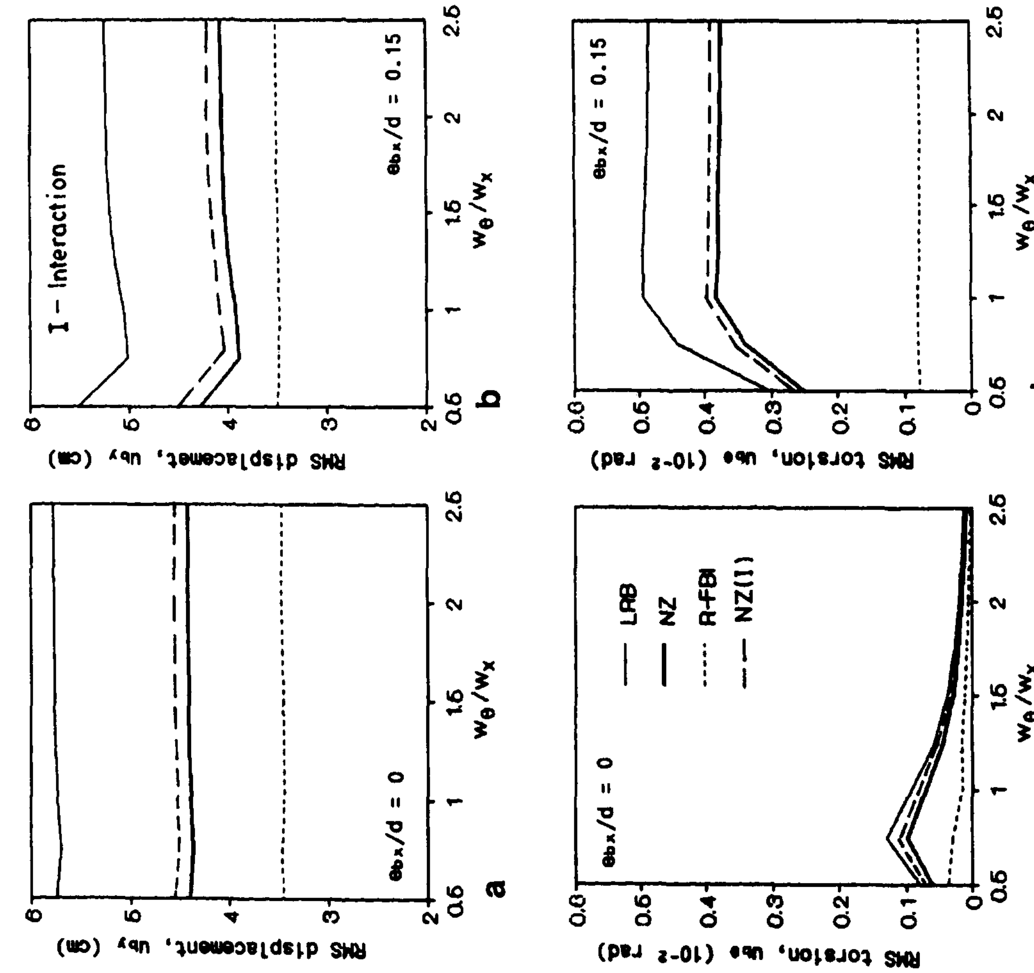


Figure 5 Variation of base displacements u_{by} and $u_{b\theta}$ against w_θ/w_x ratio; $e_x/d = 0.15$ and $w_{b\theta}/w_b = 1$

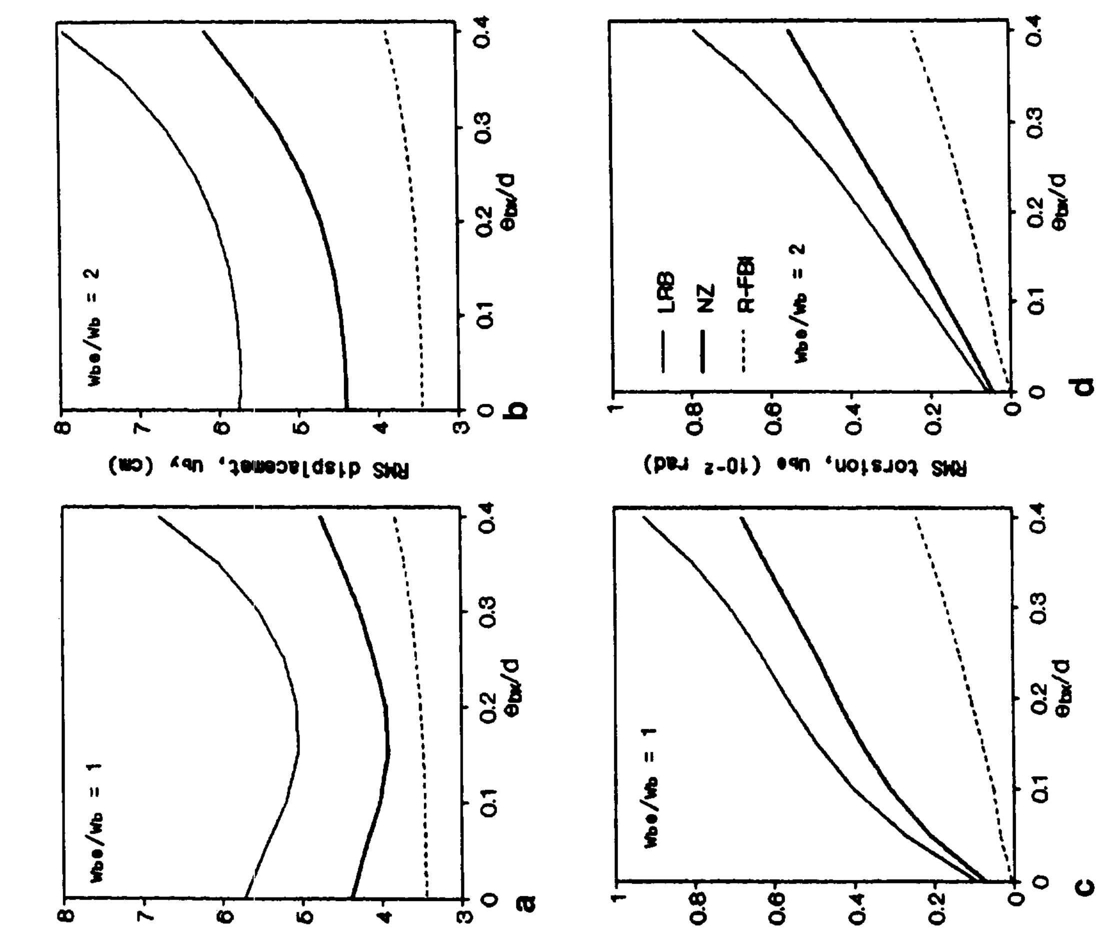


Figure 7 Variation of base displacements u_{by} and $u_{b\theta}$ against isolator eccentricity e_{bx}/d, $e_x/d = 0.15$ and $w_\theta/w_x = 1$

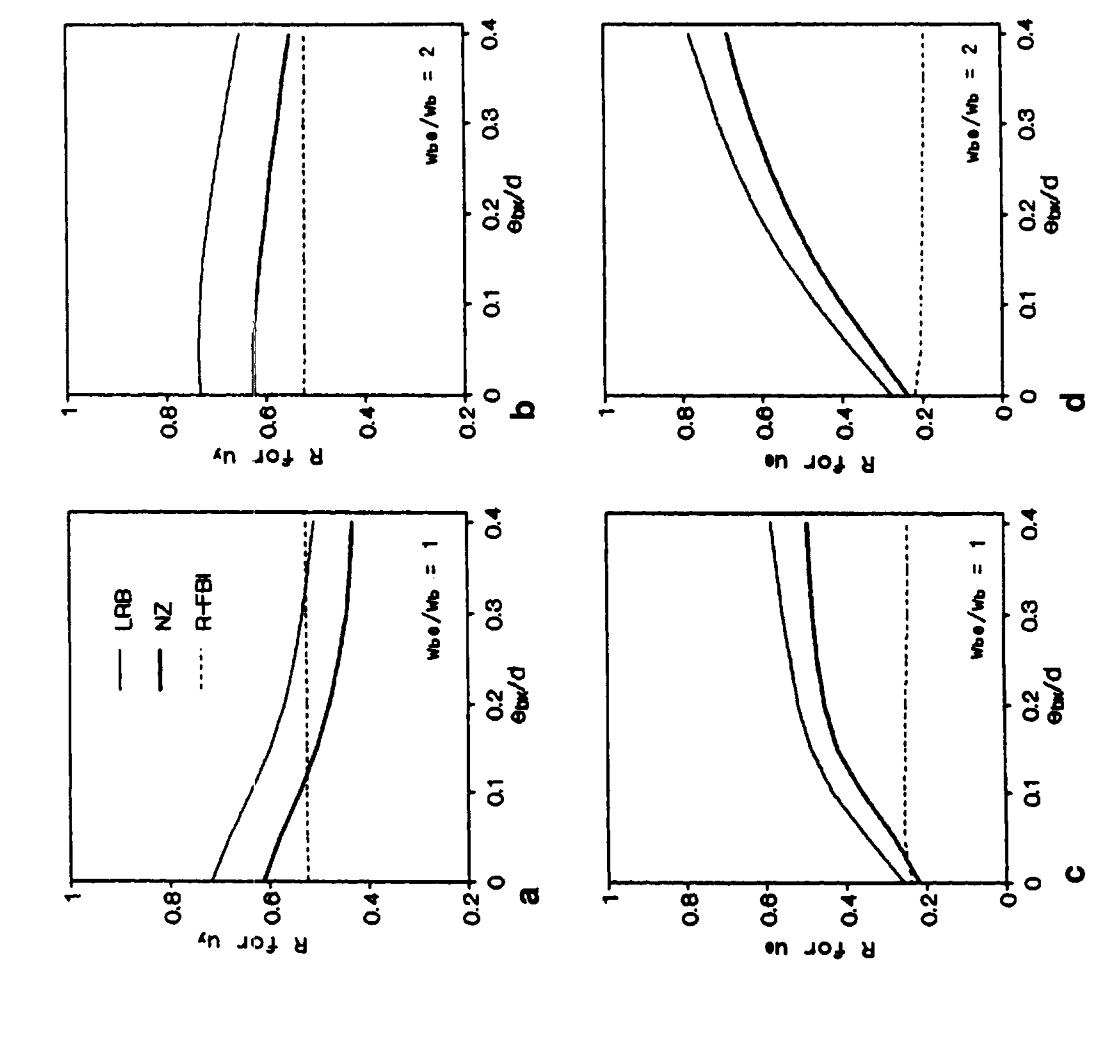


Figure 6 Variation of response ratio R against isolator eccentricity e_{bx}/d for response u_y and u_θ; $e_x/d = 0.15$ and $w_\theta/w_x = 1$

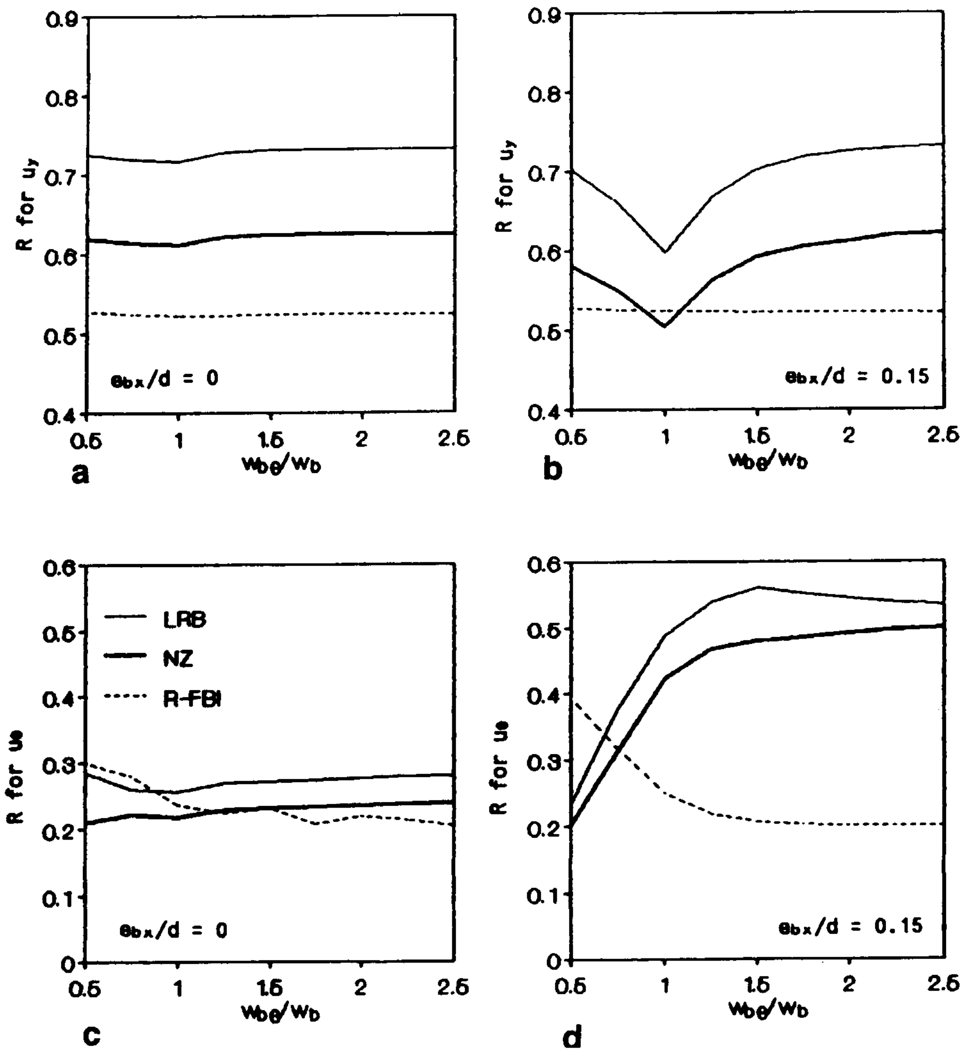


Figure 8 Variation of response ratio R against $w_{b\theta}/w_b$ ratio for response u_y and u_θ; $e_x/d = 0.15$ and $w_\theta/w_x = 1$

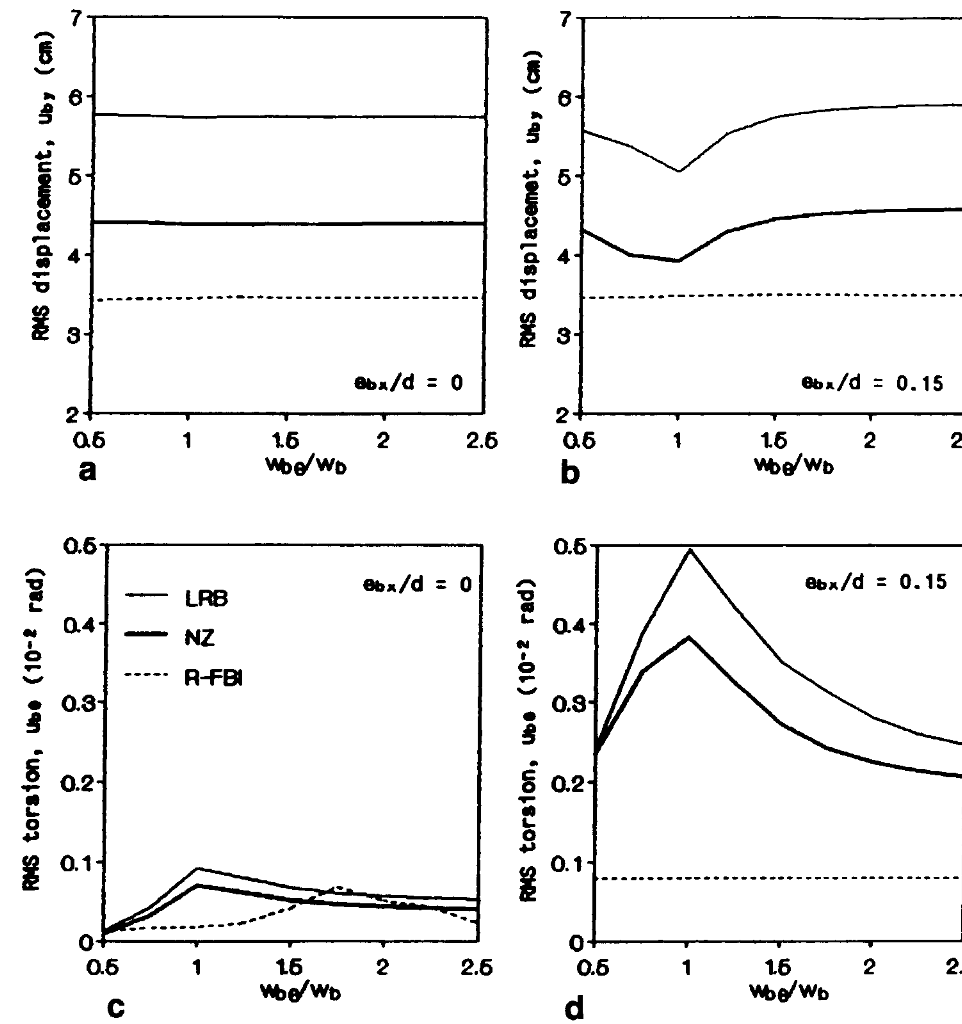


Figure 9 Variation of base displacements u_{by} and $u_{b\theta}$ against $w_{b\theta}/w_b$; $e_x/d = 0.15$ and $w_\theta/w_x = 1$

both responses is not influenced by the $w_{b\theta}/w_b$ ratio when the isolation system is symmetric (i.e. $e_{bx}/d = 0$) as shown in *Figures 8(a)* and *8(c)*. For asymmetric isolators (LRB and NZ systems) the ratio R for response u_y decreases (in the range of $w_{b\theta}/w_b \le 1$) as $w_{b\theta}/w_b$ increases. But beyond that, it increases with the increase in the $w_{b\theta}/w_b$ ratio. For the R-FBI system, it remains insensitive to the variation of $w_{b\theta}/w_b$. The ratio R for the torsional response, u_θ increases with increase in the $w_{b\theta}/w_b$ (in the range $w_{b\theta}/w_b < 1.5$). For the R-FBI system, a reverse trend of variation is observed for the response u_θ.

In *Figure 9*, the variation of the base displacement, u_{by} and base torsion, $u_{b\theta}$ is plotted against the $w_{b\theta}/w_b$ ratio. $w_{b\theta}/w_b$ does not significantly influence the base displacements for the symmetric isolator. For the asymmetric isolator, the base displacement (u_{by}) has a definite minima for $w_{b\theta}/w_b = 1$. The base torsion, $u_{b\theta}$ is significantly influenced by the $w_{b\theta}/w_b$ ratio leading to maximum amplification at $w_{b\theta}/w_b = 1$. It is to be noted that the base displacement and base torsion remain insensitive to the variation of $w_{b\theta}/w_b$ when the structure is isolated by an R-FBI system.

Conclusions

The stochastic response of a one-storey model of an asymmetric building, isolated by different base isolation systems, to random excitation is presented by directly solving the stochastic differential equation of the combined system. The responses are obtained for a stationary stochastic model of the El Centro earthquake and compared with the simulation results in order to study the effect of the linearization technique in the stochastic analysis. The parametric behaviour of the building model for a set of important parametric variations is investigated. The parametric study is limited by choosing a set of superstructure and bearing characteristics (namely time periods, damping etc.). However, from the trend of the results, the following conclusions may be drawn.

The effect of linearization on the RMS response of the systems with NZ and R-FBI isolators is not significant; the trend of variation of responses with parameters is same as that observed for simulation studies.

The bidirectional force interaction in the case of NZ bearings decreases the superstructure response and increases the base displacements. However, the effect is not very significant for the superstructure and bearing properties being used in the numerical analysis.

The stochastic response of a structure isolated by an R-FBI system is less sensitive to the parametric variations.

The effectiveness of asymmetric (torsionally coupled) base isolation is less than that of the corresponding symmetric system especially for reducing the superstructure displacement perpendicular to the direction of eccentricity.

The effectiveness of base isolation is reduced for higher eccentricity of the superstructure. However, the eccentricity of the superstructure does not have any significant influence on base displacements.

The superstructure eccentricity has less impact on the isolator response whereas the isolator eccentricity significantly influences the responses of both superstructure and isolator.

The isolator eccentricity decreases the effectiveness of isolation for torsional deformation, increases the base displacement and torsion and decreases the superstructure displacement perpendicular to the direction of eccentricity.

The ratio of torsional to lateral base isolation frequencies ($w_{b\theta}/w_b$) influences the response of the system only when the isolator system is asymmetric; the translational displacement of the base becomes a minimum for $w_{b\theta}/w_b \approx 1$.

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Appendix 1

The coefficients for equation (35) which are taken from Reference 26, are as follows

$$q_{xxi} = -\sigma z_{xi}^2\left[\frac{2\beta}{\pi}\phi_{xi} - \frac{\beta}{\pi}\sin 2\phi_{xi}\right] - A'$$

$$p_{xxi} = -\sigma \dot{u}_{xi}\sigma z_{xi}\left|\frac{4\beta}{\pi}(1-\rho\dot{u}_{xi}z_{xi}^2)^{1.5} + \left[\frac{4\beta}{\pi}\phi_{xi} - \frac{2\beta}{\pi}\sin 2\phi_{xi}\right] + \frac{2\beta}{\pi}(\cos\psi_{xi} + \psi_{xi}\rho\dot{u}_{xi}z_{xi})\right|$$

$$q_{xyi} = -\sigma z_{xi}\,\sigma z_{yi}\,\rho z_{xi}\,z_{yi}\,\beta$$

$$p_{xyi} = -\sigma u_{yi}\,\sigma z_{yi}\,\rho\dot{u}_{yi}\,z_{xi}\,\beta$$

in which

$$\phi_{xi} = \tan^{-1}[1 - \{1 - (\rho \dot{u}xizxi)^2\}^{1/2}/\rho\dot{u}xizxi]$$

$$\psi_{yi} = \sin^{-1}\rho\dot{u}_{yi}z_{yi}$$

$$\rho\dot{u}_{xi}z_{xi} = E[\dot{U}xi, Zxi]/\sigma\dot{u}_{xi}z_{xi}$$

The coefficients p_{yyi}, p_{yxi}, q_{yyi} and q_{yxi} are obtained by changing the above subscripts x and y.

Appendix 2

The matrix $[H]$ in equation (22) for the LRB system is given by

$$[H] = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ -[F_2] & -[F_1] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & H_{53} & H_{54} & H_{55} & H_{56} \\ H_{61} & H_{62} & H_{63} & H_{64} & H_{65} & H_{66} \end{bmatrix}$$

where

$$H_{53} = -[M]^{-1}[K] - [M_b]^{-1}[K],$$

$$H_{54} = [M_b]^{-1}[K_b]$$

$$H_{55} = -[M]^{-1}[C] - [M_b]^{-1}[C],$$

$$H_{56} = [M_b]^{-1}[C_b]$$

$$H_{61} = [T_g][F_2], \quad H_{62} = [T_g][F_1]$$

$$H_{63} = [M_b]^{-1}[K], \quad H_{64} = -[M_b]^{-1}[K_b]$$

$$H_{65} = [M_b]^{-1}[C] \quad \text{and} \quad H_{66} = -[M_b]^{-1}[C_b]$$

It is to be noted that 0 and I represent the null and identity matrices of appropriate size, respectively.

The system matrix $[H]$ in equation (22) for the NZ system is given by

$$[H] = \left[\begin{array}{cccccc|c} 0 & I & 0 & 0 & 0 & 0 & \\ -[F_2] & -[F_1] & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & I & 0 & H_{UZ} \\ 0 & 0 & 0 & 0 & 0 & I & \\ 0 & 0 & H_{53} & H_{54} & H_{55} & H_{56} & \\ H_{61} & H_{62} & H_{63} & H_{64} & H_{65} & H_{66} & \\ \hline & & & H_{zu} & & & H_{zz} \end{array}\right]$$

where

$$H_{53} = -[M]^{-1}[K] - [M_b]^{-1}[K],$$

$$H_{54} = \alpha k_{bi}[M_b]^{-1}\sum_i [T_{bi}]^T[T_{bi}]$$

$$H_{55} = -[M]^{-1}[C] - [M_b]^{-1}[C],$$

$$H_{56} = [M_b]^{-1}[C_b]$$

$$H_{61} = [T_g][F_2], \quad H_{62} = [T_g][F_1]$$

$$H_{63} = [M_b]^{-1}[K], \quad H_{64} = \alpha k_{bi}[M_b]^{-1}\sum_i [T_{bi}]^T[T_{bi}]$$

$$H_{65} = [M_b]^{-1}[C] \quad \text{and} \quad H_{66} = -[M_b]^{-1}[C_b]$$

$$H_{uz} = \begin{bmatrix} 0 & \cdot\cdot & 0 & \cdot\cdot & 0 \\ 0 & \cdot\cdot & 0 & \cdot\cdot & 0 \\ 0 & \cdot\cdot & 0 & \cdot\cdot & 0 \\ 0 & \cdot\cdot & 0 & \cdot\cdot & 0 \\ H_{UZ51} & \cdot\cdot & H_{UZ5i} & \cdot\cdot & H_{UZ5n} \\ H_{UZ61} & \cdot\cdot & H_{UZ6i} & \cdot\cdot & H_{UZ6n} \end{bmatrix}$$

where, $H_{UZ5i} = -H_{UZ6i} = \alpha k_{bi}[M_b]^{-1}\ \Sigma_I[T_{bi}]^T[T_{bi}]$ and n is the number NZ isolator provided.

$$H_{zu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & [Q_1][T_{b1}] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & [Q_i][T_{bi}] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & [Q_n][T_{bn}] \end{bmatrix}$$

$$H_{zz} = \begin{bmatrix} [P_1] & \cdot\cdot & 0 & \cdot\cdot & 0 \\ . & \cdot\cdot & . & \cdot\cdot & . \\ . & \cdot\cdot & . & \cdot\cdot & . \\ 0 & & [P_i] & 0 & 0 \\ . & \cdot\cdot & . & \cdot\cdot & . \\ . & \cdot\cdot & . & \cdot\cdot & . \\ 0 & \cdot\cdot & 0 & \cdot\cdot & [P_n] \end{bmatrix}$$