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Approximate analysis methods for asymmetric plan base-isolated buildings

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SUMMARY

An approximate method for linear analysis of asymmetric-plan, multistorey buildings is specialized for a single-storey, base-isolated structure. To find the mode shapes of the torsionally coupled system, the Rayleigh–Ritz procedure is applied using the torsionally uncoupled modes as Ritz vectors. This approach reduces to analysis of two single-storey systems, each with vibration properties and eccentricities (labelled 'effective eccentricities') similar to corresponding properties of the isolation system or the fixed-base structure. With certain assumptions, the vibration properties of the coupled system can be expressed explicitly in terms of these single-storey system properties. Three different methods are developed: the first is a direct application of the Rayleigh–Ritz procedure; the second and third use simplifications for the effective eccentricities, assuming a relatively stiff superstructure.

The accuracy of these proposed methods and the rigid structure method in determining responses are assessed for a range of system parameters including eccentricity and structure flexibility. For a subset of systems with equal isolation and structural eccentricities, two of the methods are exact and the third is sufficiently accurate; all three are preferred to the rigid structure method. For systems with zero isolation eccentricity, however, all approximate methods considered are inconsistent and should be applied with caution, only to systems with small structural eccentricities or stiff structures. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: asymmetric; base isolation; buildings; eccentricity; torsion; Rayleigh–Ritz

INTRODUCTION

While a detailed, non-linear, multi-degree of freedom response history analysis is now required for the seismic design of isolated buildings [1], simplified analysis procedures are desirable for preliminary design and can provide the engineer with an understanding of the response characteristics of the system. The main features of the dynamics of base-isolated buildings

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can be captured by analysing a single-storey building with a linear isolation system [2]. When the system plan is asymmetric, the lateral and torsional motions are coupled, thereby turning a two-degree-of-freedom (DOF) system into a six-DOF system. Explicit, analytical solutions that were possible for the two-DOF system are no longer feasible, thus the system must be analysed numerically, an approach that does not permit analytically relating vibration modes, frequencies, and responses to the system parameters. Another approach allows for an explicit interpretive solution to the problem but ignores the contribution of the structure, which is modelled as a rigid mass [2]. This rigid structure (RS) method gives reasonable estimates of structural forces and isolator deformations for a range of systems, however, it is inaccurate when the flexibility of the structure is significant, or when the structural eccentricity is the major source of torsion in the system. A procedure that incorporates structural effects while retaining the simplicity of an explicit method is needed.

An approximate procedure for analysis of asymmetric-plan multistorey buildings has been developed previously [3], which is based on the Rayleigh–Ritz procedure [4]. In this procedure, vibration modes and frequencies of the asymmetric-plan building are determined by using the torsionally uncoupled modes as Ritz vectors, thus reducing the eigenproblem of order $3N$ ($N =$ number of storeys) to order N, with an additional N eigenproblems each of order 3. Applying this approach to the six-DOF isolated system reduces the complexity of this problem. The vibration modes and frequencies of the torsionally uncoupled system are found by analysing a two-DOF system in each of the x-, θ- and y-directions. Next, the vibration properties of the asymmetric-plan system are determined approximately by analysing two single-storey systems, each with three DOF, arising from applications of the Rayleigh–Ritz procedure with the Ritz vectors grouped strategically. The equations can be solved directly, leading to explicit expressions for the vibration modes and frequencies in terms of given parameters of the system.

This work aims to develop the Rayleigh–Ritz procedure for the idealized six-DOF base-isolated system described above. Thereafter, two simplified procedures are developed that make additional modifications to the eccentricities that arise in solving the single-storey systems. The three methods are compared against 'exact' analysis and the RS method for a variety of systems, with varying structural flexibility and eccentricities. The purpose is twofold: to measure the accuracy of these new methods against standard procedures, and to gain a deeper understanding of the dynamics of the system through its dissection as achieved by the Rayleigh–Ritz method.

EQUATIONS OF MOTION

The system under consideration is a single-storey structure supported on isolators (Figure 1). The top mass m and base mass $m_{\rm b}$ are rigid decks supported on axially inextensible, massless columns and isolators. For simplicity, linear models of both the structure and the isolation system are adopted, with structure stiffnesses k_x, k_θ, k_y and isolation system stiffnesses k_{bx}, $k_{b\theta}$, k_{by}, in the x-, θ- and y-directions, respectively. The centre of mass (CM) of the top mass and the base mass are assumed to be vertically aligned. The planwise distributions of structural stiffnesses and isolation stiffnesses are asymmetric relative to the x- and y-axes passing through the CM. Thus, the centre of rigidities (CR) of both the structure and the isolation system are eccentric relative to the CM; structural eccentricities are e_x, e_y and isolation eccentricities are e_{bx}, e_{by} in the x- and y-directions.

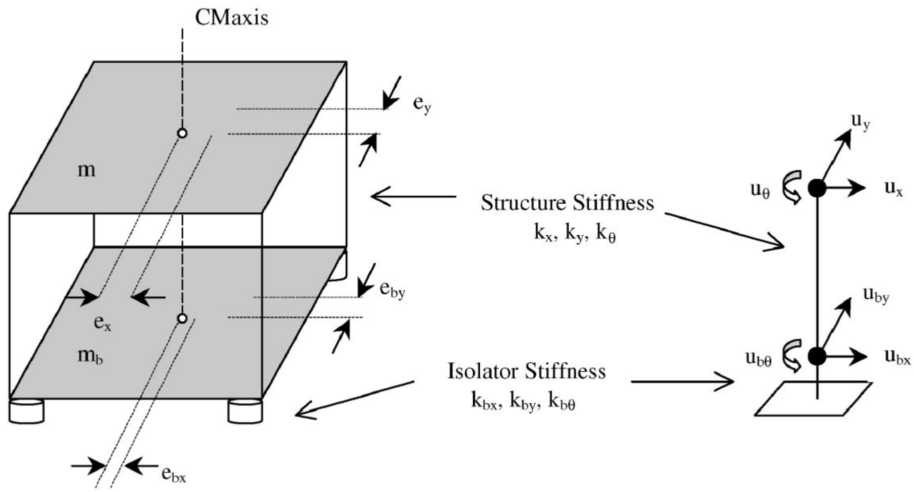

Figure 1. Simple structure on isolators.

To simplify the equations of motion, equality of the base mass and top mass is assumed ($m_b = m$). After multiplication by the inverted mass matrix, the equations of motion of the system subjected to x-direction ground acceleration may be expressed as

$$\ddot{\boldsymbol{u}} + \tilde{\boldsymbol{k}}\boldsymbol{u} = -\boldsymbol{\iota}\ddot{u}_{gx} \tag{1}$$

where

$$\boldsymbol{u} = \begin{Bmatrix} \boldsymbol{u}_x \\ r\boldsymbol{u}_\theta \\ \boldsymbol{u}_y \end{Bmatrix}, \quad \tilde{\boldsymbol{k}} = \begin{bmatrix} \tilde{\boldsymbol{k}}_{xx} & -\tilde{\boldsymbol{k}}_{x\theta} & \boldsymbol{0} \\ -\tilde{\boldsymbol{k}}_{x\theta} & \tilde{\boldsymbol{k}}_{\theta\theta} & \tilde{\boldsymbol{k}}_{y\theta} \\ \boldsymbol{0} & \tilde{\boldsymbol{k}}_{y\theta} & \tilde{\boldsymbol{k}}_{yy} \end{bmatrix}$$

$\boldsymbol{u}_x, \boldsymbol{u}_\theta$ and $\boldsymbol{u}_y$ are 2×1 displacement vectors, with the first DOF at the base and the second DOF at the roof; r is the radius of gyration, assumed to be identical for both the top mass and the base mass; and $\boldsymbol{\iota}$ the influence vector containing 1's in the x-DOF. In Equation (1), the submatrices of $\tilde{\boldsymbol{k}}$ are

$$\tilde{\boldsymbol{k}}_{xx} = \begin{bmatrix} \omega_x^2 + 2\omega_{bx}^2 & -\omega_x^2 \\ -\omega_x^2 & \omega_x^2 \end{bmatrix}, \quad \tilde{\boldsymbol{k}}_{\theta\theta} = \begin{bmatrix} \omega_\theta^2 + 2\omega_{b\theta}^2 & -\omega_\theta^2 \\ -\omega_\theta^2 & \omega_\theta^2 \end{bmatrix}, \quad \tilde{\boldsymbol{k}}_{yy} = \begin{bmatrix} \omega_y^2 + 2\omega_{by}^2 & -\omega_y^2 \\ \omega_y^2 & \omega_y^2 \end{bmatrix}$$

$$\tilde{\boldsymbol{k}}_{x\theta} = \begin{bmatrix} \frac{e_y}{r}\omega_x^2 + 2\frac{e_{by}}{r}\omega_{bx}^2 & -\frac{e_y}{r}\omega_x^2 \\ -\frac{e_y}{r}\omega_x^2 & \frac{e_y}{r}\omega_x^2 \end{bmatrix}, \quad \tilde{\boldsymbol{k}}_{y\theta} = \begin{bmatrix} \frac{e_x}{r}\omega_y^2 + 2\frac{e_{bx}}{r}\omega_{by}^2 & -\frac{e_x}{r}\omega_y^2 \\ -\frac{e_x}{r}\omega_y^2 & \frac{e_x}{r}\omega_y^2 \end{bmatrix}$$

where ω_x, ω_θ, and ω_y are the natural frequencies of the fixed-base structure for uncoupled vibration in the x-, θ- and y-directions:

$$\omega_x = \sqrt{\frac{k_x}{m}}, \quad \omega_\theta = \sqrt{\frac{k_\theta}{mr^2}}, \quad \omega_y = \sqrt{\frac{k_y}{m}}$$

and ω_{bx}, $\omega_{b\theta}$ and ω_{by} are natural frequencies of the isolation system (assuming a rigid building) for uncoupled vibration in the x-, θ- and y-directions:

$$\omega_{bx} = \sqrt{\frac{k_{bx}}{(m+m_\mathrm{b})}}, \quad \omega_{b\theta} = \sqrt{\frac{k_{b\theta}}{(m+m_\mathrm{b})r^2}}, \quad \omega_{by} = \sqrt{\frac{k_{by}}{(m+m_\mathrm{b})}}$$

DEVELOPMENT OF RAYLEIGH–RITZ METHOD

The development of an approximate method is based on a previous approach, which utilizes the Rayleigh–Ritz concept [3]. Here, equations for multistorey buildings are specialized for a single-storey isolated structure. The results of perturbation analysis suggest that the vibration frequencies and modes of the torsionally coupled system may be approximated as a linear combination of the frequencies and modes of the corresponding torsionally uncoupled system.

Natural frequencies and modes of torsionally uncoupled system

To this end, the frequencies and modes of the torsionally uncoupled system are found. In this system all eccentricities are zero, and $\tilde{\boldsymbol{k}}_{x\theta}$ and $\tilde{\boldsymbol{k}}_{y\theta}$ in Equation (1) are zero matrices. This results in a separate eigenvalue equation, yielding two modes and frequencies, for each direction. For example, the eigenvalue equation for motion in the x-direction is

$$(\tilde{\boldsymbol{k}}_{xx} - \omega^2 \boldsymbol{I})\boldsymbol{\psi}_{xj} = \boldsymbol{0} \tag{2a}$$

or

$$\begin{bmatrix} \omega_x^2 + 2\omega_{bx}^2 - \omega^2 & -\omega_x^2 \\ -\omega_x^2 & \omega_x^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} \psi_{x,1j} \\ \psi_{x,2j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2b}$$

The solution of Equation (2) provides the frequencies

$$\omega_{x1}^2 = \omega_x^2 + \omega_{bx}^2 - \sqrt{\omega_x^4 + \omega_{bx}^4}, \quad \omega_{x2}^2 = \omega_x^2 + \omega_{bx}^2 + \sqrt{\omega_x^4 + \omega_{bx}^4} \tag{3}$$

and the unnormalized modes

$$\boldsymbol{\psi}_{x1} = \begin{Bmatrix} \omega_x^2 \\ \omega_{bx}^2 + \sqrt{\omega_x^4 + \omega_{bx}^4} \end{Bmatrix}, \quad \boldsymbol{\psi}_{x2} = \begin{Bmatrix} \omega_x^2 \\ \omega_{bx}^2 - \sqrt{\omega_x^4 + \omega_{bx}^4} \end{Bmatrix} \tag{4}$$

The θ- and y-direction equations are found analogously for $\tilde{\boldsymbol{k}}_{\theta\theta}$ and $\tilde{\boldsymbol{k}}_{yy}$.

For a typical isolated structure $\omega_x \gg \omega_{bx}$; applying this assumption in Equation (3) leads to

$$\omega_{x1} \simeq \omega_{bx}, \quad \omega_{x2} \simeq \sqrt{2}\omega_x \tag{5}$$

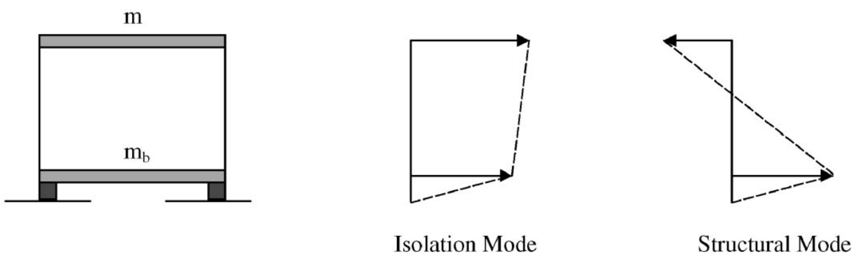

Figure 2. Mode shapes of torsionally uncoupled system.

The fundamental frequency ω_{x1} is dominated by the isolation frequency, while ω_{x2} is the structure frequency lengthened by the flexibility of the isolators. A graphical depiction of the mode shapes, shown in Figure 2, demonstrates that the first mode is primarily characterized by isolator deformation, while in the second mode the structural deformation exceeds the isolator deformation. Thus, the first mode is commonly referred to as the isolation mode and the second mode as the structural mode.

While the benefits of isolation are reduced as the relative flexibility of the structure increases, the structural mode of a well-isolated system is nearly orthogonal to the influence vector and participates only minimally in the response. That this feature is retained in most torsionally coupled systems will become apparent. Further details of the linear theory of base isolation may be found in Kelly [2].

Approximate frequencies and modes of coupled system

The isolation modes $\boldsymbol{\psi}_{x1}$ (Equation (4a)), $\boldsymbol{\psi}_{\theta 1}$, and $\boldsymbol{\psi}_{y1}$ form a set of basis vectors for the isolation modes of the torsionally coupled system; likewise for the structural modes. The torsionally coupled modes are

$$\boldsymbol{\phi}_{nj} = \begin{Bmatrix} \boldsymbol{\psi}_{xj} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} z_{x,nj} + \begin{Bmatrix} \mathbf{0} \\ \boldsymbol{\psi}_{\theta j} \\ \mathbf{0} \end{Bmatrix} z_{\theta,nj} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\psi}_{yj} \end{Bmatrix} z_{y,nj} \tag{6}$$

or

$$\boldsymbol{\phi}_{nj} = \boldsymbol{\Psi}_j \mathbf{z}_{nj} \tag{7}$$

where $j = 1$ for the isolation modes and $j = 2$ for the structural modes; $n = 1, 2, 3$ for the three modes in each triplet and

$$\boldsymbol{\Psi}_j = \begin{bmatrix} \boldsymbol{\psi}_{xj} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\psi}_{\theta j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\psi}_{yj} \end{bmatrix}, \quad \mathbf{z}_{nj} = \begin{Bmatrix} z_{x,nj} \\ z_{\theta,nj} \\ z_{y,nj} \end{Bmatrix}$$

Equations (6) and (7) imply that there is no interaction or coupling between the isolation modes and the structural modes in this approximate solution. In Equation (7), $\boldsymbol{\Psi}_j$ may be interpreted as Ritz vectors and $\boldsymbol{z}_{nj}$ as generalized co-ordinates. The eigenvalue problem in terms of the vibration modes of the uncoupled system and the generalized co-ordinates is given by the Rayleigh–Ritz method [4]:

$$(\boldsymbol{\Psi}_j^{\mathrm{T}} \tilde{\boldsymbol{k}} \boldsymbol{\Psi}_j - \omega^2 \boldsymbol{\Psi}_j^{\mathrm{T}} \boldsymbol{\Psi}_j) \boldsymbol{z}_{nj} = \mathbf{0} \tag{8}$$

Isolation-related single-storey system

Specializing Equation (8) for $j = 1$ gives the eigenvalue problem for the isolation modes

$$\begin{bmatrix} \omega_{x1}^2 - \omega^2 & -\hat{e}_{by}\omega_{x1}^2 & 0 \\ -\hat{e}_{by}\omega_{x1}^2 & \omega_{\theta 1}^2 - \omega^2 & \hat{e}_{bx}\omega_{y1}^2 \\ 0 & \hat{e}_{bx}\omega_{y1}^2 & \omega_{y1}^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} z_{x,n1} \\ z_{\theta,n1} \\ z_{y,n1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{9}$$

where

$$\hat{e}_{bx} = \frac{\boldsymbol{\psi}_{y1}^{\mathrm{T}} \tilde{\boldsymbol{k}}_{y\theta} \boldsymbol{\psi}_{\theta 1}}{\omega_{y1}^2}, \quad \hat{e}_{by} = \frac{\boldsymbol{\psi}_{x1}^{\mathrm{T}} \tilde{\boldsymbol{k}}_{x\theta} \boldsymbol{\psi}_{\theta 1}}{\omega_{x1}^2} \tag{10}$$

This eigenvalue equation is symbolic of a torsionally coupled single-storey system, referred to here as the isolation-related single-storey system. The uncoupled frequencies of this single-storey system are ω_{x1} (Equation (3a)), $\omega_{\theta 1}$ and ω_{y1}, and the eccentricities $\hat{e}_{bx}$ and $\hat{e}_{by}$ are given by Equation (10). It will be shown that $\hat{e}_{bx}$ and $\hat{e}_{by}$ are closely related to the isolation eccentricites, therefore they will be referred to as 'effective isolation eccentricities'.

Isolation frequencies and modes

Equation (9) is solved explicitly when at least two of the frequencies ω_{x1}, $\omega_{\theta 1}$, ω_{y1}, are equal. From Equation (3a), ω_{x1} depends only on ω_x and ω_{bx}, likewise for ω_{y1}. Then, $\omega_{bx} = \omega_{by}$ and $\omega_x = \omega_y$ necessarily implies $\omega_{x1} = \omega_{y1} \equiv \omega_1$. Since circular bearings have identical stiffness in all directions, $\omega_{bx} = \omega_{by}$ is well founded, however $\omega_x = \omega_y$ is valid only for a limited class of structures. Although the methods presented can be applied to any structure, the preceding assumption allows for an explicit solution.

The solution to Equation (9), with $\omega_{x1} = \omega_{y1} \equiv \omega_1$, leads to the equations for the isolation frequencies of the coupled system

$$\omega_{11}^2 = \frac{(\omega_{\theta 1}^2 + \omega_1^2)}{2} - \sqrt{\frac{(\omega_{\theta 1}^2 - \omega_1^2)^2}{4} + \omega_1^4(\hat{e}_{bx}^2 + \hat{e}_{by}^2)} \tag{11a}$$

$$\omega_{21}^2 = \omega_1^2 \tag{11b}$$

$$\omega_{31}^2 = \frac{(\omega_{\theta 1}^2 + \omega_1^2)}{2} + \sqrt{\frac{(\omega_{\theta 1}^2 - \omega_1^2)^2}{4} + \omega_1^4(\hat{e}_{bx}^2 + \hat{e}_{by}^2)} \tag{11c}$$

and for the generalized co-ordinates, interpreted as modes of the isolation-related single-storey system

$$\boldsymbol{z}_{11} = \begin{Bmatrix} \hat{e}_{by} \\ 1 - \dfrac{\omega_{11}^2}{\omega_1^2} \\ -\hat{e}_{bx} \end{Bmatrix}, \quad \boldsymbol{z}_{21} = \begin{Bmatrix} \hat{e}_{bx} \\ 0 \\ \hat{e}_{by} \end{Bmatrix}, \quad \boldsymbol{z}_{31} = \begin{Bmatrix} \hat{e}_{by} \\ 1 - \dfrac{\omega_{31}^2}{\omega_1^2} \\ -\hat{e}_{bx} \end{Bmatrix} \tag{12}$$

Specializing Equation (7) for $j = 1$ gives the isolation modes of the coupled system

$$\boldsymbol{\phi}_{n1} = \boldsymbol{\Psi}_1 \boldsymbol{z}_{n1} \tag{13}$$

Structure-related single-storey system

The eigenvalue problem for the structural modes is formed from Equation (8), with $j = 2$

$$\begin{bmatrix} \omega_{x2}^2 - \omega^2 & -\hat{e}_y \omega_{x2}^2 & 0 \\ -\hat{e}_y \omega_{x2}^2 & \omega_{\theta 2}^2 - \omega^2 & \hat{e}_x \omega_{y2}^2 \\ 0 & \hat{e}_x \omega_{y2}^2 & \omega_{y2}^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} z_{x,n2} \\ z_{\theta,n2} \\ z_{y,n2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{14}$$

where

$$\hat{e}_x = \frac{\boldsymbol{\psi}_{y2}^{\mathrm{T}} \tilde{\boldsymbol{k}}_{y\theta} \boldsymbol{\psi}_{\theta 2}}{\omega_{y2}^2}, \quad \hat{e}_y = \frac{\boldsymbol{\psi}_{x2}^{\mathrm{T}} \tilde{\boldsymbol{k}}_{x\theta} \boldsymbol{\psi}_{\theta 2}}{\omega_{x2}^2} \tag{15}$$

Analogous to the isolation-related single-storey system, the eigenvalue problem in Equation (14) represents the structure-related single-storey system. This system is defined in terms of the torsionally uncoupled structural frequencies ω_{x2} (Equation (3b), $\omega_{\theta 2}$ and ω_{y2}, and the eccentricities $\hat{e}_x$ and $\hat{e}_y$ (Equation (15)). It will be shown that these eccentricities are closely related to the structural eccentricities, and hereafter they will be referred to as 'effective structural eccentricities'.

Approximate structural frequencies and modes

The assumptions used to solve the isolation-related single-storey system can also be applied here. That is, $\omega_{bx} = \omega_{by}$ and $\omega_x = \omega_y$ leads to $\omega_{x2} = \omega_{y2} \equiv \omega_2$. Solving Equation (14), the structural frequencies of the coupled system are

$$\omega_{12}^2 = \frac{(\omega_{\theta 2}^2 + \omega_2^2)}{2} - \sqrt{\frac{(\omega_{\theta 2}^2 - \omega_2^2)^2}{4} + \omega_2^4(\hat{e}_x^2 + \hat{e}_y^2)} \tag{16a}$$

$$\omega_{22}^2 = \omega_2^2 \tag{16b}$$

$$\omega_{32}^2 = \frac{(\omega_{\theta 2}^2 + \omega_2^2)}{2} + \sqrt{\frac{(\omega_{\theta 2}^2 - \omega_2^2)^2}{4} + \omega_2^4(\hat{e}_x^2 + \hat{e}_y^2)} \tag{16c}$$

and the generalized co-ordinates are

$$\boldsymbol{z}_{12}=\left\{\begin{array}{c}\hat{e}_y\\ 1-\dfrac{\omega_{12}^2}{\omega_2^2}\\ -\hat{e}_x\end{array}\right\},\quad \boldsymbol{z}_{22}=\left\{\begin{array}{c}\hat{e}_x\\ 0\\ \hat{e}_y\end{array}\right\},\quad \boldsymbol{z}_{32}=\left\{\begin{array}{c}\hat{e}_y\\ 1-\dfrac{\omega_{32}^2}{\omega_2^2}\\ -\hat{e}_x\end{array}\right\} \tag{17}$$

The structural modes of the coupled system result from Equation (7), with $j=2$

$$\boldsymbol{\phi}_{n2}=\boldsymbol{\Psi}_2\boldsymbol{z}_{n2} \tag{18}$$

Since the torsionally coupled modes of Equations (13) and (18) preserve the uncoupled modes in the x-, θ- and y-directions, the features of isolation that contribute to its effectiveness laterally are preserved in the system with torsion.

The Rayleigh–Ritz method typically uses a set of Ritz vectors to estimate the first few frequencies and modes of the system, as is our goal in Equations (11)–(13). This solution represents the best possible result for the first three modes and frequencies, given the initial choice of Ritz vectors [4]. In particular, the fundamental frequency predicted by the method is an upper bound to the exact fundamental frequency of the system, but is less than or equal to the fundamental frequency resulting from any other combination of the Ritz vectors. This 'best approximation' feature does not necessarily hold when using the method to estimate the higher frequencies of the system.

The Rayleigh–Ritz modes and frequencies are exact, however, if the structure-isolation system complies with the following [3]:

1. Isolation eccentricities are equal to structural eccentricities ($e_{bx}=e_x,\ e_{by}=e_y$).
2. Frequency ratios, defined below in Equation (19), are all equal,

$$\Omega_x=\frac{\omega_{bx}}{\omega_x},\quad \Omega_\theta=\frac{\omega_{b\theta}}{\omega_\theta},\quad \Omega_y=\frac{\omega_{by}}{\omega_y} \tag{19}$$

Response-spectrum analysis

The peak response of the base-isolated system may be estimated by the standard response-spectrum analysis procedure. Here, the peak value of the njth mode contribution $r_{nj}(t)$ to the response $r(t)$ is given by

$$r_{nj}=r_{nj}^{\mathrm{st}}A_{nj} \tag{20}$$

where r_{nj}^{st} is the modal static response (base shear, base torque, or isolator deformation), and $A_{nj}=A(\omega_{nj},\zeta_{nj})$ is the ordinate of the pseudo-acceleration design spectrum for the frequency ω_{nj}, determined earlier, and damping ratio ζ_{nj}, given by

$$\zeta_{nj}=\frac{1}{2\omega_{nj}}\frac{\boldsymbol{\phi}_{nj}^{\mathrm{T}}\boldsymbol{c}\boldsymbol{\phi}_{nj}}{\boldsymbol{\phi}_{nj}^{\mathrm{T}}\boldsymbol{m}\boldsymbol{\phi}_{nj}} \tag{21}$$

where $\boldsymbol{m}$ and $\boldsymbol{c}$ are the mass and damping matrices of the system [4, Section 13.7]. Since we have assumed $m=m_{\mathrm{b}}$, $\boldsymbol{m}$ reduces to $m\boldsymbol{I}$, whereas the damping matrix is constructed from the damping properties of the structure and the isolation system. The peak value r of the total response $r(t)=\sum r_{nj}(t)$ is determined by combining the peak modal responses r_{nj} according to the CQC rule.

Strictly speaking, this method, based on the assumption of classical damping, may not be accurate for base-isolated systems because the damping in such systems in non-classical. However, it can provide approximate results that are sufficiently accurate for our limited objective. To justify this claim, this method and the CQC method for non-classically damped structures [5] were compared for torsionally coupled, isolated structures, over a range of damping values, frequency ratios, and eccentricities. The exact frequencies and modes of the system were utilized for this analysis. For the damping values considered in this study—2 per cent in each coupled mode of the structure and 10 per cent in each coupled mode of the isolation system—the responses computed by the two methods differ by less than 1 per cent. This result is consistent with an earlier study, based on response history analysis, which concluded that non-classical damping in isolated buildings can be ignored [6].

As mentioned earlier, some assumptions have been made for the convenience of obtaining simple expressions for the modes $\boldsymbol{\phi}_{nj}$ and frequencies ω_{nj}. Although not detailed here, with these assumptions the modal static responses r_{nj}^{st} can be expressed explicitly as well. Recognize that the preceding response analysis procedure, based on the Rayleigh–Ritz method, is valid for systems that do not satisfy these assumptions. If $\omega_{bx}\neq\omega_{by}$, or $\omega_x\neq\omega_y$, numerical solutions are required to Equations (9) and (14). If $m\neq m_{\mathrm{b}}$, explicit expressions for modes and frequencies are still possible, but they are more complicated.

SIMPLIFIED METHODS

Equations (10) and (15) for the effective eccentricities, which are functions of the properties (modes, frequencies, and eccentricities) of the torsionally uncoupled system, are too complex to provide insight into their values. Therefore, we attempt to simplify these equations. The assumption of a rigid structure in analysing the system is also considered. For the remainder of this discussion, the Rayleigh–Ritz method for torsionally coupled, isolated systems, as outlined in the previous section, will be referred to as the RR method.

Simplified eccentricity (SE) method

For a well isolated structure, $\omega_x\gg\omega_{bx}$, $\omega_\theta\gg\omega_{b\theta}$, and $\omega_y\gg\omega_{by}$; therefore Ω_x^4, Ω_θ^4 and $\Omega_y^4\simeq 0$. With this approximation, Equations (10a) and (15a) simplify to

$$\hat{e}_{bx}=\left(\frac{\omega_{by}^2}{\omega_{y1}^2}\right)\left(\frac{(e_{bx}/r)+\frac{1}{2}(e_x/r)\Omega_\theta^2}{(1+\Omega_y^2)^{1/2}(1+\Omega_\theta^2)^{1/2}}\right)\tag{22a}$$

$$\hat{e}_{x}=\left(\frac{2\omega_{y}^2}{\omega_{y2}^2}\right)\left(\frac{\frac{1}{2}(e_{bx}/r)\Omega_y^2+(e_x/r)(1-(\Omega_y^2/2))(1-(\Omega_\theta^2/2))}{(1-\Omega_y^2)^{1/2}(1-\Omega_\theta^2)^{1/2}}\right)\tag{22b}$$

while $\hat{e}_{by}$ and $\hat{e}_y$ result from switching the x- and y-indices. The simplified eccentricity (SE) method, an adaptation of the RR method, is achieved by using Equation (22) in lieu of Equations (10) and (15) to obtain the effective eccentricities.

Further simplified eccentricity (FSE) method

If the frequency ratios of the system are small enough, it is reasonable to assume that Ω_x^2, Ω_θ^2, and $\Omega_y^2 \simeq 0$; this, in conjunction with Equation (5), further simplifies Equation (22) to

$$\hat{e}_{bx} = \frac{e_{bx}}{r}, \quad \hat{e}_x = \frac{e_x}{r} \tag{23}$$

The further simplified eccentricity (FSE) method is an other simplification of the RR method, achieved by using Equation (23) for the effective eccentricities instead of Equations (10) and (15). The SE and FSE methods differ from the RR method only in the equations for effective eccentricities; the three methods will be referred to collectively as the Ritz class methods.

With the torsionally uncoupled modes as Ritz vectors, the RR method best approximates the modes and frequencies of the system. The SE and FSE methods, which represent different linear combinations of the uncoupled modes, are generally less accurate. However, the effective eccentricities of Equation (23) equal those of Equations (10a) and (15a), provided the system has equal eccentricities and equal frequency ratios (see Appendix A). Under these conditions, the FSE method provides the same results as the RR method, and also predicts the exact modes and frequencies.

Rigid structure (RS) method

The structure is often assumed to be rigid when studying the dynamics of base-isolated systems [7, 8]. The equations for the RS method can be derived directly, as detailed in Kelly [2]. However, the resulting equations can be interpreted as a special case of the isolation-related single-storey system. With the rigid structure assumption, the frequency ratios Ω_x, Ω_θ and $\Omega_y \simeq 0$, and the system is reduced to three DOF; these are the displacements of the base mass (Figure 1). Equation (9) describes the eigenvalue equation for the system, with the frequencies defined by Equation (5a), and effective isolation eccentricities defined by

$$\hat{e}_{bx} = \frac{e_{bx}}{r}, \quad \hat{e}_{by} = \frac{e_{by}}{r} \tag{24}$$

Because the structure is assumed to be rigid, the structure-related single-storey system and effective structural eccentricities are meaningless for this method.

Comparison of effective eccentricities

Effective isolation eccentricities and effective structural eccentricities obtained using the four methods are compared: RR using Equations (10) and (15), SE using Equation (22), FSE using Equation (23), and RS using Equation (24). The x-direction eccentricities were computed for systems with $\omega_{by} = \pi$, $\omega_{b\theta} = 1.25\pi$, $\Omega_\theta = \Omega_y$, $e_x/r = 0.5$ and three different values of e_{bx}/r: 0, 0.25 and 0.5. These are the minimum parameters necessary to compute x-direction effective eccentricities; the remaining parameters of the system are left undefined. Effective isolation eccentricity is presented in Figure 3 and effective structural eccentricity in Figure 4.

For the RR method (the best predictor of the modes and frequencies), effective isolation eccentricity is generally larger than the actual isolation eccentricity, and grows with increasing frequency ratio (Figure 3), while effective structural eccentricity is slightly less than the actual

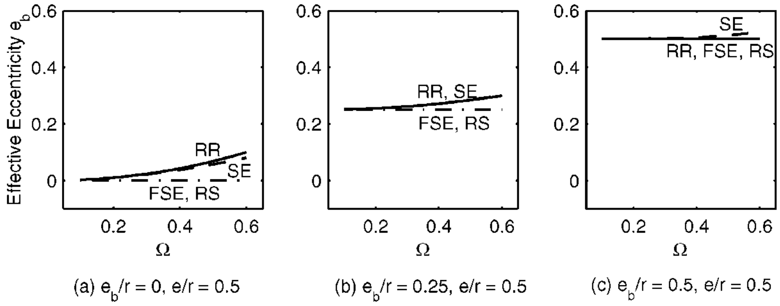

Figure 3. Effective isolation eccentricity.

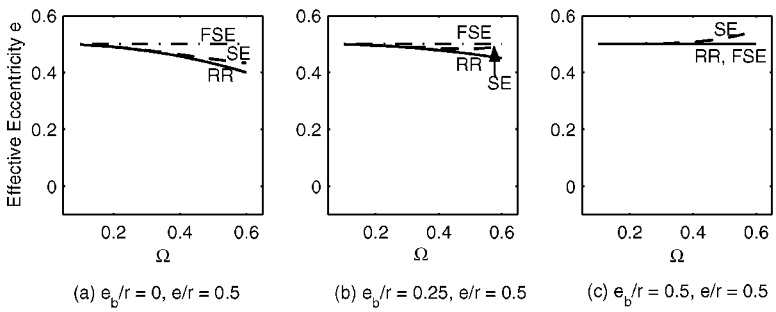

Figure 4. Effective structural eccentricity.

structural eccentricity, and decreases with increasing frequency ratio (Figures 4(a) and 4(b)). In general, the SE eccentricities differ only slightly from the RR eccentricities. For systems with structural eccentricity equal to the isolation eccentricity, exceptions are as follows: (1) The RR method gives $\hat{e}_{bx} = \hat{e}_x = e_{bx}/r = e_x/r$ (Figures 3(c) and 4(c)); and (2) effective structural eccentricity by the SE method exceeds the actual structural eccentricity (Figure 4(c)). Effective eccentricities determined by the FSE method and effective isolation eccentricity determined by the RS method are always equal to the actual eccentricities.

ACCURACY OF APPROXIMATE METHODS

In this section, the accuracy of the RR, SE, FSE, and RS methods are evaluated against the results of 'exact' analysis of base-isolated systems. The response-spectrum analysis procedure described earlier is implemented in each method to determine the earthquake response of the system. The difference among the methods is simply in how the natural vibration modes and frequencies of the system are determined. These are determined by solving the eigenvalue problem of order six in the exact method, and from Equations (11)–(13) and (16)–(18) in the Ritz class methods, with effective eccentricities varying among the methods of the Ritz class.

The ground motion is characterized by the design spectrum [4, Figure 6.9.5], scaled to peak ground acceleration of $0.5g$. The systems analysed have the following properties: $\omega_{bx} = \omega_{by} = \pi$, $\omega_{b\theta} = 1.25\pi$; $\Omega_x = \Omega_\theta = \Omega_y \equiv \Omega$; $e_{bx} = e_{by} \equiv e_b$ and $e_x = e_y \equiv e$; ω_x, ω_θ and ω_y are varied to cover a range of the frequency ratio Ω from 0.1 to 0.6. Results are presented for base shear V_{bx} and base torque T_b in the structure, and the isolator deformations u_s and u_f at the stiff and flexible edges of the plan; a square plan is selected. V_{bx} is normalized relative to $w = mg$, the weight of the superstructure, and T_b relative to wr. Also shown are the errors in each of the approximate methods relative to the exact response; positive error indicates that the approximate method overestimates the response, while a negative error indicates an underestimate. An error of 100 per cent implies that the approximate method predicts the response to be zero, but the exact response is non-zero. Note that the error can become large if the actual response is close to zero while the approximation is not.

Equal eccentricity systems

Systems with equal isolation and structural eccentricities ($e_b = e$) are examined. This represents a realistic system where the stiffness of each isolator is proportional to the stiffness of the lateral load resisting element it supports. Results are presented in Figures 5 and 6 for $e/r = 0.1$ and 0.5, respectively, and varying frequency ratio Ω. Earlier it was asserted that the RR and FSE methods predict exact results for equal eccentricity systems which satisfy $\Omega_x = \Omega_\theta = \Omega_y$; agreement of the two methods and exact analysis is shown in the figures. The responses predicted by the SE method, though not exact, are nearly held to 1–2 per cent error, with a maximum of about 5 per cent for base torque at $e/r = 0.1$ (Figure 5(b)). The error of the SE method is negligible except when the structure becomes relatively flexible. The RS method, on the other hand, gives relatively high errors, around 10–20 per cent, for base shear and base torque over all parameters (Figures 5(a), 5(b) and 6(a), 6(b)).

The consistency of the Ritz class methods shows that equal eccentricity systems are well described by separation into an isolation-related and structure-related single-storey system, and there is no interaction between the isolation and structural modes. The error of the RS method, on the other hand, is not restricted to small values, and grows quickly with increasing Ω or increasing structure period. The primary source of error in this method is neglect of the structural flexibility (recall that the RS method overlooks the structure-related single-storey system), apparent since the RS response is invariant with respect to Ω while the exact response changes. The Ritz class methods are accurate even when the structure is relatively flexible with respect to the isolation system, while the limitations of the RS method are evident.

Each method predicts torsional response with about the same degree of accuracy as it predicts lateral response; the magnitude of the base torque error is similar to the base shear error for the SE and RS methods, while the others are exact. Furthermore, the accuracy of the methods is not adversely affected by increasing eccentricity values. To support this observation, base shear and base torque with per cent error are plotted against e/r (Ω fixed at 0.4) in Figure 7. Errors in the SE and RS methods are nearly constant with respect to eccentricity, demonstrating that the methods are valid even for highly asymmetric structures.

It is of interest to note that the error magnitude of the RS method is greater for forces than deformations, an effect that is not seen in the Ritz class methods. Since the RS method neglects

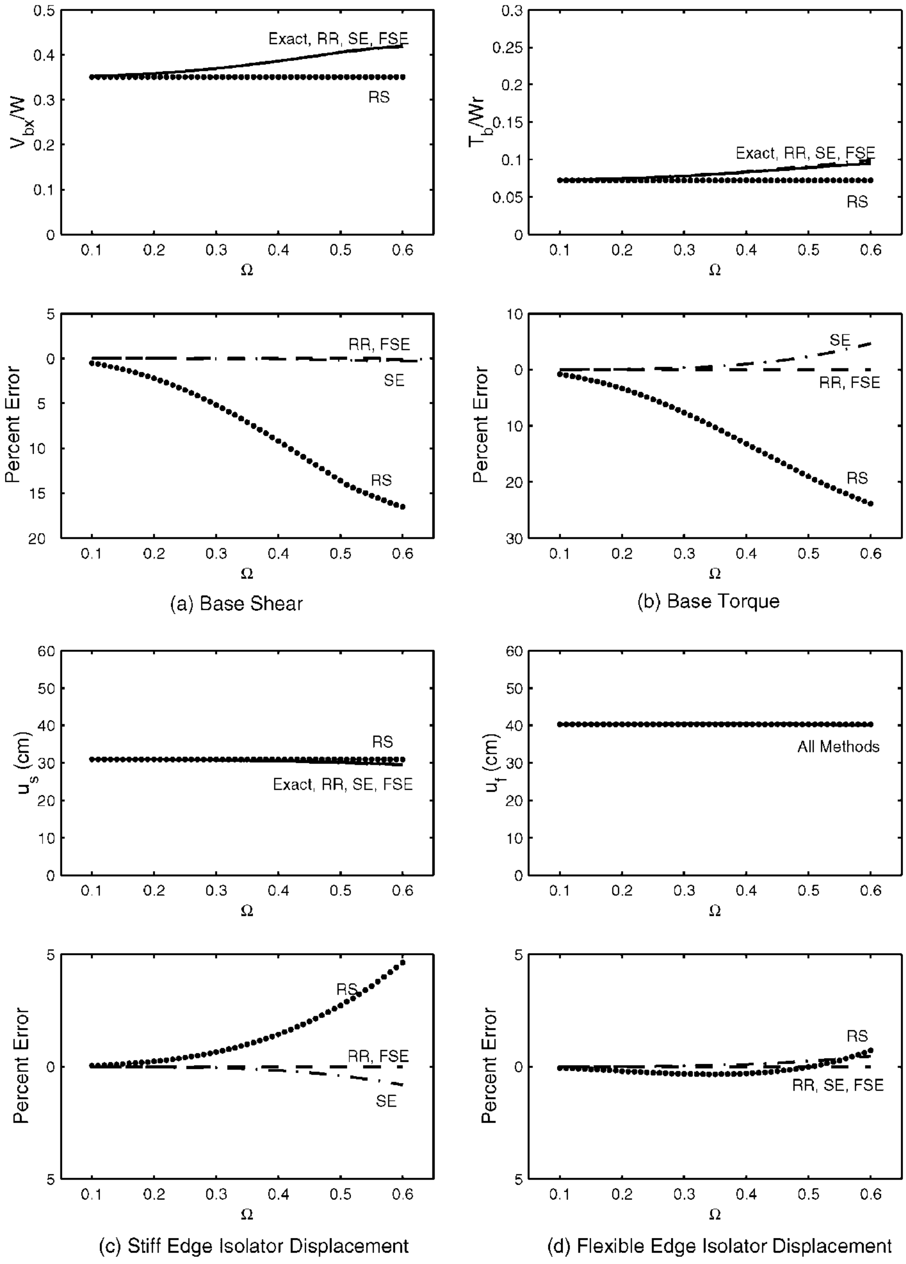

Figure 5. Responses at structural eccentricity $e/r = 0.1$, equal eccentricity systems.

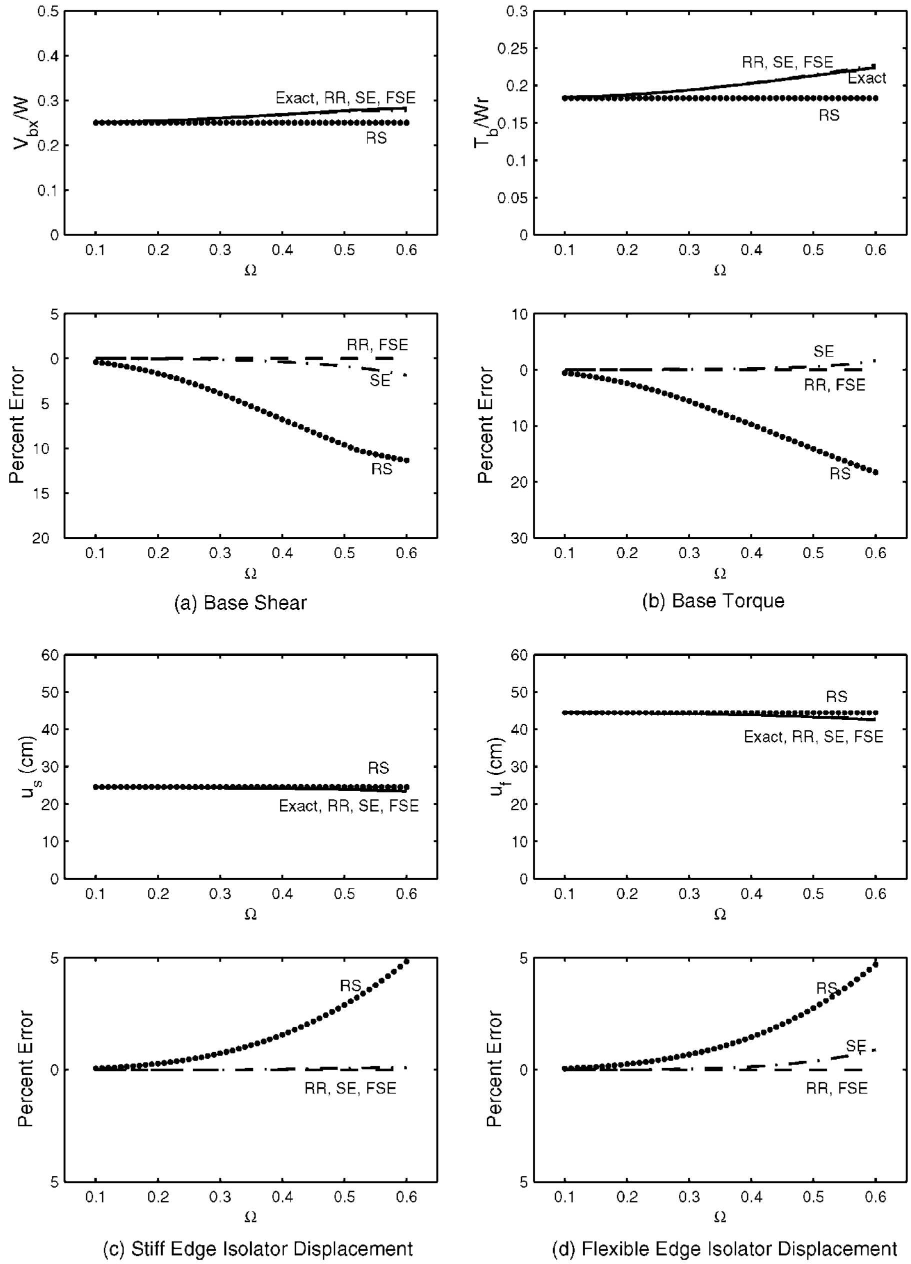

Figure 6. Responses at structural eccentricity $e/r = 0.5$, equal eccentricity systems.

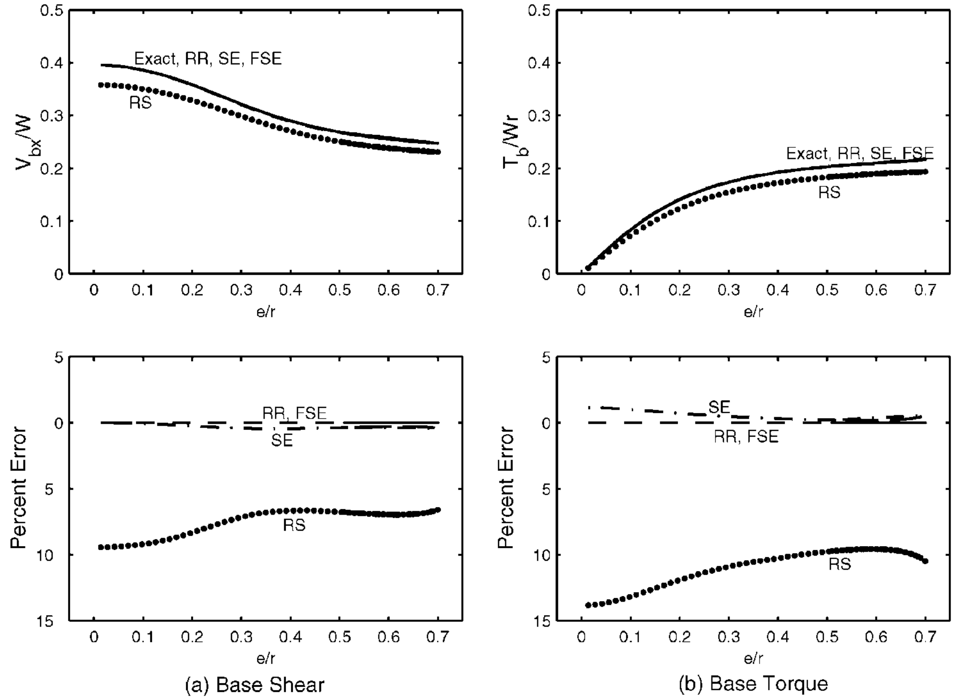

Figure 7. Structure forces at $\Omega = 0.4$, equal eccentricity systems.

Table I. Select modal responses ($\Omega = 0.4$; $e/r = 0.5$).

	Exact analysis			Rigid structure method		
Mode	A_{nj}/g	$V^{st}_{bx,nj}/m$	$u^{st}_{s,nj}$	A_{nj}/g	$V^{st}_{bx,nj}/m$	$u^{st}_{s,nj}$
11	0.259	0.367	−0.012	0.259	0.342	−0.012
21	0.359	0.536	0.050	0.359	0.500	0.050
31	0.514	0.169	0.028	0.512	0.158	0.028

the structural flexibility, it cannot estimate storey drift and also tends to underestimate forces in the structure. The effect can be seen more clearly by examining the values of the modal static response r^{st}_{nj} and psuedo-acceleration A_{nj} (Equation (20)) in the exact and RS methods. Such comparison is summarized in Table I for the base shear V_{bx} and isolator deformation u_s in a system with $\Omega = 0.4$ and $e/r = 0.5$. CQC combination of the modal responses gives $V_{bx}/w = 0.269$ for exact analysis and 0.251 for the RS method, an error of −6.7 per cent, while $u_s = 24.19$ cm for exact analysis and 24.57 cm for the RS method, an error of 1.6 per cent. Clearly, $V^{st}_{bx,nj}$ is underestimated by the RS method, while $u^{st}_{s,nj}$ is predicted accurately. The small error in u_s results from neglect of the structural modes, which are not shown in Table I.

Zero isolation eccentricity systems

Deliberate selection of the isolator placement to ensure symmetry with respect to the CM results in a system with $e_b = 0$. Results for these systems are presented in Figures 8 and 9 again for $e/r = 0.1$ and 0.5, and varying frequency ratio Ω. Errors for all the methods have increased compared to equal eccentricity systems; for the RR method in particular, the predictions for T_b and u_f at $e/r = 0.5$ err by as much as 30 per cent and 15 per cent, respectively (Figure 9(b) and 9(d)). The large error indicates that this choice of Ritz vectors is not representative of the exact solution, which contains notable coupling between the isolation and structural modes. This result is not surprising, since perturbation theory, upon which this selection of Ritz vectors is based, is exact only for equal eccentricity systems [3].

Unlike equal eccentricity systems, each Ritz class method predicts lateral response more accurately than torsional response, most apparent for $e/r = 0.5$ (Figure 9). The base torque errors are magnified in part because the exact base torque is small, but also because the Ritz vectors are not accurate, and the dominant modes of the system impart a greater torsional response than the corresponding approximate modes. While the RR method underestimates base torque by about 30 per cent, base shear is overestimated by no more than 5 per cent. The base torque errors are even larger with the SE and FSE methods, only slightly larger with the SE method, since the SE and RR methods use similar values for effective eccentricities (Figures 3(a) and 4(a)). However, for the FSE method, effective isolation eccentricity is zero and torsion results only from the structure-related single-storey system, explaining the further underestimation of the base torque relative to the RR and SE methods. The base torque is always zero with the RS method, as both structural flexibility and eccentricity are neglected; therefore it is less accurate overall, with errors on the order of 10 per cent for base shear and 15 per cent for deformation u_s.

For $e/r = 0.1$ (Figure 8), representative of small eccentricities, errors in the RR method are small, on the order of 0 per cent for base shear, 1–2 per cent for isolator deformations and 10 per cent for base torque, the last being insignificant since the exact base torque is small. The other Ritz class methods are also accurate because changes in effective eccentricity have subtle effects when structural eccentricity is small to begin with. The tendency of the Ritz class methods to underestimate torsion in the system is more apparent as the asymmetry of the structure increases. At $e/r = 0.5$ (Figure 9), base torque errors increase to 20–30 per cent for the RR and SE methods and 30–60 per cent for the FSE method; isolator deformations are adversely affected as well. The methods become increasingly disparate because large values of structural eccentricity effect larger differences in effective eccentricities.

Increasing frequency ratio or structure period leads to increasing error for all the methods. All responses except base torque are predicted within about 5 per cent by the Ritz class methods, as long as Ω is in the range of 0.2–0.4 (Figure 9). Above this value of Ω, deformation errors increase to 15 per cent for the FSE method (Figure 9(c)), and 10–15 per cent for the RR and SE methods (Figure 9(d)). Errors for base torque are relatively constant with respect to Ω, but the error is magnified for very stiff structures since the exact base torque is close to zero.

As we have seen already, the RR method is the best predictor of the system's modes and frequencies, followed by the SE and FSE methods in order—which differ only slightly from the former—and the RS method is the least accurate due to neglect of the structure flexibility and eccentricity. However, the observations do not necessarily carry over to total response,

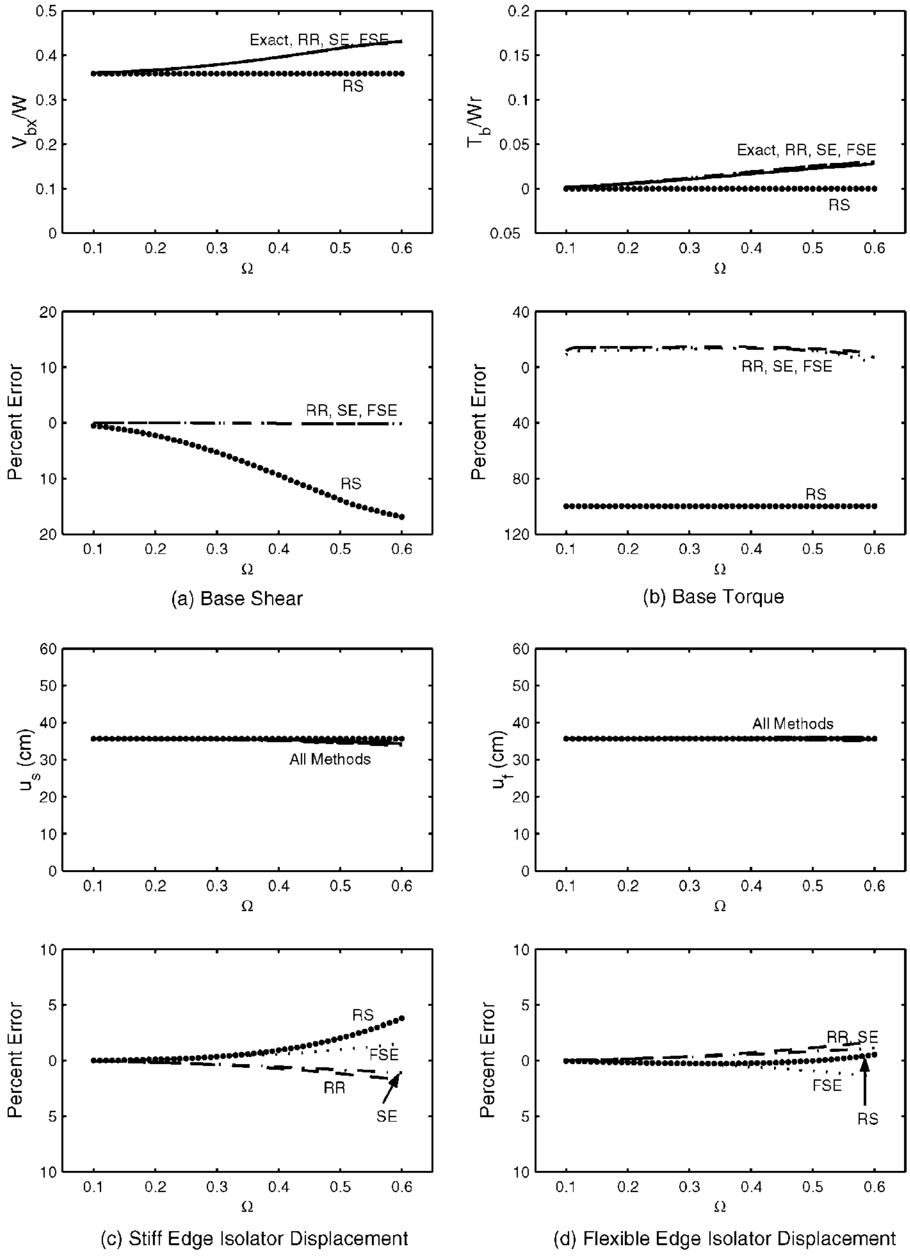

Figure 8. Responses at structural eccentricity $e/r = 0.1$, zero isolation eccentricity.

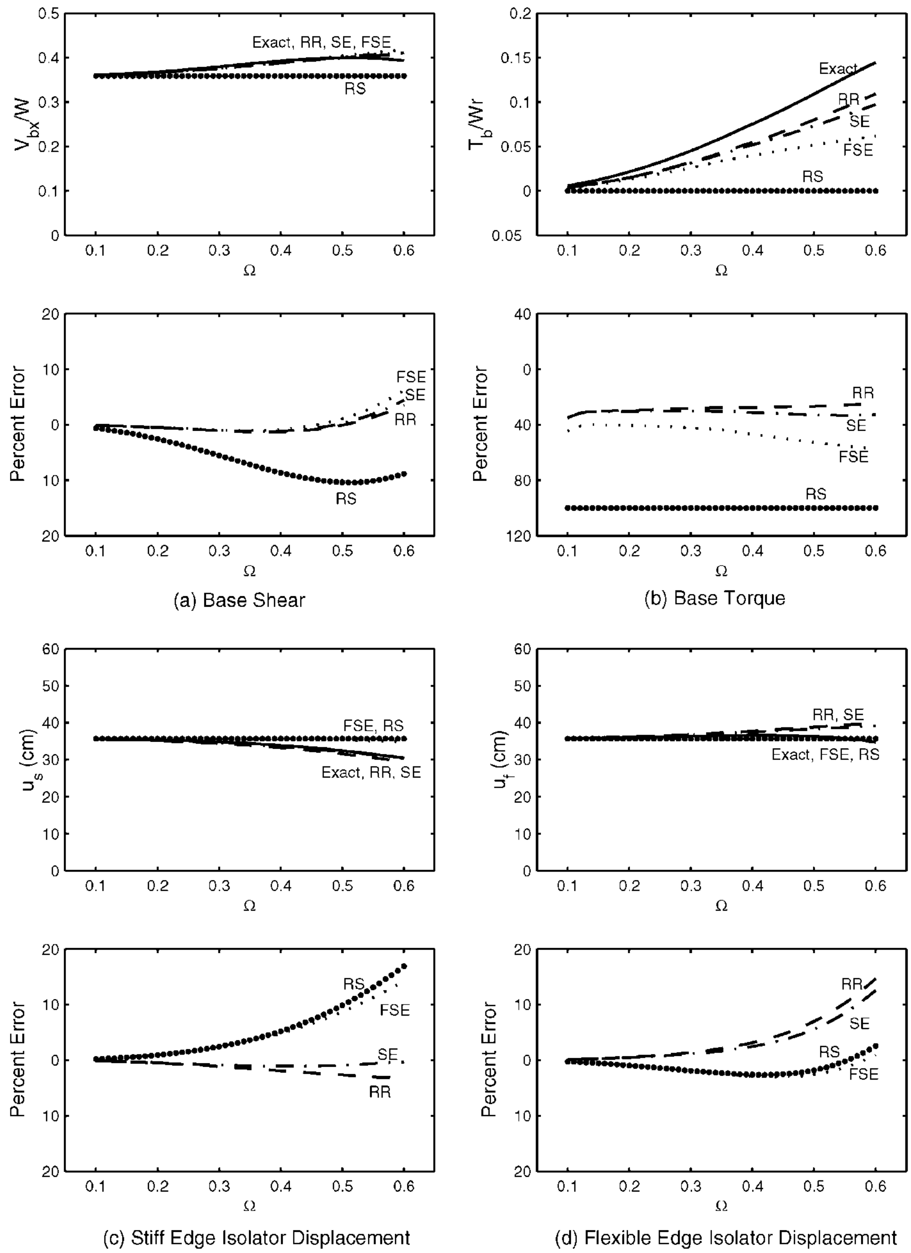

Figure 9. Responses at structural eccentricity $e/r = 0.5$, zero isolation eccentricity.

because the errors in modal responses do not combine in a systematic manner, and may be additive or cancel out. Indeed, Figures 8 and 9 indicate that the RR method is not always the best predictor of response. For example, when $e/r = 0.5$, the FSE method gives the best estimate of base shear for $\Omega < 0.45$ (Figure 9(a)), the SE method gives the best estimate of stiff edge deformation at all Ω (Figure 9(c)), and the RS method gives the best estimate of flexible edge deformation for $\Omega > 0.4$ (Figure 9(d)). Despite these anomalies, the Ritz class methods are more consistent and are preferred to the RS method. While the Ritz class methods are accurate for stiff relative to the isolation system structures or slightly asymmetric structures, a clear choice among these methods is lacking otherwise.

CONCLUSIONS

Based on perturbation analysis, the Rayleigh–Ritz procedure was specialized for asymmetric-plan, base-isolated buildings, in which the vibration modes of the system are approximated as a linear combination of the modes of the torsionally uncoupled system. Assuming $\omega_{bx} = \omega_{by}$ and $\omega_x = \omega_y$, the vibration modes and frequencies of the asymmetric-plan system were laid out explicitly. Three methods based on the Rayleigh–Ritz concept were developed. Direct application of this concept results in the Rayleigh–Ritz method with 'exact' equations for the eccentricities; simplifications of these equations, based on the assumption of small frequency ratios, leads to the simplified eccentricity and further simplified eccentricity methods. Standard response-spectrum analysis procedure is applied to estimate the peak response of the base-isolated structure using the approximate modes and frequencies.

The accuracy of the three Ritz class methods and the rigid structure method were compared separately for systems with equal isolation and structural eccentricities ($e_b = e$), and for systems with zero isolation eccentricity ($e_b = 0$). For equal eccentricity systems, all methods of the Ritz class are accurate over the considered range of parameters, with a maximum 1–2 per cent error in base shear and 5 per cent in base torque. The Rayleigh–Ritz and further simplified eccentricity methods have been shown to be exact, and errors in the simplified eccentricity method have been shown to be small, unaffected by eccentricity, and only slightly increasing at large frequency ratios. The further simplified eccentricity method, which is exact and uses simple expressions for effective eccentricity, is the method of choice for equal eccentricity systems. The rigid structure method does not account for structure flexibility, and sees errors in base shear and base torque on the order of 10–20 per cent above frequency ratios of 0.4.

All approximate methods are much less accurate for systems with zero isolation eccentricity. The choice of Ritz vectors turns out to be non-ideal for representing the modes of the system, and the Ritz class methods are inconsistent, exchanging order on a relative accuracy scale, and are particularly inaccurate in estimating base torque, with errors on the order of 20–60 per cent. These methods are recommended for small eccentricities ($e/r \simeq 0.1$) or small frequency ratios ($\Omega < 0.4$), conditions that lead to small torque. By neglecting structure eccentricity, the rigid structure method misses the most important feature of the dynamics of the asymmetric-plan system and is unable to account for any portion of the base torque; therefore, the method should be avoided.

The conclusions regarding accuracy of the methods for equal eccentricity systems are roughly applicable to systems with similar isolation and structural eccentricities. Likewise, the conclusions for zero eccentricity systems extend to systems with small isolation eccentricities relative to structural eccentricities. All conclusions are supported by a larger set of results using a multitude of eccentricity combinations; only a subset of these results have been presented for conciseness.

ACKNOWLEDGEMENT

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APPENDIX A: EQUIVALENCE OF RR AND FSE FOR EQUAL ECCENTRICITIES

It is shown here that for equal eccentricity systems, the RR equations give the following:

$$\hat{e}_{bx} = \frac{e_{bx}}{r}, \quad \hat{e}_x = \frac{e_x}{r} \tag{A1}$$

which is equivalent to the FSE method. (The same holds for y-direction eccentricities.) The claim is true only for equal frequency ratios ($\Omega_x = \Omega_\theta = \Omega_y$)

Consider the first mode in the x-direction, normalized. This differs from the unnormalized mode of Equation (4)

$$\psi_{x11} = \frac{\omega_x^2}{((\omega_{bx}^2 + \sqrt{\omega_x^4 + \omega_{bx}^4})^2 + \omega_x^4)^{1/2}}, \quad \psi_{x21} = \frac{\omega_{bx}^2 + \sqrt{\omega_x^4 + \omega_{bx}^4}}{((\omega_{bx}^2 + \sqrt{\omega_x^4 + \omega_{bx}^4})^2 + \omega_x^4)^{1/2}} \tag{A2}$$

Rewriting Equation (A1) in terms of frequency ratio

$$\psi_{x11} = \frac{1}{((\Omega_x^2 + \sqrt{1 + \Omega_x^4})^2 + 1)^{1/2}}, \quad \psi_{x21} = \frac{\Omega_x^2 + \sqrt{1 + \Omega_x^4})}{((\Omega_x^2 + \sqrt{1 + \Omega_x^4})^2 + 1)^{1/2}} \tag{A3}$$

it is seen that this mode shape depends only on the frequency ratio Ω_x. Since the frequency ratios are equal, the mode shapes in each direction are also equal ($\boldsymbol{\psi}_{x1} = \boldsymbol{\psi}_{\theta 1} = \boldsymbol{\psi}_{y1} \equiv \boldsymbol{\psi}_1$).

Equation (10) for effective isolation eccentricity is written out in terms of the new mode shape $\boldsymbol{\psi}_1$:

$$\hat{e}_{bx} = \left(\frac{1}{\omega_{y1}^2}\right) \langle \psi_{11} \quad \psi_{21} \rangle \begin{bmatrix} \frac{e_x}{r}\omega_y^2 + 2\frac{e_{bx}}{r}\omega_{by}^2 & -\frac{e_x}{r}\omega_y^2 \\ -\frac{e_x}{r}\omega_y^2 & \frac{e_x}{r}\omega_y^2 \end{bmatrix} \begin{Bmatrix} \psi_{11} \\ \psi_{21} \end{Bmatrix} \tag{A4}$$

Since the isolation eccentricity equals the structural eccentricity, the eccentricities are factored out.

$$\hat{e}_{bx} = \left(\frac{1}{\omega_{y1}^2}\right) \frac{e_{bx}}{r} \langle \psi_{11} \quad \psi_{21} \rangle \begin{bmatrix} \omega_y^2 + 2\omega_{by}^2 & -\omega_y^2 \\ -\omega_y^2 & \omega_y^2 \end{bmatrix} \begin{Bmatrix} \psi_{11} \\ \psi_{21} \end{Bmatrix} \tag{A5}$$

The 2×2 stiffness submatrix in Equation (A5) is the same matrix for which the eigenvalue equation is formed, and the mode shapes and frequencies of the torsionally uncoupled system are solved. These are the same mode shapes that appear in Equation (A5). With the help of the characteristic equation, it is shown that the first mode frequency may be substituted for the stiffness submatrix.

$$(\omega_y^2 + 2\omega_{by}^2)\psi_{11} - \omega_y^2\psi_{21} = \omega_{y1}^2\psi_{11} \tag{A6a}$$

$$-\omega_y^2\psi_{11} + \omega_y^2\psi_{21} = \omega_{y1}^2\psi_{21} \tag{A6b}$$

Substituting Equation (A6) into Equation (A5),

$$\hat{e}_{bx} = \left(\frac{1}{\omega_{y1}^2}\right)\frac{e_{bx}}{r}\langle\psi_{11} \quad \psi_{21}\rangle \begin{Bmatrix} \omega_{y1}^2\psi_{11} \\ \omega_{y1}^2\psi_{21} \end{Bmatrix} \tag{A7}$$

reduces to

$$\hat{e}_{bx} = \frac{e_{bx}}{r}(\psi_{11}^2 + \psi_{21}^2) \tag{A8}$$

which reduces to Equation (A1) because of the normalization of the modes.

This proof is easily extended to the y-direction eccentricity, and to the effective structural eccentricity using the mode shapes and frequencies of the second mode.

APPENDIX B: NOMENCLATURE

Subscripts

b	of isolation system
nj	mode number $n = 1, 2, 3$ in mode triplet $j = 1$ for isolation modes, $j = 2$ for structural modes
x, y	principal axes of resistance
θ	of rotation

A_{nj}	modal spectral acceleration
b	plan dimension
$\boldsymbol{c}$	damping matrix of torsionally coupled system
e_{bx}, e_{by}	eccentricities of isolation system
e_x, e_y	eccentricities of fixed-base structure
$\hat{e}_{bx}$, $\hat{e}_{by}$	effective isolation eccentricities
$\hat{e}_x$, $\hat{e}_y$	effective structural eccentricities
k_{bx}, $k_{b\theta}$, k_{by}	stiffnesses of isolation system, assuming a rigid structure
k_x, k_θ, k_y	stiffnesses of fixed-base structure
$\tilde{\boldsymbol{k}}$	stiffness matrix of torsionally coupled system
$\boldsymbol{k}_{xx}$, $\boldsymbol{k}_{\theta\theta}$, $\boldsymbol{k}_{yy}$ $\boldsymbol{k}_{x\theta}$, $\boldsymbol{k}_{y\theta}$	stiffness submatrices of Equation (1)
$\boldsymbol{m}$	mass matrix of torsionally coupled system
m_{b}, m	mass of base slab, mass of top slab

r	radius of gyration of structure and isolation system
r_{nj}	modal response
r_{nj}^{st}	modal static response
T_{b}	base torque at center of mass
u_{s}, u_{f}	stiff and flexible edge isolator deformation
$\boldsymbol{u}_x$, $\boldsymbol{u}_\theta$, $\boldsymbol{u}_y$	displacement subvectors
$\boldsymbol{u}$	displacement vector
$\ddot{u}_{\mathrm{g}x}$	x-direction ground acceleration
$V_{\mathrm{b}x}$	x-direction base shear
$\boldsymbol{z}_{nj}$	mode shape of isolation-related or structure-related single-storey system
$\boldsymbol{\iota}$	influence vector
ζ_{nj}	modal damping ratio
$\boldsymbol{\psi}_{xj}$, $\boldsymbol{\psi}_{\theta j}$, $\boldsymbol{\psi}_y$	mode shapes of torsionally uncoupled system
$\boldsymbol{\Psi}_j$	a triplet of modes of torsionally uncoupled system
$\boldsymbol{\phi}_{nj}$	approximate mode of torsionally coupled system
ω_{bx}, $\omega_{b\theta}$, ω_{by}	natural frequencies of isolation system, assuming a rigid structure
ω_x, ω_θ, ω_y	natural frequencies of fixed-base structure
ω_{xj}, $\omega_{\theta j}$, ω_{yj}	frequencies of torsionally uncoupled system
ω_j	refers to lateral frequency of torsionally uncoupled system when $\omega_{xj} = \omega_{yj}$
ω_{nj}	approximate frequency of torsionally coupled system
Ω_x, Ω_θ, Ω_y	frequency ratios, ratios of isolation system frequencies to structure frequencies
Ω	refers to frequency ratio when $\Omega_x = \Omega_\theta = \Omega_y$

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