CSc 120
Introduction to Computer Programming II

Recursion

Problem

How much money is in this cup?

Approach:

- We will consider a *different* way of adding up the coins
- The TAs will demonstrate!

How much money is in this cup?

If the cup is not empty:

Take out a coin. Pass the cup to the next person and ask them:

"How much money is in this cup?"

When they answer, add your coin to their answer and pass your answer back

your_answer = your_coin + their_answer

else the cup is empty:

Answer "zero" to the person who passed to you.

your_answer = 0

Challenge

Can we express that procedure in Python?

Idea:

```
>>> cup = [5, 10, 1, 5]
>>> how_much_money(cup)
21
```

Write Python code that models the cup passing example.

function: how_much_money

```
def how_much_money(cup):
    if cup == []:
        return 0
    else:
```

function: how_much_money

```
def how_much_money(cup):
    if cup == []:
        return 0
    else:
        return cup[0] + how_much_money(cup[1:])
```

5

[10, 1, 5]

Usage:

```
>>> how_much_money([5, 10, 1, 5])
21
```

Calls and returns

```
def how_much_money(cup):
    if cup == []:
        return 0
    else:
        return cup[0] + how_much_money(cup[1:])
```

```
how_much_money([5,10,1,5])
| how_much_money([10,1,5])
| | how_much_money([1,5])
| | | how_much_money([5])
| | | | how_much_money([])
| | | | how_much_money returned 0
| | | how_much_money returned 5
| | how_much_money returned 6
| how_much_money returned 16
how_much_money returned 21
```

Manual expansion of calls

```
>>> 5 + how_much_money([10, 1, 5])
21
>>> 5 + (10 + how_much_money([1,5]))
21
>>> 5 + (10 + (1 + how_much_money([5])))
21
>>> 5 + (10 + (1 + (5 + how_much_money([]))))
21
```

Recursion

A function is *recursive* if it calls itself:

```
def  how_much_money( ... ):
        ...
        how_much_money( ... )  ← recursive call
        ...
```

The call to itself is a *recursive call*

Recursion

A solution to a problem is *recursive* when it is constructed from the solution to a simpler version of the same problem.

Recursion

A solution to a problem is *recursive* when it is constructed from the solution to a simpler version of the same problem.

```
def how_much_money(cup):
    if cup == []:
        return 0
    else:
        return cup[0] + how_much_money(cup[1:])
```

simpler version of the problem

Recursion

- Recursive functions have two kinds of cases:
  - *base case(s)* :
    - *do some trivial computation and return the result*
  - *recursive case(s)* :
    - *the expression of the problem is a simpler case of the same problem*
    - *the input is reduced or the size of the problem is reduced*
- *Note*: the recursive call is given a smaller problem to work on
  - e.g., it makes progress towards the base case

recursion: base case/recursive case

```
def how_much_money(cup):
    if cup == []:
        return 0
    else:
        return cup[0] + how_much_money(cup[1:])
```

base case:
cup == []

recursive
case:
cup != []

The convention is to handle the base case(s) first.

Problem 1

Write a recursive function to count the number of coins in a cup. *The len function is not allowed.*

Usage:

```
>>> count_coins([10, 5, 1, 5])
4
```

Solution

```
def count_coins(cup):
    if cup == []:
        return 0
    else
        return 1 + count_coins(cup[1:])
```

Solution

base case:
cup == []

```
def count_coins(cup):
    if cup == []:
        return 0
    else:
        return 1 + count_coins(cup[1:])
```

recursive
case:
cup != []

recursive call is on a smaller problem

Problem 2

Write a recursive function to count the number of nickels in a cup.

Usage:

```
>>> count_nickels([10, 5, 1, 5, 1])
2
```

Solution

base case:
cup == []

recursive case:
cup != []

```
def count_nickels(cup):
    if cup == []:
        return 0
    else:
        if cup[0] == 5:
            return 1 + count_nickels(cup[1:])
        else:
            return count_nickels(cup[1:])
```

recursive call is on a smaller problem

Problem 3

Write a recursive function that returns the total length of all the elements of a list of lists (a 2-d list).

Usage:

```
>>> total_length([[1,2], [8,2,3,4], [2,2,2]])
9
```

Solution

```
def total_length(alist):
    if alist == []:
        return 0
    else:
        return len(alist[0]) + total_length(alist[1:])
```

base case:
alist == []

recursive case:
alist != []

recursive call is on a smaller problem

Problem 4

Recall that factorial is defined by the equation:

$$n! = n * (n-1) * (n-2) * (n-3) * \ldots * 2 * 1$$

and

$$0! = 1$$

Write a recursive function that computes the factorial of a number.

Usage:

```
>>> fact(4)
24
```

Solution

```
def fact(n):
   if n == 0:
      return 1
   else:
      return n *  fact(n-1)
```

base case:
n == 0

recursive case:
n != 0

recursive call is on a smaller problem

EXERCISE p. 1

Write a recursive function sumlist(L) that returns the sum of the elements in L.

Usage:

```
>>> sumlist([2,4,6,10])
22
```

EXERCISE p. 2

Write a recursive function string_len(s) that returns the length of string s.

Usage:

```
>>> >>> string_len("I wandered lonely as a cloud")
28
>>>
```

EXERCISE p. 3

Write a recursive function join_all(alist) that takes a list alist and returns a string consisting of every element of alist concatenated together.

Usage:

```
>>> join_all([1,2,3,4,5])
'12345'
>>>
>>> join_all(['aa','bb'])
'aabb'
```

EXERCISE p. 4

Write a recursive function that implements join.

That is, write a function `join(alist, sep)` that takes a list `alist` and creates a string consisting of every element of `alist` separated by the string `sep`.

Usage:

```
>>> join(['aa', 'bb' , 'cc'], '-')
'aa-bb-cc'
```

the runtime stack

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)


>>> fact(4)
24
```

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)
```

Discussion:

What does this return when `n==0` ?

What steps does it execute?

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)
```

Discussion:

What does this return when `n==1` ?

What steps does it execute?

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)
```

Insight:
The call with `n==1` has another call to `fact()` *buried inside it.*

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)
```

Discussion:

What does this return when `n==2` ?

What steps does it execute?

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)
```

Insight:

The call with `n==2` has another call (`n==1`) buried inside it.

The call with `n==1` has another call (`n==0`) buried inside it.

The call with `n==0` ends immediately.

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)
```

Insight:
There is more than one `n`, all at the same time!

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)

>>> fact(4)
24
```

We need the value of n both before and after the recursive call

∴ its value has to be saved somewhere

"somewhere" ≡ "stack frame"

How recursion works

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>>> def fact(n):
        if n == 0:
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>>> fact(4)
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```

Python's runtime system* maintains a stack:

- push a "frame" when a function is called
- pop the frame when the function returns

"frame" or "stack frame": a data structure that keeps track of variables in the function body, and their values, between the call to the function and its return

* "runtime system" = the code that Python executes to make everything work at runtime

How recursion works

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>>> def fact(n):
        if n == 0:
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>>> fact(4)
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```

Python's runtime system*
maintains a stack:

- push a "frame" when a function is called
- pop the frame when the function returns

sometimes called the "runtime stack"

* "runtime system" = the code that Python executes to make everything work at runtime

How recursion works

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>>> def fact(n):
        if n == 0:
            return 1
        else:
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>>> fact(4)
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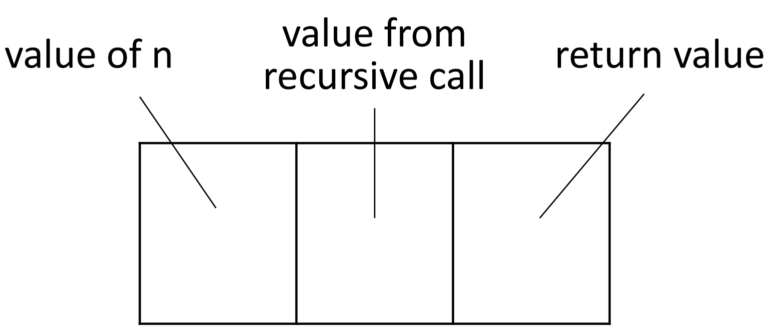
```



stack frame for fact()

Let's simulate this with `n==0`, first.

How recursion works

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        if n == 0:
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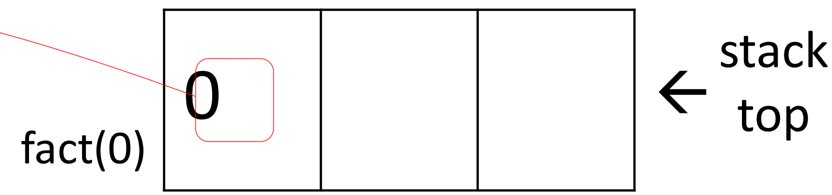
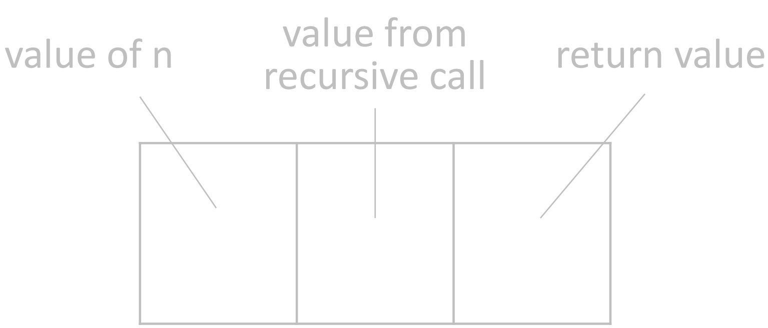
        else:
            return n * fact(n-1)


>>> fact(4)
24
```





stack frame for fact()

runtime stack

How recursion works

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        if n == 0:
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>>> fact(4)
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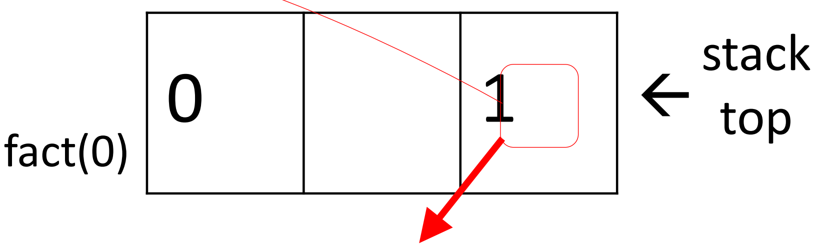

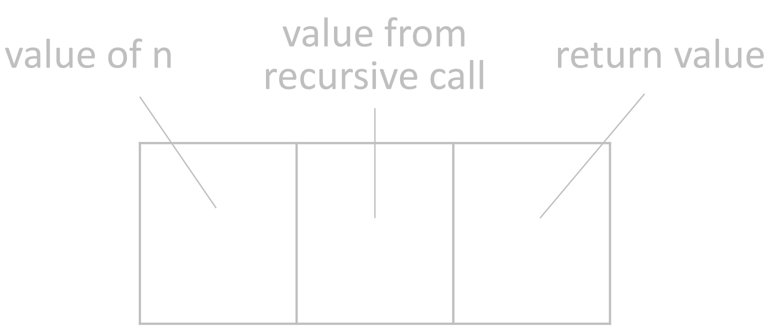

stack frame for fact()

runtime stack

How recursion works

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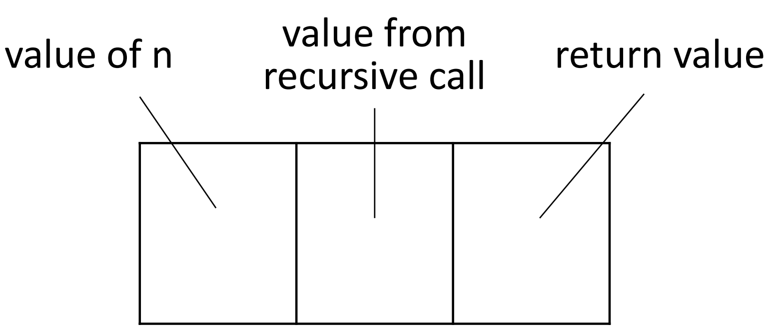
```



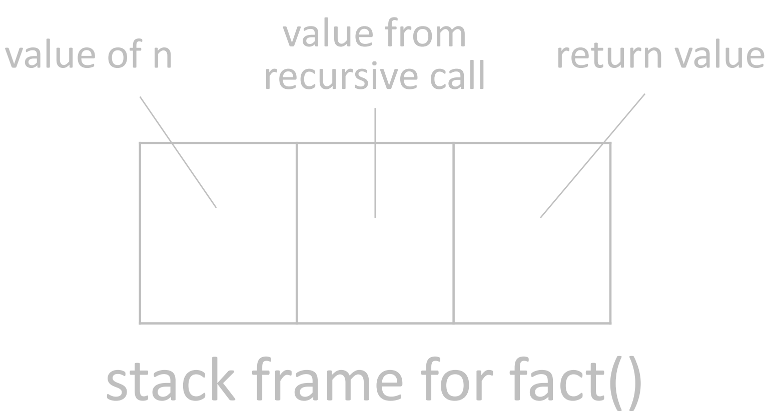
stack frame for fact()

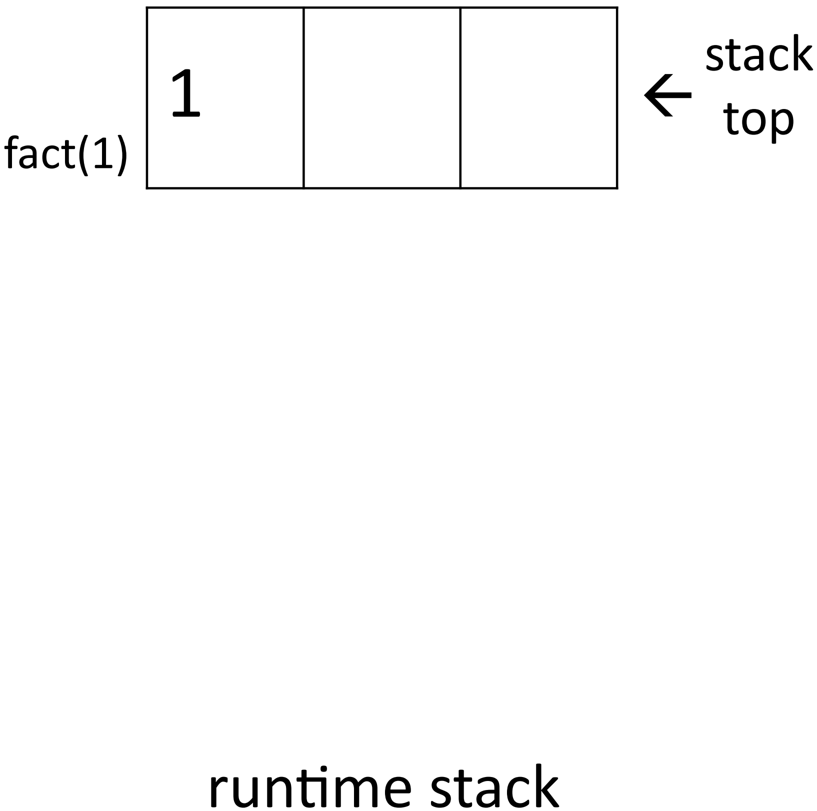
Now, let's try `n==1`.

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)


>>> fact(4)
24
```





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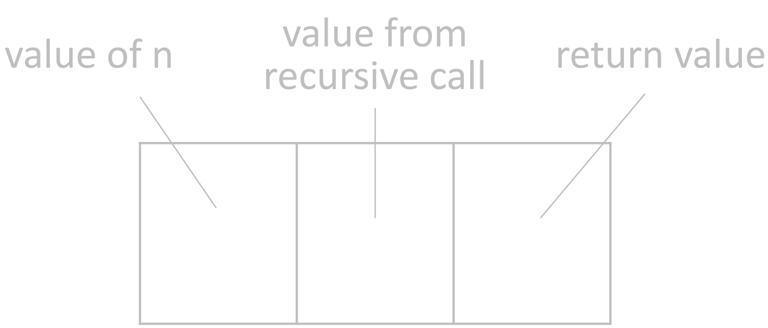

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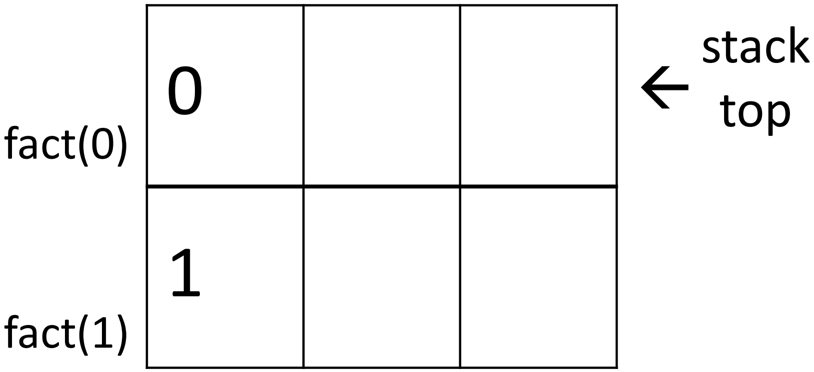

runtime stack

How recursion works

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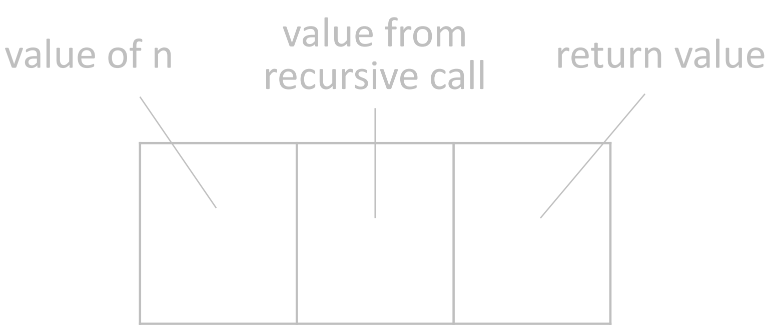
>>> fact(4)
24
```



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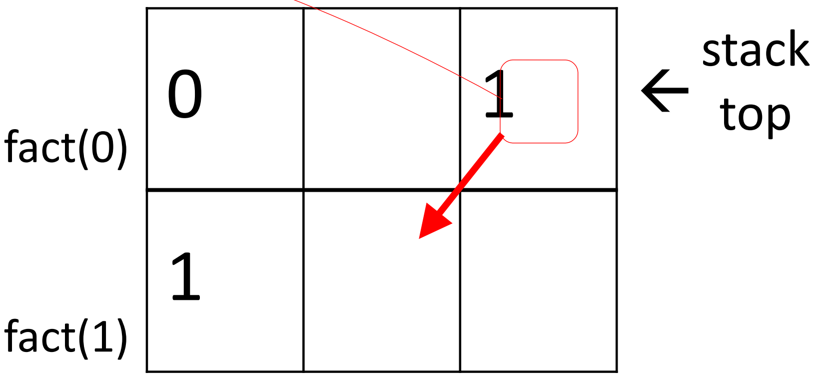

runtime stack

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>>> fact(4)
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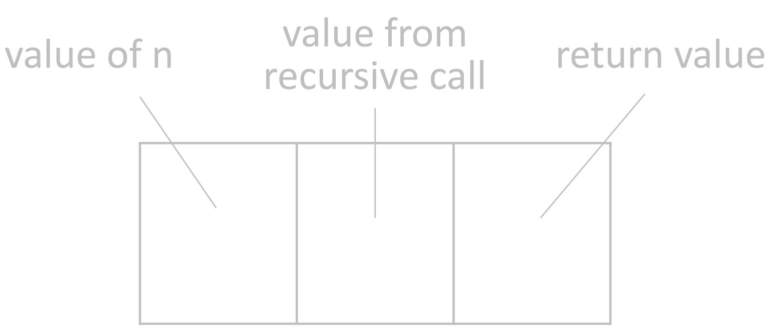
```



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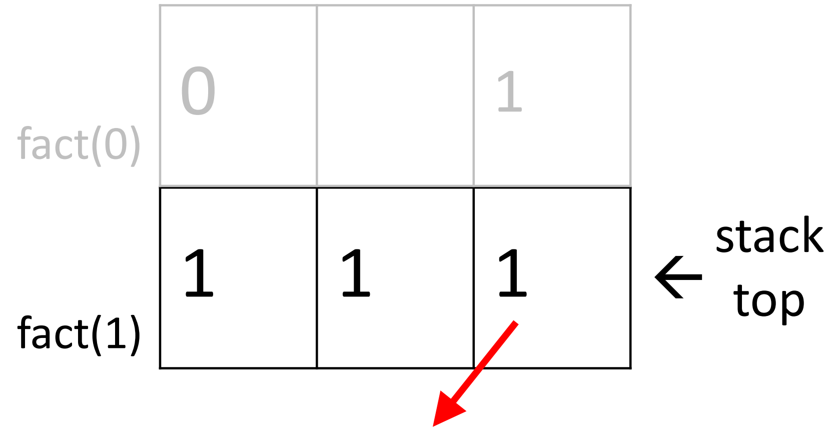

runtime stack

How recursion works

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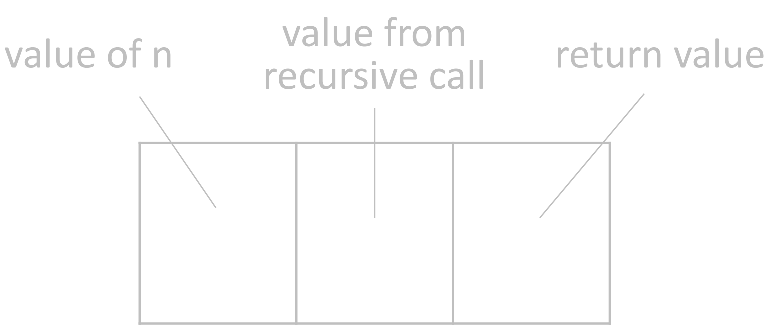
```

```
>>> fact(4)
24
```



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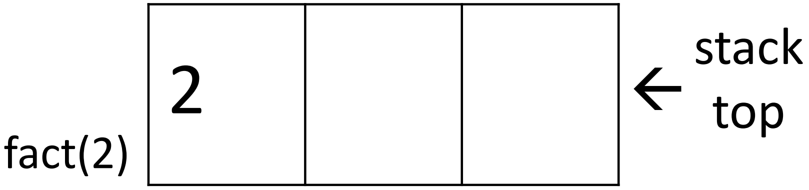

n==2

runtime stack

How recursion works

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        if n == 0:
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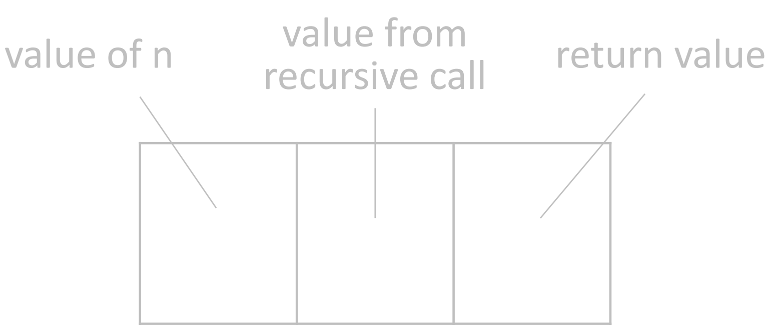
>>> fact(4)
24
```



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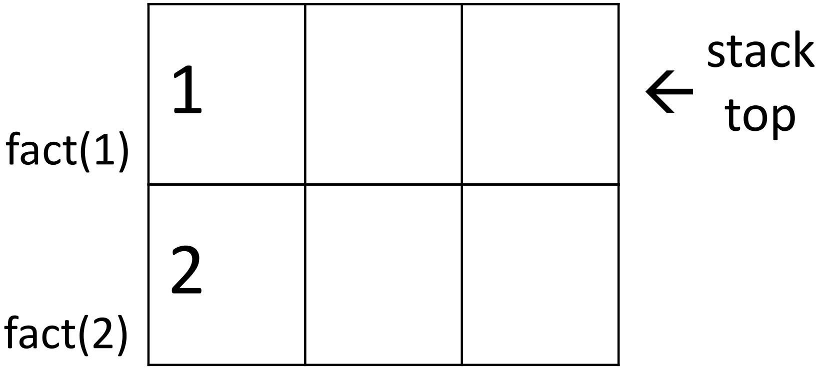

runtime stack

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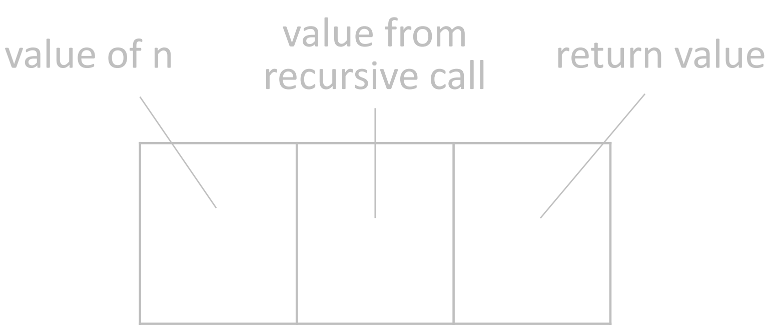

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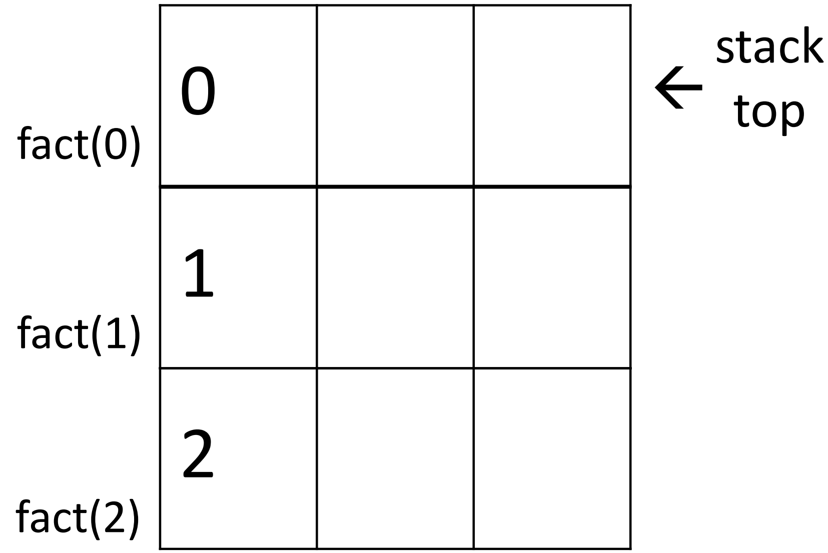

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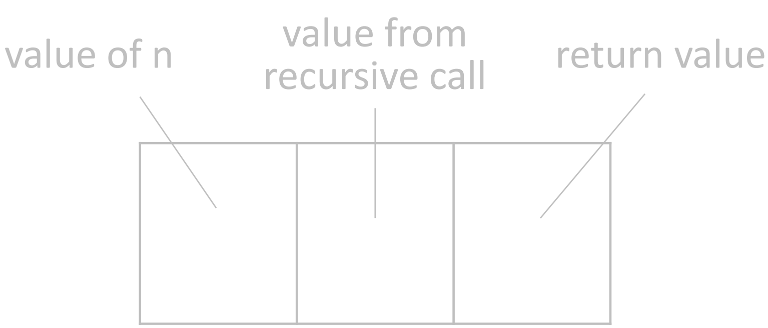

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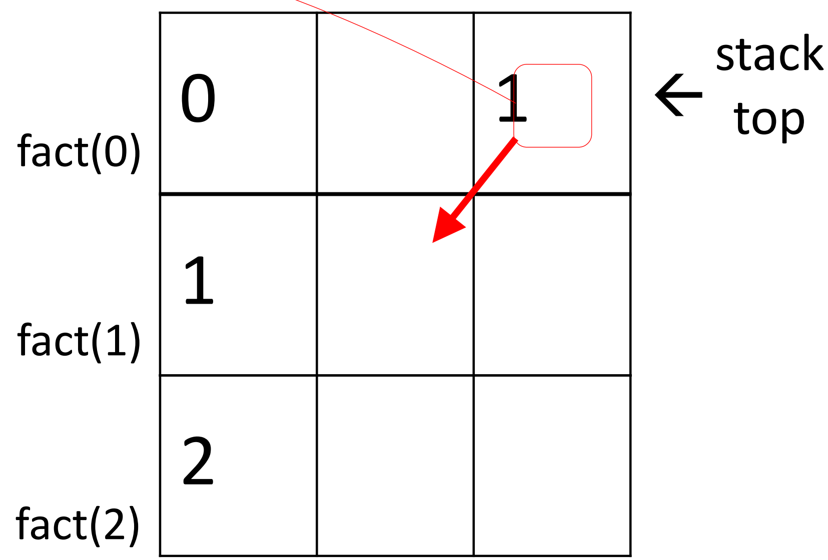

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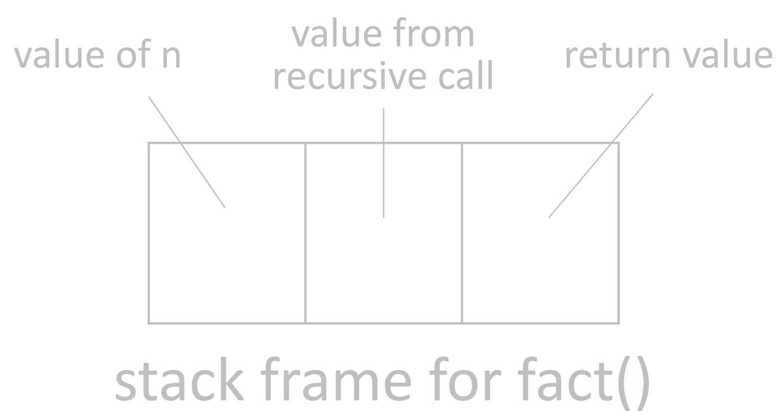

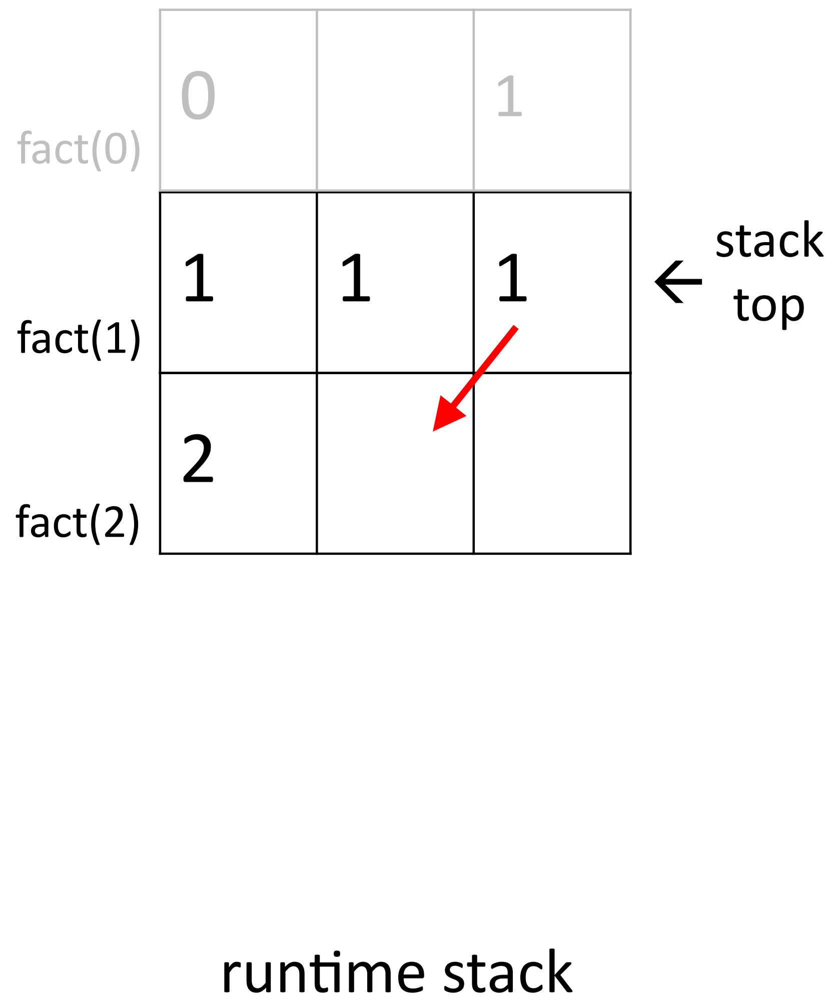

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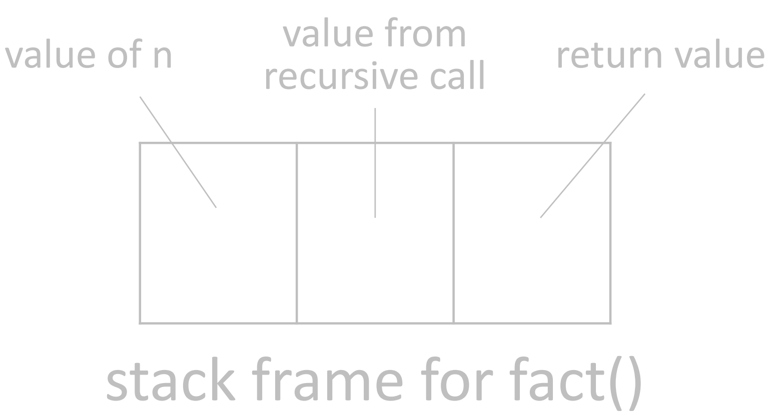

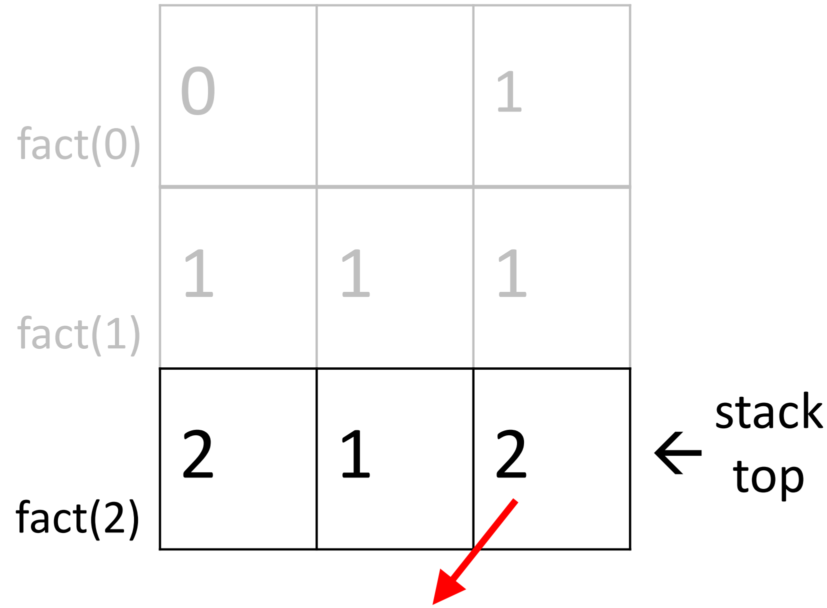

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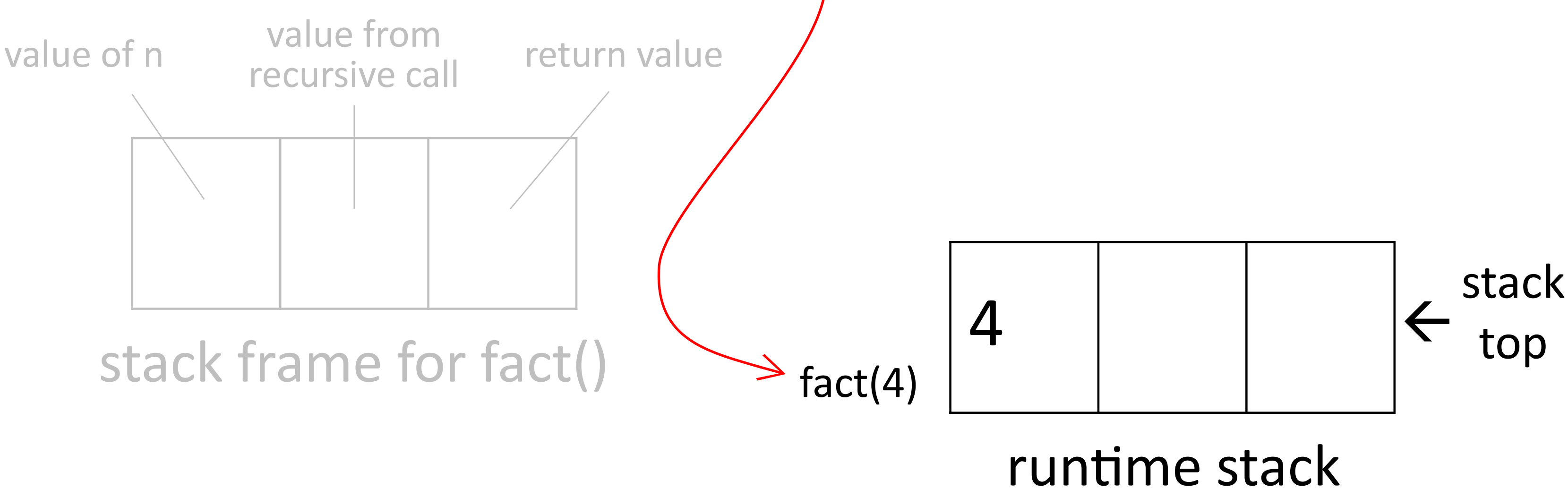

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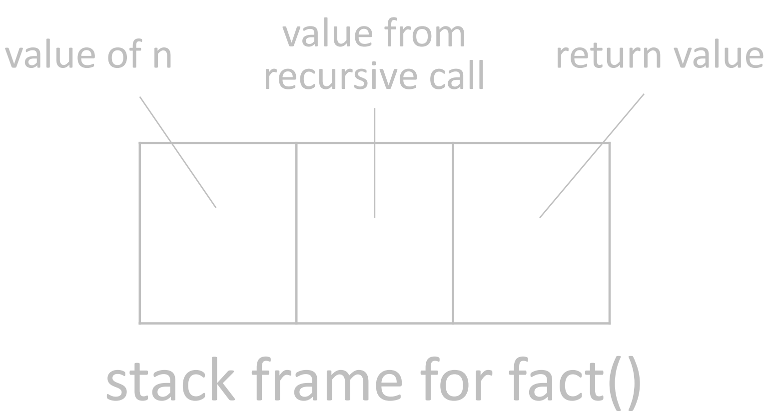

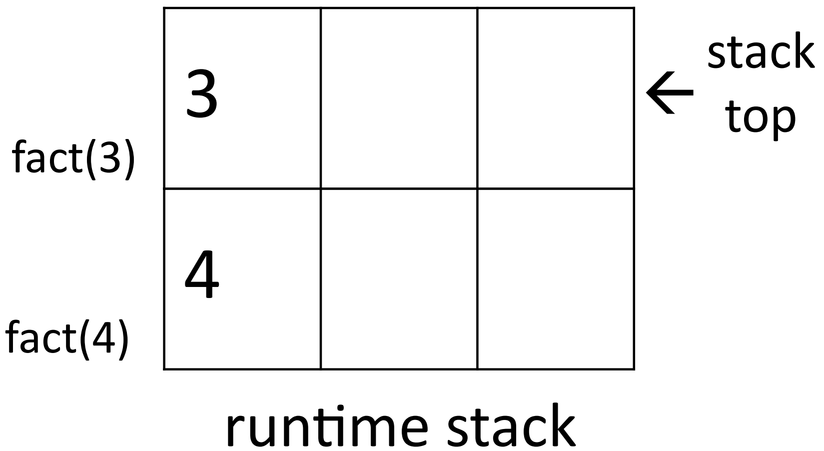

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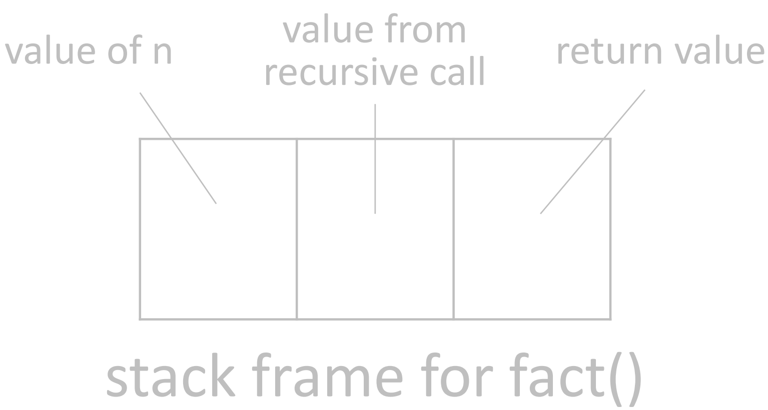

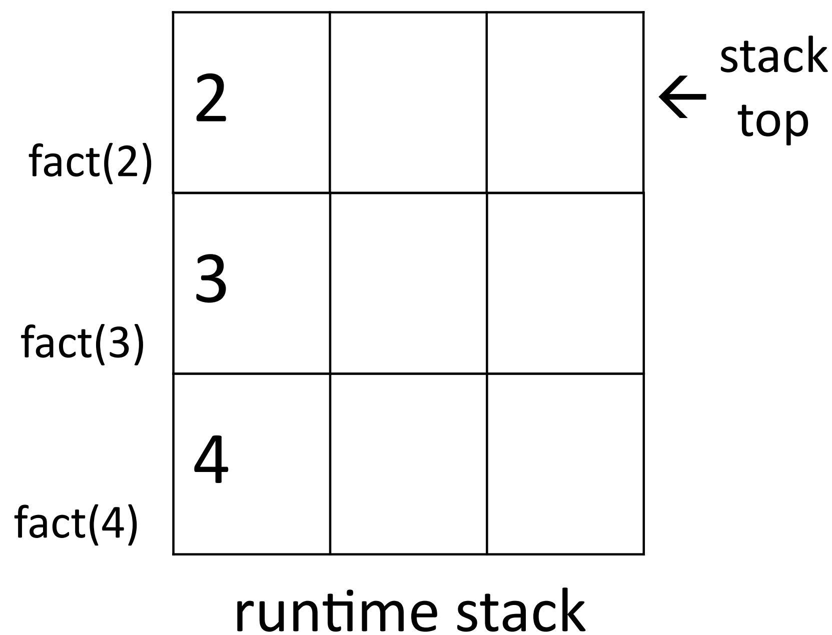

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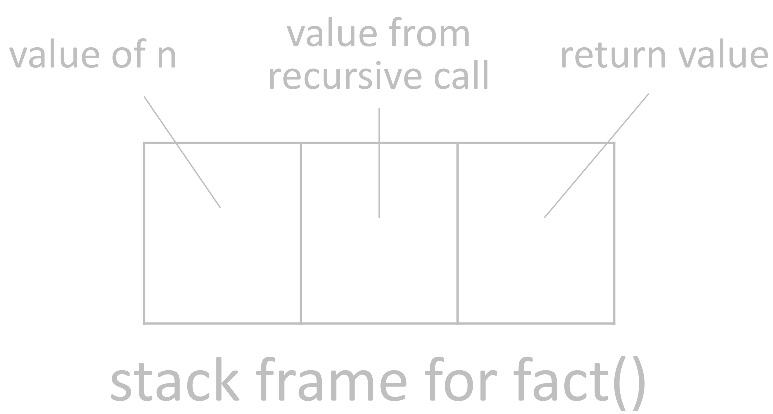

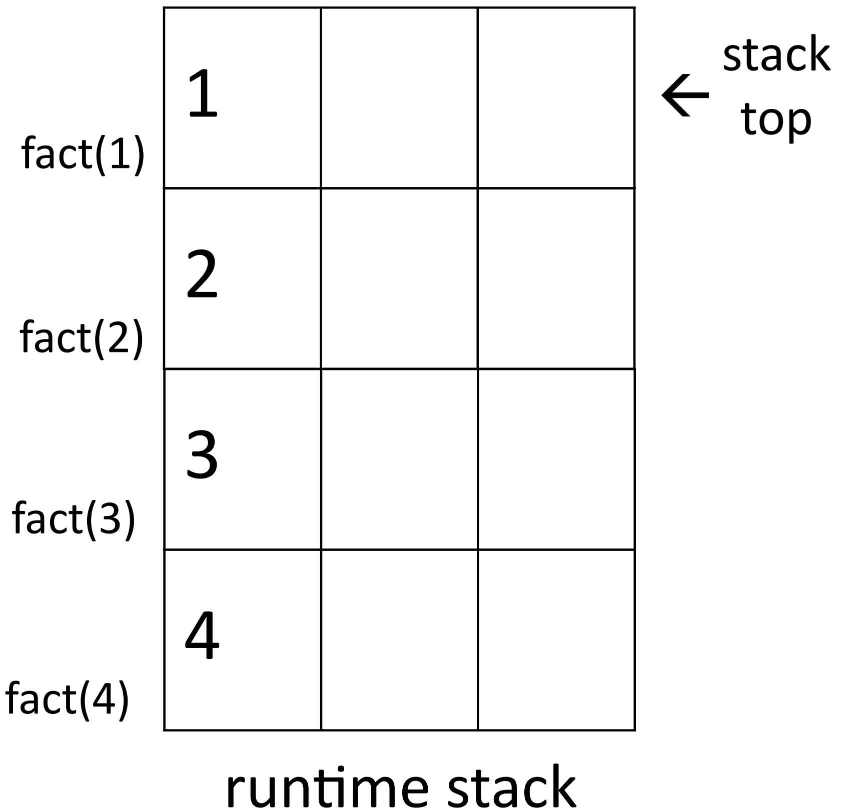

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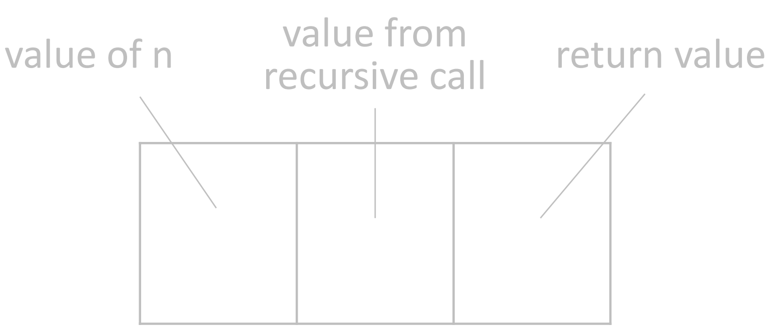
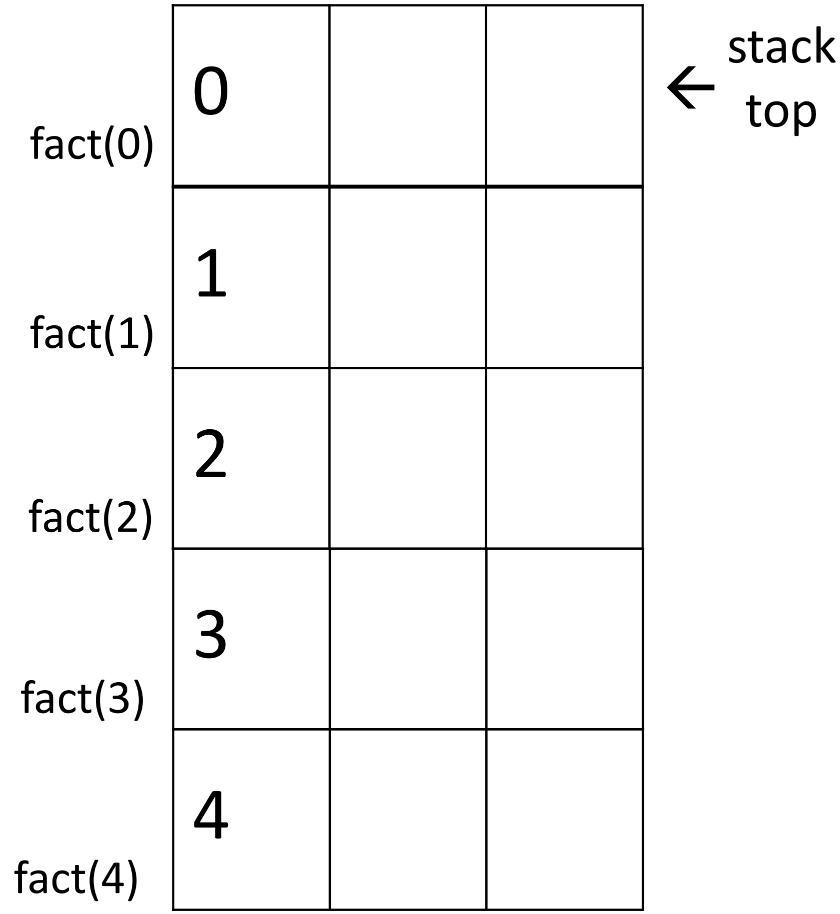


stack frame for fact()



runtime stack

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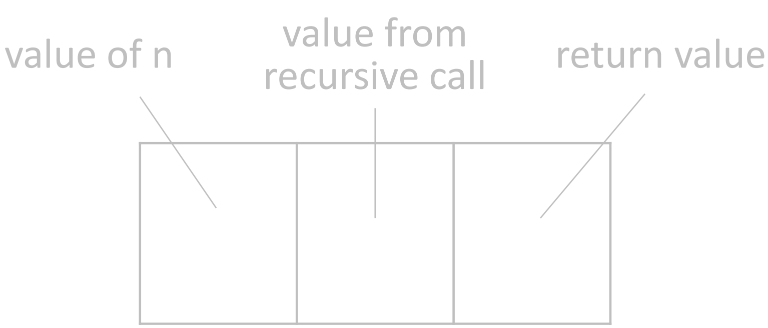
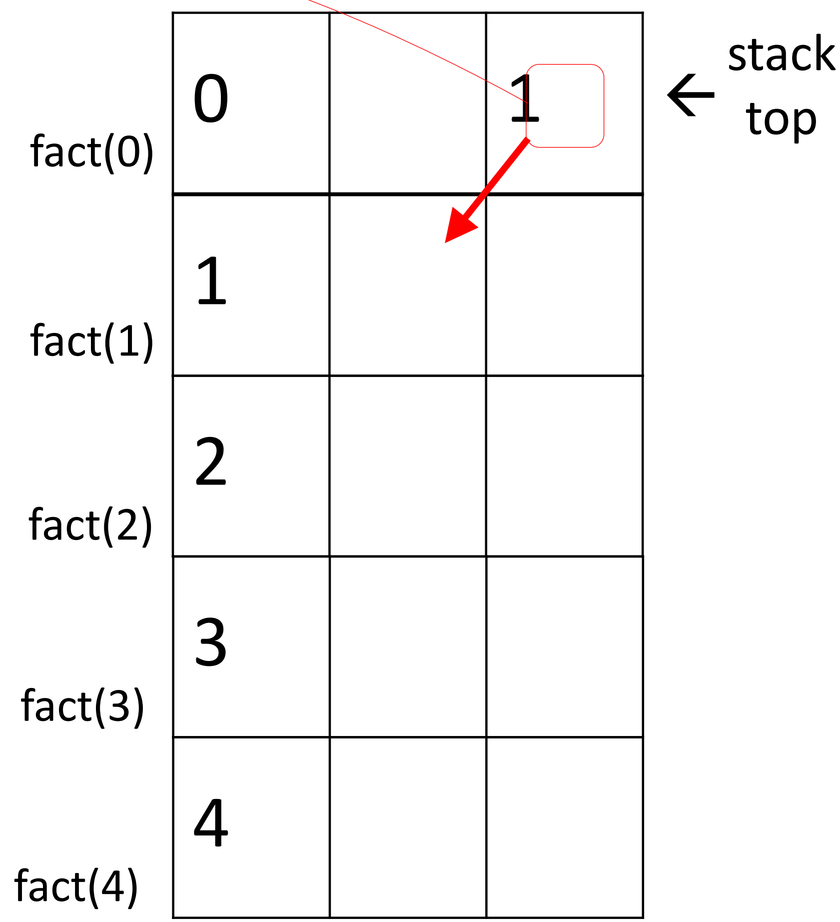


stack frame for fact()

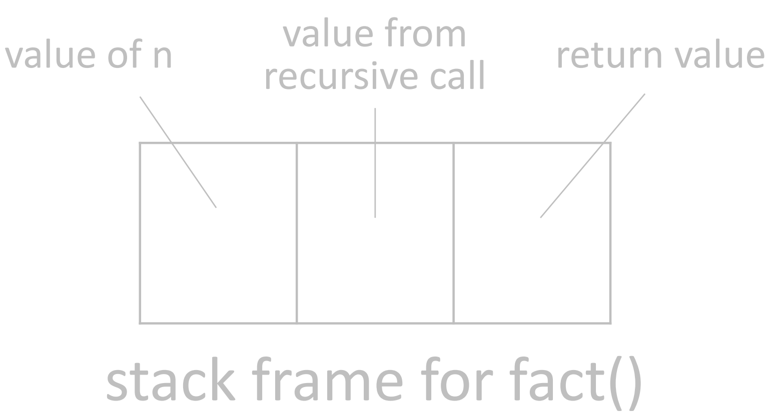


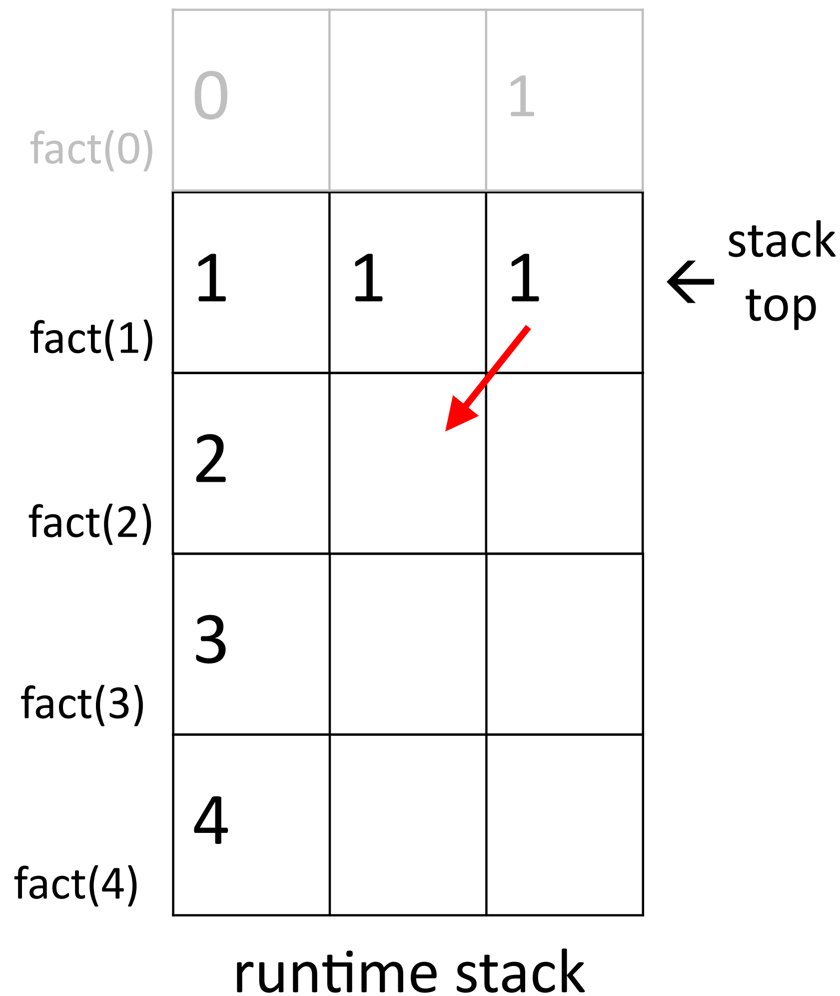
runtime stack

How recursion works

```
>>> def fact(n):
        if n == 0:
            return 1
        else:
            return n * fact(n-1)


>>> fact(4)
24
```





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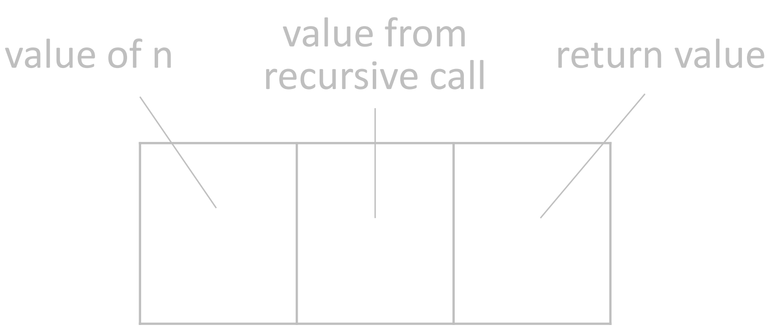
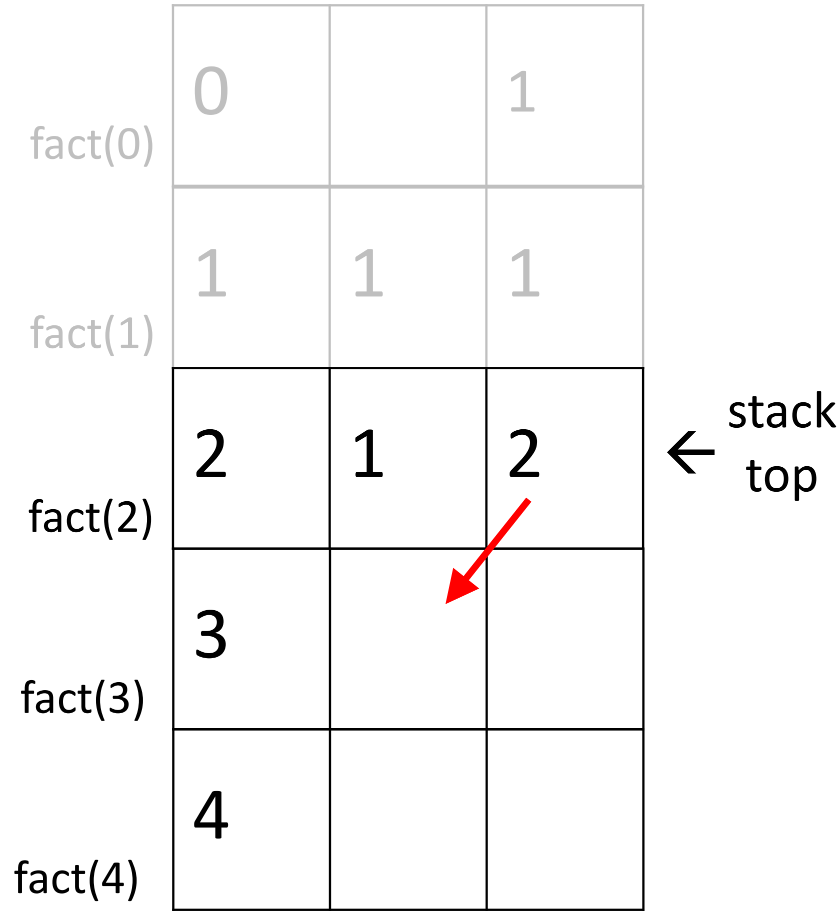


stack frame for fact()



runtime stack

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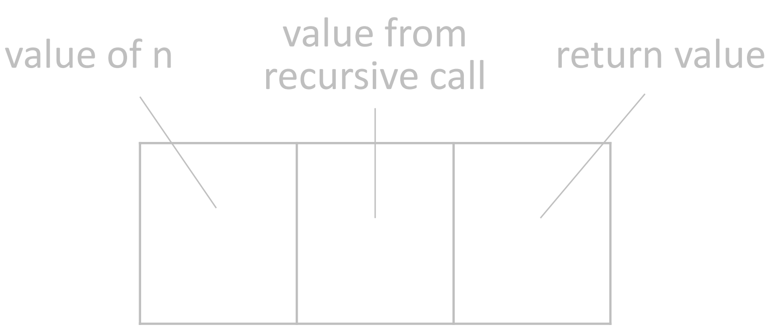
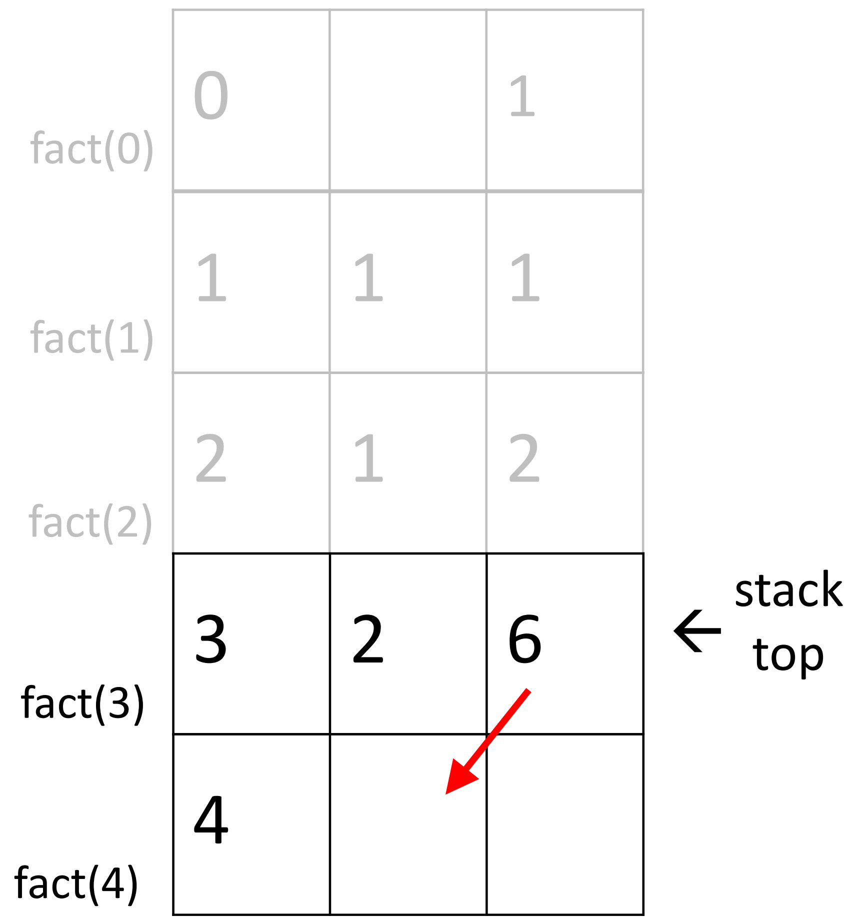


stack frame for fact()



runtime stack

How recursion works

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            return n * fact(n-1)


>>> fact(4)
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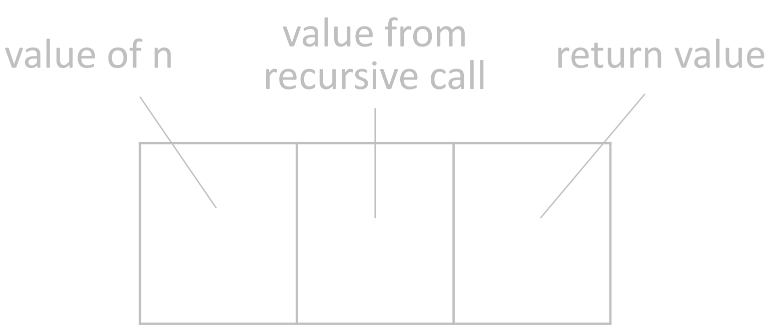
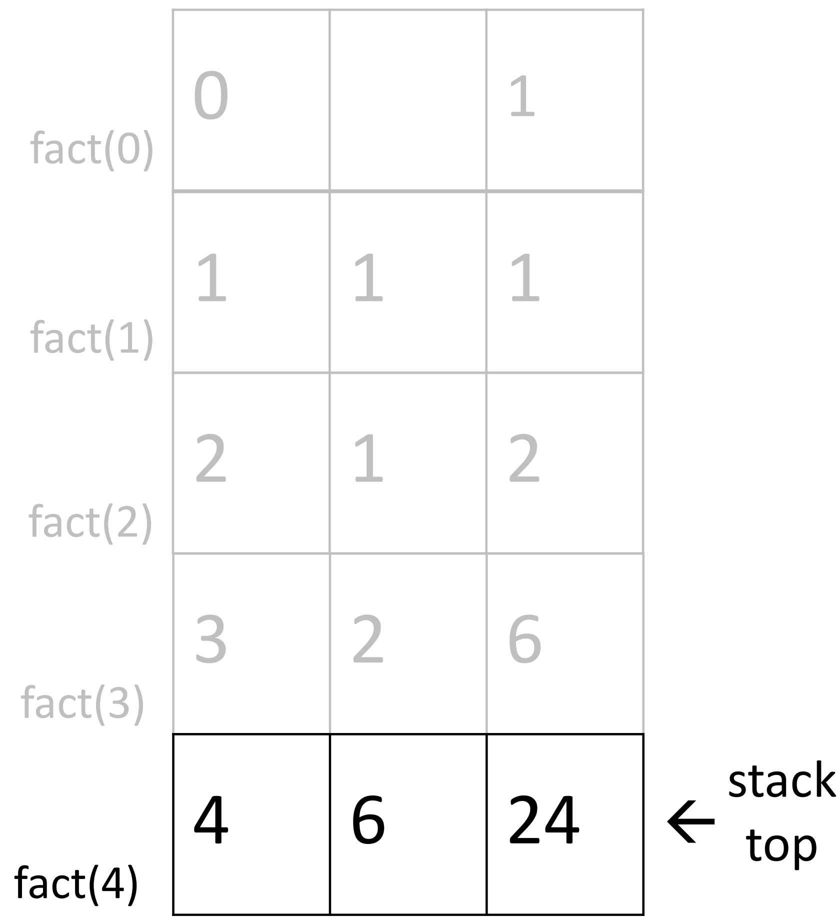


stack frame for fact()



runtime stack

How recursion works

```
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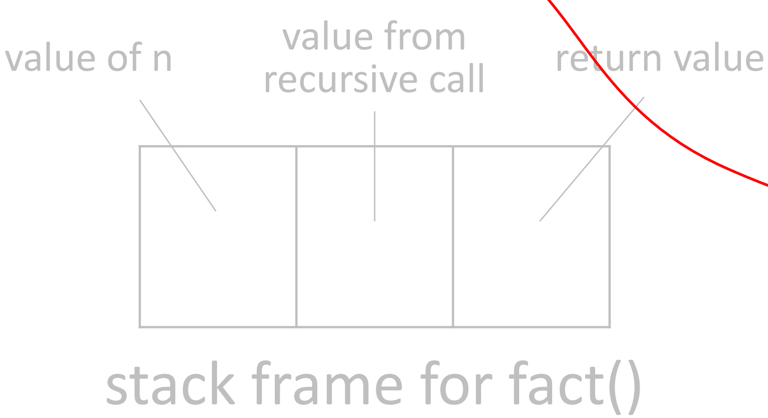
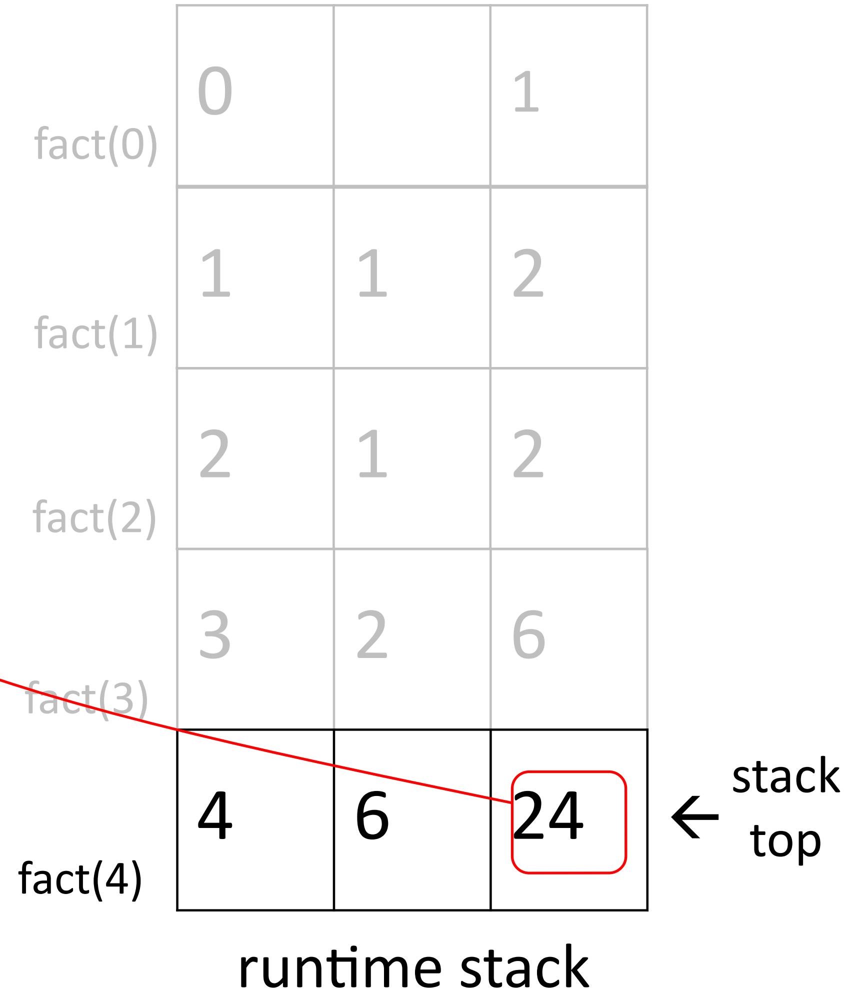
        if n == 0:
            return 1
        else:
            return n * fact(n-1)


>>> fact(4)
24
```





The runtime stack

- The use of a *runtime stack* containing *stack frames* is not specific to recursion
  - all function and method invocations use this mechanism
  - not just in Python, but other languages as well (Java, C, C++, …)

Problem 5

Write a recursive function to print the numbers from 1 through n, one per line.

Usage:

```
>>> print_n(6)
1
2
3
4
5
6
```

Solution

```
def print_n(n):
    if n == 0:
        return
    else:
        print_n(n-1)
        print(n)
```

base case:
n = 0

recursive case:
n != 0

recursive call is on a smaller value

Recursion How to

To write a recursive function, figure out:

What values are involved in the computation?

- these will be the arguments to the recursive function

- *Base case(s)*
  - when does the recursion stop?
  - what is the simple value or data that can be computed and returned?
- *Recursive case(s)*
  - what is the "smaller problem" to pass to the recursive call?
  - what does a single round of computation involve?

Recursion how to: sumlist


```
def sumlist(L):
    if len(L) == 0:
        return 0
    else:
        return L[0] + sumlist(L[1:])
```

base case

recursive case

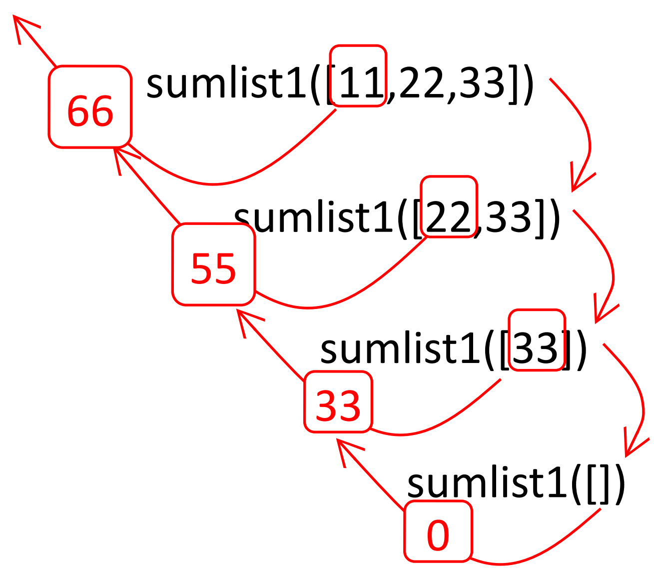
One round of computation adds L[0] and the result of the recursive call

argument to recursive call is "rest of the list" after L[0] (recurses on a smaller problem)

Recursion: flow of values

Version 1

```
def sumlist1(L):
    if len(L)  == 0:
        return 0
    else:
        return L[0] + sumlist1(L[1:])
```



EXERCISE p. 1

Write a function even_positions(s) that returns a string consisting of all the even-numbered positions of the Python string s.

Usage:

```
>>>even_positions(”abc”)
’ac'
>>>
>>> even_positions(”abcdef”)
’ace'
```

Recursion

- The recursive case can be written many ways
- Consider summing the elements in a list

```
def sumlist(L):
    if len(L)  = 0:
        return 0
    else:
        return L[0] + sumlist(L[1:])
```

- Options:
  - Recurse on L[:-1] and then add in L[-1]
  - Recurse on each half and add the results

EXERCISE p. 2

Write sumlist() in the two ways described in the exercise.

Versions of sumlist

Version 1	

```
def sumlist(L):
    if len(L) == 0:
        return 0
    else:
        return L[0] + sumlist(L[1:])
```

base case

recursive case

One round of computation adds L[0] and the result of the recursive call

argument to recursive call is "rest of the list" after L[0] (recurses on a smaller problem)

Versions of sumlist

Version 2
(variation on version 1)

```
def sumlist(L):
    n = len(L)
    if n  == 0:
        return 0
    else:
        return sumlist(L[:-1]) + L[-1]
```

argument to recursive call is "rest of the list" up to the last element (recurses on a smaller problem)

add the last element of L

Versions of sumlist

Version 2 (variation on version 1)	Version 3 ("smaller" need not be by just 1)

```
def sumlist(L):
    n = len(L)
    if n  == 0:
        return 0
    else:
        return sumlist(L[:-1]) + L[-1]
```

```
def sumlist(L):
    if len(L)  = 0:
        return 0
    elif len(L) == 1:
        return L[0]
    else:
        return sumlist(L[ :len(L)//2]) + \
                sumlist(L[len(L)//2: ])
```

better for parallel execution

argument to each recursive call is half of the current list (recurses on a smaller problem)

sumlist

```
def sumlist(L):
    if len(L)  = 0:
        return 0
    elif len(L) == 1:
        return L[0]
    else:
        return sumlist(L[ :len(L)//2]) + sumlist(L[len(L)//2: ])
```

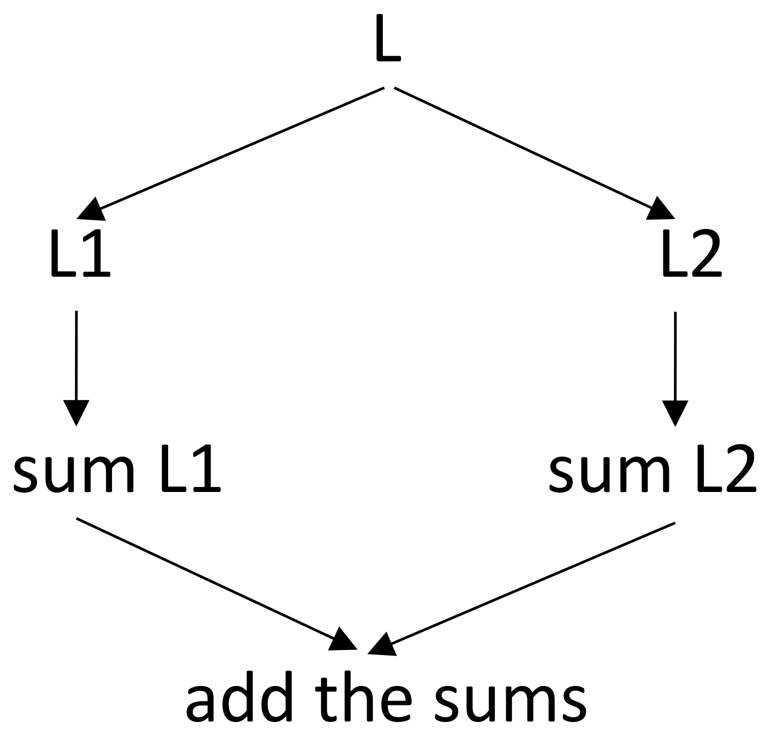
recursive sumlist

input list

split into two halves

add the halves (recursively)

return the sum of the sums



sumlist: example

```
sumlist([1,3,4,6,8])
```

sumlist: example

sumlist([1,3,4,6,8])

sumlist([1,3]) sumlist([4,6,8])

sumlist: example

sumlist([1,3,4,6,8])

sumlist([1,3]) sumlist([4,6,8])

sumlist([1]) sumlist([3])

sumlist: example

sumlist([1,3,4,6,8])

sumlist([1,3]) sumlist([4,6,8])

sumlist([1]) sumlist([3])

1 3

sumlist: example

sumlist([1,3,4,6,8])

sumlist([1,3]) sumlist([4,6,8])

sumlist([1]) sumlist([3])

1 3

4

sumlist: example

sumlist([1,3,4,6,8])

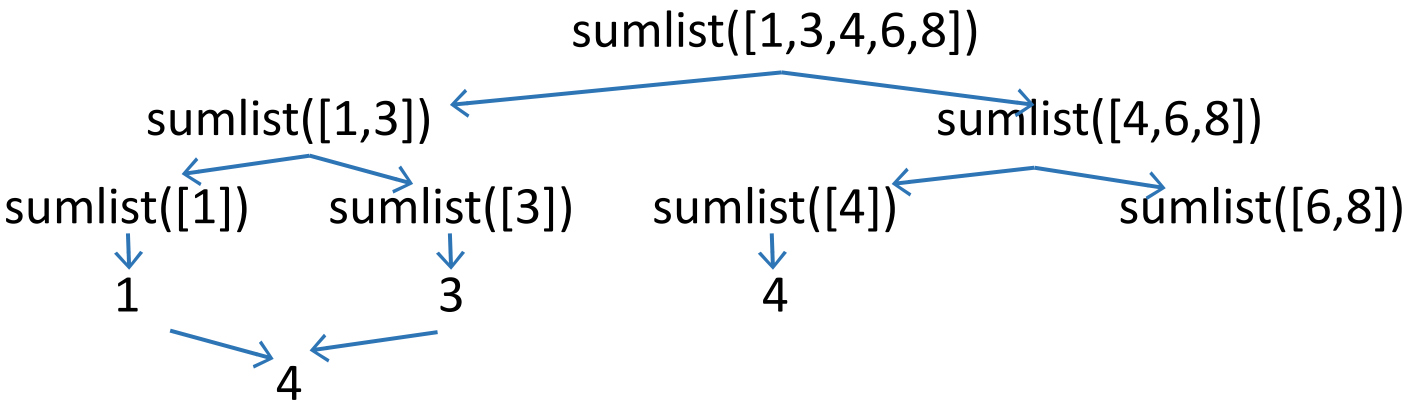
sumlist([1,3]) sumlist([4,6,8])

sumlist([1]) sumlist([3]) sumlist([4]) sumlist([6,8])

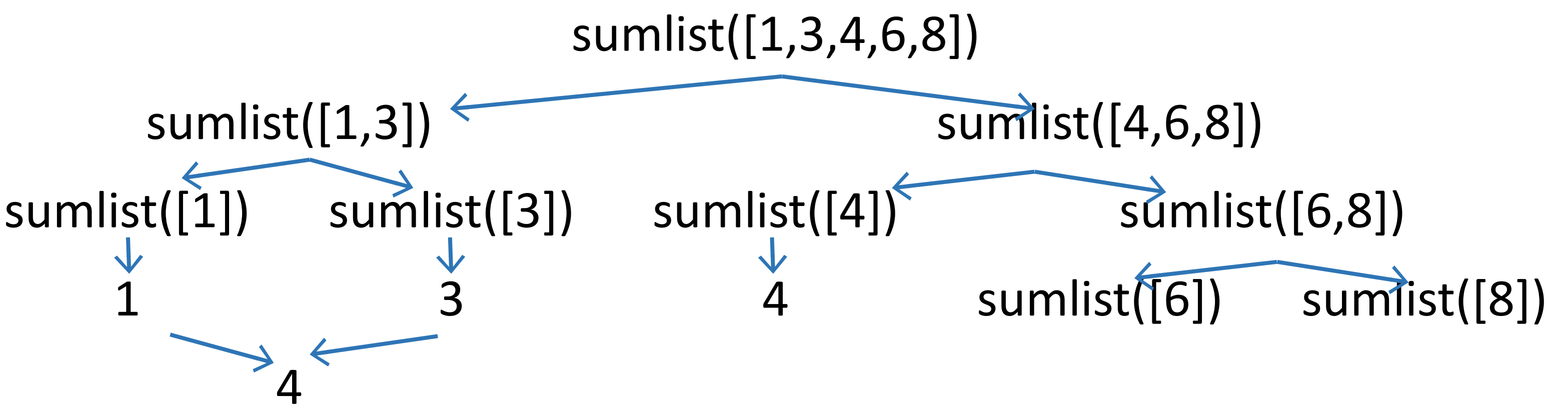
1 3

4

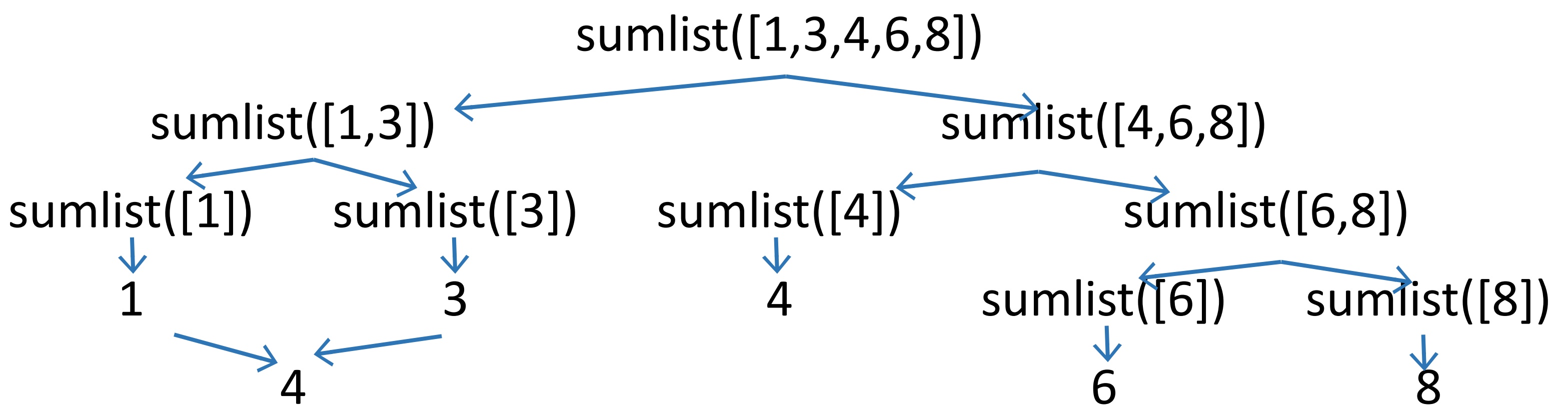
sumlist: example



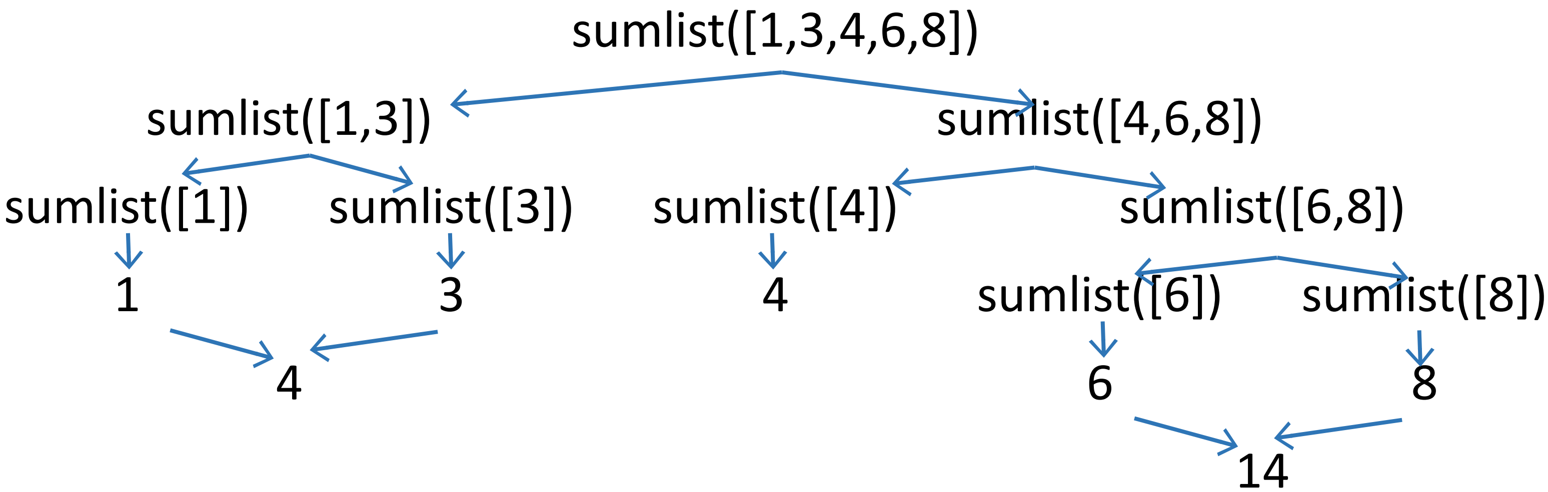
sumlist: example



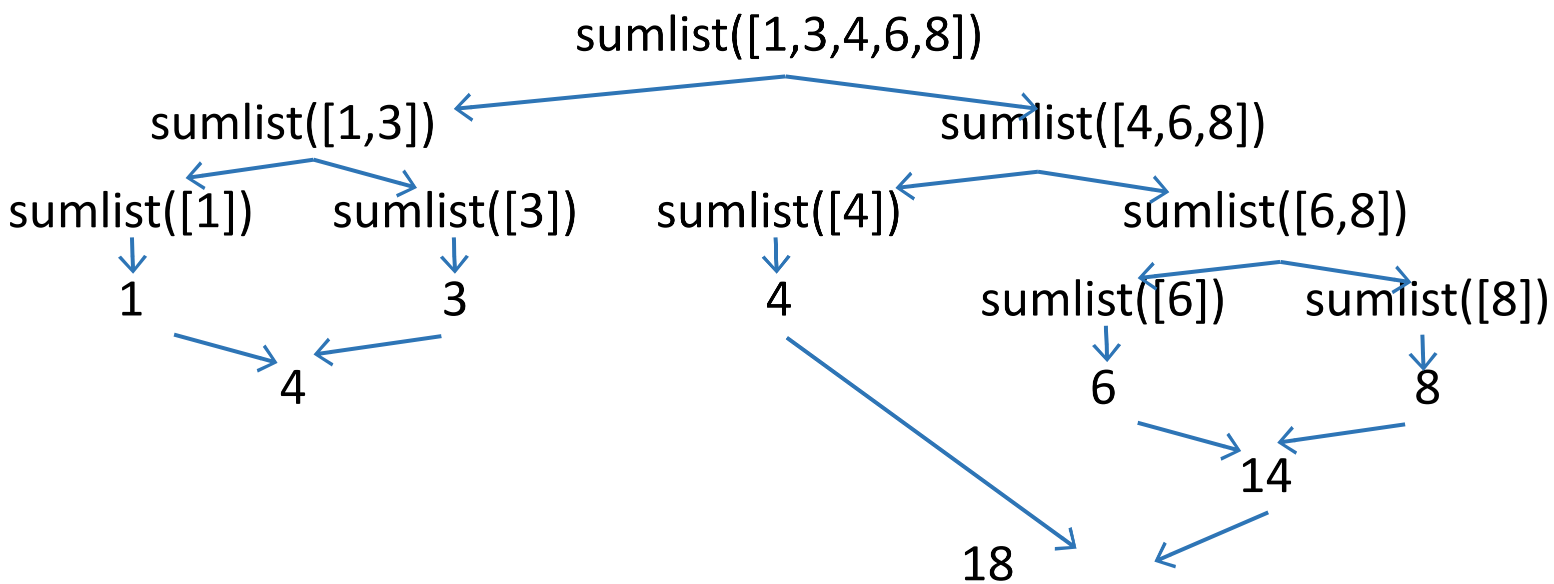
sumlist: example



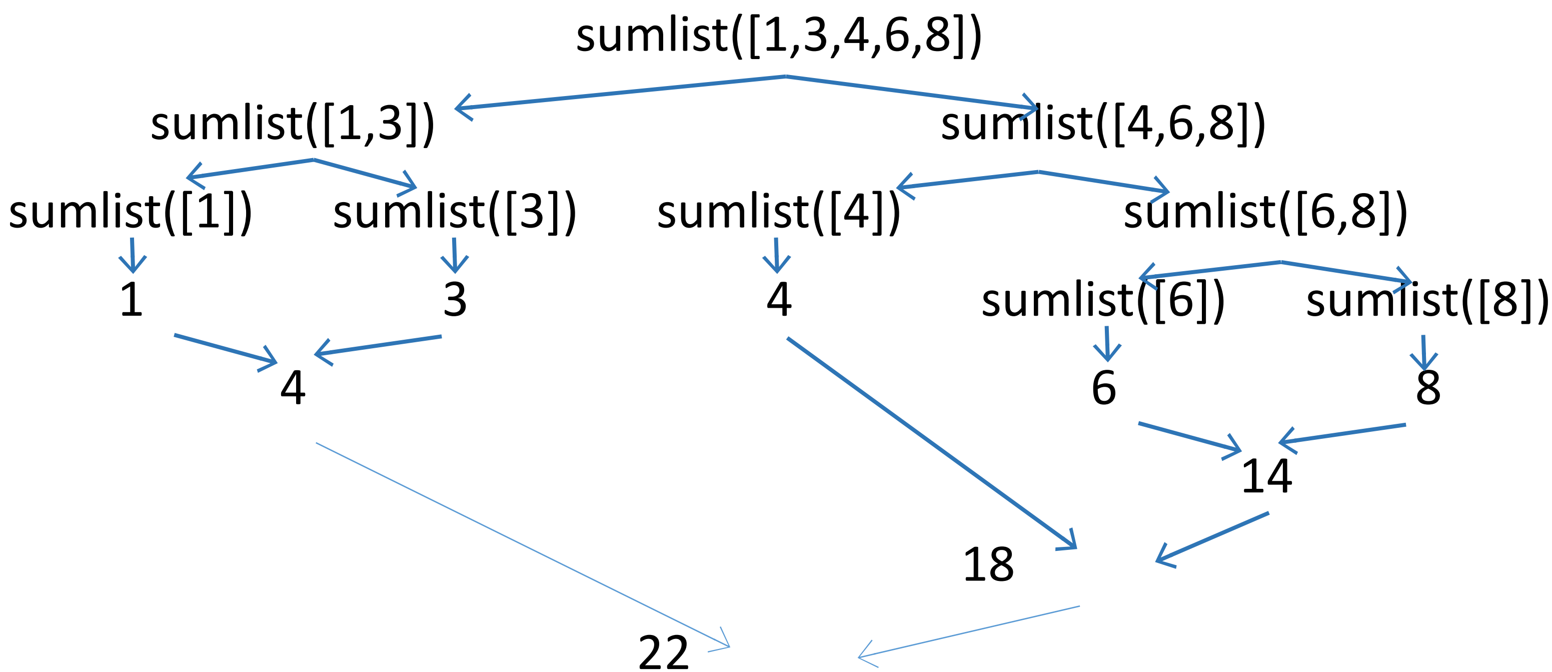
sumlist: example



sumlist: example



sumlist: example



recursion: example
binary search

Searching an unsorted list

- Problem: Given an unsorted list *L* and a value *a*, determine whether or not *a* is in *L*.

a 29

	0	1	2	3	4	5	6	7	8	9
L	3	81	2	14	64	12	8	29	14	5

Searching an unsorted list

- Problem: Given an unsorted list *L* and a value *a*, determine whether or not *a* is in *L*.

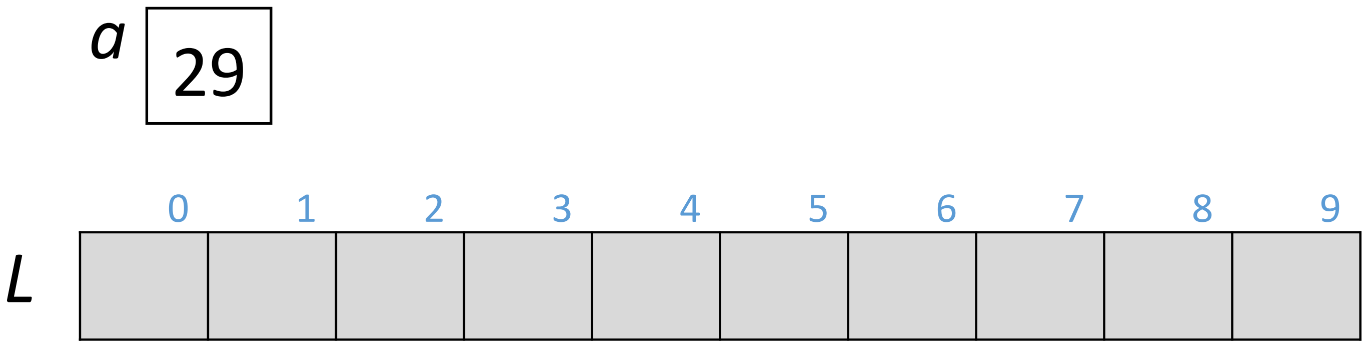
a 29

	0	1	2	3	4	5	6	7	8	9
L	3	81	2	14	64	12	8	29	14	5

- Linear search: sequentially look at (possibly) all values in the list.

Searching a sorted list

- Problem: Given a sorted list *L* and a value *a*, determine whether or not *a* is in *L*.



Searching a sorted list

- Problem: Given a sorted list L and a value a, determine whether or not a is in L.

a = 29

pick a value i in range(len(L))

say, $i = 4$

	0	1	2	3	4	5	6	7	8	9
L					14					

$a > L[4]$

Q: Can $L[3]$ be a?

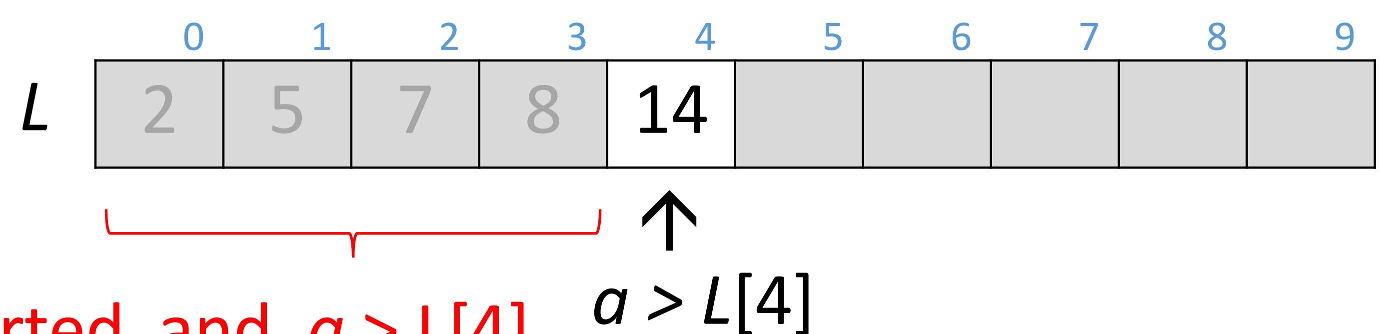
Searching a sorted list

- Problem: Given a sorted list *L* and a value *a*, determine whether or not *a* is in *L*.

a 29

pick a value *i* in range(len(*L*))

say, *i* = 4



L sorted and *a* > L[4]
means *a* cannot be any
of these elements

Binary search: recursive solution

binary search - find an item in a sorted list

```
if the list is empty
        the item is not found (return False)
look at the middle of the list
if we found the item
        then done (return True)
else
         if the item is less than the middle
                  search in the lower half of the list
         else
                   search in the upper half of the list
```

EXERCISE

Write a recursive function bin_search(alist, item) that that searches for item in alist and returns True if found and False otherwise.

Usage:

```
>>>bin_search([4, 25, 28, 33, 47, 54, 65, 83], 65)
True
>>>
```

Binary search

```
def bin_search(L, item):
    if L == []:
        return False
    mid = len(L)//2
    if L[mid] == item :
        return True
    if item < L[mid]:
        return bin_search(L[0:mid], item)
    else:
        return bin_search(L[mid+1:], item)
```

Binary search: complexity

- The size of the search area is halved at each round of repetition

Comparisons	Approx. number of items left
1	$n/2$
2	$n/4$
3	$n/8$
…	…
i	$n/2^i$

  - The number of comparisons until we are done is

    i, where $n/2^i = 1$

    solving for i gives $i = \log_2 n$
  - total no. of rounds of recursion $= \log_2(n)$

Binary search: complexity

- The size of the search area is halved at each round of repetition (recursion)
  - total no. of rounds of recursion = $\log_2(n)$

    or the number of comparisons is $log_2(n)$
- However, on each round of repetition, the work done is ***not*** a fixed amount due to slicing
  - slicing is O(n)
- Fix that by computing the indices and passing them as parameters.

Binary search: no slicing

```
def bin_search(L, item, lo, hi):
    if lo > hi:
        return False
    if lo == hi:
        return L[lo] == item

    mid = (lo+hi)//2
    if item <= L[mid]:
        return bin_search(L, item, lo, mid)
    else:
        return bin_search(L, item, mid+1, hi)
```

Binary search: complexity

- The size of the search area is halved at each round of recursion
  - total no. of rounds of recursion = $\log_2(n)$
- On each recursive step, the work done is a fixed amount
  - O(1)

∴ Overall complexity: O(log n)

EXERCISE

Write a snippet of code or a function whose complexity is O(log n).

recursion: example

Example: merging two sorted lists

Problem: Given two sorted lists L1 and L2, merge them into a single sorted list

Example: L1 = [11, 22, 33], L2 = [5, 10, 15]

- Output: [5, 10, 11, 15, 22, 33]
  - can't just concatenate the lists
  - can't alternate between the lists

Merging: values involved

Problem: Given two sorted lists L1 and L2, merge them into a single sorted list

1. Values involved in the computation in each (recursive) call ?

L1 and L2

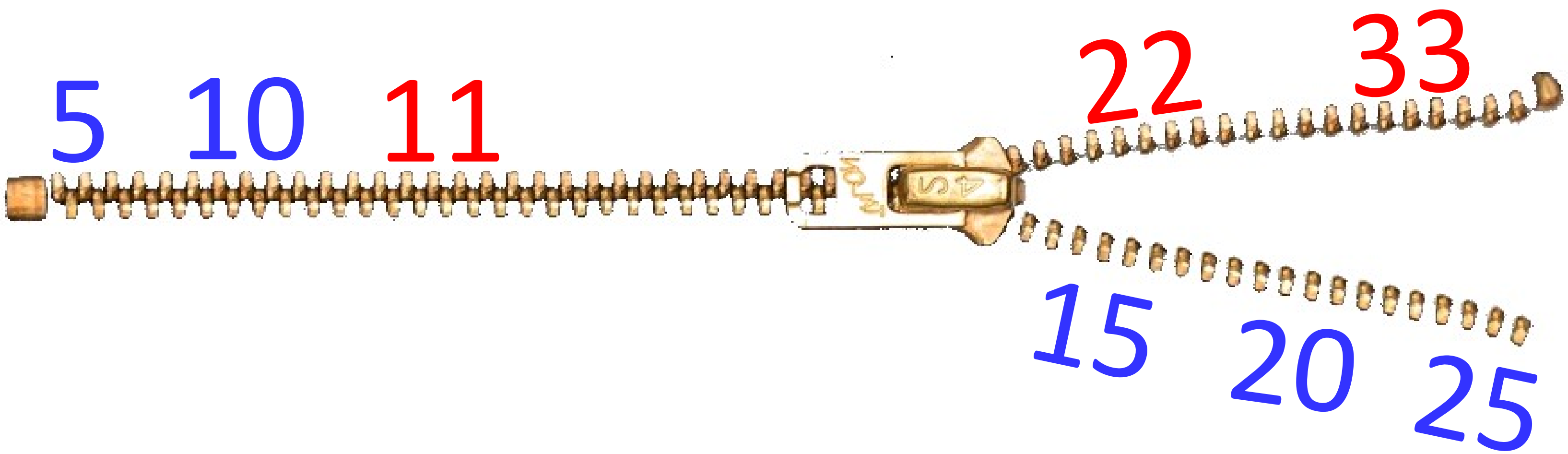
So the recursive function will look something like

```
def merge(L1, L2):   # may need another argument
    …
```

Merging: repetition

Problem: Given two sorted lists L1 and L2, merge them into a single sorted list

2. What does the computation involve in each call?



Merging: repetition

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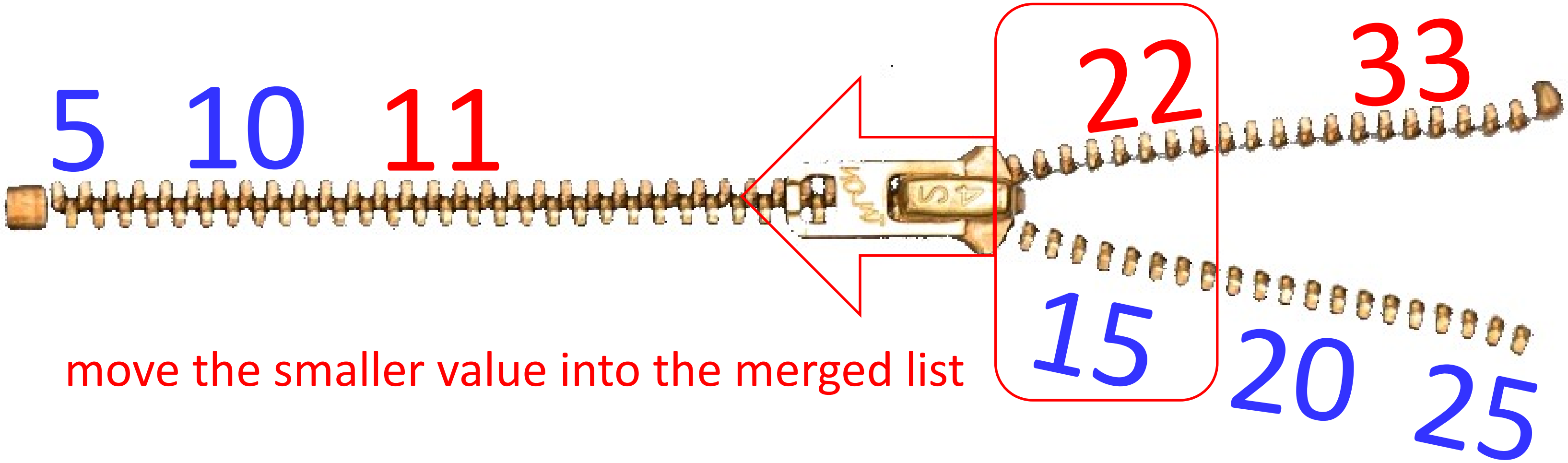
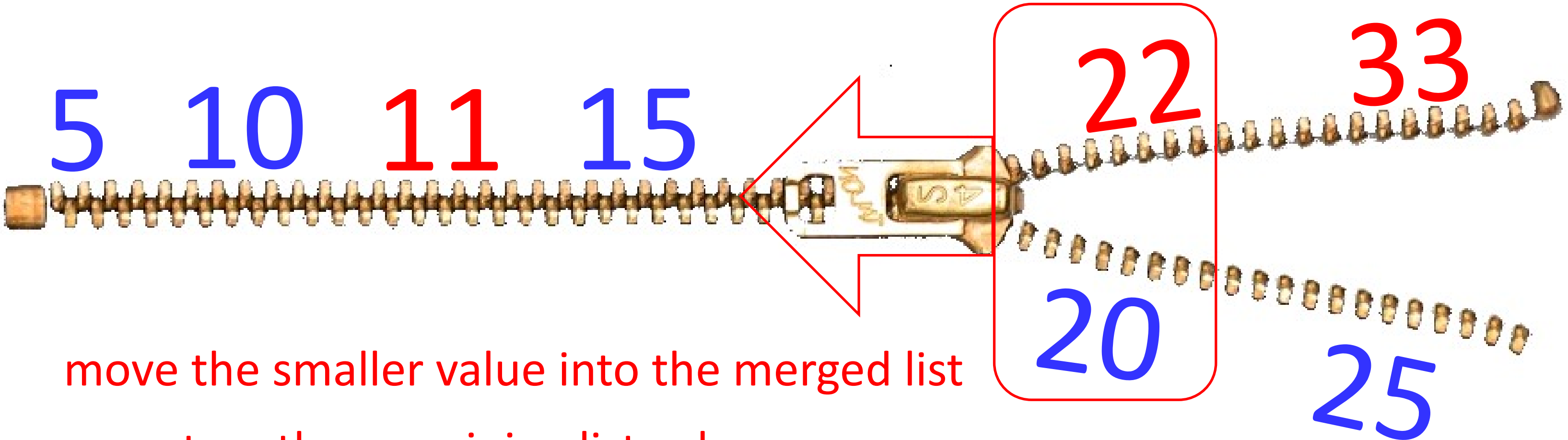


Merging: repetition

Problem: Given two sorted lists L1 and L2, merge them into a single sorted list

2. How does the problem (or data) get smaller?



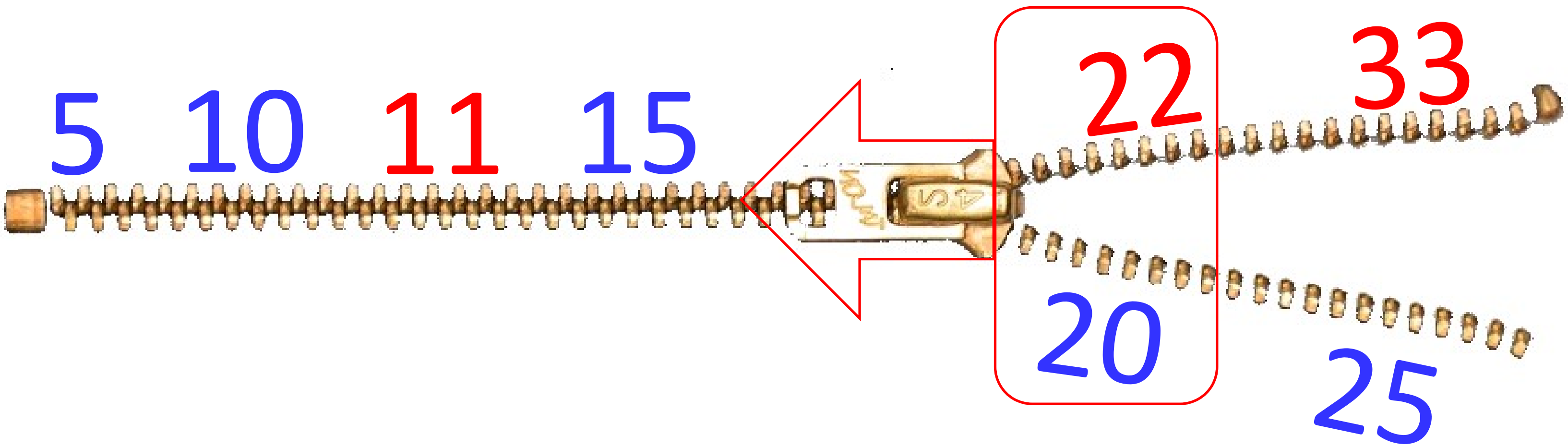
move the smaller value into the merged list

repeat on the remaining list values

Merging: base case

Problem: Given two sorted lists L1 and L2, merge them into a single sorted list

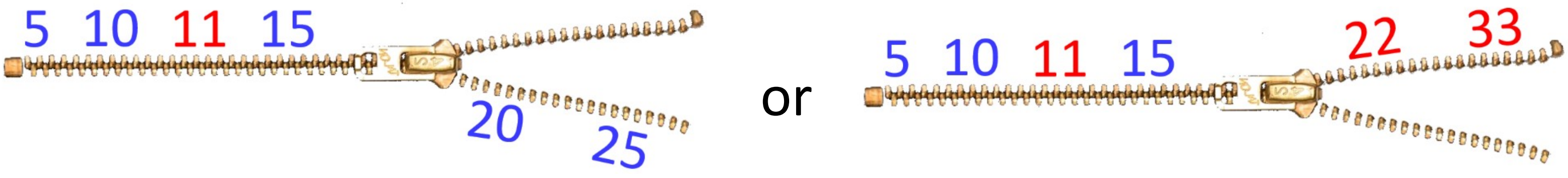
3. When can't we make the data smaller?



Merging: base case

Problem: Given two sorted lists L1 and L2, merge them into a single sorted list

3. When can't we make the data smaller?
   - when either L1 or L2 is empty



in this case, concatenate the other list into the merged list

Merging: base case

The code looks something like:

```
def  merge(L1, L2, merged):  # note the new parameter
        if L1 == []:
              return merged + L2
        elif L2 == []:
              return merged + L1
        else:
              ….
```

Merging: base case

The code looks something like:

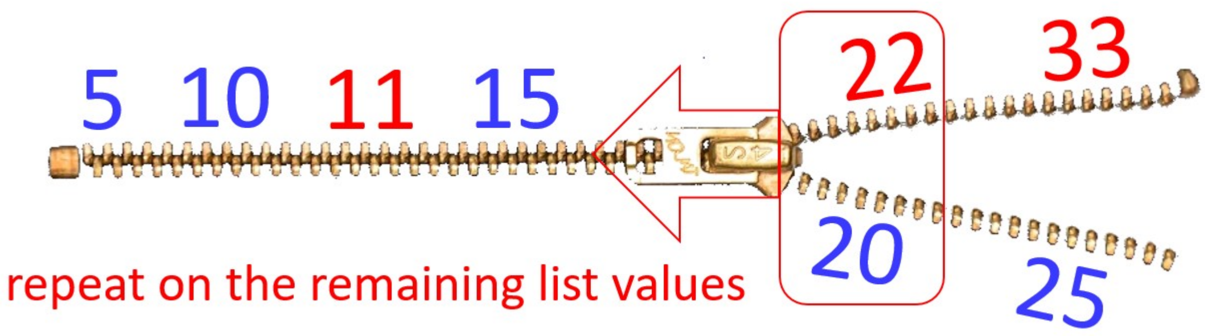
```
def  merge(L1, L2, merged): # note the new parameter
        if L1 == [] or L2 == []:
              return merged + L1 + L2
        else:
              ....
```

Merging: recursive case

Problem: Given two sorted lists L1 and L2, merge them into a single sorted list

4. What is "the rest of the computation"?

   – "repeat on the remaining list values"



EXERCISE

Given the pseudocode below, write the recursive cases for merge.

The arguments to merge are lists L1, L2, and merged

```
if L1[0] <= L2[0]
    put L1[0] into the merged list
    recursively merge using the rest of L1, L2, and merged
else
    put L2[0] into the merged list
    recursively merge using L1, the rest of L2, and merged
```

Merging: recursive case –V1

```
if L1[0] < L2[0]:
    merged.append(L1[0])
    return merge(L1[1:], L2, merged)
else:
    merged.append( L2[0] )
    return merge(L1, L2[1:], new_merged)
```

Merging: recursive case-V2

```
if L1[0] < L2[0]:
        new_merged = merged + [ L1[0] ]
        new_L1 = L1[1: ]
        new_L2 = L2
else:
        new_merged = merged + [ L2[0] ]
        new_L1 = L1
        new_L2 = L2[1: ]
return merge(new_L1, new_L2, new_merged)
```

Merging: putting it all together

base case

recursive case

```
def  merge(L1, L2, merged):
    if L1 == [] or L2 == []:
        return merged + L1 + L2
    else:
        if L1[0] < L2[0]:
            new_merged = merged + [ L1[0] ]
            new_L1 = L1[1: ]
            new_L2 = L2
        else:
            new_merged = merged + [ L2[0] ]
            new_L1 = L1
            new_L2 = L2[1: ]
        return merge(new_L1, new_L2, new_merged)
```

```
>>> def merge(L1,L2,merged):
        if L1 == [] or L2 == []:
                return merged + L1 + L2
        else:
                if L1[0] < L2[0]:
                        new_merged = merged + [L1[0]]
                        new_L1 = L1[1:]
                        new_L2 = L2
                else:
                        new_merged = merged + [L2[0]]
                        new_L1 = L1
                        new_L2 = L2[1:]
                return merge(new_L1, new_L2, new_merged)


>>> merge([11,22,33],[5,10,15,20,25],[])
[5, 10, 11, 15, 20, 22, 25, 33]
>>>
```

recursion: flow of values

Recursion: flow of values

values are computed and passed down as arguments into the recursive call

```
def merge(L1,L2,merged):
    if L1 == [] or L2 == []:
            return merged + L1 + L2
    else:
            if L1[0] < L2[0]:
                    new_merged = merged + [L1[0]]
                    new_L1 = L1[1:]
                    new_L2 = L2
            else:
                    new_merged = merged + [L2[0]]
                    new_L1 = L1
                    new_L2 = L2[1:]
            return merge(new_L1, new_L2, new_merged)
```

Recursion: flow of values

the computation of each round of repetition takes place as values are passed up as return values

```
def merge(L1,L2,merged):
    if L1 == [] or L2 == []:
            return merged + L1 + L2
    else:
            if L1[0] < L2[0]:
                    new_merged = merged + [L1[0]]
                    new_L1 = L1[1:]
                    new_L2 = L2
            else:
                    new_merged = merged + [L2[0]]
                    new_L1 = L1
                    new_L2 = L2[1:]
            return merge(new_L1, new_L2, new_merged)
```

recursion: application
merge sort

Sorting

- Problem: Given a list L, sort the elements of L into a list sortedL

- Important problem
  - arises in a wide variety of situations
  - many different algorithms, with different assumptions and characteristics
  - we will consider just one algorithm

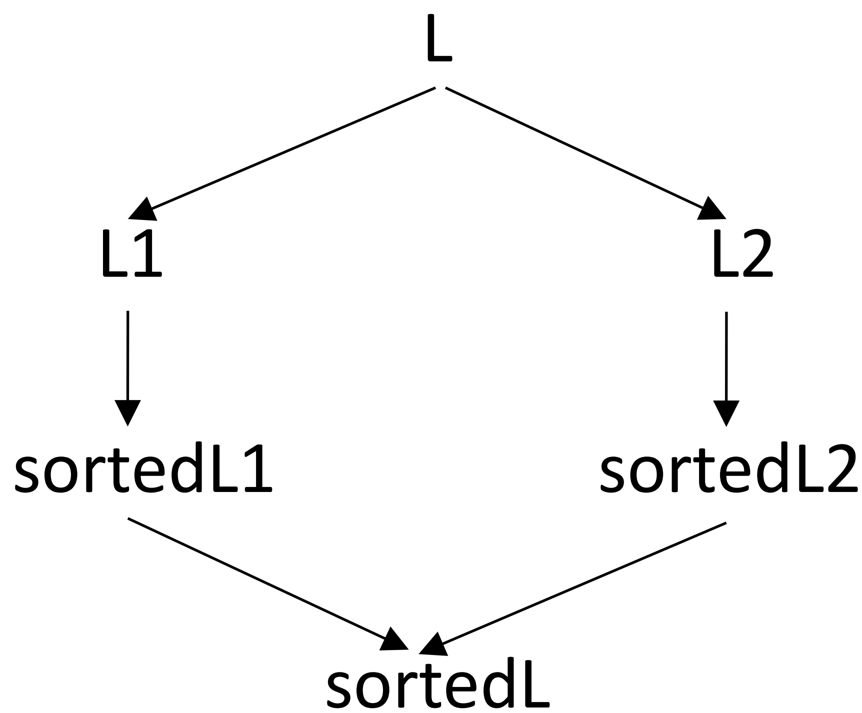
Algorithm: mergesort

input list

split into two
halves

sort the halves
recursively

merge the
sorted lists



Divide and conquer algorithm

Divide and Conquer

- An algorithm paradigm based on multi –branched recursion

  – Recursively break the problem down into two or more sub-problems (until they are trivial to solve)

  – Combine the solutions of the sub-problems to give the solution to the original problem

Mergesort

- Base case: len(L) <= 1
  - no further halving possible
- Recursive case:
  - set up the next round of computation: split the list
  - smaller problem to recurse on: a list of half the size
- Each round of computation: merging the sorted lists
  - has to be done *after* the recursive call

Mergesort

```
def msort(L):
    if len(L) <= 1:
        return L
    else:
        split_pt = len(L)//2
        L1 = L[ :split_pt]
        L2 = L[split_pt: ]
        sortedL1 = msort(L1)
        sortedL2 = msort(L2)
        return  merge(sortedL1, sortedL2,[])
```

Mergesort: example

msort([1, 3, 2, 5, 4])

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3]) msort([2, 5, 4])

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3]) msort([2, 5, 4])

msort([1]) msort([3])

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msort([1]) msort([3])

[1] [3]

Mergesort: example

msort([1, 3, 2, 5, 4])

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msort([1]) msort([3])

[1] [3]

merge([1], [3])

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3]) msort([2, 5, 4])

msort([1]) msort([3])

[1] [3]

merge([1], [3])

[1, 3]

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3]) msort([2, 5, 4])

msort([1]) msort([3])

[1] [3]

merge([1], [3])

[1, 3]

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3]) msort([2, 5, 4])

msort([1]) msort([3]) msort([2]) msort([5, 4])

[1] [3]

merge([1], [3])

[1, 3]

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3]) msort([2, 5, 4])

msort([1]) msort([3]) msort([2]) msort([5, 4])

[1] [3] [2]

merge([1], [3])

[1, 3]

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3])   msort([2, 5, 4])

msort([1])   msort([3])   msort([2])   msort([5, 4])

[1]   [3]   [2]   msort([5])   msort([4])

merge([1], [3])

[1, 3]

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3])　　msort([2, 5, 4])

msort([1])　　msort([3])　　msort([2])　　msort([5, 4])

[1]　　[3]　　[2]　　msort([5])　　msort([4])

merge([1], [3])　　[5]　　[4]

[1, 3]

Mergesort: example

msort([1, 3, 2, 5, 4])

msort([1, 3]) msort([2, 5, 4])

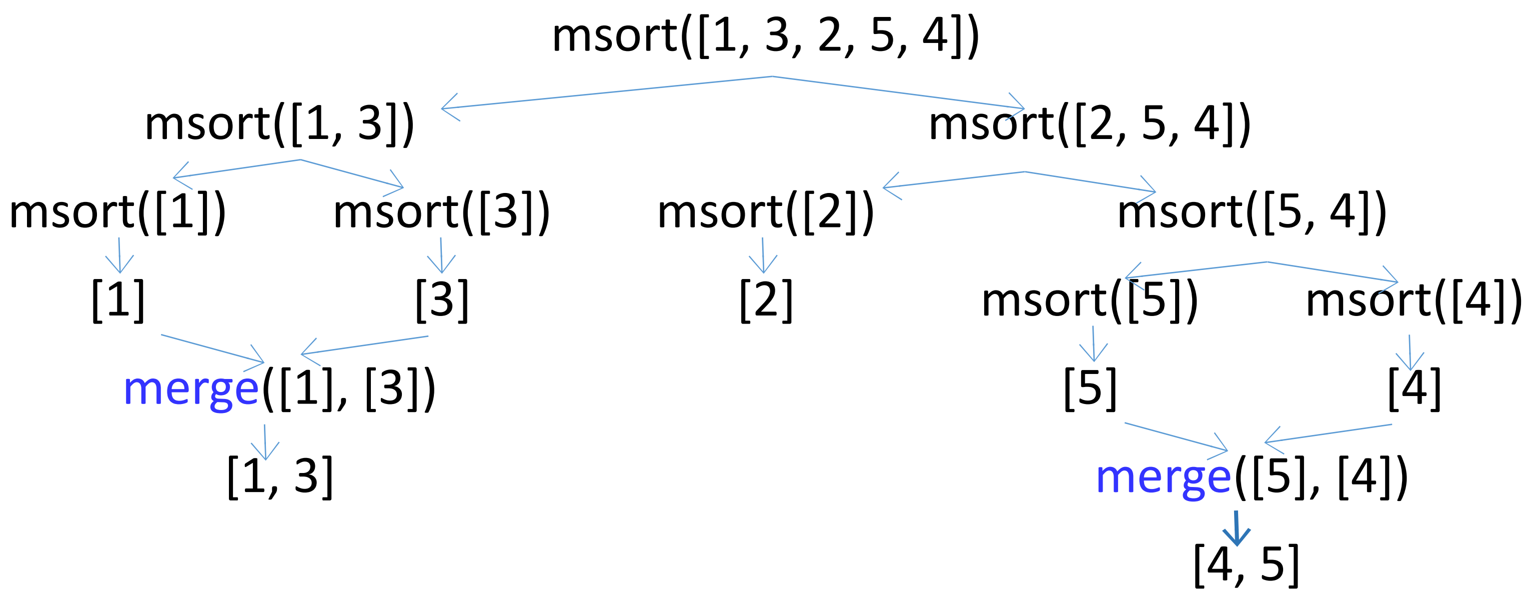
msort([1]) msort([3]) msort([2]) msort([5, 4])

[1] [3] [2] msort([5]) msort([4])

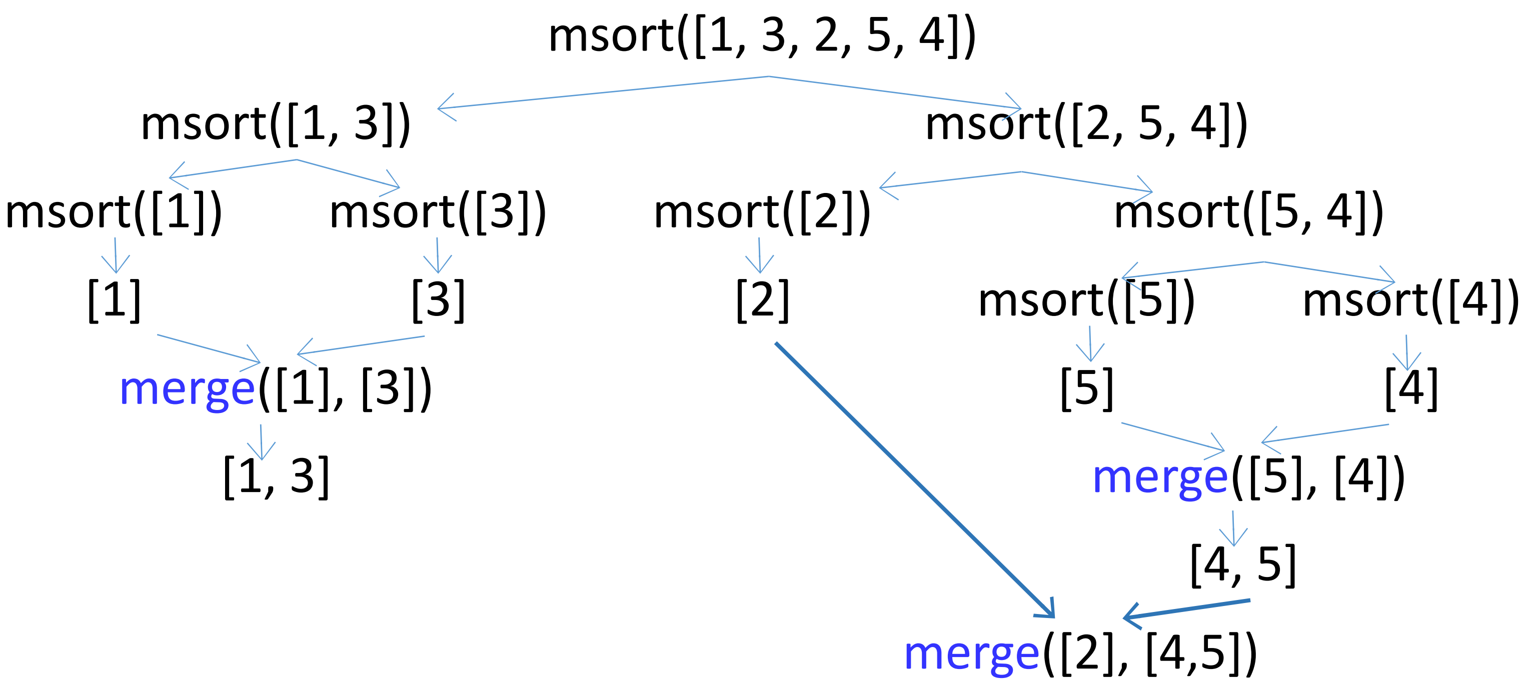
merge([1], [3]) [5] [4]

[1, 3] merge([5], [4])

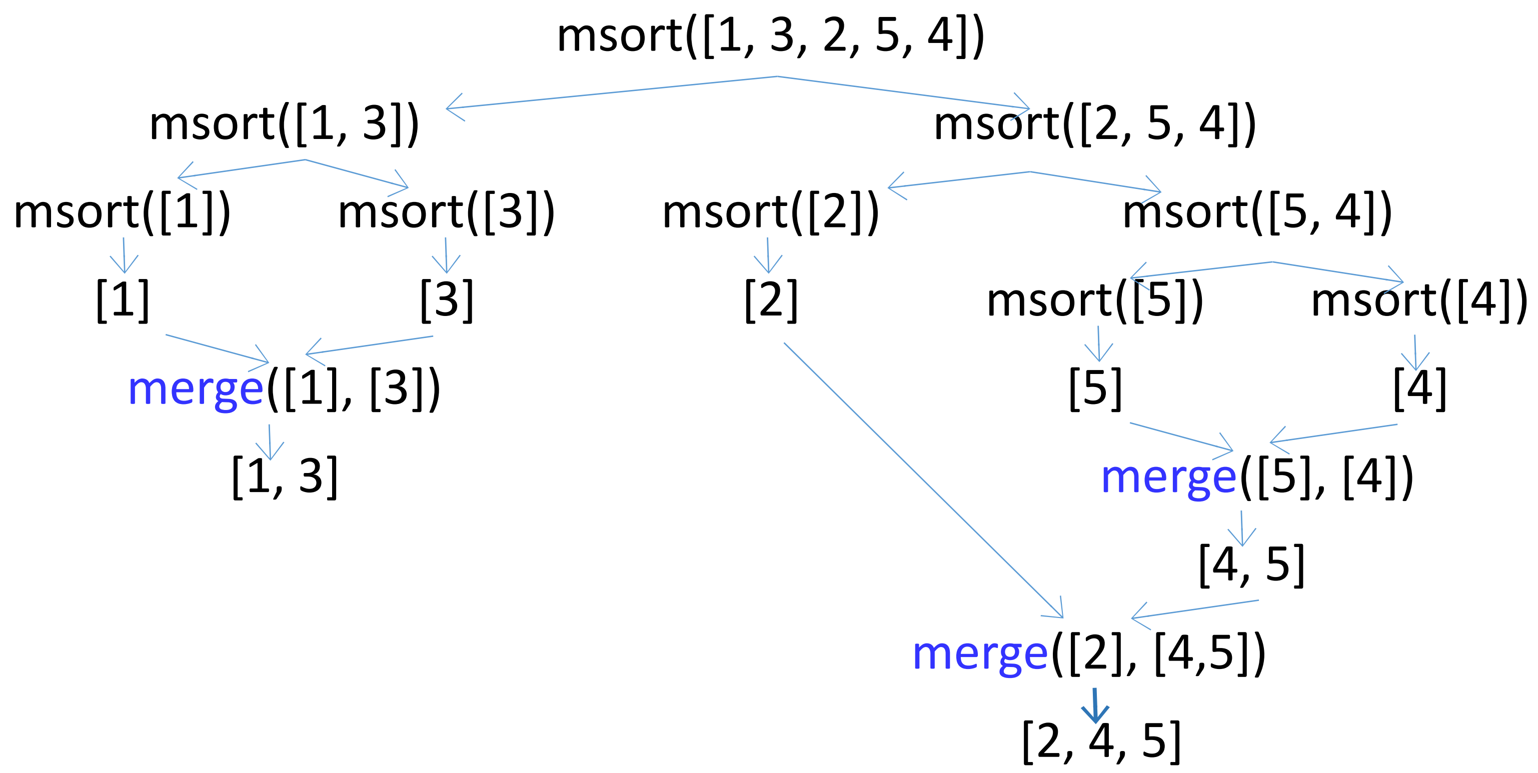
Mergesort: example



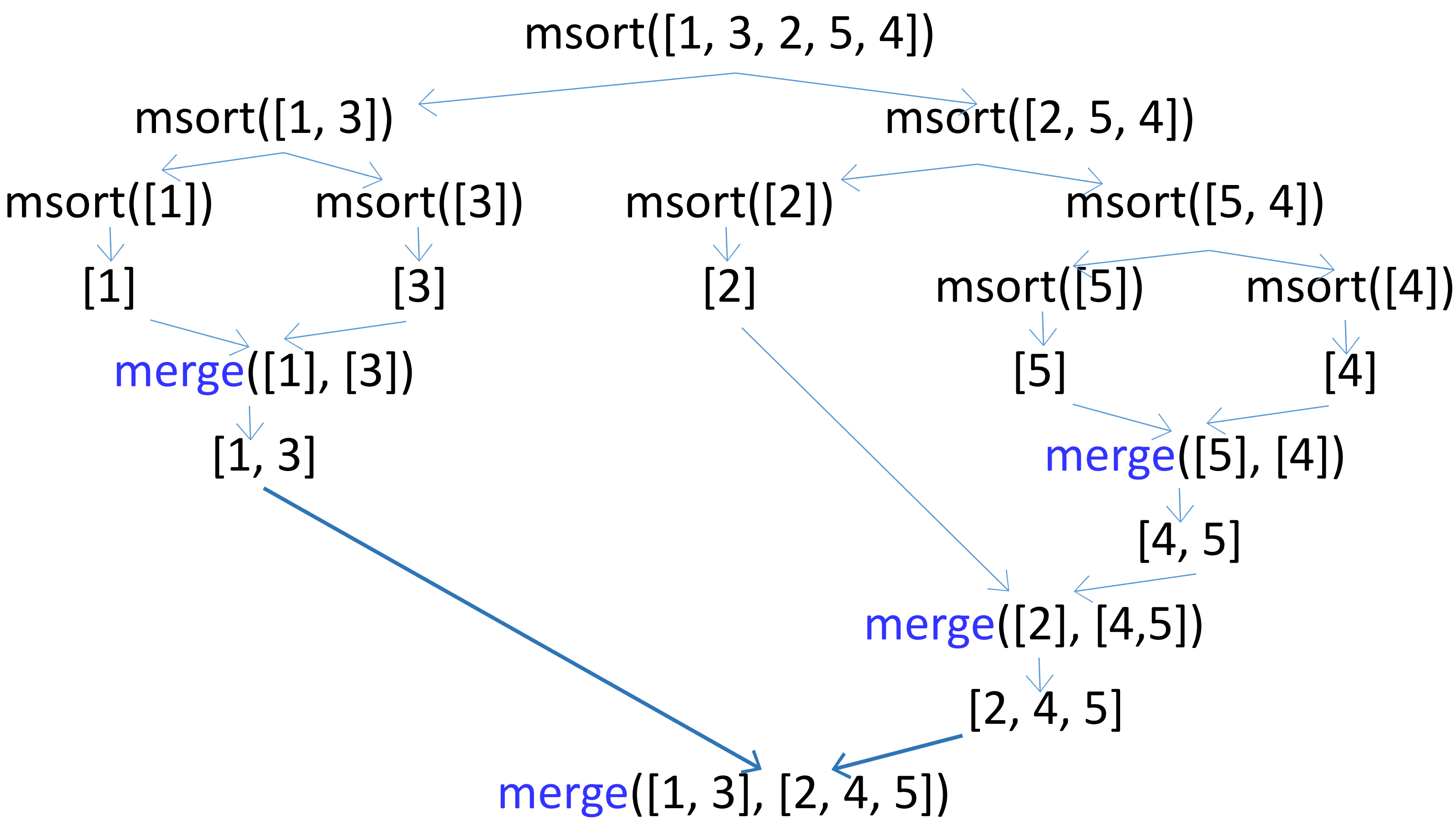
Mergesort: example



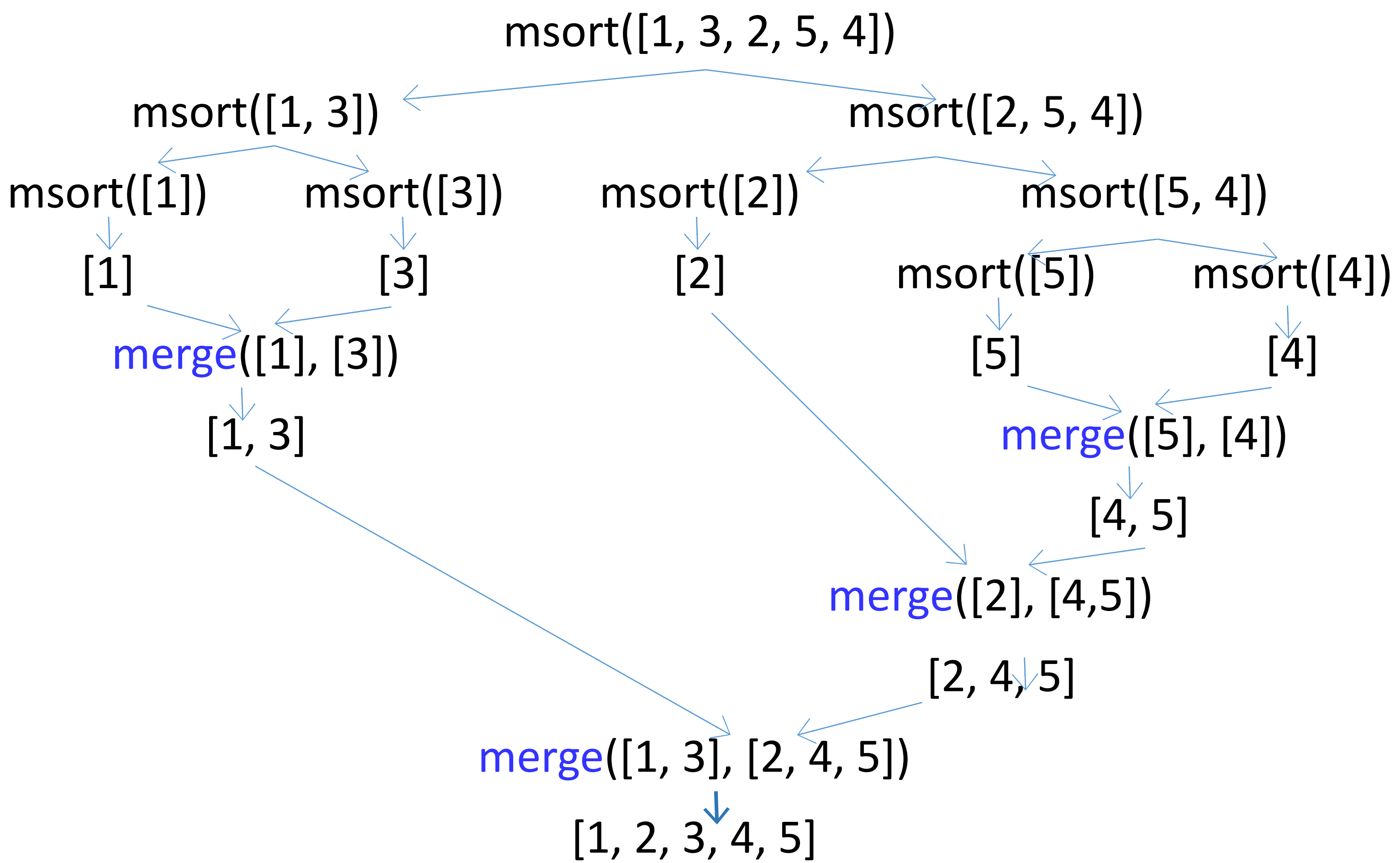
Mergesort: example



Mergesort: example



Mergesort: example



Mergesort: complexity

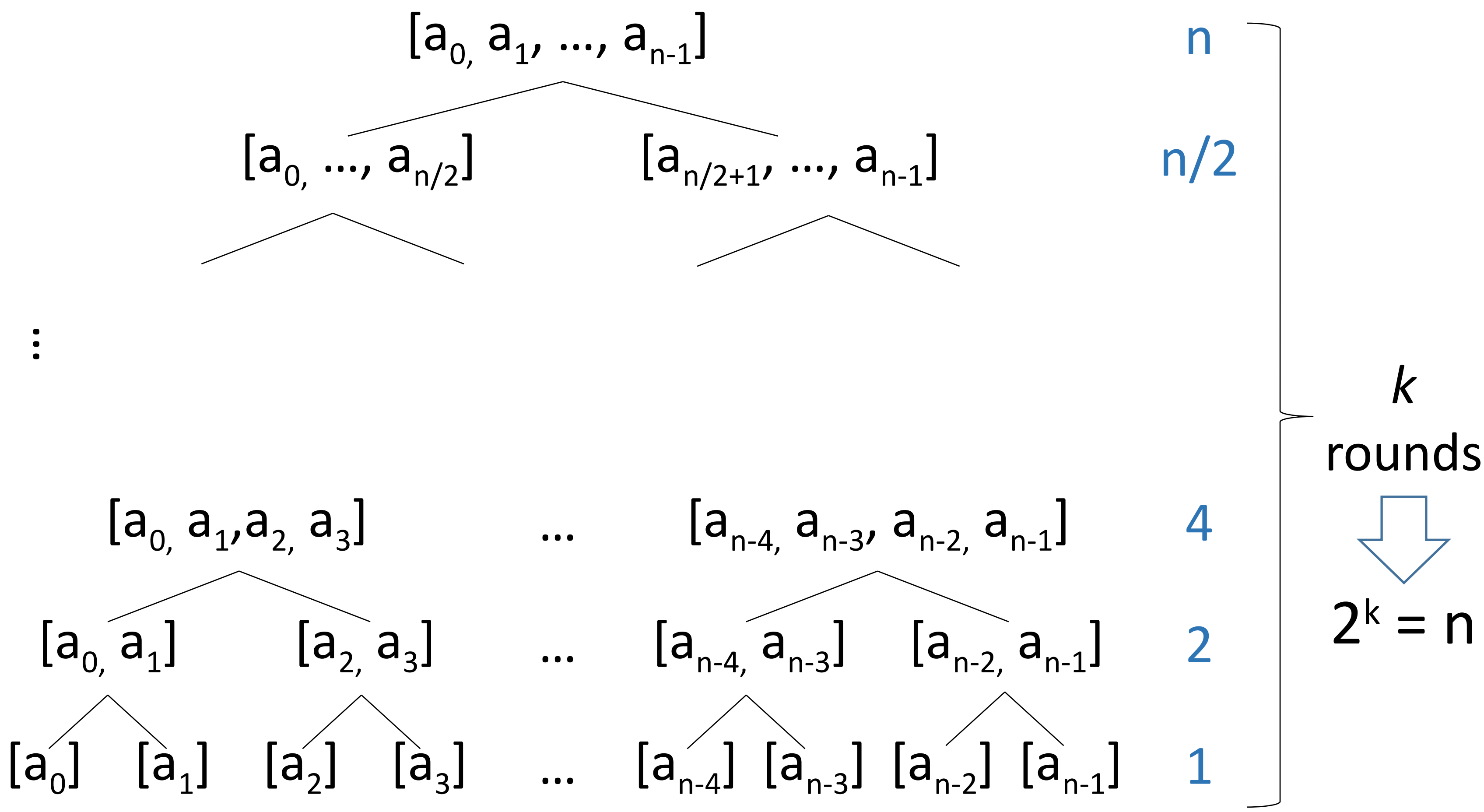
Cost = (cost per round of repetition) x (no. of rounds of repetition)$_{\text{worst case}}$

merging the sorted lists is O(n)*

???

*if slicing is removed from merge

Mergesort: complexity



Mergesort: complexity

- No. of rounds of recursion:
  - if we start with a list of size n and have k rounds of recursion, then $2^k = n$

    $\therefore \log_2(2^k) = \log_2(n)$

    $\therefore k = \log_2(n)$

- Complexity of each round of recursion: O(n)

$\Rightarrow$ Worst-case complexity of mergesort: O(n log n)*

*if slicing is removed from msort

recursion: summary

Recursion: summary

- Recursion offers a way to express repetitive computations cleanly and succinctly
- How to:
  - what are the values used in the recursive call?
  - base case: when does the recursion stop?
  - recursive case:
    - what does a single round of computation involve?
    - what is the "smaller problem" to recurse on?
- Recursion is an essential component of every good computer scientist's toolkit