Propositional Satisfiability

Methods & Tools for Software Engineering (MTSE)
Fall 2017

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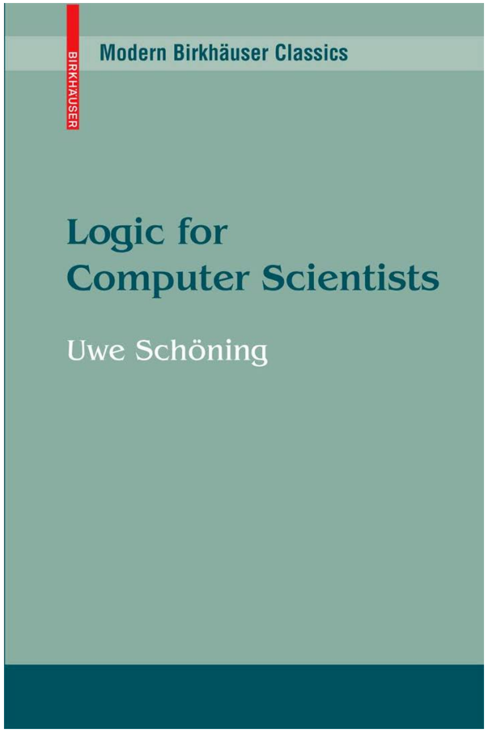
References

- Chpater 1 of Logic for Computer Scientists http://www.springerlink.com/content/978-0-8176-4762-9/



Syntax of Propositional Logic

An *atomic formula* has a form A_i , where i = 1, 2, 3 …

Formulas are defined inductively as follows:

- All atomic formulas are formulas
- For every formula F, $\neg F$ (called not F) is a formula
- For all formulas F and G, $F \wedge G$ (called and) and $F \vee G$ (called or) are formulas

Abbreviations

- use A, B, C, … instead of A_1, A_2, …
- use $F_1 \rightarrow F_2$ instead of $\neg F_1 \vee F_2$ (implication)
- use $F_1 \leftrightarrow F_2$ instead of $(F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$ (iff)

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Syntax of Propositional Logic (PL)

$$
\begin{aligned}
\text{truth\_symbol} &::= \top(\text{true}) \mid \bot(\text{false}) \\
\text{variable} &::= p, q, r, \ldots \\
\text{atom} &::= \text{truth\_symbol} \mid \text{variable} \\
\text{literal} &::= \text{atom} | \neg\text{atom} \\
\text{formula} &::= \text{literal} \mid \\
&\quad\ \neg\text{formula} \mid \\
&\quad\ \text{formula} \wedge \text{formula} \mid \\
&\quad\ \text{formula} \vee \text{formula} \mid \\
&\quad\ \text{formula} \rightarrow \text{formula} \mid \\
&\quad\ \text{formula} \leftrightarrow \text{formula}
\end{aligned}
$$

Boolean Satisfiability (CNF-SAT)

Let V be a set of variables

A *literal* is either a variable v in V or its negation ~v

A *clause* is a disjunction of literals

- e.g., (v1 || ~v2 || v3)

A Boolean formula in *Conjunctive Normal Form* (CNF) is a conjunction of clauses

- e.g., (v1 || ~v2) && (v3 || v2)

An *assignment* *s* of Boolean values to variables *satisfies* a clause *c* if it evaluates at least one literal in *c* to true

An assignment *s* *satisfies* a formula *C* in CNF if it satisfies every clause in *C*

Boolean Satisfiability Problem (CNF-SAT):

- determine whether a given CNF C is satisfiable

CNF Examples

CNF 1

- ~b
- ~a || ~b || ~c
- a
- sat: s(a) = True; s(b) = False; s(c) = False

CNF 2

- ~b
- ~a || b || ~c
- a
- ~a || c
- unsat

Algorithms for SAT

SAT is NP-complete

- solution can be checked in polynomial time
- no polynomial algorithms for finding a solution are known

DPLL (Davis-Putnam-Logemman-Loveland, ‘60)

- smart enumeration of all possible SAT assignments
- worst-case EXPTIME
- alternate between deciding and propagating variable assignments

CDCL (GRASP ‘96, Chaff ‘01)

- conflict-driven clause learning
- extends DPLL with
  - smart data structures, backjumping, clause learning, heuristics, restarts…
- scales to millions of variables
- N. Een and N. Sörensson, “An Extensible SAT-solver”, in SAT 2013.

Background Reading: SAT

http://cacm.acm.org/magazines/2009/8/34498-boolean-satisfiability-from-theoretical-h

Boolean Satisfiability: From ...

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REVIEW ARTICLES

Boolean Satisfiability: From Theoretical Hardness to Practical Success

By Sharad Malik, Lintao Zhang
Communications of the ACM, Vol. 52 No. 8, Pages 76-82
10.1145/1536616.1536637
Comments

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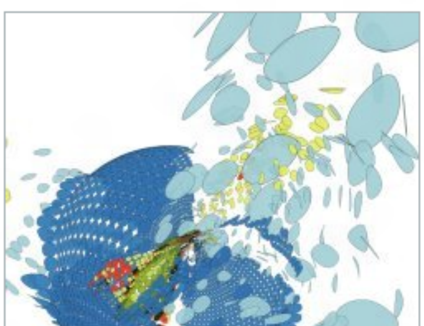

There are many practical situations where we need to satisfy several potentially conflicting constraints. Simple examples of this abound in daily life, for example, determining a schedule for a series of games that resolves the availability of players and venues, or finding a seating assignment at dinner consistent with various rules the host would like to impose. This also applies to applications in computing, for example, ensuring that a hardware/software system functions correctly with its overall behavior constrained by the behavior of its components and their composition, or finding a plan for a robot to reach a goal that is

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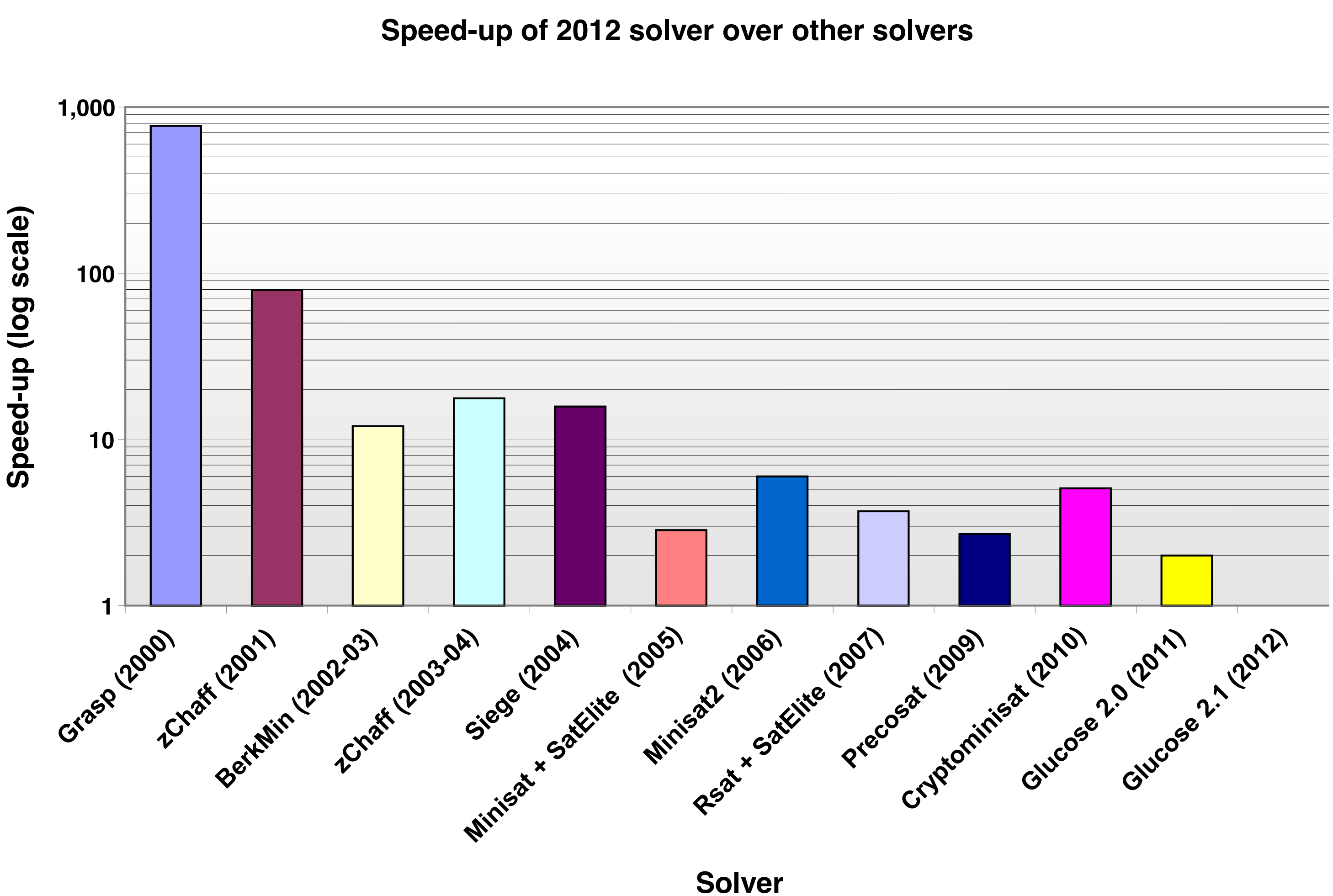
ARTICLE CONTENTS:
Introduction
Boolean Satisfiability
Theoretical hardness: SAT and NP-Completeness

Some Experience with SAT Solving



from M. Vardi, https://www.cs.rice.edu/~vardi/papers/highlights15.pdf

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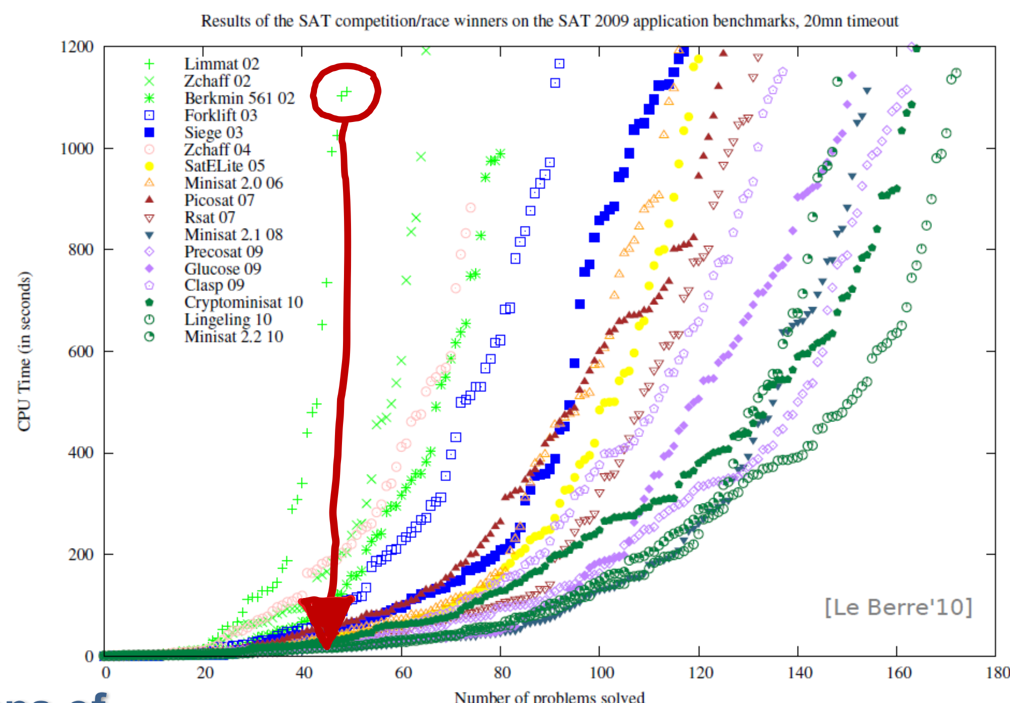
SAT - Milestones

Problems impossible 10 years ago are trivial today

year	Milestone
1960	Davis-Putnam procedure
1962	Davis-Logeman-Loveland
1984	Binary Decision Diagrams
1992	DIMACS SAT challenge
1994	SATO: clause indexing
1997	GRASP: conflict clause learning
1998	Search Restarts
2001	zChaff: 2-watch literal, VSIDS
2005	Preprocessing techniques
2007	Phase caching
2008	Cache optimized indexing
2009	In-processing, clause management
2010	Blocked clause elimination

Concept

2002 2010



Millions of variables from HW designs

Courtesy Daniel le Berre

USING A SAT SOLVER

DIMACS interface to a SAT Solver

Input:

- a CNF in DIMACS format

Output:

- SAT/UNSAT + satisfying assignment

We will use a SAT solver called MiniSAT

- available at https://github.com/agurfinkel/minisat
- written in C++
- use as a library in Assignment 4
- use via DIMACS interface today in class

DIMACS CNF File Format

Textual format to represent CNF-SAT problems

```
c start with comments
c
c
p cnf 5 3
1 -5 4 0
-1 5 3 4 0
-3 -4 0
```

Format details

- comments start with `c`
- header line: `p cnf nbvar nbclauses`
  - `nbvar` is # of variables, `nbclauses` is # of clauses
- each clause is a sequence of distinct numbers terminating with 0
  - positive numbers are variables, negative numbers are negations

MiniSat

MiniSat is one of the most famous modern SAT-solvers

- written in C++
- designed to be easily understandable and customizable
- many new SAT-solvers use MiniSAT as their base

Web page: http://minisat.se/

We will use a slightly updated version from GitHub:
https://github.com/agurfinkel/minisat

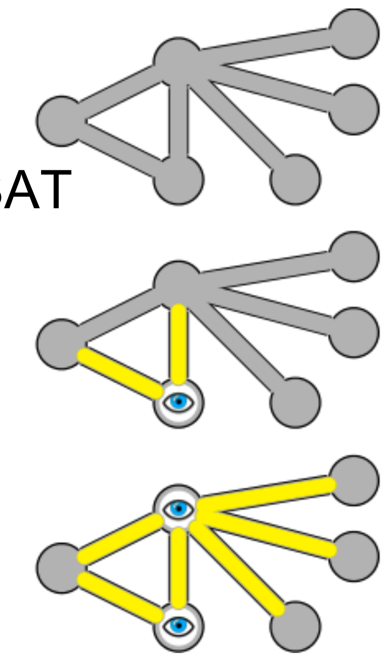
Good references for understanding SAT solving details

- MiniSat architecture: http://minisat.se/downloads/MiniSat.pdf
- Donald Knuth's SAT13 (also based on MiniSat)
  - http://www-cs-faculty.stanford.edu/~knuth/programs/sat13.w

Vertex Cover

Given a graph G=(V,E). A vertex cover of G is a subset C of vertices in V such that every edge in E is incident to at least one vertex in C

see a4_encoding.pdf for details of reduction to CNF-SAT



https://en.wikipedia.org/wiki/Vertex_cover

Exercise 33

Using resolution show that

$$A \wedge B \wedge C$$

is a consequence of

$$\neg A \vee B$$

$$\neg B \vee C$$

$$A \vee \neg C$$

$$A \vee B \vee C$$

Exercise 34

Show using resolution that F is valid

$$F = (\neg B \wedge \neg C \wedge D) \vee (\neg B \wedge \neg D) \vee (C \wedge D) \vee B$$

$$\neg F = (B \vee C \vee \neg D) \wedge (B \vee D) \wedge (\neg C \vee \neg D) \wedge \neg B$$

Davis Putnam Logemann Loveland

DPLL PROCEDURE

Decision Procedure for Satisfiability

Algorithm that in some finite amount of computation decides if a given propositional logic (PL) formula F is satisfiable

- NP-complete problem

Modern decision procedures for PL formulae are called SAT solvers

Naïve approach

- Enumerate truth table

Modern SAT solvers

- DPLL algorithm
  - Davis-Putnam-Logemann-Loveland

- Operates on Conjunctive Normal Form (CNF)

Propositional Resolution

Pivot

$$\frac{C \vee p \qquad\qquad D \vee \neg p}{C \vee D}$$

Resolvent

Res({C, p}, {D, !p}) = {C, D}

Given two clauses (C, p) and (D, !p) that contain a literal p of different polarity, create a new clause by taking the union of literals in C and D

Resolution Lemma

Lemma:

Let F be a CNF formula. Let R be a resolvent of two clauses X and Y in F. Then, $F \cup \{R\}$ is equivalent to F

Resolution Theorem

Let F be a set of clauses

$Res(F) = F \cup \{R \mid R \text{ is a resolvent of two clauses in } F\}$

$$Res^0(F) = F$$

$$Res^{n+1}(F) = Res(Res^n(F)), \text{ for } n \geq 0$$

$$Res^*(F) = \bigcup_{n \geq 0} Res^n(F)$$

Theorem: A CNF F is UNAT iff Res*(F) contains an empty clause

Exercise from LCS

For the following set of clauses determine Res^n for n=0, 1, 2

$$A \vee \neg B \vee C$$

$$B \vee C$$

$$\neg A \vee C$$

$$B \vee \neg C$$

$$\neg C$$

Proof of the Resolution Theorem

(Soundness) By Resolution Lemma, F is equivalent to $Res^i(F)$ for any i. Let n be such that $Res^{n+1}(F)$ contains an empty clause, but $Res^n(F)$ does not. Then $Res^n(F)$ must contain to unit clauses L and ¬L. Hence, it is UNSAT.

(Completeness) By induction on the number of different atomic propositions in F.

Base case is trivial: F contains an empty clause.

IH: Assume F has atomic propositions A1, … A_{n+1}

Let F_0 be the result of replacing A_{n+1} by 0

Let F_1 be the result of replacing A_{n+1} by 1

Apply IH to F_0 and F_1 . Restore replaced literals. Combine the two resolutions.

Proof System

$$P_1, \ldots, P_n \vdash C$$

An inference rule is a tuple (P_1, …, P_n, C)

- where, P_1, …, P_n, C are formulas
- P_i are called premises and C is called a conclusion
- intuitively, the rules says that the conclusion is true if the premises are

A proof system P is a collection of inference rules

A proof in a proof system P is a tree (or a DAG) such that

- nodes are labeled by formulas
- for each node *n*, (parents(n), n) is an inference rule in P

Propositional Resolution

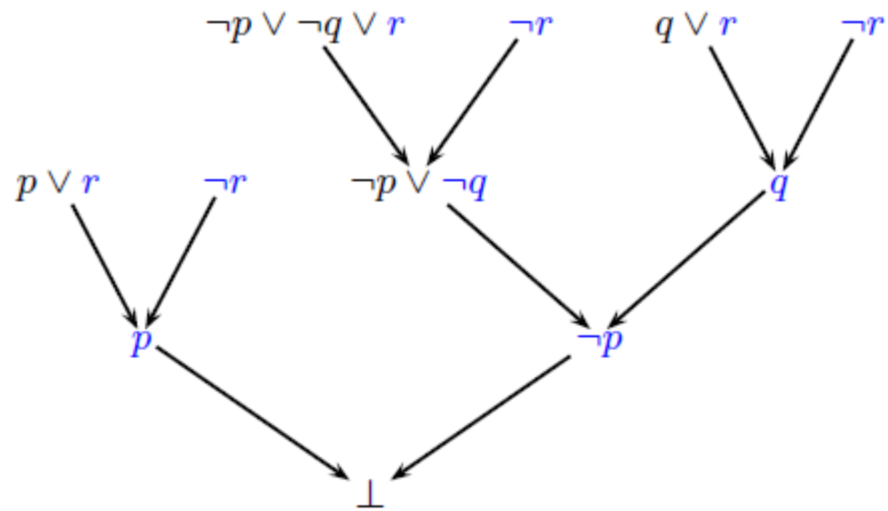
$$\frac{C \vee p \qquad\qquad D \vee \neg p}{C \vee D}$$

Propositional resolution is a sound inference rule

Proposition resolution system consists of a single propositional resolution rule

Example of a resolution proof

A refutation of $\neg p \vee \neg q \vee r$, $p \vee r$, $q \vee r$, $\neg r$:



Resolution Proof Example

Show by resolution that the following CNF is UNSAT

$$\neg b \wedge (\neg a \vee b \vee \neg c) \wedge a \wedge (\neg a \vee c)$$

$$\dfrac{\dfrac{\dfrac{\neg a \vee b \vee \neg c \qquad a}{b \vee \neg c} \qquad b}{\neg c} \qquad \dfrac{a \qquad \neg a \vee c}{c}}{\bot}$$

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Entailment and Derivation

A set of formulas F entails a set of formulas G iff every model of F and is a model of G

$$F \models G$$

A formula G is derivable from a formula F by a proof system P if there exists a proof whose leaves are labeled by formulas in F and the root is labeled by G

$$F \vdash_P G$$

Soundness and Completeness

A proof system P is sound iff

$$(F \vdash_P G) \implies (F \models G)$$

A proof system P is complete iff

$$(F \models G) \implies (F \vdash_P G)$$

Propositional Resolution

Theorem: Propositional resolution is sound and complete for propositional logic

Proof: Follows from Resolution Theorem

Exercise 33

Using resolution show that

$$A \wedge B \wedge C$$

is a consequence of

$$\neg A \vee B$$

$$\neg B \vee C$$

$$A \vee \neg C$$

$$A \vee B \vee C$$

Exercise 34

Show using resolution that F is valid

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$$\neg F = (B \vee C \vee \neg D) \wedge (B \vee D) \wedge (\neg C \vee \neg D) \wedge \neg B$$

SAT solving by resolution (DP)

Formula must be in CNF

Resolution rule: $\dfrac{C \vee p \qquad D \vee \neg p}{C \vee D}$

Example: $\dfrac{q \vee t \vee p \qquad q \vee r \vee \neg p}{q \vee t \vee r}$

The result of resolution is the resolvent (clause).
Original clauses are kept (not deleted).
Duplicate literals are deleted from the resolvent.

Note: No branching.

Termination: Only finite number of possible derived clauses.

Unit & Input Resolution

Unit resolution: $\dfrac{C \vee \ell \qquad \neg\ell}{C \qquad \neg\ell}$ ($C \vee \ell$ is subsumed by C)

Input resolution: $\dfrac{C \vee \ell \qquad D \vee \neg\ell}{C \vee D}$ ($C \vee \ell$ member of input *F*).

(Optional) Exercise:

Set of clauses *F:*

F has an input refutation iff *F* has a unit refutation.

DPLL

DPLL: David Putnam Logeman Loveland = Unit resolution + split rule.

$$\frac{F}{F,p \quad | \quad F,\neg p}\text{split} \qquad p \text{ and } \neg p \text{ are not in } F$$

$$\frac{F,\ C \vee \ell, \neg \ell}{F, C, \neg \ell}\text{unit}$$

Ingredient of most efficient SAT solvers

The original DPLL procedure

Tries to build incrementally a satisfying truth assignment M for a CNF formula F

M is grown by

- deducing the truth value of a literal from M and F, or
- guessing a truth value

If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

DPLL

M | F

Partial model

Set of clauses

DPLL

Guessing

$$p \;\; | \;\; p \vee q, \neg q \vee r$$

$$p, \neg q \; | \; p \vee q, \neg q \vee r$$

DPLL

Deducing

$p \;|\; p \vee q, \neg p \vee s$

$\Downarrow$

$p, s \;|\; p \vee q, \neg p \vee s$

DPLL

Backtracking

$p, \neg s, \; q \;\; | \;\; p \vee q, s \vee q, \neg p \vee \neg q$

$p, s \; | \; p \vee q, s \vee q, \neg p \vee \neg q$

Pure Literals

A literal is pure if only occurs positively or negatively.

Example :

$\varphi = (\boxed{\neg x_1} \vee x_2) \wedge (\boxed{x_3} \vee \neg x_2) \wedge (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$

$\neg x_1$ and x_3 are pure literals

Pure literal rule :

Clauses containing pure literals can be removed from the formula (i.e. just satisfy those pure literals)

$$\varphi_{\neg x_1, x_3} = (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$$

Preserve satisfiability, not logical equivalency !

DPLL (as a procedure)

- Standard backtrack search
- DPLL(F) :
  - Apply unit propagation
  - If conflict identified, return UNSAT
  - Apply the pure literal rule
  - If F is satisfied (empty), return SAT
  - Select decision variable x
    - If DPLL($F \wedge x$)=SAT return SAT
    - return DPLL($F \wedge \neg x$)

The Original DPLL Procedure – Example

assign	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ \boxed{1}$
Deduce 1	
1	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \boxed{\neg 1 \vee \neg 2},\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce $\neg 2$	
1, 2	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Guess 3	
1, 2, 3	$1 \vee 2,\ \boxed{2 \vee \neg 3 \vee 4},\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce 4	
1, 2, 3, 4	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Conflict	

The Original DPLL Procedure – Example

assign	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ \boxed{1}$
Deduce 1	
1	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \boxed{\neg 1 \vee \neg 2,}\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce $\neg$2	
1, 2	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Guess 3	
1, 2, 3	$1 \vee 2,\ \boxed{2 \vee \neg 3 \vee 4,}\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce 4	
1, 2, 3, 4	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Undo 3	

The Original DPLL Procedure – Example

assign	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce 1	
1	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce $\neg 2$	
1, 2	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Guess $\neg 3$	
1, 2, 3	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Model Found	

An Abstract Framework for DPLL

The DPLL procedure can be described declaratively by simple sequent-style calculi

Such calculi however cannot model meta-logical features such as backtracking, learning and restarts

We model DPLL and its enhancements as transition systems instead

A transition system is a binary relation over states, induced by a set of conditional transition rules

An Abstract Framework for DPLL

State

- **fail** or M ‖ F
- where
  - F is a CNF formula, a set of clauses, and
  - M is a sequence of annotated literals denoting a partial truth assignment

Initial State

- ∅ ‖ F, where F is to be checked for satisfiability

Expected final states:

- **fail** if F is unsatisfiable
- M ‖ G
  where
  - M is a model of G
  - G is logically equivalent to F

Transition Rules for the Original DPLL

Extending the assignment:

UnitProp $M \parallel F, C \vee l \rightarrow M\, l \parallel F, C \vee l$ $\begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$

Decide $M \parallel F, C \rightarrow M\, l^d \parallel F, C$ $\begin{cases} l \text{ or } \neg l \text{ occur in } C \\ l \text{ is undefined in } M \end{cases}$

Notation: l^d is a decision literal

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Transition Rules for the Original DPLL

Repairing the assignment:

Fail $M \parallel F, C \rightarrow \text{fail}$ $\begin{cases} M \models \neg C \\ \text{M does not contain decision literals} \end{cases}$

Backtrack $M\ l^d\ N \parallel F, C \rightarrow M\ \neg l \parallel F, C$ $\begin{cases} M\ l^d\ N \models \neg C \\ \text{l is the last decision literal} \end{cases}$

Transition Rules DPLL – Example

$\emptyset \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

UnitProp 1

$1 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

UnitProp $\neg 2$

$1, 2 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

Decide 3

$1, 2, 3^d \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

UnitProp 4

$1, 2, 3^d, 4 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

Backtrack 3

Transition Rules DPLL – Example

$\varnothing \parallel 1 \lor 2,\ 2 \lor \neg 3 \lor 4,\ \neg 1 \lor \neg 2,\ \neg 1 \lor \neg 3 \lor \neg 4,\ 1$

UnitProp 1

$1 \parallel 1 \lor 2,\ 2 \lor \neg 3 \lor 4,\ \neg 1 \lor \neg 2,\ \neg 1 \lor \neg 3 \lor \neg 4,\ 1$

UnitProp $\neg 2$

$1, 2 \parallel 1 \lor 2,\ 2 \lor \neg 3 \lor 4,\ \neg 1 \lor \neg 2,\ \neg 1 \lor \neg 3 \lor \neg 4,\ 1$

Decide 3

$1, 2, 3^d \parallel 1 \lor 2,\ 2 \lor \neg 3 \lor 4,\ \neg 1 \lor \neg 2,\ \neg 1 \lor \neg 3 \lor \neg 4,\ 1$

UnitProp 4

$1, 2, 3 \parallel 1 \lor 2,\ 2 \lor \neg 3 \lor 4,\ \neg 1 \lor \neg 2,\ \neg 1 \lor \neg 3 \lor \neg 4,\ 1$

Backtrack 3

Transition Rules for the Original DPLL

UnitProp $\quad M \parallel F, C \vee l \rightarrow M\, l \parallel F, C \vee l \quad \begin{cases} M \vDash \neg C \\ l \text{ is undefined in } M \end{cases}$

Decide $\quad M \parallel F, C \rightarrow M\, l^d \parallel F, C \quad \begin{cases} l \text{ or } \neg l \text{ occur in } C \\ l \text{ is undefined in } M \end{cases}$

Fail $\quad M \parallel F, C \rightarrow \text{fail} \quad \begin{cases} M \vDash \neg C \\ M \text{ does not contain decision literals} \end{cases}$

Backtrack $\quad M\, l^d\, N \parallel F, C \rightarrow M\, \neg l \parallel F, C \quad \begin{cases} M\, l^d\, N \vDash \neg C \\ l \text{ is the last decision literal} \end{cases}$

The Basic DPLL System – Correctness

Some terminology

- Irreducible state: state to which no transition rule applies.
- Execution: sequence of transitions allowed by the rules and starting with states of the form $\emptyset \parallel F$.
- Exhausted execution: execution ending in an irreducible state

Proposition (Strong Termination) Every execution in Basic DPLL is finite

Proposition (Soundness) For every exhausted execution starting with $\emptyset \parallel F$ and ending in $M \parallel F$, $M \models F$

Proposition (Completeness) If F is unsatisfiable, every exhausted execution starting with $\emptyset \parallel F$ ends with fail

Modern DPLL: CDCL

Conflict **D**riven **C**lause **L**earning

Two watched literals

Periodically restarting backtrack search

Clausal learning

More details at

http://gauss.ececs.uc.edu/SAT/articles/FAIA185-0131.pdf

Conflict Directed Clause Learning

Lemma learning

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s$

$\Downarrow$

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s \mid \neg p \vee \neg s$

$\Downarrow$

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s \mid \neg p \vee \neg q$

$\Downarrow$

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s \mid \neg p \vee t$

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CDCL: learns clauses by resolution

Every learned clause is obtained by resolution from clauses learned before it

$$\dfrac{\dfrac{t \vee \neg p \vee q \qquad \neg q \vee s}{t \vee \neg p \vee s} \qquad \neg p \vee \neg s}{\neg p \vee t}$$

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Modern CDCL

Initialize	$\epsilon \mid F$	F *is a set of clauses*
Decide	$M \mid F \Longrightarrow M, \ell \mid F$	ℓ *is unassigned*
Propagate	$M \mid F, C \vee \ell \Longrightarrow M, \ell^{C \vee \ell} \mid F, C \vee \ell$	C *is false under* M
Sat	$M \mid F \Longrightarrow M$	F *true under* M
Conflict	$M \mid F, C \Longrightarrow M \mid F, C \mid C$	C *is false under* M
Learn	$M \mid F \mid C \Longrightarrow M \mid F, C \mid C$	
Unsat	$M \mid F \mid \emptyset \Longrightarrow Unsat$	
Backjump	$MM' \mid F \mid C \vee \ell \Longrightarrow M\ell^{C \vee \ell} \mid F$	$\bar{C} \subseteq M, \neg\ell \in M'$
Resolve	$M \mid F \mid C' \vee \neg\ell \Longrightarrow M \mid F \mid C' \vee C$	$\ell^{C \vee \ell} \in M$
Forget	$M \mid F, C \Longrightarrow M \mid F$	C is a learned clause
Restart	$M \mid F \Longrightarrow \epsilon \mid F$	

Model

Proof

Conflict Resolution

[Nieuwenhuis, Oliveras, Tinelli J.ACM 06] customized

Conjuctive Normal Form

$$\varphi \leftrightarrow \psi \quad \Rightarrow_{\mathrm{CNF}} \quad \varphi \to \psi \land \psi \to \varphi$$

$$\varphi \to \psi \quad \Rightarrow_{\mathrm{CNF}} \quad \neg\varphi \lor \psi$$

$$\neg(\varphi \lor \psi) \quad \Rightarrow_{\mathrm{CNF}} \quad \neg\varphi \land \neg\psi$$

$$\neg(\varphi \land \psi) \quad \Rightarrow_{\mathrm{CNF}} \quad \neg\varphi \lor \neg\psi$$

$$\neg\neg\varphi \quad \Rightarrow_{\mathrm{CNF}} \quad \varphi$$

$$(\varphi \land \psi) \lor \xi \quad \Rightarrow_{\mathrm{CNF}} \quad (\varphi \lor \xi) \land (\psi \lor \xi)$$

Every propositional formula can be put in CNF

PROBLEM: (potential) exponential blowup of the resulting formula

Tseitin Transformation – Main Idea

Introduce a fresh variable e_i for every subformula G_i of F

- intuitively, e_i represents the truth value of G_i

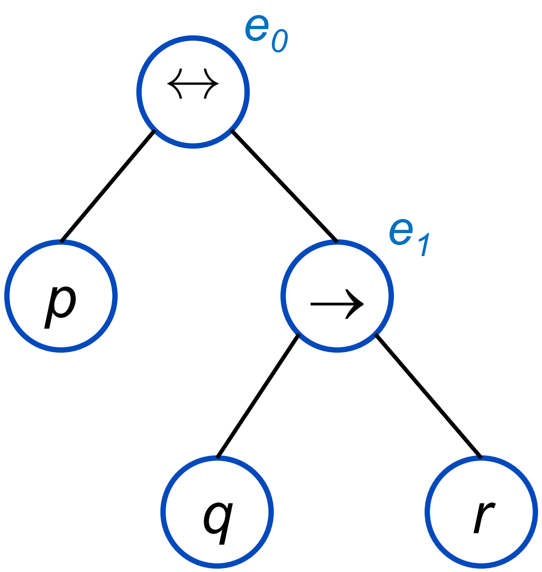
Assert that every e_i and G_i pair are equivalent

- $e_i \leftrightarrow G_i$
- and express the assertion as CNF

Conjoin all such assertions in the end

Tseitin Transformation: Example

$G : p \leftrightarrow (q \rightarrow r)$

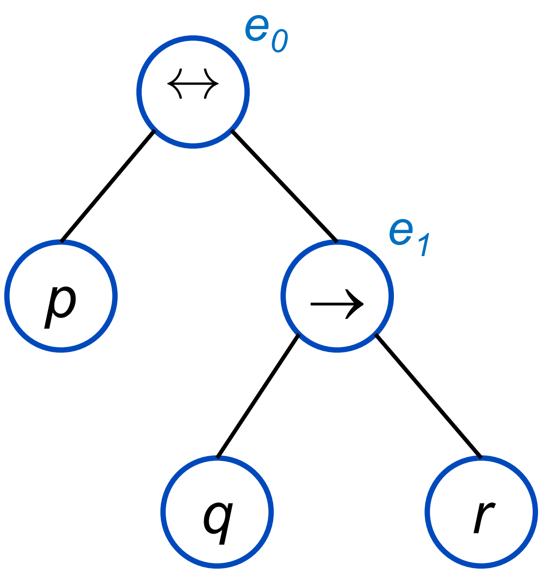


$G : e_0 \wedge (e_0 \leftrightarrow (p \leftrightarrow e_1)) \wedge (e_1 \leftrightarrow (q \rightarrow r))$

$$
\begin{aligned}
& e_1 \leftrightarrow (q \rightarrow r) \\
= \quad & (e_1 \rightarrow (q \rightarrow r)) \wedge ((q \rightarrow r) \rightarrow e_1) \\
= \quad & (\neg e_1 \vee \neg q \vee r) \wedge ((\neg q \vee r) \rightarrow e_1) \\
= \quad & (\neg e_1 \vee \neg q \vee r) \wedge (\neg q \rightarrow e_1) \wedge (r \rightarrow e_1) \\
= \quad & (\neg e_1 \vee \neg q \vee r) \wedge (q \vee e_1) \wedge (\neg r \vee e_1)
\end{aligned}
$$

Tseitin Transformation: Example

$G : p \leftrightarrow (q \rightarrow r)$

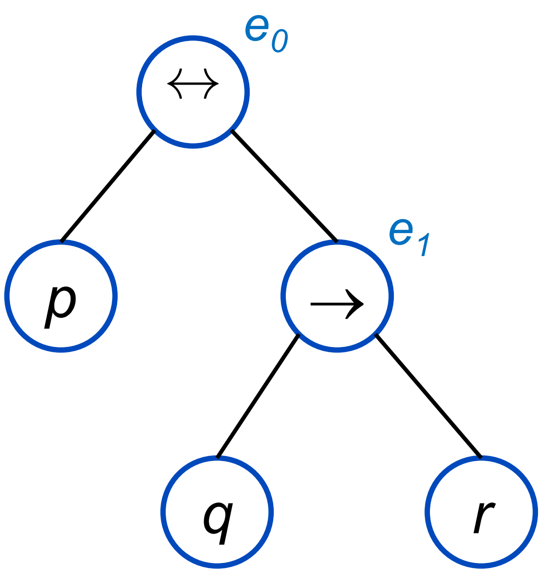


$G : e_0 \wedge (e_0 \leftrightarrow (p \leftrightarrow e_1)) \wedge (e_1 \leftrightarrow (q \rightarrow r))$

$$
\begin{aligned}
& e_0 \leftrightarrow (p \leftrightarrow e_1) \\
= \quad & (e_0 \rightarrow (p \leftrightarrow e_1)) \wedge ((p \leftrightarrow e_1)) \rightarrow e_0) \\
= \quad & (e_0 \rightarrow (p \rightarrow e_1)) \wedge (e_0 \rightarrow (e_1 \rightarrow p)) \wedge \\
& (((p \wedge e_1) \vee (\neg p \wedge \neg e_1)) \rightarrow e_0) \\
= \quad & (\neg e_0 \vee \neg p \vee e_1) \wedge (\neg e_0 \vee \neg e_1 \vee p) \wedge \\
& (\neg p \vee \neg e_1 \vee e_0) \wedge (p \vee e_1 \vee e_0)
\end{aligned}
$$

Tseitin Transformation: Example

$G : p \leftrightarrow (q \rightarrow r)$



$G : e_0 \wedge (e_0 \leftrightarrow (p \leftrightarrow e_1)) \wedge (e_1 \leftrightarrow (q \rightarrow r))$

$G : e_0 \wedge (\neg e_0 \vee \neg p \vee e_1) \wedge (\neg e_0 \vee p \vee \neg e_1) \wedge (e_0 \vee p \vee e_1) \wedge (e_0 \vee \neg p \vee \neg e_1) \wedge (\neg e_1 \vee \neg q \vee r) \wedge (e_1 \vee q) \wedge (e_1 \vee \neg r)$

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Formula to CNF Conversion

`mk_fresh_var()` returns a fresh variable not used anywhere before

```
def cnf (ϕ):
  p, F = cnf_rec (ϕ)
  return p ∧ F

def cnf_rec (ϕ):
  if is_atomic (ϕ): return (ϕ, True)
  elif ϕ == ψ ∧ ξ:
    q, F1 = cnf_rec (ψ)
    r, F2 = cnf_rec (ξ)

    p = mk_fresh_var ()
    # C is CNF for p↔(q∧r)
    C = (¬p∨q)∧(¬p∨r)∧(p∨¬q∨¬r)
    return (p, F1∧F2∧C)
  elif ϕ == ψ∨ξ:
    …
```

Exercise: Complete cases for $\phi == \psi \vee \xi$, $\phi == \neg\psi$, $\phi == \psi \leftrightarrow \xi$

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Tseitin Transformation [1968]

Used in practice

- No exponential blow-up
- CNF formula size is linear with respect to the original formula

Does not produce an equivalent CNF

However, given F, the following holds for the computed CNF F':

- F' is equisatisfiable to F
- Every model of F' can be translated (i.e., projected) to a model of F
- Every model of F can be translated (i.e., completed) to a model of F'

No model is lost or added in the conversion