Lecture Notes on Probability

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Chapter - 5

Expectation - II

Two dimensional Expectation

Consider the joint distribution of two random variables X and Y. Let $g(x, y)$ is a continuous function of x, y. Then the mean value or expected value of the random variabe $Z = g(X, Y)$ is denoted by $E\{g(X,Y)\}$ and defined by

$$E(g(X,Y)) = \sum_i \sum_j g(x_i, y_j)\, P(X = x_i, Y = y_j),$$

for discrete distribution

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, f(x, y)\, dx\, dy,$$

for continuous distribution

provided the double series or double integral is absolutely convergent.

Moments:

1. About the point (a, b) of order $k+l$
$= E[(X-a)^k (Y-b)^l]$

2. About the origin $(0,0)$ of order $k+l = E(X^l Y^k)$

3. Central moment of order $k+l = E[(X-m_x)^l (Y-m_y)^k]$,

where $m_x = E(X)$, $m_y = E(Y)$.

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Covariance: Consider the joint distribution of the random variables X and Y. Then the covariance of the random variables X and Y is denoted by $cov(X,Y)$ and defined by

$$cov(X,Y) = \mu_{11} = E[(X-m_x)(Y-m_y)]$$

Properties of two dimensional Expectation:

1. $E(c) = c$ for any constant c
2. $E(c\,g(X,Y)) = c\,E(g(X,Y))$ for any constant c
3. $E(a\,g_1(X,Y) + b\,g_2(X,Y)) = a\,E(g_1(X,Y)) + b\,E(g_2(X,Y))$, for any two constants a and b

Theorem: $cov(X,Y) = E(XY) - E(X)E(Y)$

Proof: $cov(X,Y) = E[(X-m_x)(Y-m_y)]$

$$= E[XY - m_x Y - m_y X + m_x m_y]$$

$$= E(XY) - m_x E(Y) - m_y E(X) + m_x m_y$$

$$= E(XY) - m_x m_y - m_y m_x + m_x m_y$$

$$= E(XY) - m_x m_y$$

$$= E(XY) - E(X)E(Y)$$

Theorem: If X and Y are independent random variables then $cov(X,Y) = 0$

Proof: Let the probability distribution function of X and Y is continuous having pdf $f(x,y)$.

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As X and Y are independent

$$f(x,y) = f_X(x)\, f_Y(y), \quad \text{——— } ①$$

where $f_X(x)$ and $f_Y(y)$ are, respectively, the marginal density functions of X and Y.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\, dx\, dy \quad [\text{using } ①]$$

$$= \left[\int_{-\infty}^{\infty} x f_X(x)\, dx\right]\left[\int_{-\infty}^{\infty} y f_Y(y)\, dy\right]$$

$$= m_x\, m_y = E(X)E(Y)$$

$$\Rightarrow E(XY) - E(X)E(Y) = 0$$

$$\Rightarrow \text{cov}(X,Y) = 0$$

Let the probability distribution of X and Y is discrete.

As X and Y are independent,

$$P(X=x_i, Y=y_j) = P(X=x_i)\,P(Y=y_j) \quad \text{——— } ②$$

$$E(XY) = \sum_i \sum_j x_i y_j\, P(X=x_i, Y=y_j)$$

$$= \sum_i \sum_j x_i y_j\, P(X=x_i)\,P(Y=y_j) \quad [\text{using } ②]$$

$$= \left[\sum_i x_i P(X=x_i)\right]\left[\sum_j y_j P(Y=y_j)\right]$$

$$= m_x\, m_y = E(X)E(Y) \Rightarrow E(XY) - E(X)E(Y) = 0$$

$$\Rightarrow \text{cov}(X,Y) = 0$$

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Correlation coefficient: Correlation coefficient of two random variables X and Y is denoted by $\rho_{X,Y}$ or $\rho(X,Y)$ or ρ and defined by

$$\rho(X,Y) = \text{cov}(X^*, Y^*),$$

where X^* and Y^* are, respectively, the standardised or normalised random variables corresponding to X and Y, i.e.,

$$X^* = \frac{X-m_x}{\sigma_x}, \quad Y^* = \frac{Y-m_y}{\sigma_y}$$

with $m_x = E(X)$, $m_y = E(Y)$, $Var(X) = \sigma_x^2$, $Var(Y) = \sigma_y^2$.

P1 $\rho(X,Y) = \dfrac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$ or $\text{cov}(X,Y) = \rho(X,Y)\,\sigma_x \sigma_y$

Proof: $\rho(X,Y) = \text{cov}(X^*, Y^*) = \text{cov}\left[\left(\dfrac{X-m_x}{\sigma_x}\right), \left(\dfrac{Y-m_y}{\sigma_y}\right)\right]$

$= E(X^* Y^*) - E(X^*)E(Y^*)$

$= E(X^* Y^*) - 0\cdot 0 = E(X^* Y^*)$ $[\text{As } E(X^*) = E(Y^*) = 0]$

$= E\left[\dfrac{X-m_x}{\sigma_x} \cdot \dfrac{Y-m_y}{\sigma_y}\right] = \dfrac{1}{\sigma_x \sigma_y} E[(X-m_x)(Y-m_y)]$

$= \dfrac{1}{\sigma_x \sigma_y} \text{cov}(X,Y)$

$\Rightarrow \rho(X,Y) = \dfrac{\text{cov}(X,Y)}{\sigma_x \sigma_y} \Rightarrow \text{cov}(X,Y) = \rho(X,Y)\,\sigma_x \sigma_y$

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P2 $-1 \le \rho(X,Y) \le 1 \iff |\rho(X,Y)| \le 1$

Proof: Let X^* and Y^* be the normalised variables corresponding to X and Y respectively.

$$X^* = \frac{X - m_x}{\sigma_x} \quad \text{and} \quad Y^* = \frac{Y - m_y}{\sigma_y}$$

$$E(X^*) = 0 \quad \text{and} \quad Var(X^*) = E(X^* - E(X^*))^2$$

$$= E(X^* - 0)^2 = E\left(\frac{X - m_x}{\sigma_x}\right)^2$$

$$= E\left(\frac{(X - m_x)^2}{\sigma_x^2}\right) = \frac{1}{\sigma_x^2} E(X - m_x)^2$$

$$= \frac{1}{\sigma_x^2}\sigma_x^2 = 1 \Rightarrow E[(X^*)^2] = 1$$

Ily $E(Y^*) = 0$ and $Var(Y^*) = 1$

Now $(X^* \pm Y^*)^2 \ge 0$

$$\Rightarrow E(X^{*2} + Y^{*2} \pm 2X^*Y^*) \ge 0$$

$$\Rightarrow E(X^{*2}) + E(Y^{*2}) \pm 2E(X^*Y^*) \ge 0$$

$$\Rightarrow 1 + 1 \pm 2E\left(\frac{X - m_x}{\sigma_x} \cdot \frac{Y - m_y}{\sigma_y}\right) \ge 0$$

$$\Rightarrow 2 \pm 2\frac{1}{\sigma_x \sigma_y} E((X - m_x)(Y - m_y)) \ge 0$$

$$\Rightarrow 2 \pm 2\frac{cov(X,Y)}{\sigma_x \sigma_y} \ge 0$$

$$\Rightarrow 1 \pm \rho(X,Y) \ge 0$$

$$\Rightarrow 1 + \rho(X,Y) \ge 0 \quad \text{and} \quad 1 - \rho(X,Y) \ge 0$$

$$\Rightarrow \rho(X,Y) \ge -1 \quad \text{and} \quad 1 \ge \rho(X,Y)$$

$$\Rightarrow -1 \le \rho(X,Y) \quad \text{and} \quad \rho(X,Y) \le 1 \Rightarrow -1 \le \rho(X,Y) \le 1$$

$$\Rightarrow |\rho(X,Y)| \le 1$$

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P3 $\operatorname{cov}(X,Y) = \operatorname{cov}(Y,X)$

Proof: $\operatorname{cov}(X,Y) = E[(X-m_x)(Y-m_y)] = E[(Y-m_y)(X-m_x)] = \operatorname{cov}(Y,X)$

P4 $\rho(X,Y) = \rho(Y,X)$

Proof: $\rho(X,Y) = \dfrac{\operatorname{cov}(X,Y)}{\sigma_x \sigma_y} = \dfrac{\operatorname{cov}(Y,X)}{\sigma_y \sigma_x} = \rho(Y,X)$

P5 If a, b, c, d are four constants with $a \neq 0$, $c \neq 0$ then

$$\rho(aX+b, cY+d) = \frac{ac}{|ac|}\rho(X,Y)$$

Proof: Let $U = aX+b$ and $V = cY+d$

$\therefore$ $E(U) = m_u = E(aX+b) = aE(X)+b = am_x+b$

$E(V) = m_v = E(cY+d) = cE(Y)+E(d) = cm_y+d$

$$\begin{aligned}\operatorname{Var}(U) &= E(U-m_u)^2 = E(aX+b-am_x-b)^2 \\ &= E(aX-am_x)^2 \\ &= E\{a^2(X-m_x)^2\} \\ &= a^2E(X-m_x)^2 = a^2\sigma_x^2\end{aligned}$$

$\Rightarrow \sigma_u^2 = a^2\sigma_x^2 \Rightarrow \sigma_u = \sqrt{a^2\sigma_x^2} = |a\sigma_x| = |a||\sigma_x| = |a|\sigma_x \quad [\because \sigma_x > 0]$

$\Rightarrow \sigma_u = |a|\sigma_x$

lly $\sigma_v = |c|\sigma_y$

$\rho(aX+b, cY+d) = \rho(U,V) = \dfrac{\operatorname{cov}(U,V)}{\sigma_u\sigma_v}$

$$\begin{aligned}\operatorname{Cov}(U,V) &= E(UV) - m_u m_v = E(aX+b)(cY+d) - m_u m_v \\ &= E(acXY + bcY + adX + bd) - m_u m_v \\ &= acE(XY) + bcE(Y) + adE(X) + bd - m_u m_v \\ &= acE(XY) + bcm_y + adm_x + bd - m_u m_v\end{aligned}$$

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$$\operatorname{cov}(U,V) = ac\,E(XY) + bc\,m_y + ad\,m_x + bd - m_u m_v$$

$$= ac\,E(XY) + bc\,m_y + ad\,m_x + bd - \{(a m_x + b)(c m_y + d)\}$$

$$= ac\,E(XY) + bc\,m_y + ad\,m_x + bd - (ac\,m_x m_y + bc\,m_y + ad\,m_x + bd)$$

$$= ac\,E(XY) - ac\,m_x m_y$$

$$= ac\,[E(XY) - E(X)E(Y)] = ac\operatorname{cov}(X,Y)$$

$$\rho(U,V) = \frac{\operatorname{cov}(U,V)}{\sigma_u \sigma_v} = \frac{ac\operatorname{cov}(X,Y)}{|a|\sigma_x |c| \sigma_y} = \frac{ac}{|ac|}\frac{\operatorname{cov}(X,Y)}{\sigma_x \sigma_y}$$

$$\Rightarrow \rho(U,V) = \frac{ac}{|ac|}\rho(X,Y)$$

$$\Rightarrow \rho(aX+b, cY+d) = \frac{ac}{|ac|}\rho(X,Y)$$

<u>Uncorrelated random variables:</u> Two random variables X and Y are said to be uncorrelated iff $\operatorname{cov}(X,Y)=0$

<u>P6</u> X and Y are uncorrelated iff $\rho(X,Y)=0$

<u>Proof:</u> X and Y are uncorrelated iff $\operatorname{cov}(X,Y)=0$

i.e., if and only if $\dfrac{\operatorname{cov}(X,Y)}{\sigma_x\sigma_y} = 0$

i.e., if and only if $\rho(X,Y)=0$

<u>P7</u> If X and Y are independent then X and Y are uncorrelated. But the converse is not true, i.e., if X and Y are uncorrelated then X and Y are not necessarily independent.

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Proof: Let X and Y be the independent random variables.

Therefore, for discrete case, we have

$$P(X=x_i, Y=y_j) = P(X=x_i)P(Y=y_j) \quad \forall\, i \,\&\, j$$

$$E(XY) = \sum_i \sum_j x_i y_j P(X=x_i, Y=y_j)$$

$$= \sum_i \sum_j x_i y_j P(X=x_i) P(Y=y_j)$$

$$= \sum_i \sum_j \{x_i P(X=x_i)\}\{y_j P(Y=y_j)\}$$

$$= \sum_i x_i P(X=x_i) \sum_j y_j P(Y=y_j)$$

$$= \sum_i x_i P(X=x_i) E(Y)$$

$$= E(Y) \sum_i x_i P(X=x_i) = E(Y)E(X)$$

$$\Rightarrow E(XY) - E(X)E(Y) = 0$$

$$\Rightarrow \operatorname{cov}(X,Y) = 0$$

$\Rightarrow$ X and Y are uncorrelated.

For continuous case, we have

$$f(x,y) = f_X(x) f_Y(y) \quad \forall\, x \,\&\, y$$

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$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= E(X)\, E(Y)$$

$\Rightarrow E(XY) = E(X)\,E(Y)$

$\Rightarrow E(XY) - E(X)\,E(Y) = 0$

$\Rightarrow \mathrm{cov}(X, Y) = 0$

$\Rightarrow$ X and Y are uncorrelated.

Converse of this property is not necessarily true:

Let X be the standard normal variate. So the pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

Let $Y = X^2$

$E(X) = 0$ [As X is normalised variable]

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx$$

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$$E(X^2) = \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-\frac{x^2}{2}} dx$$

Put $\frac{x^2}{2} = p$

$x^2 = 2p$

$x = \sqrt{2}\sqrt{p}$

$dx = \sqrt{2} \cdot \frac{1}{2} p^{\frac{1}{2}-1} dp$

$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{p}} dp$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty 2p e^{-p} \cdot \frac{1}{\sqrt{2}} \frac{1}{\sqrt{p}} \cdot dp$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-p} p^{1/2} dp$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-p} p^{3/2 - 1} dp$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1$$

$\Rightarrow E(X^2) = 1$

$$E(XY) = E(X \cdot X^2) = E(X^3)$$

$$= \int_{-\infty}^\infty x^3 f_X(x) dx$$

$$= \int_{-\infty}^\infty x^3 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \underbrace{x^3 e^{-\frac{x^2}{2}}}_{\text{odd function of } x} dx$$

$$= 0$$

$\therefore \operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0 - 0 \cdot 1 = 0$

$\Rightarrow$ X and Y are uncorrelated but they are not independent.

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<u>Problem-1:</u> If X and Y are two random variables such that $E(X^2)$, $E(Y^2)$ and $E(XY)$ exist finitely then show that

$$\{E(XY)\}^2 \leq E(X^2)\,E(Y^2)$$

Use this inequality to show that $|\rho| \leq 1$.

<u>Ans:</u> Let the joint distribution of X and Y is continuous having pdf $f(x,y)$.

Now $(y-kx)^2 \geq 0 \quad \forall\, x, y$ and $k \in R$

$\Rightarrow (y-kx)^2 f(x,y) \geq 0 \quad \forall\, x, y, k \in R$

$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-kx)^2 f(x,y)\,dx\,dy \geq 0 \quad \forall\, k \in R$

$\Rightarrow E((Y-kX)^2) \geq 0 \quad \forall\, k \in R$

$\Rightarrow E(Y^2 - 2kXY + k^2X^2) \geq 0 \quad \forall\, k \in R$

$\Rightarrow E(Y^2) - 2kE(XY) + k^2E(X^2) \geq 0 \quad \forall\, k \in R$

$\Rightarrow k^2E(X^2) - 2kE(XY) + E(Y^2) \geq 0 \quad \forall\, k \in R$ —— (1)

Now $x^2 \geq 0 \Rightarrow E(X^2) \geq 0$

If $E(X^2) = 0$ then $X \equiv 0$, i.e., X has one point distribution at $x = 0 \Rightarrow P(X=0) = 1$

$\Rightarrow E(X) = 0 \cdot P(X=0) = 0$

$E(XY) = \sum 0 \cdot y\, P(X=0, Y=y) = 0$

$\Rightarrow \{E(XY)\}^2 = E(X^2)\,E(Y^2)$ —— (2)

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If $E(X^2) \neq 0$ then $E(X^2) > 0$ and consequently (1) can be written in the following form:

$$k^2 - 2k \frac{E(XY)}{E(X^2)} + \frac{E(Y^2)}{E(X^2)} \geq 0$$

$$\Rightarrow k^2 - 2 \cdot k \cdot \frac{E(XY)}{E(X^2)} + \left\{\frac{E(XY)}{E(X^2)}\right\}^2 - \left\{\frac{E(XY)}{E(X^2)}\right\}^2 + \frac{E(Y^2)}{E(X^2)} \geq 0$$

$$\Rightarrow \left\{k - \frac{E(XY)}{E(X^2)}\right\}^2 - \left\{\frac{E(XY)}{E(X^2)}\right\}^2 + \frac{E(Y^2)}{E(X^2)} \geq 0 \quad \text{——} \quad (3)$$

The relation (3) is true for all values of k and consequently the inequality (3) holds good for $k = \frac{E(XY)}{E(X^2)}$. Therefore putting

$$k = \frac{E(XY)}{E(X^2)}$$

in (3), we get the following inequality

$$-\left\{\frac{E(XY)}{E(X^2)}\right\}^2 + \frac{E(Y^2)}{E(X^2)} \geq 0$$

$$\Rightarrow \frac{\{E(XY)\}^2}{\{E(X^2)\}^2} \leq \frac{E(Y^2)}{E(X^2)}$$

$$\Rightarrow \{E(XY)\}^2 \leq E(X^2)E(Y^2) \quad \text{——} \quad (4)$$

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From ② & ④, we get

$$\{E(XY)\}^2 \le E(X^2)\,E(Y^2)$$

for any two random variables X and Y

For the normalised random variables X^* and Y^*, we have

$$\{E(X^* Y^*)\}^2 \le E(X^{*2})\,E(Y^{*2})$$

$$\Rightarrow \left\{E\left[\left(\frac{X-m_x}{\sigma_x}\right)\left(\frac{Y-m_y}{\sigma_y}\right)\right]\right\}^2 \le E\left(\frac{X-m_x}{\sigma_x}\right)^2 E\left(\frac{Y-m_y}{\sigma_y}\right)^2$$

$$\Rightarrow \left\{\frac{E[(X-m_x)(Y-m_y)]}{\sigma_x\sigma_y}\right\}^2 \le \frac{1}{\sigma_x^2} E(X-m_x)^2 \cdot \frac{1}{\sigma_y^2} E(Y-m_y)^2$$

$$\Rightarrow \left\{\frac{Cov(X,Y)}{\sigma_x\sigma_y}\right\}^2 \le \frac{1}{\sigma_x^2}\sigma_x^2 \cdot \frac{1}{\sigma_y^2}\sigma_y^2 = 1$$

$$\Rightarrow \{\rho(X,Y)\}^2 \le 1$$

$$\Rightarrow |\rho(X,Y)|^2 - 1^2 \le 0$$

$$\Rightarrow (|\rho(X,Y)|+1)(|\rho(X,Y)|-1) \le 0$$

$$\Rightarrow |\rho(X,Y)| - 1 \le 0$$

$$\Rightarrow |\rho(X,Y)| \le 1$$

$$\Rightarrow -1 \le \rho(X,Y) \le 1$$

Problem-2: For any two random variables X and Y

$$Var(aX+bY) = a^2\sigma_x^2 + b^2\sigma_y^2 + 2\rho ab\,\sigma_x\sigma_y$$

provided $\rho(X,Y)(=\rho)$ exists finitely.

Answer: $Var(aX+bY) = E\left(aX+bY-E(aX+bY)\right)^2$

$= E\{aX+bY-aE(X)-bE(Y)\}^2$

$= E\left(aX+bY-am_x-bm_y\right)^2 \quad m_x = E(X),\ m_y = E(Y)$

$= E\left(a(X-m_x)+b(Y-m_y)\right)^2$

$= E\{a^2(X-m_x)^2 + b^2(Y-m_y)^2 + 2ab(X-m_x)(Y-m_y)\}$

$= a^2E(X-m_x)^2 + b^2E(Y-m_y)^2 + 2abE(X-m_x)(Y-m_y)$

$= a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\,cov(X,Y)$

$= a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\,\sigma_x\sigma_y\rho$

$\Rightarrow Var(aX+bY) = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\,\sigma_x\sigma_y\rho$

Problem-3: If X and Y are uncorrelated, then

$Var(aX+bY) = a^2\sigma_x^2 + b^2\sigma_y^2 = a^2\,Var(X) + b^2\,Var(Y)$

Answer: Follows from problem-2 along with the condition $cov(X,Y)=0 \Leftrightarrow \rho(X,Y)=0$ as the random variables X and Y are uncorrelated.

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Problem-4: If a, b, c are positive constants, then show that

$$\rho(aX+bY,\ cY) = \frac{a\,\rho(X,Y)\,\sigma_x + b\,\sigma_y}{\sqrt{a^2\sigma_x^2 + b\sigma_y^2 + 2ab\,\sigma_x\sigma_y\,\rho(X,Y)}}$$

Answer:

$$\rho(aX+bY,\ cY) = \frac{\operatorname{cov}(aX+bY,\ cY)}{\sqrt{Var(aX+bY)}\ \sqrt{Var(cY)}}$$

$$\operatorname{cov}(aX+bY,\ cY) = E\left[\{aX+bY-E(aX+bY)\}\{cY-E(cY)\}\right]$$

$$= E\left[\{a(X-E(X))+b(Y-E(Y))\}\{cY-cE(Y)\}\right]$$

$$= E\left[\{a(X-m_x)+b(Y-m_y)\}\{c(Y-m_y)\}\right]$$

$$= E\left[\{ac(X-m_x)(Y-m_y)+bc(Y-m_y)^2\}\right]$$

$$= ac\,E(X-m_x)(Y-m_y) + bc\,E(Y-m_y)^2$$

$$= ac\operatorname{cov}(X,Y) + bc\,\sigma_y^2$$

$$= ac\,\rho(X,Y)\,\sigma_x\sigma_y + bc\,\sigma_y^2$$

$$= [a\,\rho(X,Y)\,\sigma_x + b\,\sigma_y]\,c\,\sigma_y$$

$$var(aX+bY) = a^2\sigma_x^2 + b^2\sigma_y^2 + 2\rho(X,Y)\,ab\,\sigma_x\sigma_y$$

$$var(cY) = E\,(cY-E(cY))^2 = E\,(c^2(Y-m_y)^2) = c^2\sigma_y^2$$

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$$\rho(aX+bY, cY) = \frac{c\sigma_y[a\rho(X,Y)\sigma_x + b\sigma_y]}{\sqrt{a^2\sigma_x^2+b^2\sigma_y^2+2ab\rho(X,Y)\sigma_x\sigma_y}\ \sqrt{c^2\sigma_y^2}}$$

$$= \frac{c\sigma_y[a\sigma_x\rho(X,Y)+b\sigma_y]}{c\sigma_y\sqrt{a^2\sigma_x^2+b^2\sigma_y^2+2ab\sigma_x\sigma_y\rho(X,Y)}}$$

$$= \frac{a\sigma_x\rho(X,Y)+b\sigma_y}{\sqrt{a^2\sigma_x^2+b^2\sigma_y^2+2ab\sigma_x\sigma_y\rho(X,Y)}}$$

$\Rightarrow \rho(aX+bY, cY)$ is independent of c if $c>0$

Problem-5: If X and Y are uncorrelated then show that

$$\rho(a_1X+b_1Y,\ a_2X+b_2Y) = \frac{a_1a_2\sigma_x^2 + b_1b_2\sigma_y^2}{\sqrt{a_1^2\sigma_x^2+b_1^2\sigma_y^2}\ \sqrt{a_2^2\sigma_x^2+b_2^2\sigma_y^2}}$$

Problem-6: If the random variables X and Y are connected by the relation $aX+bY+c=0$ then show that

$$\rho(X,Y) = \begin{cases} -1 & \text{if } ab>0 \\ 1 & \text{if } ab<0 \end{cases}$$

Answer: $aX+bY+c=0 \Rightarrow c = -(aX+bY)$ —— (1)

We take $\boxed{ab\neq 0}$

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$$aX + bY + c = 0$$

$$\Rightarrow E(aX + bY + c) = 0$$

$$\Rightarrow E(aX) + E(bY) + E(c) = 0$$

$$\Rightarrow aE(X) + bE(Y) + c = 0$$

$$\Rightarrow aE(X) + bE(Y) - (aX + bY) = 0$$

$$\Rightarrow a(X - E(X)) + b(Y - E(Y)) = 0$$

$$\Rightarrow a(X - m_x) = -b(Y - m_y)$$

$$\Rightarrow a^2(X - m_x)^2 = b^2(Y - m_y)^2$$

$$\Rightarrow E(a^2(X - m_x)^2) = E(b^2(Y - m_y)^2)$$

$$\Rightarrow a^2 E(X - m_x)^2 = b^2 E(Y - m_y)^2$$

$$\Rightarrow a^2 \sigma_x^2 = b^2 \sigma_y^2$$

$$\Rightarrow \sigma_y^2 = \frac{a^2 \sigma_x^2}{b^2}$$

$$\Rightarrow \sigma_y = \sqrt{\frac{a^2 \sigma_x^2}{b^2}} = \left|\frac{a\sigma_x}{b}\right| = \left|\frac{a}{b}\right| \sigma_x$$

$$\Rightarrow \sigma_y = \left|\frac{a}{b}\right| \sigma_x$$

$$\operatorname{cov}(X, Y) = E(X - m_x)(Y - m_y)$$

$$= E\left[\left(-\frac{bY + c}{a} - m_x\right)(Y - m_y)\right]$$

$$= E\left[-\left(\frac{bY + c}{a} + m_x\right)(Y - m_y)\right]$$

$$= E\left[-\frac{bY + c + am_x}{a}(Y - m_y)\right]$$

$$= -\frac{1}{a} E(bY + c + am_x)(Y - m_y)$$

$$\text{cov}(X,Y) = -\frac{1}{a} E\{bY^2 + cY + am_xY - bm_yY - cm_y - am_xm_y\}$$

$$= -\frac{1}{a}\left[bE(Y^2) + cE(Y) + am_xE(Y) - bm_yE(Y) - E(cm_y) - E(am_xm_y)\right]$$

$$= -\frac{1}{a}\left[bE(Y^2) + \underline{cm_y} + \underline{am_xm_y} - bm_y^2 - \underline{cm_y} - \underline{am_xm_y}\right]$$

$$= -\frac{1}{a}\left[b\{E(Y^2) - m_y^2\}\right]$$

$$= -\frac{b}{a}\left[E(Y^2) - \{E(Y)\}^2\right] = -\frac{b}{a}\sigma_y^2$$

$$= -\frac{b}{a}\sigma_y^2 = -\frac{b}{a}\sigma_y^2$$

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_x\sigma_y} = -\frac{b}{a}\frac{\sigma_y^2}{\sigma_x\sigma_y} = -\frac{b}{a}\frac{\sigma_y}{\sigma_x}$$

$$= -\frac{b}{a}\cdot\left|\frac{a}{b}\right|\sigma_x\cdot\frac{1}{\sigma_x} = -\frac{b}{a}\left|\frac{a}{b}\right|$$

If $ab > 0$ then $\frac{ab}{b^2} > 0$ (as $b^2 > 0$)

$\Rightarrow \frac{a}{b} > 0 \Rightarrow \left|\frac{a}{b}\right| = \frac{a}{b}$

$\Rightarrow \rho(X,Y) = -\frac{b}{a}\cdot\frac{a}{b} = -1$

If $ab < 0$ then $\frac{ab}{b^2} < 0$ (as $b^2 > 0$)

$\Rightarrow \frac{a}{b} < 0 \Rightarrow \left|\frac{a}{b}\right| = -\frac{a}{b}$

$\Rightarrow \rho(X,Y) = -\frac{b}{a}\cdot\left(-\frac{a}{b}\right) = 1$

$$\therefore \quad \rho(X,Y) = \begin{cases} -1 & \text{if } ab > 0 \\ 1 & \text{if } ab < 0 \end{cases}$$

Problem-7: Let X and Y be two random variables. Show that the random variables

$U = X\cos\alpha + Y\sin\alpha$ & $V = -X\sin\alpha + Y\cos\alpha$

are uncorrelated if

$$\alpha = \frac{1}{2}\tan^{-1}\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}, \quad \rho = \rho(X,Y)$$

Probability AB JE