Combinatorics

Ujjwal Maulik

Department of Computer Science & Engineering

Jadavpur University, Kolkata

Definition
(Ack: Wikipedia)

- Combinatorics is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures.

Permutation and Combination

1. How many strings of length 4 can be formed using 10 letters ?
   - without repetitions - 10(P)4
   - with repetitions - 10^4

2. How any ways 4 friends can be chosen from 10 friends ?
   - 10(C)4

3. Let us assume there are 10 balls among which 3 are red, 4 are green and 3 are yellow. How many ways 4 balls can be chosen from ?

Pigeon Hole Principle

- Pigeon Hole Principle
  - If *$n > 1$* pigeons are put into m holes *$(n>m)$* then at least one hole will have more than one pigeons.

- Generalized Pigeon Hole Principle
  - If *$n > 1$* pigeons are put into *m* holes *$(n>m)$* such that *$n = km+1$* where *k* is an integer then at least one hole will have *$k+1$* pigeons

Recurrence Relation

(Searching Algorithms worst case # of comparisons for n elements)

- Sequential Search
  - Recurrence Relation T(n) = 1+T(n-1) and termination criteria T(1) = 1
  - Solution:

  T(n) = 1+T(n-1) = 1+1+T(n-2)=1+1+1+T(n-3)=1+1+1 … (n-1) times + T(1)= n

- Binary Search
  - Recurrence Relation T(n) = 1+T(n/2) and termination criteria T(1) = 1
  - Solution:

  T(n) = 1+T(n/2^1) = 1+1+t(n/2^2)= 1+1+1+T(n/2^3)= 1+1+1 … k times +T(n/2^k)

  when (n/2^k) = 1 then 2^k =n thus k= log (n)

  Therefore T(n) = 1+1+1+ …. Log(n) times +T(1) = log(n) + 1

(Write Code for Binary Search – Lab Exercise)

Recurrence Relation

(Sorting Algorithms worst/best case # of comparisons for n elements)

- Bubble Sort/Selection Sort
  - Recurrence Relation T(n) = (n-1)+T(n-1) and termination criteria T(1) = 0
  - Solution:

  T(n) = (n-1)+T(n-1) = (n-1)+(n-2)+T(n-2)=(n-1)+(n-2)+ ….(n-1) times +T(1)
  Therefore, T(n)=)=(n-1)+(n-2)+(n-3)+(n-4)+……2+1= n(n+1)/2

- Merger Sort (Best and Worst Case)/Quick Sort(Best Case)
  - Recurrence Relation T(n) = (n-1)+T(n/2) and termination criteria T(1) = 0
  - Solution:

  T(n) = n+2T(n/2^1) = n+n+4T(n/2^2)=n+n+n+ …. K times +2^kT(n/2^k)

  when (n/2^k) = 1 then 2^k =n thus k= log (n)
  Therefore T(n) = n+n…. log(n) times +T(1) = n*log(n)

  - Quick Sort(Worst Case)
    - Same as Bubble/Selection sort → n(n+1)/2

(Write Code for Merge Sort and Quick Sort - Lab Exercise)

Recurrence Relation

- If n lines each intersect with other and no 3 lines are intersection in the same point then how may regions they will form

  - Recurrence Relation R(n) = n+R(n-1) and termination criteria R(1) = 2
  - Solution:

R(n) = n+R(n-1) = n+(n-1)+R(n-2)=n+(n-1)+(n-2) …. +3+2+R(1)

=n+(n-1)+(n-2) …. +3+2+**2**=n+(n-1)+(n-2) …. +3+2**+1+1** = n(n+1)/2 + 1

  - If there are 4/9/11 lines how may regions will be formed

For 4 lines total regions will be 4*5/2+1 = 11

For 9 lines total regions will be 9*10/2+1 = 46

For 11 lines total regions will be 11*12/2+1 = 67

Recurrence Relation

- If n circles each intersect with other and no 3 circles are intersecting in the same point then how may regions they will form

  - Recurrence Relation R(n) = 2(n-1)+R(n-1) and termination criteria R(1) = 2
  - Solution:

R(n) = 2(n-1)+ 2(n-2)+R(n-2) = 2(n-1)+2(n-2)+2(n-3)+….2(n-n+1)+R(1)=
2[(n-1)+(n-2) …. +3+1]+**2**
=2 [n(n+1)/2 – n] + 2 = 2[n(n+1)-2n]/2 + 2 = n(n-1) + 2

  - If there are 3/4/9/11 circles how may regions will be formed

For 3 circles total regions will be = 3*(3-1)+2=8
For 4 circles total regions will be = 4*(4-1)+2=14
For 9 circles total regions will be = 9*(9-1)+2=74
For 11 circles total regions will be = 11*(11-1)+2=112

Tower of Hanoi

- Recurrence Relation $M(n) = 2\ M(n\text{-}1) + 1$ and

  termination criteria M(1) = 1
- Solution
  - M(n) = 2 [2 M(n-2) +1] +1

    = 2^2 M(n-2) + 2 + 1 = 2^2 [2 M(n-3) +1] + 2^1 +2^0

    = 2^3 M(n-3) + 2^2 + 2^1 + 2^0 = 2^3 [2M(n-4)+1]+2^2+ 2^1 + 2^0

    ………….

    = 2^(n-1) [M{n-(n-1)}]+2^(n-2)+2^(n-3)+……+2^2+2^1 + 2^0

    = 2^(n-1)+2^(n-2)+2^(n-3)+……+2^2+2^1 + 2^0 (Sum G. P. Series)

    = 2^0 [2^n-1]/[2-1] = 2^n- 1

Generative Function

- It is a (possibly infinite) polynomial whose coefficients correspond to terms in a sequence of numbers.
- It encode an infinite sequence of numbers (a_n) by treating them as the coefficients of a formal power series. This series is called the generating function of the sequence
  - An ordinary generating function can be represent as

$f(x) = \sum a_n X^n$ where n can vary from 0 to ∞.

Let $S = 1 + x^1 + x^2 + x^3 + x^4 + \ldots\ldots..$ (all an = 1)

$Sx = x^1 + x^2 + x^3 + x^4 + x^5 \ldots\ldots..$

So $S(1-x) = 1$ Thus $S = 1/(1-x)$.

- Therefore, the generating function of 1, 1, 1, 1 …. is 1/(1-x)
- Replacing "x" by "-x" the generating function of 1, -1, 1, -1 …. is 1/(1+x)
- Similarly replacing "x" by "3x"the generating function of 1, 3, 9, 27 …. is 1/(1-3x)

Generative Function for Face of a Die

Examples:

1. Find the generating function for the face value of 1 die.

$R1(x)=x^1 + x^2 + x^3 + x^4 + x^5 + x^6$

Here, the exponent represents the score on the die face; the coefficient is the number of ways each score can be obtained

2. Find the generating function for the sum of faces values of 2 dice.

$R2(x)= (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^2 = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$

Partition of Integers

Example:

Partition of 2 → {2}, {1 1}

Partition of 3 → {3}, {2 1}, {1 1 1}

Partition of 4 → {4}, {3 1, 2 2}, {2 1 1}, {1 1 1 1}

Partition of 5 → {5}, {4 1, 3 2}, {2 2 1, 3 1 1}, {2 1 1 1}, {1 1 1 1 1}

…………..

……………

Partitions of Integers

Notations:

- p(n, k) = number of partitions of n with k parts
- p(n) = total number of partitions of n

- Example
  - Partitions of 5: (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), and (1, 1, 1, 1, 1).
  - Thus p(5) = 7, p(5, 1) = 1, p(5, 2) = 2, p(5, 3) = 2, p(5, 4) = 1, and p(5, 5) = 1,

Generative Function for Partitions of Integers

- Although there is no simple formula for Pn we can have a generative function.

$(1+x^1 + x^2 + x^3 +\ldots)\ (1+ x^2 + x^4 + x^6 + \ldots)\ (1+ x^3 + x^6 + x^9 + \ldots)\ \ldots..\ (1+ x^k + x^{2k} + x^{3k} + \ldots)$

= product of $\sum X^{ik}$ (sum over "i" and product over "k". "i" and "k" can vary from 0 to ∞)

- Find P4

$(1+x^1 + x^2 + x^3 + x^4)\ (1+ x^2 + x^4)\ (1+ x^3)\ (1+ x^4) \Rightarrow$ (removing all term more than x^4)

$[\ (1+x^1 + x^2 + x^3 + x^4)\ (1+x^2 + x^4 \ldots.)\ (1+x^3+ x^4 \ldots.) \Rightarrow$ (removing all term more than x^4)

$(1+x^1 + x^2 + x^3 + x^4)\ (1+x^2 + x^3 + 2x^4 \ldots) \Rightarrow$ (removing all term more than x^4)

$(1+x^1 + x^2 + x^3 + x^4 + x^2 + x^3 + x^4 + x^3 + x^4 + 2x^4 \ldots) \Rightarrow$

$(1+\mathbf{1}x^1 + \mathbf{2}x^2 + \mathbf{3}x^3 + \mathbf{5}x^4 \ldots) \Rightarrow$ **So P4 = 5**

Five partition of P4

{4}, {3+1, 2+2}, {2+1+1}, {1+1+1+1}

References

- Björner, Anders; and Stanley, Richard P.; (2010); *A Combinatorial Miscellany*
- Bóna, Miklós; (2011); *A Walk Through Combinatorics (3rd Edition)*. ISBN 978-981-4335-23-2, 978-981-4460-00-2
- Stanley, Richard P. (1997, 1999); *Enumerative Combinatorics, Volumes 1 and 2*, Cambridge University Press. ISBN 0-521-55309-1, 0-521-56069-1
- van Lint, Jacobus H.; and Wilson, Richard M.; (2001); *A Course in Combinatorics*, 2nd Edition, Cambridge University Press. ISBN 0-521-80340-3

- Rittaud, Benoît; Heeffer, Albrecht (2014). "The pigeonhole principle, two centuries before Dirichlet". *The Mathematical Intelligencer*. **36** (2): 27–29. doi:10.1007/s00283-013-9389-1