# Lecture Notes on Probability

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

## Chapter-6

## Convergence in Probability

**Tchebycheff's Inequality:** If $X$ is a random variable having finite variance $\sigma^2$ then for any $\varepsilon > 0$,

$$P(|X-m| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

where $m = E(X)$

**Proof:** Let $X$ be a discrete random variable.

Let $\tau = \{x_i : |x_i - m| \geq \varepsilon\}$

$$\therefore P(|X-m| \geq \varepsilon) = \sum_{x_i \in \tau} P(X = x_i) \quad \text{——— (1)}$$

Now $x_i \in \tau \Rightarrow |x_i - m| \geq \varepsilon$

$$\Rightarrow |x_i - m|^2 \geq \varepsilon^2$$

$$\Rightarrow (x_i - m)^2 \geq \varepsilon^2$$

$$\therefore (x_i - m)^2 \geq \varepsilon^2 \quad \forall x_i \in \tau$$

$$\Rightarrow (x_i - m)^2 P(X = x_i) \geq \varepsilon^2 P(X = x_i) \quad \forall x_i \in \tau$$

$$\Rightarrow \sum_{x_i \in \tau} (x_i - m)^2 P(X = x_i) \geq \sum_{x_i \in \tau} \varepsilon^2 P(X = x_i)$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$\Rightarrow \sum_{x_i \in \tau} (x_i - m)^2 P(X = x_i) \geq \varepsilon^2 \sum_{x_i \in \tau} P(X = x_i)$$

$$\Rightarrow \sum_{x_i \in \tau} (x_i - m)^2 P(X = x_i) \geq \varepsilon^2 P(|X - m| \geq \varepsilon) \quad \text{—— (2)}$$

Now
$$\sigma^2 = \sum_i (x_i - m)^2 P(X = x_i)$$

$$= \sum_{x_i \in \tau} (x_i - m)^2 P(X = x_i) + \sum_{x_i \in \bar{\tau}} (x_i - m)^2 P(X = x_i)$$

$$\geq \sum_{x_i \in \tau} (x_i - m)^2 P(X = x_i)$$

$$\geq \varepsilon^2 P(|X - m| \geq \varepsilon) \quad [\text{using } (2)]$$

$$\Rightarrow P(|X - m| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

Let $X$ be a continuous random variable.

Let $\tau = \{x : |x - m| \geq \varepsilon\}$

$\therefore \quad |x - m| \geq \varepsilon \quad \forall x \in \tau$

$\Rightarrow |x - m|^2 \geq \varepsilon^2 \quad \forall x \in \tau$

$\Rightarrow (x - m)^2 \geq \varepsilon^2 \quad \forall x \in \tau$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$\Rightarrow (x-m)^2 f(x) \geq \varepsilon^2 f(x) \quad \forall\, x \in \tau$$

$$\Rightarrow \int_{x\in\tau} (x-m)^2 f(x)\,dx \geq \int_{x\in\tau} \varepsilon^2 f(x)\,dx$$

$$\Rightarrow \int_{x\in\tau} (x-m)^2 f(x)\,dx \geq \varepsilon^2 \int_{x\in\tau} f(x)\,dx = \varepsilon^2 P(X\in\tau)$$

$$\Rightarrow \int_{x\in\tau} (x-m)^2 f(x)\,dx \geq \varepsilon^2 P(|X-m| \geq \varepsilon) \quad \text{--------- ③}$$

Now $\sigma^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$

$$= \int_{x\in\tau} (x-m)^2 f(x)\,dx + \int_{x\in\bar{\tau}} (x-m)^2 f(x)\,dx$$

$$= \int_{x\in\tau} (x-m)^2 f(x)\,dx + \int_{x\in\bar{\tau}} (x-m)^2 f(x)\,dx$$

$$\geq \int_{x\in\tau} (x-m)^2 f(x)\,dx \geq \varepsilon^2 P(|X-m| \geq \varepsilon)$$

$$\Rightarrow \varepsilon^2 P(|X-m| \geq \varepsilon) \leq \sigma^2$$

$$\Rightarrow P(|X-m| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

$\therefore$ from Tchebycheff's inequality, we have

$$P(|X-m| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$\Rightarrow 1 - P(|X-m| < \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$$

$$\Rightarrow 1 - \frac{\sigma^2}{\varepsilon^2} \le P(|X-m| < \varepsilon)$$

$$\Rightarrow P(|X-m| < \varepsilon) \ge 1 - \frac{\sigma^2}{\varepsilon^2}$$

$$\Rightarrow P(-\varepsilon < X-m < \varepsilon) \ge 1 - \frac{\sigma^2}{\varepsilon^2}$$

$$\Rightarrow P(m-\varepsilon < X < m+\varepsilon) \ge 1 - \frac{\sigma^2}{\varepsilon^2}$$

$$\Rightarrow 1 \ge P(m-\varepsilon < X < m+\varepsilon) \ge 1 - \frac{\sigma^2}{\varepsilon^2} \quad \forall \varepsilon > 0$$

If $\sigma$ is small then the total probability is concentrated in a neighbourhood of $m = E(X)$.

<u>Defn</u>: A sequence of random variables $\{X_n\}$ is said to converge to a constant $a$ in probability if for any $\varepsilon > 0$

$$\lim_{n \to \infty} P(|X_n - a| < \varepsilon) = 1$$

or $$\lim_{n \to \infty} P(|X_n - a| \ge \varepsilon) = 0$$

In this case we write $X_n \xrightarrow[\text{in } p]{} a$ as $n \to \infty$ and we say that $\{X_n\}$ converges to $a$ in probability

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

Defn: A sequence of random variables $\{X_n\}$ is said to converge to random variable $X$ if the sequence $\{X_n - X\}$ converges to 0 in probability, i.e., $X_n - X \xrightarrow[\text{in } p]{} 0$ as $n \to \infty$

Theorem: Let $X_n \xrightarrow[\text{in } p]{} a$ and $Y_n \xrightarrow[\text{in } p]{} b$ as $n \to \infty$ then

T1) $X_n - a \xrightarrow[\text{in } p]{} 0$ as $n \to \infty$

T2) $c X_n \xrightarrow[\text{in } p]{} ca$ as $n \to \infty$ for any constant $c$

T3) $X_n \pm Y_n \xrightarrow[\text{in } p]{} a \pm b$ as $n \to \infty$

T4) $X_n^2 \xrightarrow[\text{in } p]{} a^2$ as $n \to \infty$

T5) $X_n Y_n \xrightarrow[\text{in } p]{} ab$ as $n \to \infty$

T6) $\dfrac{X_n}{Y_n} \xrightarrow[\text{in } p]{} \dfrac{a}{b}$ as $n \to \infty$ provided $b \neq 0$

Proof of T1: Let $Z_n = X_n - a$

$X_n \xrightarrow[\text{in } p]{} a$ as $n \to \infty$

$\Rightarrow$ Given any $\varepsilon > 0$, $\lim_{n \to \infty} P(|X_n - a| < \varepsilon) = 1$

$\Rightarrow \lim_{n\to\infty} P(|X_n - a| < \varepsilon) = 1$ for any $\varepsilon > 0$

$\Rightarrow \lim_{n\to\infty} P(|Z_n| < \varepsilon) = 1$ for any $\varepsilon > 0$

$\Rightarrow \lim_{n\to\infty} P(|Z_n - 0| < \varepsilon) = 1$ for any $\varepsilon > 0$

$\Rightarrow Z_n \xrightarrow[\text{in } p]{} 0$ as $n \to \infty$

$\Rightarrow X_n - a \xrightarrow[\text{in } p]{} 0$ as $n \to \infty$

Proof of T2: Here we have the following two cases:

case 1:– $c = 0$ & case 2 – $c \neq 0$

case 1 : $c = 0$

Let $A_n = \{\omega \in S : |c(X_n(\omega) - a)| < \varepsilon\}$ for $\varepsilon > 0$

$\Rightarrow A_n \subseteq S$ ——— (1)

Again $\omega \in S \Rightarrow |c(X_n(\omega) - a)| = |0(X_n(\omega) - a)| = 0 < \varepsilon$

$\Rightarrow |c(X_n(\omega) - a)| < \varepsilon \Rightarrow \omega \in A_n$

$\Rightarrow S \subseteq A_n$ ——— (2)

From (1) & (2), we get

$A_n = S \quad \forall n \in N$

$\Rightarrow P(A_n) = P(S) = 1 \quad \forall n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} P(A_n) = 1$

$\Rightarrow \lim_{n\to\infty} P(|c(X_n - a)| < \varepsilon) = 1$

$\Rightarrow \lim_{n\to\infty} P(|cX_n - ca| < \varepsilon) = 1$

$\Rightarrow cX_n \xrightarrow[\text{in } p]{} ca$ as $n \to \infty$

Case 2: $c \neq 0 \Rightarrow |c| > 0 \Rightarrow \frac{\varepsilon}{|c|} = \varepsilon_1 > 0$ for any $\varepsilon > 0$

$X_n \xrightarrow[\text{in } p]{} a$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} P(|X_n - a| < \varepsilon_1) = 1$ for any $\varepsilon_1 > 0$

$\Rightarrow \lim_{n\to\infty} P\left(|X_n - a| < \frac{\varepsilon}{|c|}\right) = 1$ for any $\varepsilon > 0$

$\Rightarrow \lim_{n\to\infty} P(|c||X_n - a| < \varepsilon) = 1$ for any $\varepsilon > 0$

$\Rightarrow \lim_{n\to\infty} P(|c(X_n - a)| < \varepsilon) = 1$ for any $\varepsilon > 0$

$\Rightarrow \lim_{n\to\infty} P(|cX_n - ca| < \varepsilon) = 1$ for any $\varepsilon > 0$

$\Rightarrow cX_n \xrightarrow[\text{in } p]{} ca$ as $n \to \infty$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

Therefore, in any case, we have

$$cX_n \xrightarrow[\text{in } p]{} ca \quad \text{as } n \to \infty$$

Proof of T3: Let $\varepsilon > 0$ be any real number.

Consider the following events:

$$A_n = \{\omega \in S : |X_n(\omega) - a| < \varepsilon/2\} = (|X_n - a| < \varepsilon/2)$$

$$B_n = \{\omega \in S : |Y_n(\omega) - b| < \varepsilon/2\} = (|Y_n - b| < \varepsilon/2)$$

$$C_n = \{\omega \in S : |X_n(\omega) + Y_n(\omega) - (a+b)| < \varepsilon\}$$

$$= (|X_n + Y_n - (a+b)| < \varepsilon)$$

$$D_n = \{\omega \in S : |X_n(\omega) - Y_n(\omega) - (a-b)| < \varepsilon\}$$

$$= (|X_n - Y_n - (a-b)| < \varepsilon)$$

Now $\omega \in A_n B_n \Rightarrow \omega \in A_n$ and $\omega \in B_n$

$$\Rightarrow |X_n(\omega) - a| < \varepsilon/2 \text{ and } |Y_n(\omega) - b| < \varepsilon/2$$

$$\Rightarrow |X_n(\omega) + Y_n(\omega) - (a+b)|$$

$$= |X_n(\omega) - a + Y_n(\omega) - b| \leq |X_n(\omega) - a| + |Y_n(\omega) - b| < \varepsilon/2 + \varepsilon/2$$

$$\Rightarrow |X_n(\omega) + Y_n(\omega) - (a+b)| < \varepsilon$$

$$\Rightarrow \omega \in C_n$$

$$\therefore \omega \in A_n B_n \Rightarrow \omega \in C_n$$

$$\Rightarrow A_n B_n \subseteq C_n \quad \text{——— (1)}$$

Again $\omega \in A_n B_n \Rightarrow \omega \in A_n$ & $\omega \in B_n$

$$\Rightarrow |X_n(\omega) - a| < \varepsilon/2 \text{ and } |Y_n(\omega) - b| < \varepsilon/2$$

$$\Rightarrow |X_n(\omega) - Y_n(\omega) - (a-b)|$$

$$= |X_n(\omega) - a - \{Y_n(\omega) - b\}|$$

$$\leq |X_n(\omega) - a| + |Y_n(\omega) - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\Rightarrow \omega \in D_n$$

$\therefore \omega \in A_n B_n \Longrightarrow \omega \in D_n$

$\therefore A_n B_n \subseteq D_n$ ——————— (2)

From (1), we have

$$A_n B_n \subseteq C_n \Rightarrow \overline{C_n} \subseteq \overline{A_n B_n} = \overline{A_n} + \overline{B_n}$$

$$\Rightarrow P(\overline{C_n}) \leq P(\overline{A_n} + \overline{B_n}) \leq P(\overline{A_n}) + P(\overline{B_n})$$

$$\Rightarrow 0 \leq P(\overline{C_n}) \leq P(\overline{A_n}) + P(\overline{B_n}) \quad \text{——————— (3)}$$

Now $X_n \xrightarrow[\text{in } p]{} a$ as $n \to \infty$

$$\Rightarrow \lim_{n \to \infty} P(|X_n - a| < \varepsilon) = 1 \text{ for any } \varepsilon > 0$$

$$\Rightarrow \lim_{n \to \infty} P(|X_n - a| < \varepsilon/2) = 1$$

$$\Rightarrow \lim_{n \to \infty} P(A_n) = 1$$

$$\Rightarrow \lim_{n \to \infty} \{1 - P(\overline{A_n})\} = 1 \Rightarrow 1 - \lim_{n \to \infty} P(\overline{A_n}) = 1$$

$$\Rightarrow \lim_{n \to \infty} P(\overline{A_n}) = 0$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$\Rightarrow \lim_{n\to\infty} P(\bar{A}_n) = 0$$

$$\left.\begin{array}{l}\Rightarrow P(\bar{A}_n) \longrightarrow 0 \quad \text{as } n \to \infty \\ \text{Ily } P(\bar{B}_n) \longrightarrow 0 \quad \text{as } n \to \infty\end{array}\right\} \text{——— (4)}$$

Making $n \longrightarrow \infty$ in (3) and using (4) we get

$$0 \le P(\bar{C}_n) \le 0 + 0 \quad \text{as } n \longrightarrow \infty$$

$$\Rightarrow 0 \le P(\bar{C}_n) \le 0 \quad \text{as } n \longrightarrow \infty$$

$$\Rightarrow \lim_{n\to\infty} P(\bar{C}_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} [1 - P(C_n)] = 0$$

$$\Rightarrow \lim_{n\to\infty} P(C_n) = 1$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n + Y_n - (a+b)| < \varepsilon) = 1 \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow X_n + Y_n \xrightarrow[\text{in } P]{} a + b \quad \text{as } n \longrightarrow \infty$$

Again from (2)

$$A_n B_n \subseteq D_n \Rightarrow \bar{D}_n \subseteq \overline{A_n B_n} = \bar{A}_n + \bar{B}_n$$

$$\Rightarrow P(\bar{D}_n) \le P(\bar{A}_n + \bar{B}_n) \le P(\bar{A}_n) + P(\bar{B}_n)$$

$$\Rightarrow 0 \le P(\bar{D}_n) \le P(\bar{A}_n) + P(\bar{B}_n)$$

$$\Rightarrow 0 \le P(\bar{D}_n) \le 0 + 0 \quad \text{as } n \longrightarrow \infty$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$\Rightarrow \quad 0 \leq P(\bar{D}_n) \leq 0 \quad$ as $n \to \infty$

$\Rightarrow \quad \lim_{n\to\infty} P(\bar{D}_n) = 0$

$\Rightarrow \lim_{n\to\infty} [1 - P(D_n)] = 0$

$\Rightarrow \lim_{n\to\infty} P(D_n) = 1$

$\Rightarrow \lim_{n\to\infty} P(|X_n - Y_n - (a-b)| < \varepsilon) = 1$

$\Rightarrow \quad X_n - Y_n \xrightarrow[\text{in } p]{} a-b \quad$ as $n \to \infty$

$\therefore \quad X_n \pm Y_n \xrightarrow[\text{in } p]{} a \pm b \quad$ as $n \to \infty$

<u>**Proof of T4:**</u> $\quad X_n^2 = (X_n - a + a)^2 = (Z_n + a)^2, \quad Z_n = X_n - a$

$$= Z_n^2 + 2aZ_n + a^2 \qquad \text{——— (1)}$$

<u>From T1</u> $\quad X_n \xrightarrow[\text{in } p]{} a \quad$ as $n \to \infty$

$\Rightarrow \quad X_n - a \xrightarrow[\text{in } p]{} 0 \quad$ as $n \to \infty$

$\Rightarrow \quad Z_n \xrightarrow[\text{in } p]{} 0 \quad$ as $n \to \infty$

$\Rightarrow \quad \lim_{n\to\infty} P(|Z_n - 0| < \varepsilon_1) = 1 \quad$ for any $\varepsilon_1 > 0$

$\Rightarrow \lim_{n\to\infty} P(|Z_n| < \sqrt{\varepsilon}) = 1 \quad$ for any $\varepsilon > 0$, $\varepsilon_1 = \sqrt{\varepsilon}$

$$\Rightarrow \lim_{n\to\infty} P(|Z_n| - \sqrt{\varepsilon} < 0) = 1 \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow \lim_{n\to\infty} P\left( (|Z_n| - \sqrt{\varepsilon})(|Z_n| + \sqrt{\varepsilon}) < 0 \right) = 1 \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow \lim_{n\to\infty} P\left( |Z_n|^2 - (\sqrt{\varepsilon})^2 < 0 \right) = 1 \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow \lim_{n\to\infty} P(|Z_n^2| < \varepsilon) = 1 \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow \lim_{n\to\infty} P(|Z_n^2 - 0| < \varepsilon) = 1 \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow Z_n^2 \xrightarrow[\text{in } p]{} 0 \text{ as } n \to \infty$$

$$Z_n \xrightarrow[\text{in } p]{} 0 \text{ as } n \to \infty$$

$$\Downarrow$$

$$2aZ_n \xrightarrow[\text{in } p]{} 0 \text{ as } n \to \infty$$

$$\therefore Z_n^2 + 2aZ_n \xrightarrow[\text{in } p]{} 0 \text{ as } n \to \infty$$

$$\Rightarrow X_n^2 - a^2 \xrightarrow[\text{in } p]{} 0 \text{ as } n \to \infty \quad (\text{from } ①)$$

$$\Rightarrow X_n^2 \xrightarrow[\text{in } p]{} a^2 \text{ as } n \to \infty$$

**Proof of T5 :**

From T3

$$X_n \pm Y_n \xrightarrow[m\,p]{} a \pm b \quad \text{as } n \to \infty$$

$$\Downarrow \text{ [using T4]}$$

$$(X_n \pm Y_n)^2 \xrightarrow[m\,p]{} (a \pm b)^2 \quad \text{as } n \to \infty$$

$$\Downarrow \text{ [using T3]}$$

$$(X_n + Y_n)^2 - (X_n - Y_n)^2 \xrightarrow[m\,p]{} (a+b)^2 - (a-b)^2 \quad \text{as } n \to \infty$$

$$\Downarrow$$

$$4 X_n Y_n \xrightarrow[m\,p]{} 4ab \quad \text{as } n \to \infty$$

$$\Downarrow \text{ [using T2]}$$

$$\frac{1}{4} \cdot 4 X_n Y_n \xrightarrow[m\,p]{} \frac{1}{4} \cdot 4ab \quad \text{as } n \to \infty$$

$$\Downarrow$$

$$X_n Y_n \xrightarrow[in\,p]{} ab \quad \text{as } n \to \infty$$

**Proof of T6 :** To prove T6, we shall first of all prove that

$$\frac{1}{Y_n} \xrightarrow[in\,p]{} \frac{1}{b} \quad \text{as } n \to \infty \quad \text{provided } b \neq 0$$

Let $A_n = \{\omega \in S : |Y_n(\omega) - b| < |b|\} = (|Y_n - b| < b)$

$B_n = \{\omega \in S : \left|\frac{1}{Y_n(\omega)} - \frac{1}{b}\right| < \varepsilon\}$ for any $\varepsilon > 0$

$= \left(\left|\frac{1}{Y_n} - \frac{1}{b}\right| < \varepsilon\right)$ for any $\varepsilon > 0$

$\therefore \overline{B}_n = \{\omega \in S : \left|\frac{1}{Y_n(\omega)} - \frac{1}{b}\right| \geq \varepsilon\}$ for $\varepsilon > 0$

$= \left(\left|\frac{1}{Y_n} - \frac{1}{b}\right| \geq \varepsilon\right)$ for $\varepsilon > 0$

Now $\omega \in \overline{B}_n \Rightarrow \left|\frac{1}{Y_n(\omega)} - \frac{1}{b}\right| \geq \varepsilon$

$\Rightarrow \left|\frac{b - Y_n(\omega)}{b\, Y_n(\omega)}\right| \geq \varepsilon$

$\Rightarrow |b - Y_n(\omega)| \geq \varepsilon |b| |Y_n(\omega)|$

$\Rightarrow |Y_n(\omega) - b| \geq \varepsilon |b| |Y_n(\omega)|$

$\Rightarrow |Y_n(\omega) - b| \geq \varepsilon |b| |Y_n(\omega) - b + b|$

$\geq \varepsilon |b| |Y_n(\omega) - b - (-b)|$

$\geq \varepsilon |b| \big| |Y_n(\omega) - b| - |(-b)| \big|$

$\Rightarrow |Y_n(\omega) - b| \geq \varepsilon |b| \big| |Y_n(\omega) - b| - |b| \big|$ ——— (1)

Let $\omega \in A_n \overline{B}_n$

$\omega \in A_n \overline{B}_n$

$\Rightarrow \omega \in A_n$ and $\omega \in \overline{B}_n$

$\Rightarrow |Y_n(\omega) - b| < |b|$ and

$|Y_n(\omega) - b| \geq \varepsilon |b| \, \big| |Y_n(\omega) - b| - |b| \big|$ [from ①]

$\Rightarrow |Y_n(\omega) - b| - |b| < 0$ and

$|Y_n(\omega) - b| \geq \varepsilon |b| \, \big| |Y_n(\omega) - b| - |b| \big|$

$\Rightarrow \big| |Y_n(\omega) - b| - |b| \big| = -\left(|Y_n(\omega) - b| - |b|\right)$ and

$|Y_n(\omega) - b| \geq \varepsilon |b| \, \big| |Y_n(\omega) - b| - |b| \big|$

$\Rightarrow \big| |Y_n(\omega) - b| - |b| \big| = |b| - |Y_n(\omega) - b|$ and

$|Y_n(\omega) - b| \geq \varepsilon |b| \, \big| |Y_n(\omega) - b| - |b| \big|$

$\Rightarrow |Y_n(\omega) - b| \geq \varepsilon |b| \left( |b| - |Y_n(\omega) - b| \right)$

$\Rightarrow |Y_n(\omega) - b| + \varepsilon |b| \, |Y_n(\omega) - b| \geq \varepsilon |b|^2$

$\Rightarrow (1 + \varepsilon |b|) \, |Y_n(\omega) - b| \geq \varepsilon |b|^2$

$$\Rightarrow |Y_n(\omega) - b| \geq \frac{\varepsilon |b|^2}{1 + \varepsilon |b|} \quad \text{——— ②}$$

Let $C_n = \left\{ \omega \in S : |Y_n(\omega) - b| \geq \dfrac{\varepsilon |b|^2}{1 + \varepsilon |b|} \right\}$

$$= \left( |Y_n - b| \geq \frac{\varepsilon |b|^2}{1 + \varepsilon |b|} \right)$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

From (2) it follows that

$$\omega \in A_n \bar{B}_n \Rightarrow \omega \in C_n$$

$$\Rightarrow A_n \bar{B}_n \subseteq C_n$$

$$\bar{B}_n = S \bar{B}_n = (A_n + \bar{A}_n)\bar{B}_n = A_n \bar{B}_n + \bar{A}_n \bar{B}_n$$

$$\Rightarrow \bar{B}_n = A_n \bar{B}_n + \bar{A}_n \bar{B}_n \subseteq C_n + \bar{A}_n$$

$$\Rightarrow P(\bar{B}_n) \le P(C_n + \bar{A}_n) \le P(C_n) + P(\bar{A}_n)$$

$$\Rightarrow 0 \le P(\bar{B}_n) \le P(C_n) + P(\bar{A}_n) \quad \text{———} \quad (3)$$

$$Y_n \xrightarrow[\text{in } p]{} b \text{ as } n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} P(|Y_n - b| < \varepsilon_1) = 1 \quad \text{for any } \varepsilon_1 > 0$$

$$\Rightarrow \lim_{n\to\infty} P(|Y_n - b| < \varepsilon_1) = 1 \quad \text{for any } \varepsilon_1 > 0$$

$$\Rightarrow \lim_{n\to\infty} [1 - P(|Y_n - b| \ge \varepsilon_1)] = 1 \quad \text{for any } \varepsilon_1 > 0$$

$$\Rightarrow \lim_{n\to\infty} P(|Y_n - b| \ge \varepsilon_1) = 0 \quad \text{for any } \varepsilon_1 > 0 \quad \text{———} \quad (4)$$

For $\varepsilon_1 = |b|$, we get from (4)

$$\lim_{n\to\infty} P(|Y_n - b| \ge |b|) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(\bar{A}_n) = 0 \quad \text{———} \quad (5)$$

For $\varepsilon_1 = \dfrac{|b|^2 \varepsilon}{1 + \varepsilon |b|}$ (for $\varepsilon > 0$), we get from ④

$$\lim_{n\to\infty} P\left( |Y_n - b| \geq \frac{|b|^2 \varepsilon}{1+\varepsilon|b|} \right) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(C_n) = 0 \quad \text{———} \quad ⑥$$

Making $n \to \infty$ in ③ and using ⑤ and ⑥, we get

$$0 \leq P(\bar{B}_n) \leq 0 + 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} P(\bar{B}_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} [1 - P(B_n)] = 0$$

$$\Rightarrow 1 - \lim_{n\to\infty} P(B_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(B_n) = 1$$

$$\Rightarrow \lim_{n\to\infty} \left( P\left( \left| \frac{1}{Y_n} - \frac{1}{b} \right| < \varepsilon \right) \right) = 1 \text{ for any } \varepsilon > 0$$

$$\Rightarrow \frac{1}{Y_n} \xrightarrow[\text{in } p]{} \frac{1}{b} \text{ as } n \to \infty \text{ if } b \neq 0.$$

$$\therefore X_n \xrightarrow[\text{in } p]{} a \text{ as } n \to \infty \quad \text{and}$$

$$\frac{1}{Y_n} \xrightarrow[\text{in } p]{} \frac{1}{b} \text{ as } n \to \infty \text{ if } b \neq 0$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$\Rightarrow \dfrac{X_n}{Y_n} \xrightarrow[\text{in } p]{} \dfrac{a}{b}$ as $n \to \infty$ provided $b \neq 0$

<u>Tchebycheff's Theorem</u>: Let $\{X_n\}$ be a sequence of random variables such that the mean $m_n = E(X_n)$ and the standard deviation $\sigma_n = \sqrt{Var(X_n)}$ exist finitely for all $n$. If $\sigma_n \to 0$ as $n \to \infty$ then $X_n - m_n \xrightarrow[\text{in } p]{} 0$ as $n \to \infty$.

<u>Proof</u>: From Tchebycheff's inequality, we have

$$0 \leq P(|X_n - m_n| \geq \varepsilon) \leq \frac{Var(X_n)}{\varepsilon^2} \quad \text{for any } \varepsilon > 0$$

$\Downarrow$

$$0 \leq P(|Y_n| \geq \varepsilon) \leq \frac{\sigma_n^2}{\varepsilon^2} \quad \text{for any } \varepsilon > 0, \; Y_n = X_n - m_n$$

$\Downarrow$

$$0 \leq P(|Y_n - 0| \geq \varepsilon) \leq \frac{\sigma_n^2}{\varepsilon^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\Downarrow$

$$0 \leq \lim_{n \to \infty} P(|Y_n - 0| \geq \varepsilon) \leq 0 \quad \text{for any } \varepsilon > 0$$

$\Downarrow$

$$\lim_{n \to \infty} P(|Y_n - 0| \geq \varepsilon) = 0 \quad \text{for any } \varepsilon > 0$$

$\Downarrow$

$Y_n \xrightarrow[\text{in } p]{} 0$ as $n \to \infty \Rightarrow X_n - m_n \xrightarrow[\text{in } p]{} 0$ as $n \to \infty$

**Law of Large numbers:** Let $\{X_n\}$ be a sequence of random variables such that mean $M_n$ and the standard deviation $\sigma_n$ of $S_n = X_1 + X_2 + \cdots + X_n$ exist finitely for all $n$.

If $\frac{\sigma_n}{n} \longrightarrow 0$ as $n \longrightarrow \infty$ then

$$\frac{S_n - M_n}{n} \xrightarrow[\text{in } p]{} 0 \quad \text{as } n \longrightarrow \infty$$

**Proof:** Let $Y_n = \frac{S_n - M_n}{n}$

$E(S_n) = M_n$ & $Var(S_n) = \sigma_n^2$

$$E(Y_n) = E\left(\frac{S_n - M_n}{n}\right) = \frac{1}{n} E(S_n - M_n)$$

$$= \frac{1}{n}\{E(S_n) - E(M_n)\} = \frac{1}{n}\{M_n - M_n\} = 0$$

$$Var(Y_n) = E(Y_n - E(Y_n))^2 = E(Y_n - 0)^2 = E(Y_n^2)$$

$$= E\left(\frac{S_n - M_n}{n}\right)^2 = \frac{1}{n^2} E(S_n - E(S_n))^2$$

$$= \frac{1}{n^2} Var(S_n) = \frac{1}{n^2}\sigma_n^2 = \frac{\sigma_n^2}{n^2}$$

Therefore, from Tchebycheff's inequality, we have

$$0 \le P(|Y_n - E(Y_n)| \ge \epsilon) \le \frac{Var(Y_n)}{\epsilon^2} \quad \text{for any } \epsilon > 0$$

$$\Rightarrow 0 \leq P(|Y_n - 0| \geq \varepsilon) \leq \frac{(\sigma_n/n)^2}{\varepsilon^2} \quad \text{for any } \varepsilon > 0$$

$$\Downarrow$$

$$0 \leq \lim_{n\to\infty} P(|Y_n - 0| \geq \varepsilon) \leq 0 \quad \text{for any } \varepsilon > 0 \ \left(\text{as } \frac{\sigma_n}{n} \to 0\right)$$

$$\Downarrow$$

$$\lim_{n\to\infty} P(|Y_n - 0| \geq \varepsilon) = 0$$

$$\Downarrow$$

$$Y_n \xrightarrow[\text{in } p]{} 0 \quad \text{as } n \to \infty$$

$$\Downarrow$$

$$\frac{S_n - M_n}{n} \xrightarrow[\text{in } p]{} 0 \quad \text{as } n \to \infty$$

**Law of Large numbers for the case of equal components:** If $\{X_n\}$ be a sequence of mutually independent random variables such that each $X_n$ has the same distribution as $X$. If the variance of $X$ exists finitely then $\bar{X} \xrightarrow[\text{in } p]{} m = E(X)$ as $n \to \infty$, where

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

**Proof:** As each $X_i$ has the same distribution as $X$, we have

$$E(X_i) = E(X) = m \text{ (say) } \forall i$$

$$Var(X_i) = Var(X) = \sigma^2 \text{ (say) } \forall i$$

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

$\Downarrow$

$$E(\bar{X}) = \frac{1}{n} E(X_1 + X_2 + \cdots + X_n)$$

$$= \frac{1}{n}\{E(X_1) + E(X_2) + \cdots + E(X_n)\}$$

$$= \frac{1}{n}\{m + m + \cdots + m\} = \frac{1}{n} \times mn = m$$

$$\Rightarrow E(\bar{X}) = m$$

$$Var(\bar{X}) = E(\bar{X} - E(\bar{X}))^2 = E\left(\frac{X_1 + X_2 + \cdots + X_n}{n} - m\right)^2$$

$$= E\left(\frac{X_1 + X_2 + \cdots + X_n - mn}{n}\right)^2$$

$$= \frac{1}{n^2} E(X_1 - m + X_2 - m + \cdots + X_n - m)^2$$

$$= \frac{1}{n^2} E(Y_1 + Y_2 + \cdots + Y_n)^2, \quad Y_j = X_j - m$$

$$= \frac{1}{n^2} E\left\{Y_1^2 + Y_2^2 + \cdots + Y_n^2 + \sum_{i \neq j}\sum Y_i Y_j\right\}$$

$$= \frac{1}{n^2}\left\{E(Y_1^2) + E(Y_2^2) + \cdots + E(Y_n^2) + \sum_{i \neq j}\sum E(Y_i Y_j)\right\}$$

$$E(Y_j^2) = E(X_j - m)^2 = E(X_j - E(X_j))^2$$

$$= Var(X_j) = Var(X) = \sigma^2$$

$$E(Y_i Y_j) = E((X_i - m)(X_j - m))$$

$$= E((X_i - E(X_i))(X_j - E(X_j)))$$

$$= Cov(X_i, X_j)$$

$$= 0 \quad \forall i \neq j$$ [as $X_i$ and $X_j$ are independent $\forall i \neq j$]

$$\therefore Var(\bar{X}) = \frac{1}{n^2}\{\sigma^2 + \sigma^2 + \cdots + \sigma^2\} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

From Tchebycheff's inequality, we have

$$0 \leq P(|\bar{X} - E(\bar{X})| \geq \varepsilon) \leq \frac{Var(\bar{X})}{\varepsilon^2} \quad \text{for any } \varepsilon > 0$$

$$\Downarrow$$

$$0 \leq P(|\bar{X} - m| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \quad \text{for any } \varepsilon > 0$$

$$\Downarrow$$

$$0 \leq \lim_{n\to\infty} P(|\bar{X} - m| \geq \varepsilon) \leq 0 \quad \text{for any } \varepsilon > 0$$

$$\Downarrow$$

$$\lim_{n\to\infty} P(|\bar{X} - m| \geq \varepsilon) = 0 \quad \text{for any } \varepsilon > 0$$

$$\Rightarrow \bar{X} \xrightarrow[in\, p]{} m \quad \text{as } n \longrightarrow \infty$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University


**Bernoulli's Theorem:** If $X_n = B(n, p)$ then show that $\frac{X_n}{n} \xrightarrow[\text{in } p]{} p$ as $n \to \infty$

**Proof:** Let $Y_n = \frac{X_n}{n}$

$$\therefore E(Y_n) = E\left(\frac{X_n}{n}\right) = \frac{1}{n} E(X_n) = \frac{1}{n} \cdot np = p$$

$$Var(Y_n) = E(Y_n - E(Y_n))^2 = E\left(\frac{X_n}{n} - p\right)^2$$

$$= E\left(\frac{X_n - np}{n}\right)^2 = \frac{1}{n^2} E(X_n - E(X_n))^2$$

$$= \frac{1}{n^2} Var(X_n) = \frac{1}{n^2} \cdot npq = \frac{pq}{n}$$

From Tchebycheff's inequality, we get

$$P(|Y_n - E(Y_n)| \geq \varepsilon) \leq \frac{Var(Y_n)}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$\Downarrow$

$$0 \leq P(|Y_n - p| \geq \varepsilon) \leq \frac{pq}{n\varepsilon^2} \quad \forall \varepsilon > 0$$

$\Downarrow$

$$0 \leq \lim_{n \to \infty} P(|Y_n - p| \geq \varepsilon) \leq 0 \quad \forall \varepsilon > 0$$

$\Downarrow$

$$\lim_{n \to \infty} P(|Y_n - p| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$$

$\Rightarrow Y_n \xrightarrow[\text{in } p]{} p$ as $n \to \infty \Rightarrow \frac{X_n}{n} \xrightarrow[\text{in } p]{} p$ as $n \to \infty$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

Axiomatic definition of probability

$\Rightarrow$ Statistical definition of probability.

Proof: Let $A$ be an event connected with a random experiment $E$. Let $n(A)$ denotes the number of occurrence of the event $A$ out of $n$ repetitions of the experiment $E$.

The frequency ratio or the relative frequency of the event $A$ is given by

$$f(A) = \frac{n(A)}{n}$$

Let $X_n$ be the random variable which denotes the number of occurrence of the event $A$ out of $n$ repetitions of $E$, i.e.,

$X_n = n(A)$

If we consider the occurrence of the event $A$ in any trial of $E$ as a success then

$X_n = B(n, P(A))$,

when $P(A)$ = the probability of success
= the probability of $A = P(A)$

Therefore, according to Bernoulli's theorem, we have

$$\frac{X_n}{n} \xrightarrow[\text{in } p]{} P(A) \text{ as } n \to \infty$$

$$\Downarrow$$

$$\frac{n(A)}{n} \xrightarrow[\text{in } p]{} P(A) \text{ as } n \to \infty$$

$$\Downarrow$$

$$f(A) \xrightarrow[\text{in } p]{} P(A) \text{ as } n \to \infty$$

$$\Downarrow$$

$$\lim_{n\to\infty} P(|f(A) - P(A)| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$$

$$\Downarrow$$

$$\lim_{n\to\infty} [1 - P(|f(A) - P(A)| < \varepsilon)] = 0 \quad \forall \varepsilon > 0$$

$$\Downarrow$$

$$\lim_{n\to\infty} P(|f(A) - P(A)| < \varepsilon) = 1$$

$$\Downarrow$$

$$f(A) \in (P(A) - \varepsilon, P(A) + \varepsilon) \text{ as } n \to \infty$$

$$\Downarrow$$

$f(A) \approx P(A)$ for small value of $\varepsilon$ and large $n$.

and this gives the statistical definition of probability.

Problem -1: Show that in 2000 throws of a coin the probability that the number of heads lying between 900 and 1100 is at least 19/20.

Answer: Let $X$ be the random variable which denotes the number of occurrence of heads out of 2000 throws of a coin.

$\therefore X = B(2000, \frac{1}{2})$

$E(X) = 2000 \times \frac{1}{2} = 1000$

$Var(X) = npq = 2000 \times \frac{1}{2} \times \frac{1}{2} = 500$

From Tchebycheff's inequality, we get

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow P(|X - 1000| \geq \varepsilon) \leq \frac{500}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow 1 - P(|X - 1000| < \varepsilon) \leq \frac{500}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow 1 - \frac{500}{\varepsilon^2} \leq P(|X - 1000| < \varepsilon) \quad \forall \varepsilon > 0$$

$$\Rightarrow P(|X - 1000| < \varepsilon) \geq 1 - \frac{500}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow P(1000 - \varepsilon < X < 1000 + \varepsilon) \geq 1 - \frac{500}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow P(1000-\varepsilon < x < 1000+\varepsilon) \geq 1 - \frac{500}{\varepsilon^2} \quad \forall \varepsilon > 0 \qquad \text{—— (1)}$$

Put $\varepsilon = 100$ in (1),

$$P(1000-100 < x < 1000+100) \geq 1 - \frac{500}{100 \times 100}$$

$$\Downarrow$$

$$P(900 < x < 1100) \geq 1 - \frac{5}{100} = 1 - \frac{1}{20} = \frac{19}{20}$$

$$\Downarrow$$

$$P(900 < x < 1100) \geq \frac{19}{20}.$$

<u>Problem-2</u> Let $X$ be a Gamma $n$ variate.

Show that $P(0 < X < 2n) \geq \frac{n-1}{n}$.

Answer :

$X = \Gamma(n) \Rightarrow E(X) = n$ & $Var(X) = n$.

From Tchebycheff's inequality, we get

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow P(|X - n| \geq \varepsilon) \leq \frac{n}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow 1 - P(|X-n| < \varepsilon) \leq \frac{n}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Rightarrow 1 - \frac{n}{\varepsilon^2} \leq P(|X-n| < \varepsilon)$$

$$\Rightarrow P(|X-n| < \varepsilon) \geq 1 - \frac{n}{\varepsilon^2}$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$\Rightarrow P(n-\varepsilon < X < n+\varepsilon) \geq 1 - \frac{n}{\varepsilon^2} \quad \forall \varepsilon > 0 \qquad \text{--- (1)}$$

Put $\varepsilon = n$ in (1),

$$P(n-n < X < n+n) \geq 1 - \frac{n}{n^2} = 1 - \frac{1}{n} = \frac{n-1}{n}$$

$$\Downarrow$$

$$P(0 < X < 2n) \geq \frac{n-1}{n}$$