Lecture Notes on Probability

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Chapter-4

Expectation-I

Definition: Let X be a random variable and $y = g(x)$ be a continuous function. Then the expectation of the random variable $Y = g(X)$ is denoted by $E(g(X))$ and defined by

$$E(g(X)) = \begin{cases} \sum_{i=-\infty}^{\infty} g(x_i)\, P(X = x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

provided the series or the infinite integral converges absolutely.

Properties of Expectation:

P1 $E(a) = a$ for any constant a.

P2 $E(a\, g(X)) = a\, E(g(X))$ for any constant a.

P3 $E(a\, g_1(X) + b\, g_2(X)) = a\, E(g_1(X)) + b\, E(g_2(X))$.

P4 $|E(g(X))| \leq E(|g(X)|)$.

P5 If $g(x) \geq 0$ for all x, then $E(g(X)) \geq 0$.

P6 If $g(x) \geq 0$ for all x and $E(g(X)) = 0$ then $g(X) = 0$.

Proof of P1: Let X be continuous with pdf $f(x)$, then

$$E(a) = \int_{-\infty}^{\infty} a f(x)\,dx = a \int_{-\infty}^{\infty} f(x)\,dx = a \cdot 1 = a$$

Let X be discrete, then

$$E(a) = \sum_{i=-\infty}^{\infty} a\,P(X=x_i) = a \sum_{i=-\infty}^{\infty} P(X=x_i) = a \cdot 1 = a$$

Proof of P2: Let X be continuous random variable, then

$$E(a\,g(X)) = \int_{-\infty}^{\infty} a\,g(x) f(x)\,dx = a \int_{-\infty}^{\infty} g(x) f(x)\,dx = a\,E(g(X))$$

Let X be a discrete random variable, then

$$E(a\,g(X)) = \sum_{i=-\infty}^{\infty} a\,g(x_i)\,P(X=x_i)$$

$$= a \sum_{i=-\infty}^{\infty} g(x_i)\,P(X=x_i)$$

$$= a\,E(g(X))$$

Proof of P3: Let X be a continuous random variable.

Then

$$E(a\,g_1(X) + b\,g_2(X)) = \int_{-\infty}^{\infty} (a\,g_1(x) + b\,g_2(x)) f(x)\,dx$$

$$= a \int_{-\infty}^{\infty} g_1(x) f(x)\,dx + b \int_{-\infty}^{\infty} g_2(x) f(x)\,dx = a\,E(g_1(X)) + b\,E(g_2(X))$$

Let X be a discrete random variable.

$$E\left(a g_1(X) + b g_2(X)\right)$$

$$= \sum_{i=-\infty}^{\infty} \left(a g_1(x_i) + b g_2(x_i)\right) P(X = x_i)$$

$$= \sum_{i=-\infty}^{\infty} a g_1(x_i) P(X=x_i) + \sum_{i=-\infty}^{\infty} b g_2(x_i) P(X=x_i)$$

$$= a \sum_{i=-\infty}^{\infty} g_1(x_i) P(X=x_i) + b \sum_{i=-\infty}^{\infty} g_2(x_i) P(X=x_i)$$

$$= a E(g_1(X)) + b E(g_2(X))$$

<u>Proof of P4:</u> Let X be a continuous random variable.

$$|E(g(X))| = \left| \int_{-\infty}^{\infty} g(x) f(x)\, dx \right|$$

$$\leq \int_{-\infty}^{\infty} |g(x) f(x)|\, dx$$

$$\leq \int_{-\infty}^{\infty} |g(x)|\,|f(x)|\, dx$$

$$\Rightarrow |E(g(X))| \le \int_{-\infty}^{\infty} |g(x)| f(x)\,dx \quad \left[\begin{array}{l}\text{As } f(x) \ge 0 \\ \Rightarrow |f(x)| = f(x)\end{array}\right]$$

$$\Rightarrow |E(g(X))| \le E(|g(X)|)$$

Let X be a discrete random variable.

$$|E(g(X))| = \left| \sum_{i=-\infty}^{\infty} g(x_i) P(X = x_i) \right|$$

$$\le \sum_{i=-\infty}^{\infty} |g(x_i) P(X = x_i)|$$

$$\le \sum_{i=-\infty}^{\infty} |g(x_i)|\,|P(X = x_i)|$$

$$\le \sum_{i=-\infty}^{\infty} |g(x_i)| P(X = x_i)$$

$$\le E(|g(X)|)$$

$$\Rightarrow |E(g(X))| \le E(|g(X)|)$$

<u>Proof of P5:</u> Let X be a continuous random variable.

$$E(g(X)) = \int_{-\infty}^{\infty} f(x) g(x)\,dx \ge 0 \quad \text{as } f(x) \ge 0 \;\forall x$$

$$\Rightarrow E(g(X)) \ge 0$$

Let X be a discrete random variable.

$$E(g(X)) = \sum_{i=-\infty}^{\infty} g(x_i) P(X = x_i) \geq 0$$

[as $g(x_i) \geq 0 \ \forall i$ and $P(X = x_i) \geq 0$ for all i]

Proof of P6: Let X be a continuous random variable.

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} g(x) f(x)\, dx = 0 \quad \text{———} \quad (1)$$

$\Rightarrow g(x) f(x) = 0 \ \forall x$ because if $g(x) f(x) \neq 0$ for all x then there exists a point x_0 such that $g(x) f(x) \neq 0$ at $x = x_0$

$\Rightarrow g(x_0) f(x_0) \neq 0$

$\Rightarrow g(x_0) f(x_0) > 0$ [as $g(x) \geq 0, f(x) \geq 0 \ \forall x$
$\Rightarrow g(x) f(x) \geq 0 \ \forall x$]

$\Rightarrow \exists$ a neighbourhood $N(x_0, \delta)$ such that

$g(x) f(x) > 0 \ \forall x \in N(x_0, \delta)$ [as $g(x)$ and $f(x)$ are continuous $\forall x$
$\Rightarrow g(x) f(x)$ is continuous $\forall x$]

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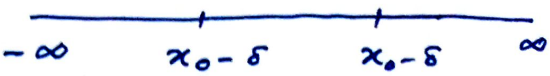
$$\therefore \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$



$$= \underbrace{\int_{-\infty}^{x_0-\delta} g(x) f(x)\, dx}_{\geq 0} + \underbrace{\int_{x_0-\delta}^{x_0+\delta} g(x) f(x)\, dx}_{> 0} + \underbrace{\int_{x_0+\delta}^{\infty} g(x) f(x)\, dx}_{\geq 0}$$

$$\Rightarrow \int_{-\infty}^{\infty} g(x)\, f(x)\, dx > 0$$

which contradicts ①. Therefore, our assumption is wrong and consequently

$$g(x)\, f(x) = 0 \quad \forall x$$

$$\Downarrow$$

$$g(x) = 0 \quad \forall x \quad [\text{as } f(x) \not\equiv 0 \text{ in } (-\infty, \infty)]$$

$$\Downarrow$$

$$g(X) = 0$$

$$\Downarrow$$

$g(X)$ has a one point distribution at $x = 0$

By similar argument, we can prove this property for the case of discrete random variable.

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Mean: The mean of the random variable X is generally denoted by $m(x)$ or m_x or m and defined by

$$m = E(X)$$

Physically the mean represents the centre of mass of the probability mass distribution. This indicates the average value of the random variable X. This gives a measure of the central tendency of the distribution which is also known as a measure of location. The mean is not the only measure of location. There are also other measures of location.

Moments: Let X be a random variable and k be a non-negative integer. Then the moment of order k (or the kth order moment) of X about the point a is denoted by M_{ak} and defined by

$$M_{ak} = E\left((X-a)^k\right)$$

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$E(|X-a|^k)$ is known as the k-th order absolute moment of X about the fixed point a.

Raw moment: k-th order raw moment of the random variable X is generally denoted by α_k or $\alpha_k(X)$ or α_{kX} and defined by

α_k = k th order raw moment of X

= k-th order moment of X about the origin

$= E((X-0)^k) = E(X^k)$

$\Rightarrow \boxed{\alpha_0 = 1}$ & $\boxed{\alpha_1 = E(X) = m}$

Central Moment: k-th order central moment of the random variable X is generally denoted by $\mu_k(X)$ or μ_{kX} or μ_k and defined by

μ_k = k th order central moment of X

= k-th order central moment of X about its mean

$= E(X - E(X))^k = E(X-m)^k$, $m = E(X)$

$\Rightarrow \boxed{\mu_0 = 1}$ $\quad \mu_1 = E(X-m) = E(X) - E(m) = m - m = 0$

$\Rightarrow \boxed{\mu_1 = 0}$

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Theorem : $\mu_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \alpha_{k-i}\, m^i$

Proof : $\mu_k = E(x-m)^k$

$$= E\left\{ \sum_{i=0}^{k} \binom{k}{i} x^{k-i} (-m)^i \right\}$$

$$= E\left\{ \sum_{i=0}^{k} (-1)^i \binom{k}{i} m^i x^{k-i} \right\}$$

$$= \sum_{i=0}^{k} E\left\{ (-1)^i \binom{k}{i} m^i x^{k-i} \right\}$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} m^i E(x^{k-i})$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} m^i \alpha_{k-i}$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \alpha_{k-i}\, m^i$$

$\Rightarrow \mu_2 = \alpha_2 - m^2$

$\mu_3 = \alpha_3 - 3\alpha_2 m + 2m^3$

$\mu_4 = \alpha_4 - 4\alpha_3 m + 6\alpha_2 m^2 - 3m^4$

<u>Theorem</u>: $\mu_k(ax+b) = a^k \mu_k(x)$

Proof: $\mu_k(ax+b) = E\left(ax+b-E(ax+b)\right)^k$

$$= E\left(ax+b-E(ax)-E(b)\right)^k$$

$$= E\left(ax+b-aE(x)-b\right)^k$$

$$= E\left(ax-aE(x)\right)^k$$

$$= E\left(a(x-E(x))\right)^k$$

$$= E\left(a^k(x-E(x))^k\right)$$

$$= a^k E\left(x-E(x)\right)^k$$

$$= a^k \mu_k(x)$$

<u>Variance</u>: The second order central moment $\mu_2(x)$ or μ_{2x} or μ_2 of the random variable x is known as the variance of x. This is also denoted by $Var(x)$, i.e.,

$$Var(x) = E\left(x-E(x)\right)^2$$

Physically the variance of a distribution represents the moment of inertia of the probability mass distribution about a line through the mean perpendicular to the line of the distribution.

The variance is also known as a measure of dispersion, i.e., it describes the deviation of the probability masses about the mean. So, it can be regarded as an inverse measure of concentration of the probability masses about the mean

P1 $Var(X) = 0 \Rightarrow$ the total probability mass is concentrated at the mean, i.e., X has a one point distribution at $x = m$.

Proof: As $(x-m)^2 \geq 0 \; \forall x$, $Var(X) = 0 \Rightarrow x = m \Rightarrow X = m \Rightarrow P(X = m) = 1 \Rightarrow$ the total probability mass is concentrated at $x = m$.

P2 The second moment about any point is minimum when taken about the mean.

Proof: $E(X-a)^2 = E(X - E(X) + E(X) - a)^2$

$$= E\left\{(X - E(X))^2 + 2(X - E(X))(E(X) - a) + (E(X) - a)^2\right\}$$

$$= E(X - E(X))^2 + 2(E(X) - a)\, E(X - E(X)) + E(E(X) - a)^2$$

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$$\Rightarrow E(x-a)^2 = E(x-E(x))^2 + 2(E(x)-a)\,E(x-E(x)) + E(E(x)-a)^2$$

$$= \mu_2 + 2(E(x)-a)\{E(x) - E(E(x))\} + (E(x)-a)^2$$

$$= \mu_2 + 2(E(x)-a)(E(x) - E(x)) + (E(x)-a)^2$$

$$= \mu_2 + 2(E(x)-a)\cdot 0 + (E(x)-a)^2$$

$$= \mu_2 + (E(x)-a)^2$$

$$\Rightarrow E(x-a)^2 - \mu_2 = (E(x)-a)^2 \geq 0$$

$$\Rightarrow E(x-a)^2 \geq \mu_2 = E(x-E(x))^2$$

$\Rightarrow$ The second moment about any point is minimum when taken about the mean

<u>P3</u>: $Var(x) = \sigma^2 = \sigma^2(x) = \alpha_2 - m^2 = E(x^2) - (E(x))^2$

$Var(x) = E(x-m)^2, \quad m = E(x)$

$$= E(x^2 - 2mx + m^2)$$

$$= E(x^2) - 2m\,E(x) + E(m^2)$$

$$= \alpha_2 - 2m\cdot m + m^2 = \alpha_2 - m^2 = E(x^2) - (E(x))^2$$

P4 $Var(x) = E(x(x-1)) - E(x)(E(x)-1)$

$$= E(x(x-1)) - m(m-1), \quad m = E(x)$$

Proof: $Var(x) = E(x-m)^2$

$$= E(x^2 - 2mx + m^2)$$

$$= E(x^2 - x + x - 2mx + m^2)$$

$$= E(x^2 - x) + E(x) - 2mE(x) + E(m^2)$$

$$= E(x(x-1)) + m - 2mm + m^2$$

$$= E(x(x-1)) + m - m^2$$

$$= E(x(x-1)) - m(m-1)$$

$$= E(x(x-1)) - E(x)(E(x)-1)$$

P5 $Var(ax+b) = a^2 Var(x)$

Proof: $Var(ax+b) = \mu_2(ax+b) = a^2 \mu_2(x)$

$$= a^2 E(x - E(x))^2 = a^2 Var(x)$$

Putting $a=0$, we get

$Var(b) = 0$

$\Rightarrow$ Variance of a constant $= 0$

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Standard deviation: The standard deviation of the random variable X is generally denoted by $\sigma(X)$ or σ_X or σ and defined by

$$\sigma = +\sqrt{Var(X)}$$

Standardised or Normalised random variable:

For any random variable X, the random variable

$$X^* = \frac{X-m}{\sigma}$$ is called the Normalised random variable,

where $m = E(X)$ and $\sigma = +\sqrt{Var(X)}$. Therefore, one can define normalised random variable iff $var(X)$ exists finitely. It is also important to note that the normalised random variable is a dimensionless quantity.

Again, it is simple to check that

① $E(X^*) = 0$ & ② $\sigma(X^*) = 1 \Leftrightarrow Var(X^*) = 1$

③ the normalised random variable corresponding to $aX+b = \frac{a}{|a|} \times$ (the normalised random variable corresponding to X)

Third Central Moment: This measures the symmetry of the distribution.

Symmetrical distribution:

① A discrete probability distribution is said to be symmetrical about a point c if

$p(c-h) = p(c+h) \quad \forall h$

where

$$p(x) = \begin{cases} P(X = x_i) & \text{if } x = x_i \\ 0 & \text{elsewhere} \end{cases}$$

② A continuous probability distribution is said to be symmetrical about a point c if

$f(c+h) = f(c-h) \quad \forall h$

where $f(x)$ is the pdf of the random variable.

Theorem: If a continuous probability distribution is symmetrical about a point c then

$c = E(X) = m$.

Proof: As the distribution is symmetrical about c,

$f(c+u) = f(c-u) \quad \forall u$ ——————— ①

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$$m = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \text{Put } x = c+u$$

$$\Rightarrow m = \int_{-\infty}^{\infty} (c+u) f(c+u)\, du \qquad \text{———} (2)$$

Again $m = \int_{-\infty}^{\infty} x f(x)\, dx$ Put $x = c-u$

$$\Rightarrow m = \int_{\infty}^{-\infty} (c-u) f(c-u)(-du)$$

$$\Rightarrow m = \int_{-\infty}^{\infty} (c-u) f(c-u)\, du$$

$$\Rightarrow m = \int_{-\infty}^{\infty} (c-u) f(c+u)\, du \qquad [\text{using } (1)] \text{—} (3)$$

(2) + (3) $\Rightarrow$

$$2m = \int_{-\infty}^{\infty} (c+u+c-u) f(c+u)\, du$$

$$\Rightarrow 2m = 2c \int_{-\infty}^{\infty} f(c+u)\, du$$

$$\Rightarrow 2m = 2c \int_{-\infty}^{\infty} f(x)\, dx \qquad \text{Putting } x = c+u$$

$$\Rightarrow m = c \cdot 1 = c$$

$$\Rightarrow c = m = E(x)$$

<u>Theorem</u>: If a continuous distribution is symmetrical, then every odd order central moment is zero, i.e., $\mu_n = 0$ for odd n.

<u>Proof</u>: Let the distribution is symmetrical about the point c, then we have

$c = m = E(X)$,

where X be the random variable having pdf $f(x)$

Therefore, the distribution function is symmetrical about $m \Rightarrow f(m-u) = f(m+u) \quad \forall u$ ——— (1)

$$\mu_n = E(x-m)^n = \int_{-\infty}^{\infty} (x-m)^n f(x)\,dx$$

Put $x - m = u$

$$= \int_{-\infty}^{\infty} u^n f(u+m)\,du$$

$$= \int_{-\infty}^{\infty} u^n f(m-u)\,du$$

Put $m - u = x$

$$= \int_{\infty}^{-\infty} (m-x)^n f(x)(-dx)$$

$$= \int_{-\infty}^{\infty} (m-x)^n f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \{(-1)(x-m)\}^n f(x)\,dx = (-1)^n \mu_n$$

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$\Rightarrow \mu_n = (-1)^n \mu_n \quad \forall\, n = 1, 2, \ldots$

For odd n,

$\mu_n = -\mu_n \Rightarrow 2\mu_n = 0 \Rightarrow \mu_n = 0$

$\Rightarrow$ odd order central moment $= 0$ for symmetrical distribution

Again $\mu_1 = E(X-m) = E(X) - E(m) = m - m = 0$ for any distribution. But $\mu_n = 0$ for odd n only when the distribution is symmetrical. This property is the basic characteristic of the symmetrical distribution and consequently except μ_1 one can take the first non vanishing odd order central moment as a measure of asymmetry (or symmetry) of the distribution.

If μ_k is the first non vanishing odd order central moment then we take μ_k as a measure of asymmetry but to make it dimensionless we take it $\dfrac{\mu_k}{\sigma^k}$ as a measure of asymmetry and this is known as the coefficient of skewness of the distribution and is denoted by γ_1, i.e.,

$\gamma_1 = \dfrac{\mu_k}{\sigma^k}$, where μ_k is the first non vanishing odd order central moment.

But irrespective of the distribution, we take the coefficient of skewness of the distribution as

$$\gamma_1 = \frac{\mu_3}{\sigma^3}$$

* A distribution is said to be skewed to the right or positively skewed if $\gamma_1 > 0$
* A distribution is said to be skewed to the left or negatively skewed if $\gamma_1 < 0$
* $\beta_1 = \gamma_1^2$ is used to measure to skewness of the distribution regardless of the fact that the distribution is skewed to the right or to the left.
* the coefficient of skewness is the third central moment of the normalised or standardised random variable because

for $X^* = \dfrac{X-m}{\sigma}$, $E(X^*) = 0$, $Var(X^*) = 1$

$$\begin{aligned}
\mu_3(X^*) &= E(X^* - E(X^*))^3 \\
&= E(X^*)^3 = E\left(\frac{X-m}{\sigma}\right)^3 \\
&= \frac{E(X-m)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3}.
\end{aligned}$$

Fourth central moment: This describes the peakedness of the density curve of a continuous distribution or the graph of the probability mass function for a discrete random variable. The corresponding dimensionless quantity is the fourth central moment of the normalised or standardised random variable which describes the degree of peakedness of the density curve of a continuous distribution or the graph of the probability mass function for a discrete distribution. This property is known as Kurtosis and the degree of Kurtosis or the degree of peakedness is known as the coefficient of kurtosis. Therefore the coefficient of kurtosis is denoted by β_2 and defined by

β_2 = coefficient of Kurtosis which measures the degree of peakedness

= Fourth central moment of the normalised random Variable

$$= E\left(\frac{X-m}{\sigma} - E\left(\frac{X-m}{\sigma}\right)\right)^4 = E\left(\frac{X-m}{\sigma}\right)^4 \quad \left[\text{As } E\left(\frac{X-m}{\sigma}\right)=0\right]$$

$$= \frac{1}{\sigma^4} E(X-m)^4 = \frac{\mu_4}{\sigma^4}$$

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Measure of kurtosis with respect to the normal distribution is known as the coefficient of excess of kurtosis and this is defined as

γ_2 = coefficient of excess of kurtosis

$= \beta_2 - \beta_2(N(m,\sigma))$

where $\beta_2(N(m,\sigma))$ = coefficient of kurtosis for the Normal (m,σ) distribution

For the Normal (m,σ) distribution it can be proved $\beta_2 = 3$

$\therefore\ \gamma_2$ = Coefficient of excess of kurtosis for any distribution

$$= \beta_2 - 3 = \frac{\mu_4}{\sigma^4} - 3$$

Now we have the following two cases:

(I) If $\beta_2 > 3$ then the density curve has more sharp peak than the normal density curve

(II) If $\beta_2 < 3$ then the peak of the density curve is more flat than the normal density curve.

The distribution function is said to be mesotokurtic, leptokurtic and platykurtic according to whether $\beta_2 = 3$, $\beta_2 > 3$ and $\beta_2 < 3$ respectively.

Moment Generating Function: The moment generating function of a random variable X is denoted by $\psi_X(t)$ or $\psi(t)$ and defined by

$$\psi(t) = E(e^{tX})$$

$$\Rightarrow \psi(t) = E\left(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right)$$

$$= E\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k\right)$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k)$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \alpha_k$$

$$= \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k$$

$$\Rightarrow \frac{\alpha_k}{k!} = \text{coefficient of } t^k \text{ in } \psi(t)$$

$$\Rightarrow \alpha_k = k! \times (\text{coefficient of } t^k \text{ in } \psi(t))$$

Therefore, If $\psi(t)$ exists finitely then we can find all moments about 0.

Characteristic Function: The characteristic function of a random variable X is generally denoted by $\chi_X(t)$ or $\chi(t)$ and defined by

$$\chi(t) = E(e^{itX})$$

Theorem: $\chi(t)$ exists for all values of t and for all distribution.

Proof: Let X be a continuous random variable with pdf $f(x)$

$$\therefore \chi(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

$$\Rightarrow |\chi(t)| = \left| \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \right|$$

$$\leq \int_{-\infty}^{\infty} |e^{itx} f(x)|\, dx$$

$$\leq \int_{-\infty}^{\infty} |e^{itx}|\, |f(x)|\, dx$$

$$\leq \int_{-\infty}^{\infty} f(x)\, dx \qquad [\text{as } |e^{itx}| = 1 \text{ and } |f(x)| = f(x)]$$

$$\leq 1$$

$\Rightarrow |\chi(t)| \leq 1 \Rightarrow \chi(t)$ exists for all values of t and for all distributions

Let X be a discrete random variable.

$$\chi(t) = E(e^{itX}) = \sum_{j=-\infty}^{\infty} e^{itx_j} P(X = x_j)$$

$$\Rightarrow |\chi(t)| = \left| \sum_{j=-\infty}^{\infty} e^{itx_j} P(X = x_j) \right|$$

$$\leq \sum_{j=-\infty}^{\infty} \left| e^{itx_j} P(X = x_j) \right|$$

$$\leq \sum_{j=-\infty}^{\infty} \left| e^{itx_j} \right| \left| P(X = x_j) \right|$$

$$\leq \sum_{j=-\infty}^{\infty} 1 \cdot P(X = x_j) \quad \left[\because \left| e^{itx_j} \right| = 1 \ \& \ P(X = x_j) \geq 0 \right]$$

$$\leq \sum_{j=-\infty}^{\infty} P(X = x_j) = 1$$

$$\Rightarrow |\chi(t)| \leq 1$$

$\Rightarrow \chi(t)$ exists for all t and for all distributions

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Now $\chi(t) = E(e^{itx})$

$$= E\left(\sum_{k=0}^{\infty} \frac{(itx)^k}{k!}\right)$$

$$= E\left(\sum_{k=0}^{\infty} \frac{(it)^k}{k!} x^k\right)$$

$$= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E(x^k)$$

$$= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \alpha_k$$

$$= \sum_{k=0}^{\infty} \frac{i^k \alpha_k}{k!} t^k$$

$\Rightarrow \frac{i^k \alpha_k}{k!} =$ coefficient of t^k in $\chi(t)$

$\Rightarrow \alpha_k = \frac{k!}{i^k}$ coefficient of t^k in $\chi(t)$

$= \frac{i^k k!}{i^{2k}}$ coefficient of t^k in $\chi(t)$

$= \frac{i^k k!}{(-1)^k}$ coefficient of t^k in $\chi(t)$

$= (-1)^k i^k k!$ coefficient of t^k in $\chi(t)$

① Moment generating function (MGF) of $B(n,p)$

$X = B(n,p) \Rightarrow P(X=j) = \binom{n}{j} p^j q^{n-j}, \quad q = 1-p, \ j = 0(1)n$

$$\psi_X(t) = E(e^{tX}) = \sum_{j=0}^{n} e^{tj} P(X=j)$$

$$= \sum_{j=0}^{n} e^{tj} \binom{n}{j} p^j q^{n-j}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (pe^t)^j q^{n-j}$$

$$= (pe^t + q)^n$$

$\therefore$ MGF of $B(n,p) = (pe^t + q)^n$

② MGF of $P(\lambda)$

$X = P(\lambda) \Rightarrow P(X=j) = \dfrac{e^{-\lambda}\lambda^j}{j!}, \ j = 0, 1, 2, \ldots$

$$\psi_X(t) = E(e^{tX}) = \sum_{j=0}^{\infty} e^{tj} P(X=j)$$

$$= \sum_{j=0}^{\infty} e^{tj} \frac{e^{-\lambda}\lambda^j}{j!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda e^t)^j}{j!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

③ MGF of Normal (m, σ) : $X = N(m, \sigma)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 + tx\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}\left\{(x-m)^2 - 2\sigma^2 t x\right\}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}\left\{x^2 - 2mx + m^2 - 2\sigma^2 t x\right\}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}\left\{x^2 - 2(m+\sigma^2 t)x + m^2\right\}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}\left\{(x-(m+\sigma^2 t))^2 - (m+\sigma^2 t)^2 + m^2\right\}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}\left\{(x-m_1)^2 - 2m\sigma^2 t - \sigma^4 t^2\right\}\right] dx$$

$$[m_1 = m + \sigma^2 t]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-m_1)^2}{2\sigma^2} + mt + \frac{\sigma^2 t^2}{2}\right] dx$$

$$= \frac{\exp\left[mt + \frac{1}{2}\sigma^2 t^2\right]}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2\right] dx$$

$$\psi(t) = \frac{\exp[mt + \frac{1}{2}\sigma^2 t^2]}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x - m_1}{\sigma}\right)^2\right] dx$$

$$= \exp[mt + \tfrac{1}{2}\sigma^2 t^2] \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x - m_1}{\sigma}\right)^2\right] dx$$

$$= \exp[mt + \tfrac{1}{2}\sigma^2 t^2] \cdot 1$$

$$\Rightarrow \psi(t) = \exp[mt + \tfrac{1}{2}\sigma^2 t^2]$$

because we know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] dx = 1$$

for <u>any value of m</u> with $\sigma > 0$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2\right] dx = 1$$

<u>Problem-1:</u> Find the characteristic function of the $B(n,p)$, $P(\mu)$ and $N(m,\sigma)$ variates.

<u>Answer:</u>

For $B(n,p)$, $\chi(t) = (pe^{it} + q)^n$

For $P(\mu)$, $\chi(t) = e^{\mu(e^{it} - 1)}$

For $N(m,\sigma)$, $\chi(t) = e^{imt - \frac{1}{2}\sigma^2 t^2}$

Problem-2: Find the mean and variance of the Binomial (n, p) distribution, Poisson (μ) distribution, Negative Binomial (r, p) distribution, Geometric (p) distribution, Normal (m, σ) distribution, Uniform distribution, Gamma distribution, Beta distribution of first kind and Beta distribution of second kind.

Answer: For the discrete distribution we shall use the following formula to derive the formula for variance.

$$Var(X) = E(X(X-1)) - m(m-1), \quad m = E(X)$$

① Binomial (n, p) distribution:

$X = B(n, p)$

$$P(X=i) = \binom{n}{i} p^i q^{n-i}, \quad i = 0(1)n, \quad q = 1-p, \; 0<p<1$$

$$m = E(X) = \sum_{i=0}^{n} i\, P(X=i)$$

$$= \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i}$$

$$= \sum_{i=1}^{n} i \frac{n!}{i!(n-i)!} p^i q^{n-i}$$

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$$\Rightarrow m = \sum_{i=1}^{n} \frac{n!}{(i-1)!\,(n-i)!} p^i q^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n(n-1)!}{(i-1)!\,(n-i)!} p^i q^{n-i}$$

Put $n-1 = N$

$i-1 = j$

$N-j = n-i$

$$= \sum_{j=0}^{N} \frac{(N+1)\,N!}{j!\,(N-j)!} p^{j+1} q^{N-j}$$

$$= (N+1)p \sum_{j=0}^{N} \binom{N}{j} p^j q^{N-j}$$

$$= (N+1)p\,(p+q)^N = (N+1)p \cdot 1^N = (N+1)p = np$$

$$\Rightarrow m = E(X) = np$$

$$E(X(X-1)) = \sum_{i=0}^{n} i(i-1)\,P(X=i)$$

$$= \sum_{i=2}^{n} i(i-1)\,P(X=i)$$

$$= \sum_{i=2}^{n} i(i-1) \frac{n!}{i!\,(n-i)!} p^i q^{n-i}$$

$$= \sum_{i=2}^{n} \frac{n!}{(i-2)!\,(n-i)!} p^i q^{n-i}$$

Put

$i-2 = j$

$n-2 = M$

$n-i = M-j$

$$= \sum_{j=0}^{M} \frac{(M+2)!}{j!\,(M-j)!} p^{j+2} q^{M-j}$$

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$$\Rightarrow E(X(X-1)) = \sum_{j=0}^{M} \frac{(M+2)!}{j!\,(M-j)!} p^{j+2} q^{M-j}$$

$$= \sum_{j=0}^{M} \frac{(M+2)(M+1)\,M!}{j!\,(M-j)!} p^{j} p^{2} q^{M-j}$$

$$= (M+2)(M+1)p^2 \sum_{j=0}^{M} \binom{M}{j} p^j q^{M-j}$$

$$= (M+2)(M+1)p^2 (p+q)^M$$

$$= (M+2)(M+1)p^2 = n(n-1)p^2$$

$$\therefore \ Var(X) = E(X(X-1)) - m(m-1)$$

$$= n(n-1)p^2 - np(np-1)$$

$$= n^2p^2 - np^2 - n^2p^2 + np$$

$$= np - np^2 = np(1-p) = npq$$

$\therefore$ $E(X) = np$ and $Var(X) = npq \Leftrightarrow \sigma_X = \sqrt{npq}$

2 Poisson (μ) distribution

$$P(X=i) = \frac{e^{-\mu}\mu^i}{i!}, \quad i = 0, 1, 2, \ldots$$

$$m = E(X) = \sum_{i=0}^{\infty} i\, P(X=i) = \sum_{i=1}^{\infty} i\, P(X=i)$$

$$= \sum_{i=1}^{\infty} i \frac{e^{-\mu}\mu^i}{i!} = \sum_{i=1}^{\infty} \frac{e^{-\mu}\mu^i}{(i-1)!}$$

$$\Rightarrow m = \sum_{i=1}^{\infty} \frac{e^{-\mu}\mu^{i}}{(i-1)!} \qquad \text{Put } i-1=j$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\mu}\mu^{j+1}}{j!}$$

$$= e^{-\mu}.\mu \sum_{j=0}^{\infty} \frac{\mu^{j}}{j!}$$

$$= \mu e^{-\mu}.e^{\mu} = \mu$$

$$\Rightarrow m = E(X) = \mu$$

$$E(X(X-1)) = \sum_{i=0}^{\infty} i(i-1)\, P(X=i)$$

$$= \sum_{i=2}^{\infty} i(i-1)\, P(X=i)$$

$$= \sum_{i=2}^{\infty} i(i-1) \frac{e^{-\mu}\mu^{i}}{i!}$$

$$= \sum_{i=2}^{\infty} \frac{e^{-\mu}\mu^{i}}{(i-2)!} \qquad \text{Put } i-2=j$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\mu}\mu^{j+2}}{j!} = e^{-\mu}\mu^{2} \sum_{j=0}^{\infty} \frac{\mu^{j}}{j!}$$

$$= e^{-\mu}\mu^{2}e^{\mu} = \mu^{2}$$

$$\Rightarrow E(X(X-1)) = \mu^{2}$$

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$$Var(X) = E(X(X-1)) - m(m-1)$$

$$= \mu^2 - \mu(\mu - 1) = \mu^2 - \mu^2 + \mu = \mu$$

$\therefore$ $m = E(X) = \mu$ & $Var(X) = \mu \Leftrightarrow \sigma_X = \sqrt{\mu}$

③ Negative Binomial (r, p) distribution:

$X = NB(r, p)$

$$P(X=i) = \binom{i+r-1}{i}(1-p)^i p^r, \quad i = 0, 1, 2, \ldots, \quad 0<p<1, r \geq 1$$

$$= \frac{(i+r-1)!}{i!\,(r-1)!}(1-p)^i p^r, \quad i = 0, 1, 2, \ldots$$

Again, $\displaystyle\sum_{i=0}^{\infty} P(X=i) = 1$

$$\Rightarrow \sum_{i=0}^{\infty} \frac{(i+r-1)!}{i!\,(r-1)!}(1-p)^i p^r = 1$$

$$\Rightarrow \frac{p^r}{(r-1)!}\sum_{i=0}^{\infty} \frac{(i+r-1)!}{i!}(1-p)^i = 1$$

$$\Rightarrow \sum_{i=0}^{\infty} \frac{(i+r-1)!}{i!}(1-p)^i = \frac{(r-1)!}{p^r} = (r-1)!\,p^{-r}$$

$$\Rightarrow \sum_{i=0}^{\infty} \frac{(i+r-1)!}{i!}(1-p)^i = (r-1)!\,p^{-r} \quad \text{——— ①}$$

We shall use this formula.

$$E(X) = \sum_{i=0}^{\infty} i\, P(X=i)$$

$$= \sum_{i=1}^{\infty} i\, P(X=i)$$

$$= \sum_{i=1}^{\infty} i \frac{(i+r-1)!}{i!\,(r-1)!}(1-p)^i p^r$$

$$= \frac{p^r}{(r-1)!} \sum_{i=1}^{\infty} \frac{(i+r-1)!}{(i-1)!}(1-p)^i \qquad \boxed{i-1=j}$$

$$= \frac{p^r}{(r-1)!} \sum_{j=0}^{\infty} \frac{(j+1+r-1)!}{j!}(1-p)^{j+1}$$

$$= \frac{p^r}{(r-1)!} \sum_{j=0}^{\infty} \frac{(j+r)!}{j!}(1-p)^{j}(1-p)$$

$$= \frac{p^r}{(r-1)!}(1-p) \sum_{j=0}^{\infty} \frac{(j+r_0-1)!}{j!}(1-p)^{j} \qquad \boxed{r = r_0 - 1}$$

$$= \frac{p^r}{(r-1)!}(1-p)\cdot(r_0-1)!\, p^{-r_0} \qquad [\text{using } 1]$$

$$= \frac{p^r}{(r-1)!}(1-p)\cdot r!\, p^{-(r+1)}$$

$$= \frac{p^{-1}(1-p)\, r\,(r-1)!}{(r-1)!} = \frac{rq}{p} \qquad \boxed{q = 1-p}$$

$$\Rightarrow \boxed{m = E(X) = \frac{rq}{p}}$$

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$$E(X(X-1)) = \sum_{i=0}^{\infty} i(i-1)\, P(X=i)$$

$$= \sum_{i=2}^{\infty} i(i-1)\, P(X=i)$$

$$= \sum_{i=2}^{\infty} i(i-1) \frac{(i+r-1)!}{i!\,(r-1)!} (1-p)^i p^r$$

$$= \frac{p^r}{(r-1)!} \sum_{i=2}^{\infty} \frac{(i+r-1)!}{(i-2)!} (1-p)^i$$

Put $\boxed{j = i-2}$

$$= \frac{p^r}{(r-1)!} \sum_{j=0}^{\infty} \frac{(j+2+r-1)!}{j!} (1-p)^{j+2}$$

$$= \frac{p^r}{(r-1)!} (1-p)^2 \sum_{j=0}^{\infty} \frac{(j+r_1-1)!}{j!} (1-p)^j, \quad \boxed{r_1 = 2+r}$$

$$= \frac{p^r}{(r-1)!} (1-p)^2 (r_1-1)!\, p^{-r_1}$$

$$= \frac{p^r}{(r-1)!} (1-p)^2 (2+r-1)!\, p^{-2-r}$$

$$= \frac{p^{-2}(1-p)^2 (r+1)!}{(r-1)!} = \frac{(r+1)r(1-p)^2}{p^2} = \frac{(r+1)rq^2}{p^2}$$

$$Var(X) = E(X(X-1)) - m(m-1)$$

$$= \frac{(r+1)rq^2}{p^2} - \frac{rq}{p}\left(\frac{rq}{p} - 1\right)$$

$$\Rightarrow Var(X) = \frac{r(r+1)q^2}{p^2} - \frac{rq}{p}\left(\frac{rq}{p} - 1\right)$$

$$= \frac{r(r+1)q^2}{p^2} - \frac{r^2q^2}{p^2} + \frac{rq}{p}$$

$$= \frac{rq^2}{p^2}(r+1-r) + \frac{rq}{p}$$

$$= \frac{rq^2}{p^2} + \frac{rq}{p} = \frac{rq}{p}\left(\frac{q}{p} + 1\right)$$

$$= \frac{rq}{p}\left(\frac{q+p}{p}\right) = \frac{rq}{p} \cdot \frac{1}{p} = \frac{rq}{p^2}$$

$$= \frac{r(1-p)}{p^2}$$

$$\therefore\ m = E(X) = \frac{rq}{p} = \frac{r(1-p)}{p} \ \&$$

$$Var(X) = \frac{rq}{p^2} = \frac{r(1-p)}{p^2} \Leftrightarrow \sigma_X = \sqrt{\frac{r(1-p)}{p^2}}$$

4 Geometric (p) distribution:

$X = G(p)$

$P(X=i) = pq^i, \quad i = 0, 1, 2, \ldots, \quad 0 < p < 1, \quad q = 1-p$

$$\sum_{i=0}^{\infty} P(X=i) = 1 \Rightarrow \sum_{i=0}^{\infty} pq^i = 1$$

$$\Rightarrow p\sum_{i=0}^{\infty} q^i = 1 \Rightarrow \sum_{i=0}^{\infty} q^i = p^{-1} \quad \text{——— (1)}$$

We shall use this formula.

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$$m = E(x) = \sum_{i=0}^{\infty} i\, P(x=i)$$

$$= \sum_{i=1}^{\infty} i\, P(x=i)$$

$$= \sum_{i=1}^{\infty} i\, p q^{i}$$

$$= p \sum_{i=1}^{\infty} i\, q^{i} \quad \text{(2)} \qquad \text{put } j = i-1$$

$$= p \sum_{j=0}^{\infty} (j+1)\, q^{j+1}$$

$$= p \left[\sum_{j=0}^{\infty} j\, q^{j+1} + \sum_{j=0}^{\infty} q^{j+1} \right]$$

$$= p \left[\sum_{i=0}^{\infty} i\, q^{i+1} + \sum_{i=0}^{\infty} q^{i+1} \right]$$

$$= pq \sum_{i=0}^{\infty} i\, q^{i} + pq \sum_{i=0}^{\infty} q^{i}$$

$$= pq \sum_{i=1}^{\infty} i\, q^{i} + pq \cdot p^{-1} \quad [\text{using } (1)]$$

$$= q \left(p \sum_{i=1}^{\infty} i\, q^{i} \right) + q$$

$$= q m + q \quad [\text{using } (2)]$$

$\Rightarrow m = qm + q$

$\Rightarrow (1-q)m = q$

$\Rightarrow m = \frac{q}{1-q} = \frac{q}{p}$

$\Rightarrow \boxed{m = E(X) = \frac{q}{p}}$

$$E(X(X-1)) = \sum_{i=0}^{\infty} i(i-1)P(X=i) \quad \text{———} \quad (3)$$

$$= \sum_{i=2}^{\infty} i(i-1)pq^i \quad \text{———} \quad (4)$$

$$= p\sum_{i=2}^{\infty} i(i-1)q^i \qquad \boxed{\text{Put } j = i-2}$$

$$= p\sum_{j=0}^{\infty} (j+2)(j+1)q^{j+2}$$

$$= p\sum_{j=0}^{\infty} (j^2+3j+2)q^j q^2$$

$$= pq^2\sum_{j=0}^{\infty} \{j(j-1+1)+3j+2\}q^j$$

$$= pq^2\sum_{j=0}^{\infty} \{j(j-1)+4j+2\}q^j$$

$$= pq^2\sum_{i=0}^{\infty} \{i(i-1)+4i+2\}q^i \qquad \boxed{i \leftrightarrow j}$$

$$\Rightarrow E(x(x-1)) = pq^2\left[\sum_{i=0}^{\infty} i(i-1)q^i + 4\sum_{i=0}^{\infty} iq^i + 2\sum_{i=0}^{\infty} q^i\right]$$

$$= pq^2\left[\sum_{i=2}^{\infty} i(i-1)q^i + 4\sum_{i=1}^{\infty} iq^i + 2\sum_{i=0}^{\infty} q^i\right]$$

$$= q^2\left[\sum_{i=2}^{\infty} i(i-1)pq^i + 4\sum_{i=1}^{\infty} ipq^i + 2\sum_{i=0}^{\infty} pq^i\right]$$

$$= q^2\left[E(x(x-1)) + 4m + 2\right] \quad [\text{using } 4, 2 \ \& \ 1]$$

$$\Rightarrow \frac{E(x(x-1))}{q^2} = E(x(x-1)) + 4m + 2$$

$$\Rightarrow \frac{E(x(x-1))}{q^2} - E(x(x-1)) = 4m+2$$

$$\Rightarrow E(x(x-1))\left[\frac{1}{q^2} - 1\right] = 4m+2$$

$$\Rightarrow E(x(x-1))\,\frac{1-q^2}{q^2} = 4m+2$$

$$\Rightarrow E(x(x-1)) = \frac{(4m+2)q^2}{1-q^2}$$

$$\mathrm{var}(x) = E(x(x-1)) - m(m-1)$$

$$= \frac{(4m+2)q^2}{1-q^2} - m(m-1)$$

$$= \frac{2q^2}{1-q^2} + \frac{4mq^2}{1-q^2} - m^2 + m$$

$$= \frac{2q^2}{1-q^2} + m\left[\frac{4q^2}{1-q^2} + 1\right] - m^2$$

$$= \frac{2q^2}{1-q^2} + m\,\frac{1+3q^2}{1-q^2} - m^2$$

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$$Var(x) = \frac{2q^2}{1-q^2} + \frac{q}{p}\cdot\frac{1+3q^2}{1-q^2} - \frac{q^2}{p^2}$$

$$= \frac{2q^2}{p(1+q)} + \frac{q}{p^2}\,\frac{1+3q^2}{1+q} - \frac{q^2}{p^2}$$

$$= \frac{q}{p^2}\left\{\frac{2qp}{1+q} + \frac{1+3q^2}{1+q}\right\} - \frac{q^2}{p^2}$$

$$= \frac{q}{p^2}\left\{\frac{2q(1-q)+1+3q^2}{1+q}\right\} - \frac{q^2}{p^2}$$

$$= \frac{q}{p^2}\left\{\frac{2q-2q^2+1+3q^2}{1+q}\right\} - \frac{q^2}{p^2}$$

$$= \frac{q}{p^2} \times \frac{q^2+2q+1}{1+q} - \frac{q^2}{p^2}$$

$$= \frac{q}{p^2} \times \frac{(1+q)^2}{1+q} - \frac{q^2}{p^2}$$

$$= \frac{q}{p^2}(1+q) - \frac{q^2}{p^2}$$

$$= \frac{q}{p^2}\{1+q-q\}$$

$$= \frac{q}{p^2} = \frac{1-p}{p^2}$$

$\therefore\ m = E(x) = \frac{q}{p} = \frac{1-p}{p}$ & $Var(x) = \frac{q}{p^2} = \frac{1-p}{p^2}$

These can also be derived from $NB(r,p)$ distribution because we know that $G(p)$ distribution is a particular case of $NB(r,p)$ for $r=1$

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Therefore, for $r=1$, from $NB(r,p)$, we get

$$m = \frac{r(1-p)}{p} = \frac{1-p}{p} = \frac{q}{p}$$

$$Var(x) = \frac{rq}{p^2} = \frac{1\cdot q}{p^2} = \frac{q}{p^2} = \frac{1-p}{p^2}$$

⑤ <u>Normal (m,σ) distribution:</u> $X = N(m,\sigma)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right], \quad -\infty < x < \infty$$

μ_r = rth order central moment of X

$\therefore$ $\mu_0 = 1$ and $\mu_1 = 0$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-m+m) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-m) f(x)\,dx + \int_{-\infty}^{\infty} m f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-m) f(x)\,dx + m \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-m) f(x)\,dx + m \quad \text{———} \quad (1)$$

$$\left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

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$$\int_{-\infty}^{\infty} (x-m) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-m) \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] dx$$

Put $x-m=y$

$$= \int_{-\infty}^{\infty} y \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right] dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \underbrace{y \exp\left(-\frac{y^2}{2\sigma^2}\right)}_{\text{odd function of } y} dy$$

$$= 0 \text{ ———— } (2)$$

From (1) & (2), we get

$$\boxed{E(X) = m}$$

$$\mu_r = E(X - E(X))^r = E(X-m)^r$$

$$= \int_{-\infty}^{\infty} (x-m)^r f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-m)^r \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] dx$$

[Put $x-m=y$]

$$= \int_{-\infty}^{\infty} y^r \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right] dy$$

$$\mu_r = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} y^r \exp\left[-\frac{y^2}{2\sigma^2}\right] dy \quad \text{——— (1)}$$

If r is odd, then $y^r \exp\left[-\frac{y^2}{2\sigma^2}\right]$ is an odd function of y and consequently

$$\int_{-\infty}^{\infty} y^r \exp\left[-\frac{y^2}{2\sigma^2}\right] dy = 0$$

$\Rightarrow \mu_r = 0$ when r is odd

If r is even, then $y^r \exp\left[-\frac{y^2}{2\sigma^2}\right]$ is an even function of y and consequently

$$\int_{-\infty}^{\infty} y^r \exp\left[-\frac{y^2}{2\sigma^2}\right] dy = 2\int_{0}^{\infty} y^r \exp\left[-\frac{y^2}{2\sigma^2}\right] dy$$

$$\Rightarrow \mu_r = \frac{2}{\sqrt{2\pi}\sigma} \int_0^{\infty} y^r \exp\left[-\frac{y^2}{2\sigma^2}\right] dy$$

Put $\frac{y^2}{2\sigma^2} = p$

$\Downarrow$

$y^2 = (\sqrt{2}\sigma\sqrt{p})^2$

$\Downarrow$

$y = \sqrt{2}\sigma\sqrt{p}$ (as $y>0$)

$\Downarrow$

$dy = \sqrt{2}\sigma \cdot \frac{1}{2}\frac{1}{\sqrt{p}} dp$

$= \frac{\sigma}{\sqrt{2}} \cdot \frac{1}{\sqrt{p}} dp$

$$= \frac{2}{\sqrt{2\pi}\sigma} \int_0^{\infty} (\sqrt{2}\sigma\sqrt{p})^r \exp(-p) \cdot \frac{\sigma}{\sqrt{2}} \cdot \frac{1}{\sqrt{p}} dp$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} (2)^{r/2} \sigma^r p^{\frac{r}{2}-\frac{1}{2}} \exp(-p)\, dp$$

$$= \frac{2^{r/2}\sigma^r}{\sqrt{\pi}} \int_0^{\infty} p^{\frac{r}{2}+\frac{1}{2}-1} e^{-p}\, dp$$

$$= \frac{2^{r/2}\sigma^r}{\sqrt{\pi}} \Gamma\left(\frac{r}{2}+\frac{1}{2}\right)$$

$$\left.\begin{aligned} \mu_r &= 0 \quad \text{for odd } r \\ &= \frac{2^{r/2}\sigma^r}{\sqrt{\pi}} \Gamma\left(\frac{r}{2}+\frac{1}{2}\right) \quad \text{for even } r \end{aligned}\right\} \text{— } (*)$$

Variance of $X = Var(X) = \mu_2$

$$= \frac{2^{2/2}\sigma^2}{\sqrt{\pi}} \cdot \Gamma\left(1+\frac{1}{2}\right)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \sigma^2$$

$\Rightarrow$ $Var(X) = \sigma^2$

$\therefore$ $E(X) = m$ & $Var(X) = \sigma^2$

Put $r = 2k$ in $(*)$

$$\mu_{2k} = \frac{2^k \sigma^{2k}}{\sqrt{\pi}} \Gamma\left(k+\frac{1}{2}\right)$$

$$= \frac{2^{k-1}\sigma^{2k-2}}{\sqrt{\pi}} \Gamma\left(k-1+\frac{1}{2}\right) = \frac{2^{k-1}\sigma^{2k-2}}{\sqrt{\pi}} \Gamma\left(k-\frac{1}{2}\right)$$

$$\frac{\mu_{2k}}{\mu_{2k-2}} = \frac{2^k\sigma^{2k}\Gamma\left(k+\frac{1}{2}\right)}{2^{k-1}\sigma^{2k-2}\Gamma\left(k-\frac{1}{2}\right)} = 2\sigma^2 \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k-\frac{1}{2}\right)}$$

$$\frac{\mu_{2k}}{\mu_{2k-2}} = 2\sigma^2 \frac{\Gamma(k-\frac{1}{2}+1)}{\Gamma(k-\frac{1}{2})} = 2\sigma^2 (k-\tfrac{1}{2})$$

$$\Rightarrow \frac{\mu_{2k}}{\mu_{2k-2}} = (2k-1)\sigma^2$$

$$\Rightarrow \frac{\mu_{2r}}{\mu_{2r-2}} = (2r-1)\sigma^2 \quad \text{———} \quad (**)$$

Put $r = 1, 2, \dots, k$ in $(**)$, we get

$$\frac{\mu_2}{\mu_0} = 1\cdot\sigma^2$$

$$\frac{\mu_4}{\mu_2} = 3\cdot\sigma^2$$

$$\frac{\mu_6}{\mu_4} = 5\cdot\sigma^2$$

$$\cdots\cdots\cdots$$

$$\frac{\mu_{2k}}{\mu_{2k-2}} = (2k-1)\cdot\sigma^2$$

$$\frac{\mu_{2k}}{\mu_0} = 1\cdot 3\cdot 5\cdots(2k-1)(\sigma^2)^k$$

$$\Rightarrow \mu_{2k} = 1\cdot 3\cdot 5\cdots(2k-1)\,\sigma^{2k}$$

6) Uniform distribution: $X = U(a, b)$

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$
$$= 0, \quad \text{elsewhere}$$

$$\alpha_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{a} x^r f(x)\,dx}_{0} + \int_{a}^{b} x^r f(x)\,dx + \underbrace{\int_{b}^{\infty} x^r f(x)\,dx}_{0}$$

$$= \int_{a}^{b} x^r \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \left.\frac{x^{r+1}}{r+1}\right|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^{r+1} - a^{r+1}}{r+1}$$

$$\alpha_1 = E(X) = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{1}{b-a} \cdot \frac{(b-a)(b+a)}{2}$$

$$= \frac{b+a}{2}$$

$$\Rightarrow m = E(X) = \frac{b+a}{2}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \alpha_2 - (E(X))^2 = \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + b^2 + 2ab}{4}$$

$$\Rightarrow Var(X) = \frac{b^2+ab+a^2}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

$$\therefore\ m = E(X) = \frac{b+a}{2} \ \& \ Var(X) = \frac{(b-a)^2}{12}$$

(7) <u>Gamma distribution</u>: $X = \text{Gamma}(l)$

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{l-1}}{\Gamma(l)}, & 0 < x < \infty,\ l > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\alpha_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{0} x^r f(x)\,dx}_{0} + \int_{0}^{\infty} x^r f(x)\,dx$$

$$= \int_0^\infty x^r \frac{e^{-x} x^{l-1}}{\Gamma(l)}\,dx$$

$$= \int_0^\infty \frac{e^{-x} x^{l+r-1}}{\Gamma(l)}\,dx$$

$$= \frac{1}{\Gamma(l)} \int_0^\infty e^{-x} x^{l+r-1}\,dx$$

$$= \frac{1}{\Gamma(l)} \Gamma(l+r) = \frac{\Gamma(l+r)}{\Gamma(l)}$$

$$m = E(X) = \alpha_1 = \frac{\Gamma(l+1)}{\Gamma(l)} = \frac{l\,\Gamma(l)}{\Gamma(l)} = l$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \alpha_2 - l^2$$

$$= \frac{\Gamma(l+2)}{\Gamma(l)} - l^2 = \frac{(l+1)\,l\,\Gamma(l)}{\Gamma(l)} - l^2$$

$$= l(l+1) - l^2 = l$$

$$\boxed{m = E(X) = l \quad \& \quad \mathrm{Var}(X) = l}$$

(8) **Beta distribution of first kind:** $X = \beta_1(l, m)$

$$f(x) = \begin{cases} \dfrac{x^{l-1}(1-x)^{m-1}}{B(l,m)}, & 0 < x < 1, \quad l, m > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\alpha_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{0} x^r f(x)\,dx}_{0} + \int_0^1 x^r f(x)\,dx + \underbrace{\int_1^{\infty} x^r f(x)\,dx}_{0}$$

$$= \int_0^1 x^r \frac{x^{l-1}(1-x)^{m-1}}{B(l,m)}\,dx$$

$$= \int_0^1 \frac{x^{l+r-1}(1-x)^{m-1}}{B(l,m)}\,dx$$

$$= \frac{B(l+r, m)}{B(l,m)}$$

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$$\alpha_r = E(X^r) = \frac{B(l+r, m)}{B(l, m)}$$

$$= \frac{\Gamma(l+r)\Gamma(m)}{\Gamma(l+r+m)} \cdot \frac{\Gamma(l+m)}{\Gamma(l)\Gamma(m)}$$

$$= \frac{\Gamma(l+r)\Gamma(l+m)}{\Gamma(l+r+m)\Gamma(l)}$$

$$m = E(X) = \alpha_1 = \frac{\Gamma(l+1)\Gamma(l+m)}{\Gamma(l+m+1)\Gamma(l)}$$

$$= \frac{l\,\Gamma(l)\Gamma(l+m)}{(l+m)\Gamma(l+m)\Gamma(l)} = \frac{l}{l+m}$$

$$\alpha_2 = E(X^2) = \frac{\Gamma(l+2)\Gamma(l+m)}{\Gamma(l+m+2)\Gamma(l)}$$

$$= \frac{(l+1)\,l\,\Gamma(l)\Gamma(l+m)}{(l+m+1)(l+m)\Gamma(l+m)\Gamma(l)}$$

$$= \frac{l(l+1)}{(l+m)(l+m+1)}$$

$$Var(X) = E(X^2) - (E(X))^2 = \alpha_2 - m^2$$

$$= \frac{l(l+1)}{(l+m)(l+m+1)} - \frac{l^2}{(l+m)^2}$$

$$= \frac{l}{l+m}\left\{\frac{l+1}{l+m+1} - \frac{l}{l+m}\right\}$$

$$= \frac{l}{l+m}\left\{\frac{(l+1)(l+m) - l(l+m+1)}{(l+m+1)(l+m)}\right\}$$

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$$Var(X) = \frac{l}{l+m}\left\{\frac{l^2+lm+l+m-l^2-lm-l}{(l+m+1)(l+m)}\right\}$$

$$= \frac{l}{l+m}\cdot\frac{m}{(l+m+1)(l+m)}$$

$$= \frac{lm}{(l+m)^2(l+m+1)}$$

$$\therefore \boxed{E(X) = \frac{l}{l+m} \quad \& \quad Var(X) = \frac{lm}{(l+m)^2(l+m+1)}}$$

⑨ **Beta distribution of second kind:** $X = \beta_2(m,n)$

$$f(x) = \begin{cases} \dfrac{x^{m-1}}{B(m,n)(1+x)^{m+n}}, & 0<x<\infty,\ m>0, n>0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\alpha_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{0} x^r f(x)\,dx}_{0} + \int_{0}^{\infty} x^r f(x)\,dx$$

$$= \int_0^\infty x^r \cdot \frac{x^{m-1}}{B(m,n)(1+x)^{m+n}}\,dx$$

$$= \frac{1}{B(m,n)}\int_0^\infty \frac{x^{m+r-1}}{(1+x)^{m+n}}\,dx$$

$$\alpha_r = \frac{1}{B(m,n)} \int_0^\alpha \frac{x^{m+r-1}}{(1+x)^{m+r+n-r}}\,dx$$

$$= \frac{1}{B(m,n)} B(m+r, n-r) \quad \text{provided } n-r>0$$

$$= \frac{B(m+r, n-r)}{B(m,n)} \quad \text{provided } n-r>0$$

$$= \frac{\Gamma(m+r)\,\Gamma(n-r)}{\Gamma(m+r+n-r)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \quad \text{provided } n-r>0$$

$$= \frac{\Gamma(m+r)\,\Gamma(n-r)}{\Gamma(m)\,\Gamma(n)} \quad \text{provided } n-r>0$$

$$\therefore\ E(X) = \alpha_1 = \frac{\Gamma(m+1)\,\Gamma(n-1)}{\Gamma(m)\,\Gamma(n)} \quad \text{provided } n-1>0$$

$$= \frac{m\Gamma(m)\,\Gamma(n-1)}{\Gamma(m)\,(n-1)\,\Gamma(n-1)} \quad \text{provided } n>1$$

$$= \frac{m}{n-1} \quad \text{provided } n>1$$

$\therefore$ Mea of the distribution exists if $n>1$ and in this case

$$m = E(X) = \frac{m}{n-1}$$

$$\alpha_2 = E(x^2) = \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m)\Gamma(n)} \quad \text{provided } n-2>0$$

$$= \frac{m(m+1)\Gamma(m)\Gamma(n-2)}{\Gamma(m)(n-1)(n-2)\Gamma(n-2)} \quad \text{provided } n-2>0$$

$$= \frac{m(m+1)}{(n-1)(n-2)} \quad \text{provided } n-2>0$$

$$Var(x) = \alpha_2 - (E(x))^2 = \frac{m(m+1)}{(n-1)(n-2)} - \frac{m^2}{(n-1)^2} \quad \text{provided } n-2>0$$

$$= \frac{m}{n-1}\left\{\frac{m+1}{n-2} - \frac{m}{n-1}\right\} \quad \text{provided } n-2>0$$

$$= \frac{m}{n-1} \frac{(m+1)(n-1) - m(n-2)}{(n-1)(n-2)} \quad \text{provided } n-2>0$$

$$= \frac{m}{n-1} \cdot \frac{mn+n-m-1-mn+2m}{(n-1)(n-2)} \quad \text{if } n>2$$

$$= \frac{m}{n-1} \cdot \frac{m+n-1}{(n-1)(n-2)} \quad \text{if } n>2$$

$$= \frac{m(m+n-1)}{(n-1)^2(n-2)} \quad \text{if } n>2$$

If $n>2$, then $Var(x)$ exists and

$$var(x) = \frac{m(m+n-1)}{(n-1)^2(n-2)} \quad \text{provided } n>2$$

If $n>2$, both $E(x)$ and $V(x)$ exist finitely.

Median: μ is said to be the median of the distribution of a random variable X if

$F(\mu-0) \le \frac{1}{2}$ and $F(\mu) \ge \frac{1}{2}$ ——— ①

where F is the distribution function of X.

We have following two cases:

case 1: Let X be the discrete random variable. Then ① determines μ uniquely except for the case for which

$F(x) = \frac{1}{2}$ for $x_K \le x < x_{K+1}$

In this case, ① is satisfied for any $\mu \in [x_K, x_{K+1})$ and consequently for this case, we define

$$\mu = \frac{x_K + x_{K+1}}{2}$$

case 2: Let X be a continuous random variable.

$\therefore \quad F(\mu-0) = F(\mu)$

$\Rightarrow \quad F(\mu) = F(\mu-0)$

$\Rightarrow \frac{1}{2} \le F(\mu) = F(\mu-0) \le \frac{1}{2}$

$\Rightarrow \frac{1}{2} \le F(\mu) \le \frac{1}{2} \Rightarrow F(\mu) = \frac{1}{2} \Rightarrow \int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{2}$

This is an equation for μ and this equation determines μ uniquely.

<u>Theorem</u>: For a continuous symmetrical distribution Median is equal to mean of the distribution if the mean exists finitely.

<u>Proof:</u> Let $f(x)$ be the pdf of X with $F(x)$ is the probability distribution function. Let $m = E(X)$ and μ be the median of the distribution.

$$\therefore F(\mu) = \frac{1}{2} \Rightarrow \int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{2} \quad \text{———} \quad (1)$$

Let the distribution function is symmetrical about the point c. Therefore, we have

$c = m$

Again as $F(x)$ symmetrical about $c = m$, we get

$$f(m+u) = f(m-u) \quad \forall u \quad \text{———} \quad (2)$$

From (1)

$$\int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\mu - m} f(m+u)\,du = \frac{1}{2} \quad \text{———} \quad (3)$$

Put $x = m + u$
$\Rightarrow dx = du$
when $x \to -\infty$, $u \to -\infty$
when $x = \mu$, $u = \mu - m$

Again $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \frac{1}{2} + \int_{\mu}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{\mu}^{\infty} f(x)\,dx = \frac{1}{2}$$

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$$\Rightarrow \int_{\mu}^{\infty} f(x)\,dx = \frac{1}{2}$$

Put $x = m - u$

$dx = -du$

when $x = \mu$, $u = m - \mu$

when $x \to \infty$, $u \to -\infty$

$$\Rightarrow \int_{m-\mu}^{-\infty} f(m-u)(-du) = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{m-\mu} f(m-u)\,du = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{m-\mu} f(m+u)\,du = \frac{1}{2} \quad [\text{using } (2)]$$

$$\Rightarrow \int_{-\infty}^{m-\mu} f(m+u)\,du = \frac{1}{2} = \int_{-\infty}^{\mu-m} f(m+u)\,du$$

$$\Rightarrow \int_{-\infty}^{m-\mu} f(m+u)\,du = \int_{-\infty}^{\mu-m} f(m+u)\,du$$

$$\Rightarrow m - \mu = \mu - m \Rightarrow 2m = 2\mu \Rightarrow \boxed{m = \mu}$$

Theorem: The first absolute moment about any point is minimum when taken about the median, i.e., $E(|X-c|) \geq E(|X-\mu|)$ for any point c, where μ is the median of the distribution.

Proof: Let X be a continuous random variable with pdf $f(x)$. Let $F(x)$ be the probability distribution function of X.

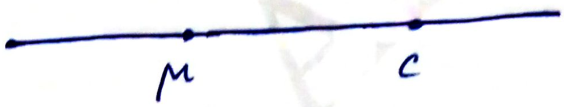
Therefore

$$F(\mu) = \frac{1}{2} \Leftrightarrow \int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{2} \Leftrightarrow \boxed{2F(\mu) = 1}$$

Again $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{\infty} f(x)\,dx = 1$

$$\Rightarrow F(\mu) + \int_{\mu}^{\infty} f(x)\,dx = 1 \Rightarrow \boxed{\int_{\mu}^{\infty} f(x)\,dx = 1 - F(\mu)}$$

Let $c > \mu$



$$E(|X-c|) = \int_{-\infty}^{\infty} |x-c|\, f(x)\,dx$$

$$= \int_{-\infty}^{c} \underbrace{|x-c|\, f(x)\,dx}_{x<c} + \int_{c}^{\infty} \underbrace{|x-c|\, f(x)\,dx}_{x>c}$$

$$= \int_{-\infty}^{c} (c-x) f(x)\,dx + \int_{c}^{\infty} (x-c) f(x)\,dx$$

$$= \int_{-\infty}^{c} (c-\mu+\mu-x) f(x)\,dx + \int_{c}^{\infty} (x-\mu+\mu-c) f(x)\,dx$$

$$= (c-\mu)\int_{-\infty}^{c} f(x)\,dx + \int_{-\infty}^{c} (\mu-x) f(x)\,dx$$

$$+ \int_{c}^{\infty} (x-\mu) f(x)\,dx + (\mu-c)\int_{c}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{c} (\mu-x) f(x)\,dx + \int_{c}^{\infty} (x-\mu) f(x)\,dx$$

$$+ (c-\mu)\left\{\int_{-\infty}^{c} f(x)\,dx - \int_{c}^{\infty} f(x)\,dx\right\} = I_1 + (c-\mu) I_2 \quad \text{—} (1)$$

where

$$I_1 = \int_{-\infty}^{c} (\mu - x) f(x)\, dx + \int_{c}^{\infty} (x-\mu) f(x)\, dx$$

$$I_2 = \int_{-\infty}^{c} f(x)\, dx - \int_{c}^{\infty} f(x)\, dx$$

Now

$$I_2 = \int_{-\infty}^{c} f(x)\, dx - \int_{c}^{\infty} f(x)\, dx$$

$$= \int_{-\infty}^{\mu} f(x)\, dx + \int_{\mu}^{c} f(x)\, dx - \left\{ \int_{\mu}^{\infty} f(x)\, dx - \int_{\mu}^{c} f(x)\, dx \right\}$$

$$= F(\mu) + 2\int_{\mu}^{c} f(x)\, dx - (1 - F(\mu))$$

$$= 2F(\mu) - 1 + 2\int_{\mu}^{c} f(x)\, dx = 2\int_{\mu}^{c} f(x)\, dx \quad [\text{as } 2F(\mu) = 1]$$

$$I_2 = 2\int_{\mu}^{c} f(x)\, dx$$

Now $I_1 = \int_{-\infty}^{c} (\mu - x) f(x)\, dx + \int_{c}^{\infty} (x-\mu) f(x)\, dx$

$$= \int_{-\infty}^{\mu} (\mu - x) f(x)\, dx + \int_{\mu}^{c} (\mu - x) f(x)\, dx$$

$$+ \int_{\mu}^{\infty} (x-\mu) f(x)\, dx - \int_{\mu}^{c} (x-\mu) f(x)\, dx$$

$$= \int_{-\infty}^{\mu} (\mu - x) f(x)\, dx + \int_{\mu}^{\infty} (x-\mu) f(x)\, dx$$

$$+ \int_{\mu}^{c} (\mu - x) f(x)\, dx + \int_{\mu}^{c} (\mu - x) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} |x - \mu| f(x)\, dx + 2\int_{\mu}^{c} (\mu - x) f(x)\, dx$$

$$\Rightarrow I_1 = E(|X-M|) + 2\int_M^c (M-x) f(x)\,dx$$

$$\therefore\ E(|X-c|) = E(|X-M|) + 2\int_M^c (M-x) f(x)\,dx + 2(c-M)\int_M^c f(x)\,dx$$

$$= E(|X-M|) + 2\int_M^c (M-x+c-M) f(x)\,dx$$

$$= E(|X-M|) + 2\int_M^c (c-x) f(x)\,dx$$

$$\Rightarrow E(|X-c|) - E(|X-M|) = 2\int_M^c (c-x) f(x)\,dx \geq 0$$

$$\Rightarrow E(|X-c|) \geq E(|X-M|)$$

Similarly, it can be proved that

$E(|X-c|) \geq E(|X-M|)$ for $c<M$

$$\therefore\ E(|X-c|) \geq E(|X-M|)$$

This result holds good for the case of discrete random variable.

* When we take the <u>median</u> as a measure of <u>central tendency</u> then the measure of <u>dispersion</u> will be the <u>first order absolute moment about the median</u>, i.e., $E(|X-M|)$ will be the measure of dispersion

Quantile: A real number ξ_p is said to be a Quantile of order p $(0<p<1)$ of the random variable X if

$$F(\xi_p - 0) \le p \text{ and } F(\xi_p) \ge p \quad \text{—— (1)}$$

where F is the probability distribution function of X.

Here we have the following two cases:

case 1: Let X be a discrete random variable.

Then (1) determines ξ_p uniquely except for the case

$$F(\xi_p) = p \text{ for } x_k \le \xi_p < x_{k+1}$$

In this case, we take

$$\xi_p = \frac{x_k + x_{k+1}}{2}$$

case 2: Let X be a continuous random variable.

$$F(\xi_p - 0) = F(\xi_p)$$

$$\Rightarrow F(\xi_p) = F(\xi_p - 0)$$

$$\Rightarrow p \le F(\xi_p) = F(\xi_p - 0) \le p \text{ [from (1)]}$$

$$\Rightarrow p \le F(\xi_p) \le p \Rightarrow F(\xi_p) = p$$

$$\Rightarrow \int_{-\infty}^{\xi_p} f(x)\,dx = p$$

This is an equation of ξ_p for given value of p and this equation determines ξ_p uniquely.

<u>Decile of order k</u> : If $p = \frac{k}{10}$, $k = 1, 2, 3, \ldots, 9$ then the Quantile is known as a Decile of order k

<u>Percentile of order k</u> : If $p = \frac{k}{100}$, $k = 1, 2, \ldots, 99$ then the Quantile is known as a percentile of order k.

<u>Mode</u> : This is a third measure of central tendency. Mode of a distribution can be defined as follows:

<u>continuous distribution</u> : Any point ξ (or M) for which $f(x)$ has a maximum value is known as a Mode, where $f(x)$ is the pdf of the distribution. The distribution is said to be unimodal, bimodal or multimodal according to whether $f(x)$ has one, two or many points of maximum.

<u>Discrete distribution</u> : Let the spectrum of the random variable X is as follows:

$$X = \{\ldots x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$$

where

$$\ldots < x_{-3} < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots$$

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Then x_k is said to be a mode if

$f_k > f_{k-1}$ and $f_k > f_{k+1}$

where $f_k = P(X = x_k)$

Then clearly there may be more than one mode for the discrete distribution also because a mode is point of the spectrum having relatively tallest ordinate in the probability graph.