Lecture Notes on Probability

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Chapter - 3

Two-dimensional distribution

Probability distribution function of two random variables or joint probability distribution function:

Let X and Y be two random variables defined on the same event space S connected with a random experiment E. Then the joint probability distribution function of the random variables X and Y is denoted by $F_{X,Y}(x, y)$ or $F(x,y)$ and defined by

$$F_{X,Y}(x,y) = F(x,y) = P(\underbrace{-\infty < X \le x, -\infty < Y \le y}) $$

$$\Downarrow$$

$$P((-\infty < X \le x) \cap (-\infty < Y \le y))$$

Properties of $F_{X,Y}(x,y)$:

P1: $F(x, \infty) = F_{X,Y}(x, \infty) = F_X(x)$

$$F_{X,Y}(x,y) = P(-\infty < X \le x, -\infty < Y \le y)$$

Making $y \to \infty$, we get

$$F_{X,Y}(x, \infty) = F(x, \infty) = P(-\infty < X \le x, -\infty < Y < \infty)$$

$$= P((-\infty < X \le x) \cap (-\infty < Y < \infty))$$

$$= P((-\infty < X \le x) \cap S)$$

$$= P(-\infty < X \le x)$$

$$= F_X(x)$$

$\Rightarrow F_{X,Y}(x, \infty) = F_X(x)$

✓ Here $F_X(x)$ is known as Marginal distribution of X

<u>P2</u>: $F_{X,Y}(\infty, y) = F_Y(y)$

$$F_{X,Y}(x, y) = P(-\infty < X \le x, -\infty < Y \le y)$$

Making $x \to \infty$, we get

$$F_{X,Y}(\infty, y) = P(-\infty < X < \infty, -\infty < Y \le y)$$

$$= P((-\infty < X < \infty) \cap (-\infty < Y \le y))$$

$$= P(S \cap (-\infty < Y \le y))$$

$$= P(-\infty < Y \le y) = F_Y(y)$$

$\Rightarrow F_{X,Y}(\infty, y) = F_Y(y)$

✓ $F_Y(y)$ is also a marginal Distribution.

P3 $P(a < X \le b, -\infty < Y \le d) = F(b,d) - F(a,d)$

Proof:

$F(b,d) = P(-\infty < X \le b, -\infty < Y \le d)$

$= P\big((-\infty < X \le a, -\infty < Y \le d)$

$\quad \cup (a < X \le b, -\infty < Y \le d)\big)$

$= P(-\infty < X \le a, -\infty < Y \le d)$

$\quad + P(a < X \le b, -\infty < Y \le d)$

$= F(a,d) + P(a < X \le b, -\infty < Y \le d)$

$\Rightarrow \boxed{P(a < X \le b, -\infty < Y \le d) = F(b,d) - F(a,d)}$

P4 $P(-\infty < X \le b, c < Y \le d) = F(b,d) - F(b,c)$

$F(b,d) = P(-\infty < X \le b, -\infty < Y \le d)$

$= P\big((-\infty < X \le b, -\infty < Y \le c) \cup (-\infty < X \le b, c < Y \le d)\big)$

$= P(-\infty < X \le b, -\infty < Y \le c)$

$\quad + P(-\infty < X \le b, c < Y \le d)$

$= F(b,c) + P(-\infty < X \le b, c < Y \le d)$

$\Rightarrow \boxed{P(-\infty < X \le b, c < Y \le d) = F(b,d) - F(b,c)}$

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<u>P5</u>: $F(x, y)$ is monotonic non decreasing in both the variables.

<u>Proof</u>: Let $x_1 \geq x_2$

If $x_1 = x_2$ then $F(x_1, y) = F(x_2, y)$ ——— (1)

If $x_1 > x_2$ then from <u>P3</u> we get

$F(x_1, y) - F(x_2, y) = P(x_2 < X \leq x_1, -\infty < Y \leq y) \geq 0$

$\Rightarrow F(x_1, y) \geq F(x_2, y)$ ——— (2)

Combining (1) and (2), we get

$F(x_1, y) \geq F(x_2, y) \quad \forall\ x_1 \geq x_2$ and $\forall y$

✓ $\Rightarrow$ F is monotonically non-decreasing wrt the variable x

Let $y_1 \geq y_2$

If $y_1 = y_2$ then $F(x, y_1) = F(x, y_2)$ ——— (3)

If $y_1 > y_2$ then using <u>P4</u> we get

$F(x, y_1) - F(x, y_2) = P(-\infty < X \leq x, y_2 < Y \leq y_1) \geq 0$

$\Rightarrow F(x, y_1) \geq F(x, y_2)$ ——— (4)

Combining (3) & (4), we get

$F(x, y_1) \geq F(x, y_2) \quad \forall\ y_1 \geq y_2$ and $\forall x$

$\Rightarrow$ F is monotonically non-decreasing wrt the variable y.

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P6: $P(a < X \leq b, c < Y \leq d) = F(b,d) + F(a,c) - F(a,d) - F(b,c)$

Proof: $F(b,d) + F(a,c) - F(a,d) - F(b,c)$

$= F(b,d) - F(a,d) + F(a,c) - F(b,c)$

$= F(b,d) - F(a,d) - \{F(b,c) - F(a,c)\}$

$= P(a < X \leq b, -\infty < Y \leq d) - P(a < X \leq b, -\infty < Y \leq c)$ [P3]

$= P((a < X \leq b, -\infty < Y \leq c) \cup (a < X \leq b, c < Y \leq d))$

$\quad - P(a < X \leq b, -\infty < Y \leq c)$

$= P(a < X \leq b, -\infty < Y \leq c) + P(a < X \leq b, c < Y \leq d)$

$\quad - P(a < X \leq b, -\infty < Y \leq c)$

$= P(a < X \leq b, c < Y \leq d)$

$\Rightarrow \boxed{P(a < X \leq b, c < Y \leq d) = F(b,d) + F(a,c) - F(a,d) - F(b,c)}$

✓ We can prove this property by considering P4

$F(b,d) + F(a,c) - F(a,d) - F(b,c)$

$= F(b,d) - F(b,c) - \{F(a,d) - F(a,c)\}$

$= P(-\infty < X \leq b, c < Y \leq d) - P(-\infty < X \leq a, c < Y \leq d)$

$= P((-\infty < X \leq a, c < Y \leq d) \cup (a < X \leq b, c < Y \leq d))$

$\quad - P(-\infty < X \leq a, c < Y \leq d)$

$= P(-\infty < X \le a, c < Y \le d) + P(a < X \le b, c < Y \le d)$

$- P(-\infty < X \le a, c < Y \le d)$

$= P(a < X \le b, c < Y \le d)$

$$\Rightarrow \boxed{P(a < X \le b, c < Y \le d) = F(b,d) + F(a,c) - F(a,d) - F(b,c)}$$

P7: $F(-\infty, y) = 0 = F(x, -\infty)$

Proof:

$$F(x,y) = P(-\infty < X \le x, -\infty < Y \le y)$$

$$= P((-\infty < X \le x) \cap (-\infty < Y \le y))$$

Consider a sequence of events $\{A_n\}$ defined by

$A_n = (-\infty < X \le -n) \cap (-\infty < Y \le y)$

It is simple to check that $\{A_n\}$ is a contracting sequence and

$\lim A_n = \Phi$, ——————— (1)

where we have used the same argument as given in chapter 2 to derive (1).

$\therefore \; P(\lim A_n) = P(\Phi) = 0$

$\Rightarrow \lim P(A_n) = 0$

$\Rightarrow \lim P((-\infty < X \leq -n) \cap (-\infty < Y \leq y)) = 0$

$\Rightarrow \lim P(-\infty < X \leq -n, -\infty < Y \leq y) = 0$

$\Rightarrow \lim F(-n, y) = 0$

$\Rightarrow \boxed{F(-\infty, y) = 0}$

Similarly, considering the sequence $\{B_n\}$ defined by

$B_n = (-\infty < X \leq x) \cap (-\infty < Y \leq -n)$

It is simple to prove that $\{B_n\}$ is contracting and

$\lim B_n = \Phi$, ——————— (2)

where we have used the same argument as given in chapter 2 to derive (2)

$\therefore P(\lim B_n) = P(\Phi) = 0$

$\Rightarrow \lim P(B_n) = 0$

$\Rightarrow \lim P((-\infty < X \leq x) \cap (-\infty < Y \leq -n)) = 0$

$\Rightarrow \lim P(-\infty < X \leq x, -\infty < Y \leq -n) = 0$

$\Rightarrow \lim F(x, -n) = 0$

$\Rightarrow \boxed{F(x, -\infty) = 0}$

P8: $F(a+0, c) = F(a, c+0) = F(a, c)$

Consider two sequences of event $\{A_n\}$ and $\{B_n\}$ such that

$$A_n = (a < X \leq a + \tfrac{1}{n}, -\infty < Y \leq c)$$

$$B_n = (-\infty < X \leq a, \quad c < Y \leq c + \tfrac{1}{n})$$

It is simple to check that the sequences $\{A_n\}$ and $\{B_n\}$ both are contracting and

$$\lim A_n = \Phi \text{ ——— } (1)$$

$$\lim B_n = \Phi \text{ ——— } (2)$$

From (1)

$$P(\lim A_n) = P(\Phi) = 0$$

$$\Rightarrow \lim P(A_n) = 0$$

$$\Rightarrow \lim P(a < X \leq a + \tfrac{1}{n}, -\infty < Y \leq c) = 0$$

$$\Rightarrow \lim \left[F(a + \tfrac{1}{n}, c) - F(a, c)\right] = 0$$

$$\Rightarrow F(a+0, c) - F(a, c) = 0$$

$$\Rightarrow F(a+0, c) = F(a, c) \text{ ——— } (3)$$

Ily from (2) we get $F(a, c+0) = F(a, c)$ — (4)

From (3) & (4), we get $F(a+0, c) = F(a, c+0) = F(a, c)$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

i.e., the distribution function is continuous from right for both the variables.

P9 $P(X=b, c<Y\leq d) = F(b,d) + F(b-0,c)$
$- F(b-0,d) - F(b,c)$

Proof: Define

$$A_n = \left(b-\tfrac{1}{n} < X \leq b,\ c<Y\leq d\right)$$

Then it is simple to check that

1) $\{A_n\}$ is contracting and

2) $\lim A_n = (X=b,\ c<Y\leq d)$

$\therefore\ P(\lim A_n) = \lim P(A_n)$

$\Rightarrow \lim P(A_n) = P(X=b,\ c<Y\leq d)$

$\Rightarrow \lim P\left(b-\tfrac{1}{n} < X\leq b,\ c<Y\leq d\right) = P(X=b,\ c<Y\leq d)$

$$\Rightarrow P(X=b,\ c<Y\leq d) = \lim\left[F(b,d) + F\left(b-\tfrac{1}{n},c\right) - F\left(b-\tfrac{1}{n},d\right) - F(b,c)\right]$$

$$= F(b,d) + F(b-0,c) - F(b-0,d) - F(b,c)$$

$$\Rightarrow P(X=b,\ c<Y\leq d) = F(b,d) + F(b-0,c) - F(b-0,d) - F(b,c)$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

P10: $P(a < X \leq b, Y = d) = F(b,d) + F(a, d-0) - F(b, d-0) - F(a,d)$

Proof: Define

$$A_n = \left(a < X \leq b,\ d - \tfrac{1}{n} < Y \leq d\right)$$

Then it is simple to check that

1) $\{A_n\}$ is contracting

2) $\lim A_n = (a < X \leq b,\ Y = d)$

$$\therefore P(\lim A_n) = \lim P(A_n)$$

$$= \lim P\left(a < X \leq b,\ d - \tfrac{1}{n} < Y \leq d\right)$$

$$= \lim \left[F(b,d) + F\left(a, d - \tfrac{1}{n}\right) - F\left(b, d - \tfrac{1}{n}\right) - F(a,d)\right]$$

$$= F(b,d) + F(a, d-0) - F(b, d-0) - F(a,d)$$

$$\Rightarrow P(a < X \leq b, Y = d) = F(b,d) + F(a, d-0) - F(b, d-0) - F(a,d)$$

P11: $P(X = b, Y = d) = F(b,d) + F(b-0, d-0) - F(b-0, d) - F(b, d-0)$

Define

$A_n = (b-\frac{1}{n} < X \le b,\ d-\frac{1}{n} < Y \le d)$

Then it is simple to check that

① $\{A_n\}$ is contracting

② $\lim A_n = (X=b,\ Y=d)$

$\therefore\ P(\lim A_n) = \lim P(A_n)$

$= \lim P(b-\frac{1}{n} < X \le b,\ d-\frac{1}{n} < Y \le d)$

$= \lim [F(b,d) + F(b-\frac{1}{n}, d-\frac{1}{n}) - F(b-\frac{1}{n}, d) - F(b, d-\frac{1}{n})]$

$= F(b,d) + F(b-0, d-0) - F(b-0, d) - F(b, d-0)$

$\Rightarrow P(X=b,\ Y=d) = F(b,d) + F(b-0,d-0) - F(b-0,d) - F(b,d-0)$

P12 $F(\infty,\infty) = 1$

Define $A_n = (-\infty < X \le n,\ -\infty < Y \le n)$

It is simple to check that

① $\{A_n\}$ is expanding

② $\lim A_n = (-\infty < X < \infty,\ -\infty < Y < \infty)$

$= (-\infty < X < \infty) \cap (-\infty < Y < \infty)$

$= S \cap S = S$

$\Rightarrow \lim A_n = S$

$\therefore \; P(\lim A_n) = \lim P(A_n)$

$\Rightarrow P(S) = \lim P(-\infty < X \le n, -\infty < Y \le n)$

$= \lim F(n, n)$

$= F(\infty, \infty)$

$\Rightarrow F(\infty, \infty) = P(S) = 1$

$\Rightarrow F(\infty, \infty) = 1$

Def$^{\underline{n}}$: Two random variables X and Y are said to be independent if

$$P(-\infty < X \le x, -\infty < Y \le y) = P(-\infty < X \le x) \times P(-\infty < Y \le y)$$

Theorem: A necessary and sufficient condition for the independence of the random variables X and Y is that their joint distribution function can be written as the product of a function of x alone and a function of y alone.

Proof! Condition is necessary: Let X and Y are independent. Therefore, we get

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$P(-\infty < X \le x, -\infty < Y \le y) = P(-\infty < X \le x)\, P(-\infty < Y \le y)$

$\Rightarrow F(x,y) = F_X(x)\, F_Y(y)$

$\Rightarrow$ joint probability distribution function $F(x,y)$ can be written as a product of two functions $F_X(x), F_Y(y)$, where $F_X(x)$ is a function of x alone and $F_Y(y)$ is a function of y alone.

Condition is sufficient: Let $F(x,y)$ be the joint probability distribution function of X and Y, and this distribution function can be written as a product of two functions $g(x)$ and $h(y)$ such that $g(x)$ is a function of x alone and $h(y)$ is a function of y alone

$\therefore F(x,y) = g(x)\, h(y)$ ——— (1)

Making $x \to \infty$ in (1), we get

$F(\infty, y) = g(\infty)\, h(y)$ ——— (2)

Making $y \to \infty$ in (1), we get

$$F(x, \infty) = g(x) h(\infty) \quad \text{———} \quad (3)$$

Making $x \to \infty$ in (3), we get

$$F(\infty, \infty) = g(\infty) h(\infty)$$

$$\Rightarrow g(\infty) h(\infty) = F(\infty, \infty) = 1$$

$$\Rightarrow g(\infty) h(\infty) = 1 \quad \text{———} \quad (4)$$

From (2) & (3), we get

$$F(\infty, y) F(x, \infty) = g(\infty) h(y) g(x) h(\infty)$$

$$= g(x) h(y) g(\infty) h(\infty)$$

$$= g(x) h(y) \quad [\text{using } (4)]$$

$$= F(x, y) \quad [\text{using } (1)]$$

$$\Rightarrow F(x,y) = F(\infty, y) F(x, \infty)$$

$$= F_Y(y) F_X(x)$$

$$\Rightarrow F(x,y) = F_X(x) F_Y(y)$$

$\Rightarrow$ X and Y are independent.

Theorem: If X and Y are independent random variables then

$$P(a < X \leq b, c < Y \leq d) = P(a < X \leq b) P(c < Y \leq d)$$

Proof: We know that

$P(a < X \le b, c < Y \le d) = F(b,d) + F(a,c) - F(b,c) - F(a,d)$

$= F(b,d) + F(a,c) - F(b,c) - F(a,d)$

$= \underline{F_X(b) F_Y(d)} + F_X(a) F_Y(c) - \underline{F_X(b) F_Y(c)} - F_X(a) F_Y(d)$

$= F_X(b) \{F_Y(d) - F_Y(c)\} - F_X(a) \{F_Y(d) - F_Y(c)\}$

$= \{F_X(b) - F_X(a)\} \{F_Y(d) - F_Y(c)\}$

$= P(a < X \le b) P(c < Y \le d)$

$\therefore P(a < X \le b, c < Y \le d) = P(a < X \le b) P(c < Y \le d)$

<u>Theorem</u>: If X and Y are two independent random variables then

$P(X = b, Y = d) = P(X = b) P(Y = d)$

<u>Proof</u>: We know the following result for two independent random variables X and Y.

$P(a < X \le b, c < Y \le d) = P(a < X \le b) P(c < Y \le d)$

Let $a = b - \frac{1}{n}$ and $c = d - \frac{1}{n}$ for $n = 1, 2, \dots$

$$P(b - \tfrac{1}{n} < X \le b, d - \tfrac{1}{n} < Y \le d) = P(b - \tfrac{1}{n} < X \le b) \times P(d - \tfrac{1}{n} < Y \le d) \quad \text{--- (1)}$$

Consider the folling sequences of events $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ defined by

$A_n = \left(b - \frac{1}{n} < X \le b,\ d - \frac{1}{n} < Y \le d\right)$

$B_n = \left(b - \frac{1}{n} < X \le b\right)$

$C_n = \left(d - \frac{1}{n} < Y \le d\right)$

It is simple to check that

(1) $\{A_n\}, \{B_n\}, \{C_n\}$ are all contracting

(2) $\lim A_n = (X = b, Y = d)$

$\lim B_n = (X = b)$

$\lim C_n = (Y = d)$

Making limt as $n \to \infty$, we get, from (1),

$P(X = b, Y = d) = P(X = b)\, P(Y = d)$

Continuous random variables or continuous distribution for two dimensional random variables: The joint probability distribution function of two random variables X and Y is said to be continuous if

(1) $F(x, y)$ is continuous everywhere,

(2) First and second order partial derivatives are piecewise continuous.

Discrete distributions: The joint distribution of two dimensional random variable (X, Y) is said to be discrete if

① the distribution function $F(x, y)$ is a step function

② the height of step at the point (x_i, y_j) is $P(X = x_i, Y = y_j) = f_{ij}$ for $i, j = 0, \pm 1, \pm 2, \cdots$

Therefore,

$$F(x, y) = \sum_{i=-\infty}^{r} \sum_{j=-\infty}^{s} f_{ij} \quad \text{for } x_r \leq x < x_{r+1} \text{ and } y_s \leq y < y_{s+1}$$

✓ $P(X = x_i, Y = y_j)$ is known as the the probability mass at the point (x_i, y_j)

Properties of continuous two-dimensional distribution $F(x, y)$:

CP1 $P(X = b, Y = d) = 0$

i.e., Probability at a point $= 0$

Proof: From the property of two-dimensional distribution function we get

$$P(X=b, Y=d) = F(b,d) + F(b-0, d-0) - F(b, d-0) - F(b-0, d) \quad \text{—(1)}$$

As $F(x,y)$ is continuous, we have

$$F(b-0, d-0) = F(b, d-0) = F(b-0, d) = F(b,d) \quad \text{—(2)}$$

From (1), using (2), we get

$$P(X=b, Y=d) = F(b,d) + F(b,d) - F(b,d) - F(b,d)$$

$\Rightarrow P(X=b, Y=d) = 0$

$\Rightarrow$ Probability mass at a point is equal to zero for any two dimensional continuous distribution.

CP2: $P(a < X \leq b, c < Y \leq d) = \int_a^b \int_c^d \frac{\partial^2 F}{\partial x \partial y} \, dx \, dy$

Proof: $\int_a^b \int_c^d \frac{\partial^2 F}{\partial x \partial y} \, dx \, dy$

$$= \int_a^b \int_c^d \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) dx \, dy$$

$$= \int_c^d \left\{ \int_a^b \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) dx \right\} dy$$

$$= \int_c^d \left\{ \frac{\partial F}{\partial y} \Big|_a^b \right\} dy$$

$$= \int_c^d \left\{ \frac{\partial F(b,y)}{\partial y} - \frac{\partial F(a,y)}{\partial y} \right\} dy$$

$$= \int_c^d \frac{\partial F(b,y)}{\partial y} dy - \int_c^d \frac{\partial F(a,y)}{\partial y} dy$$

$$= F(b,y) \Big|_c^d - F(a,y) \Big|_c^d$$

$$= F(b,d) - F(b,c) - \{ F(a,d) - F(a,c) \}$$

$$= F(b,d) + F(a,c) - F(b,c) - F(a,d)$$

$$= P(a < X \le b, c < Y \le d)$$

$$\therefore P(a < X \le b, c < Y \le d) = \int_a^b \int_c^d \frac{\partial^2 F}{\partial x \partial y} dx\, dy$$

Defn: For continuous distribution

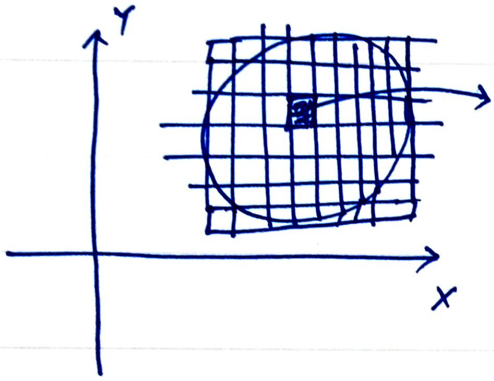
$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

is known as the joint probability density function.

CP3: $$P((X,Y)\in D) = \iint_D f(x,y)\,dx\,dy$$

$$= \iint_{(x,y)\in D} f(x,y)\,dx\,dy$$

where D is a region in R^2.



$$D_{ij} = (x_i < X \le x_{i+1},\ y_j < Y \le y_{j+1})$$

$$P(x_i < X \le x_{i+1},\ y_j < Y \le y_{j+1}) = \iint_{D_{ij}} f(x,y)\,dx\,dy$$

$$\Rightarrow P((X,Y)\in D_{ij}) = \iint_{D_{ij}} f(x,y)\,dx\,dy$$

$$\Rightarrow \sum_i \sum_j P((X,Y)\in D_{ij}) = \sum_i \sum_j \iint_{D_{ij}} f(x,y)\,dx\,dy$$

$$\Rightarrow P((X,Y)\in D) = \iint_D f(x,y)\,dx\,dy$$

CP4 $f(x,y) \geq 0 \quad \forall\, x, y$

This result is true because $F(x,y)$ is non-decreasing function wrt both the variables.

CP5 $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Proof: From **CP2**, we get

$$P(a < X \leq b,\ c < Y \leq d) = \int_a^b \int_c^d f(x,y)\,dx\,dy \quad \text{——} \ (1)$$

Making $a \to -\infty$, $c \to \infty$ in (1), we get

$$P(-\infty < X \leq b,\ -\infty < Y \leq d) = \int_{-\infty}^{b} \int_{-\infty}^{d} f(x,y)\,dx\,dy \quad \text{——} \ (2)$$

Making $b \to \infty$ and $d \to \infty$ in (2), we get

$$P(-\infty < X < \infty,\ -\infty < Y < \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow P\left((-\infty < X < \infty)\ (-\infty < Y < \infty)\right) = \iint_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow P(S \cap S) = \iint_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow P(S) = \iint_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow 1 = \iint_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow \iint_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

<u>CP6</u>: $F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$

Proof: From <u>CP2</u>, we have

$$P(a < X \le b,\ c < Y \le d) = \int_{c}^{d}\int_{a}^{b} f(x,y)\,dx\,dy \quad \text{———} \; (1)$$

Putting $b = x$ and $d = y$, we get

$$P(a < X \le x,\ c < Y \le y) = \int_{c}^{y}\int_{a}^{x} f(x,y)\,dx\,dy \quad \text{———} \; (2)$$

Making $a \to -\infty$, $c \to -\infty$ in (2), we get

$$P(-\infty < X \le x,\ -\infty < Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$

$$\Rightarrow F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$

<u>CP7</u>: $F_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y)\,dx\,dy$

$$= \int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)dx$$

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$$F_Y(y) = \int_{-\infty}^{y} \left(\int_{-\infty}^{\infty} f(x,y)\,dx \right) dy$$

Proof: From CP6, we have

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\,dx\,dy \quad \text{——— (1)}$$

$$\Rightarrow F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\,dy\,dx \quad \text{——— (2)}$$

Making $y \longrightarrow \infty$ in (2), we get

$$F(x,\infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

$$\Rightarrow F_X(x) = \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f(x,y)\,dy \right) dx$$

$$\Rightarrow F_X(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} f(x,y)\,dx \right) dy$$

Making $x \longrightarrow \infty$ in (1), we get

$$F(\infty,y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow F_Y(y) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{y} f(x,y)\,dy \right) dx$$

CP8 : $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Proof : From CP7, we get

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \quad \text{——— (1)}$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \quad \text{——— (2)}$$

Differentiating (1) wrt x, we get

$$F_X'(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Differentiating (2) wrt y, we get

$$F_Y'(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Theorem: A necessary and sufficient condition for the independence of two random variables X and Y is

$f(x,y) = f_X(x)\, f_Y(y)$

for the case of continuous random variables

Proof: Condition is necessary: Let X and Y are independent

$\therefore F(x,y) = F_X(x) F_Y(y)$

$$\Rightarrow \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(F_X(x) F_Y(y)\right)$$

$$= \left\{\frac{d}{dx}\left(F_X(x)\right)\right\} F_Y(y) = F_X'(x) F_Y(y)$$

$$\Rightarrow \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial y}\left\{F_X'(x) F_Y(y)\right\}$$

$$= F_X'(x) \frac{d}{dy}\left(F_Y(y)\right) = F_X'(x) F_Y'(y)$$

$$\Rightarrow f(x,y) = f_X(x) f_Y(y)$$

Condition is sufficient:

Let $f(x,y) = f_X(x) f_Y(y)$

$$\Rightarrow \int_{-\infty}^{x} f(x,y)\,dx = \int_{-\infty}^{x} f_X(x) f_Y(y)\,dx$$

$$= f_Y(y) \int_{-\infty}^{x} f_X(x)\,dx$$

$$= f_Y(y) F_X(x)$$

$$\Rightarrow \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy = \int_{-\infty}^{y} f_Y(y) F_X(x)\,dy$$

$$\Rightarrow F(x,y) = F_X(x) \int_{-\infty}^{y} f_Y(y)\, dy$$

$$\Rightarrow F(x,y) = F_X(x)\, F_Y(y)$$

$\Rightarrow$ X and Y are indebendent

Conditional distribution:

Discrete case:

$$P(X = x_i \mid Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$$

$$P(Y = y_j \mid X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)}$$

Continuous case:

$$P(a < X \le b \mid y < Y \le y + \Delta y)$$

$$= \frac{P(a < X \le b,\ y < Y \le y + \Delta y)}{P(y < Y \le y + \Delta y)}$$

$$= \frac{\int_a^b \int_y^{y+\Delta y} f(x,y)\, dy\, dx}{\int_y^{y+\Delta y} f_Y(y)\, dy}$$

$$= \frac{\int_a^b \left(\int_y^{y+\Delta y} f(x,y)\, dy\right) dx}{\int_y^{y+\Delta y} f_Y(y)\, dy} = \frac{\int_a^b \Delta y\, f(x, \eta_1)\, dx}{\Delta y\, f_Y(\eta_2)}$$

where we have used the MVT of integral calculus and $y \le \eta_1, \eta_2 \le y + \Delta y$

$\therefore P(a < X \leq b \mid y < Y \leq y + \Delta y)$

$$= \frac{\int_a^b f(x, \eta_1)\, dx}{f_Y(\eta_2)} \quad \text{———} \quad (1)$$

Making $\Delta y \longrightarrow 0$, we get $\eta_1 = \eta_2 = y$ and consequently (1) assumes the following form:

$$P(a < X \leq b \mid Y = y) = \frac{\int_a^b f(x, y)\, dx}{f_Y(y)}$$

$$\Rightarrow P(a < X \leq b \mid Y = y) = \int_a^b \frac{f(x, y)}{f_Y(y)}\, dx$$

$$\Rightarrow P(a < X \leq b \mid Y = y) = \int_a^b f_X(x \mid y)\, dx$$

where

$$f_X(x \mid y) = \frac{f(x, y)}{f_Y(y)} \quad \text{———} \quad (2)$$

is known as the conditional density function of X on the basis of the hypothesis $Y = y$.

Similarly we can define

$$f_Y(y \mid x) = \frac{f(x, y)}{f_X(x)} \quad \text{———} \quad (3)$$

as the conditional probability density function of Y on the hypothesis $X=x$.

Therefore

$$P(c<Y\le d \mid X=x) = \int_c^d f_Y(y|x)\,dy$$

Therefore, the conditional distribution function of X on the hypothesis $Y=y$ is given by

$$F_X(x|y) = P(-\infty < X \le x \mid Y=y)$$

$$= \int_{-\infty}^{x} f_X(x|y)\,dx$$

Similarly, the conditional distribution function of Y on the hypothesis $X=x$ is given by

$$F_Y(y|x) = P(-\infty < Y \le y \mid X=x)$$

$$= \int_{-\infty}^{y} f_Y(y|x)\,dy$$

From (2) & (3), we get

✓ Theorem: $f(x,y) = f_Y(y)\, f_X(x|y) = f_X(x)\, f_Y(y|x)$

✓ $f(x,y) = f_Y(y)\, f_X(x|y) = f_X(x)\, f_Y(y|x)$

Theorem: If the random variables X and Y are independent then

$$f_Y(y) = f_Y(y|x) \text{ and } f_X(x) = f_X(x|y)$$

Proof: As the random variables X and Y are independent, we have

$$f(x,y) = f_X(x)\, f_Y(y) \quad \text{——— (1)}$$

Again from the conditional probability density function, we have

$$f(x,y) = f_X(x)\, f_Y(y|x) = f_Y(y)\, f_X(x|y) \quad \text{——— (2)}$$

From (1) & (2), we get

$$f_Y(y) = f_Y(y|x) \text{ and } f_X(x) = f_X(x|y)$$

Transformation of two-dimensional random variable (X, Y) to (U, V)

Consider the transformation $(x,y) \longrightarrow (u,v)$ under the law given by

$u = u(x,y)$ & $v = v(x,y)$

such that

either $\frac{\partial(u,v)}{\partial(x,y)} > 0 \quad \forall x, y$

or $\frac{\partial(u,v)}{\partial(x,y)} < 0 \quad \forall x, y$

Then the density function for (U,V) is given by

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|,$$

where $f_{U,V}(u,v)$ is the density of (U,V) and $f_{X,Y}(x,y)$ is the density of (X,Y).

<u>Proof:</u>

$P(u < U \leq u + \Delta u,\ v < V \leq v + \Delta v)$

$= P(x < X \leq x + \Delta x,\ y < Y \leq y + \Delta y)$

$$\Rightarrow \int_u^{u+\Delta u} \int_v^{v+\Delta v} f_{U,V}(u,v)\, du\, dv$$

$$= \int_x^{x+\Delta x} \int_y^{y+\Delta y} f_{X,Y}(x,y)\, dx\, dy$$

$$\Rightarrow f_{U,V}(u,v)\, \Delta u\, \Delta v = f_{X,Y}(x,y)\, \Delta x\, \Delta y$$

For small increment of x and y, $f_{X,Y}(x,y)$ can be considered as a constant and similarly $f_{U,V}(u,v)$ can be considered as a constant, i.e.,

$f_{X,Y}(x,y)$ can be regarded as a constant in $[x, x+\Delta x] \times [y, y+\Delta y]$ whereas

$f_{U,V}(u,v)$ can be regarded as a constant in $[u, u+\Delta u] \times [v, v+\Delta v]$

Again $\Delta x \, \Delta y = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$

$\therefore f_{U,V}(u,v)\, \Delta u \, \Delta v = f_{X,Y}(x,y) \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$

$\Rightarrow f_{U,V}(u,v) = f_{X,Y}(x,y) \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right|$

Some important continuous distributions:

① Uniform distribution in 2D:

Let D is a bounded subset of $R \times R$.

$$f(x,y) = \begin{cases} k & \text{for } x,y \in D \subseteq R^2 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a constant.

As $f(x,y)$ is a density function, we have the following property of $f(x,y)$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

$$\Rightarrow \iint_{D} f(x,y)\, dx\, dy + \iint_{(x,y)\in \bar{D}} f(x,y)\, dx\, dy = 1$$

$$\Rightarrow \iint_{(x,y)\in D} k\cdot dx\, dy + \iint_{(x,y)\in \bar{D}} 0\cdot dx\, dy = 1$$

$$\Rightarrow k \iint_{(x,y)\in D} dx\, dy = 1$$

$$\Rightarrow k\,|D| = 1$$

$$\Rightarrow k = \frac{1}{|D|}$$

where $|D|$ is the area of D.

Bivariate Normal distribution :-

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\Bigg[-\frac{1}{2(1-\rho^2)}$$

$$\times \left\{\left(\frac{x-m_x}{\sigma_x}\right)^2 - 2\rho\, \frac{x-m_x}{\sigma_x}\cdot\frac{y-m_y}{\sigma_y} + \left(\frac{y-m_y}{\sigma_y}\right)^2\right\}\Bigg],$$

$$-\infty < x < \infty, \quad -\infty < y < \infty$$

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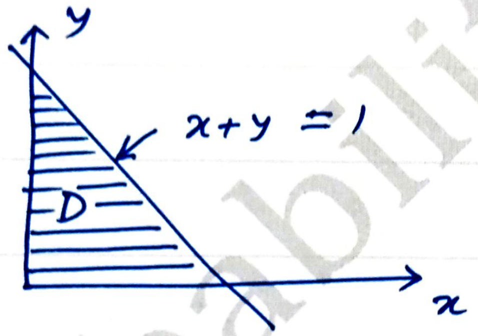
Here m_x, m_y, $\sigma_x > 0$, $\sigma_y > 0$, ρ $(|\rho|<1)$ are the five parameters of the distribution.

<u>Problem-1</u>: The joint pdf of the random variables X and Y is given by

$$f(x,y) = \begin{cases} k(1-x-y), & x>0,\ y>0,\ x+y<1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the value of $\underline{k}$ and $\underline{P(X<\frac{1}{2}, Y>\frac{1}{4})}$.

<u>Answer:</u>



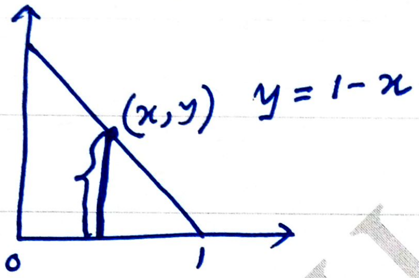
$$f(x,y) = \begin{cases} k(1-x-y), & (x,y) \in D \\ 0, & (x,y) \in \overline{D} \end{cases}$$

As $f(x,y)$ is a pdf,

$$\iint_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \iint_D f(x,y)\,dx\,dy + \underbrace{\iint_{\overline{D}} f(x,y)\,dx\,dy}_{0} = 0$$

$$\Rightarrow \iint_D f(x,y)\,dx\,dy = 1$$



$$\Rightarrow \int_0^1 \int_0^{1-x} k(1-x-y)\,dy\,dx = 1$$

$$\Rightarrow k\int_0^1 \left\{\int_0^{1-x} (1-x-y)\,dy\right\} dx = 1$$

$$\Rightarrow k\int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx = 1$$

$$\Rightarrow k\int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx = 1$$

$$\Rightarrow k\int_0^1 \frac{(1-x)^2}{2}\,dx = 1$$

$$\Rightarrow \frac{k}{2}\int_0^1 (1-x)^2\,dx = 1$$

$$\Rightarrow \frac{k}{2}\int_0^1 (x-1)^2\,dx = 1$$

$$\Rightarrow \frac{k}{2}\int_0^1 (1+x^2-2x)\,dx = 1$$

$$\Rightarrow \frac{k}{2}\left[x + \frac{x^3}{3} - 2\cdot\frac{x^2}{2}\right]\Big|_0^1 = 1$$

$$\Rightarrow \frac{k}{2}\left[1 + \frac{1}{3} - 1\right] = 1$$

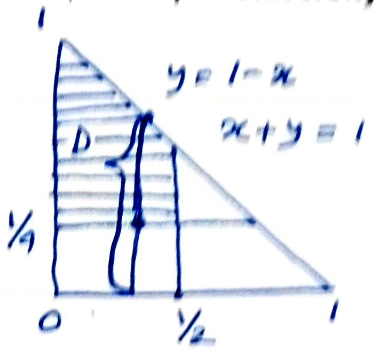
$$\Rightarrow \frac{k}{2}\cdot\frac{1}{3} = 1 \Rightarrow \boxed{k=6}$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$P(X < \frac{1}{2}, Y > \frac{1}{4})$

$= P(0 < X < \frac{1}{2}, \frac{1}{4} < Y < 1-x)$

$= P((X,Y) \in D)$



$= \int_0^{1/2} \int_{1/4}^{1-x} f(x,y)\, dy\, dx$

$= \int_0^{1/2} \left\{ \int_{1/4}^{1-x} k(1-x-y)\, dy \right\} dx$

$= k \int_0^{1/2} \left[(1-x)y - \frac{y^2}{2} \right]_{1/4}^{1-x} dx$

$= k \int_0^{1/2} \left[(1-x)\left\{1-x-\frac{1}{4}\right\} - \frac{1}{2}\left\{(1-x)^2 - \left(\frac{1}{4}\right)^2\right\}\right] dx$

$= k \int_0^{1/2} \left[(1-x)^2 - \frac{1}{4}(1-x) - \frac{1}{2}(1-x)^2 + \frac{1}{32} \right] dx$

$= k \int_0^{1/2} \left[\frac{(1-x)^2}{2} - \frac{1}{4}(1-x) + \frac{1}{32} \right] dx$

$= k \int_1^{1/2} \left(\frac{p^2}{2} - \frac{p}{4} + \frac{1}{32} \right)(-dp)$

Put $1-x = p$

$\Downarrow$

$dx = -dp$

when $x = 0, p = 1$

when $x = 1/2, p = 1/2$

$= k \int_{1/2}^{1} \left(\frac{p^2}{2} - \frac{p}{4} + \frac{1}{32} \right) dp$

$= k \left[\frac{1}{2}\frac{p^3}{3} - \frac{1}{4}\cdot\frac{p^2}{2} + \frac{1}{32}p \right]_{1/2}^{1}$

$$= k\left[\frac{1}{6}\left\{1^3-\left(\tfrac{1}{2}\right)^3\right\} - \frac{1}{8}\left\{1^2-\left(\tfrac{1}{2}\right)^2\right\} + \frac{1}{32}\left\{1-\frac{1}{2}\right\}\right]$$

$$= k\left[\frac{1}{6}\left(1-\frac{1}{8}\right) - \frac{1}{8}\left(1-\frac{1}{4}\right) + \frac{1}{32}\cdot\frac{1}{2}\right]$$

$$= k\left[\frac{7}{6\times 8} - \frac{3}{8\times 4} + \frac{1}{32\times 2}\right]$$

$$= \frac{k}{16}\left[\frac{7}{3} - \frac{3}{2} + \frac{1}{4}\right]$$

$$= \frac{k}{16}\times\frac{28-18+3}{12}$$

$$= \frac{k}{16}\times\frac{13}{12}$$

$$= \frac{6}{16}\times\frac{13}{12}$$

$$= \frac{13}{16\times 2} = \frac{13}{32}$$

<u>Problem-2</u>: Two points P and Q are chosen at random on a line segment of length a. What is the probability that the distance between P and Q is less than b $(<a)$?

<u>Answer</u>:

Let AB be a line segment of length a,

$\overline{AP} = X$ and $\overline{AQ} = Y$

Therefore, X and Y both are uniformly distributed in $(0, a)$ and consequently the pdf of X and Y are given by

$$f_X(x) = \begin{cases} \frac{1}{a}, & 0 < x < a \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{a}, & 0 < y < a \\ 0, & \text{elsewhere} \end{cases}$$

As X and Y are independent, the joint probability density function is given by

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} \frac{1}{a^2}, & 0 < x < a, 0 < y < a \\ 0, & \text{elsewhere} \end{cases}$$

The required probability

$= P(X \sim Y < b) = P(|X - Y| < b)$

$= P(-b < X - Y < b)$

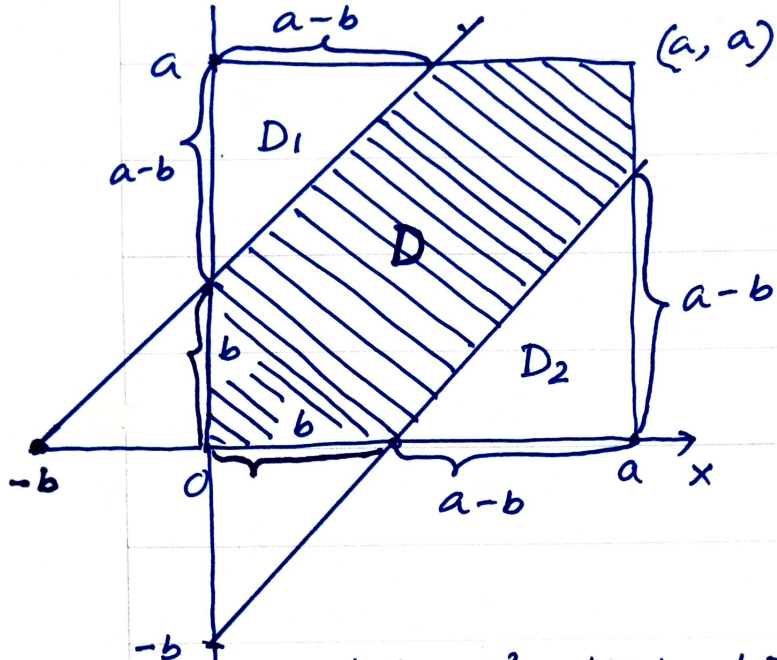
$= P((X, Y) \in D)$

$\left[\text{where } D = \{(x,y) \mid 0 < x < a,\ 0 < y < a,\ -b < x - y < b\}\right]$

$$= \iint_{(x,y)\in D} f(x,y)\,dx\,dy = \iint_D \frac{1}{a^2}\,dx\,dy$$

$$= \frac{1}{a^2} \iint_D dx\,dy = \frac{1}{a^2} |D| = \frac{|D|}{a^2}$$

where $|D| =$ area of D



$x - y = b$

$\Rightarrow \frac{x}{b} + \frac{y}{-b} = 1$

$x - y = -b$

$\frac{x}{-b} + \frac{y}{b} = 1$

$x - y < b$

$x - y > -b$

$|D| = a^2 - |D_1| - |D_2|$

$\Downarrow$

$|D| = a^2 - 2 \cdot \frac{1}{2}(a-b)^2$

$= a^2 - (a-b)^2 = (a - a + b)(a + a - b)$

$= b(2a - b)$

$\therefore \; P(|X - Y| < b) = \frac{b(2a-b)}{a^2} = \frac{b}{a^2}(2a - b)$

<u>Problem-3</u>: Two points are chosen at random independently in $(0, 1)$. Show that the probability that the distance between them is less than a fixed number c $(0 < c < 1)$ is $c(2 - c)$

<u>Answer:</u> This problem is exactly same as <u>Problem-2</u>; where $a=1$ and $b=c$.

<u>Problem-4:</u> A straight line segment AB is divided into two parts AC and CB whose lengths are 'a' and 'b' respectively with $a>b$. If two points P and Q are independently chosen at random on AC and CB respectively, find the probability that AP, PQ, QB can form a triangle.

<u>Answer:</u>

Let $\overline{AP} = X$ and $\overline{AQ} = Y \Rightarrow \overline{PQ} = Y - X$, $\overline{QB} = a+b-Y$

$$\therefore f_X(x) = \begin{cases} \frac{1}{a}, & 0<x<a \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{b}, & a<y<a+b \\ 0, & \text{elsewhere} \end{cases}$$

$$f(x,y) = f_X(x) f_Y(y)$$

$$= \begin{cases} \frac{1}{ab}, & 0<x<a,\ a<y<a+b \\ 0, & \text{elsewhere} \end{cases}$$

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The required probability

= P(AP, PQ, QB form the sides of a triangle)

Now AP, PQ, QB form the sides of a triangle

iff $AP+PQ > QB$, $PQ+QB > AP$ and $QB+AP > PQ$

i.e., iff $\underline{x} + Y - \underline{x} > a+b-Y$, $\underline{Y} - x + a + b - \underline{Y} > x$ and

$a+b-Y+x > Y-x$

i.e., iff $2Y > a+b$, $-2x+a+b > 0$ and

$a+b > 2Y-2x$

i.e., iff $Y > \frac{a+b}{2}$, $x < \frac{a+b}{2}$, $Y - x < \frac{a+b}{2}$

The required probability

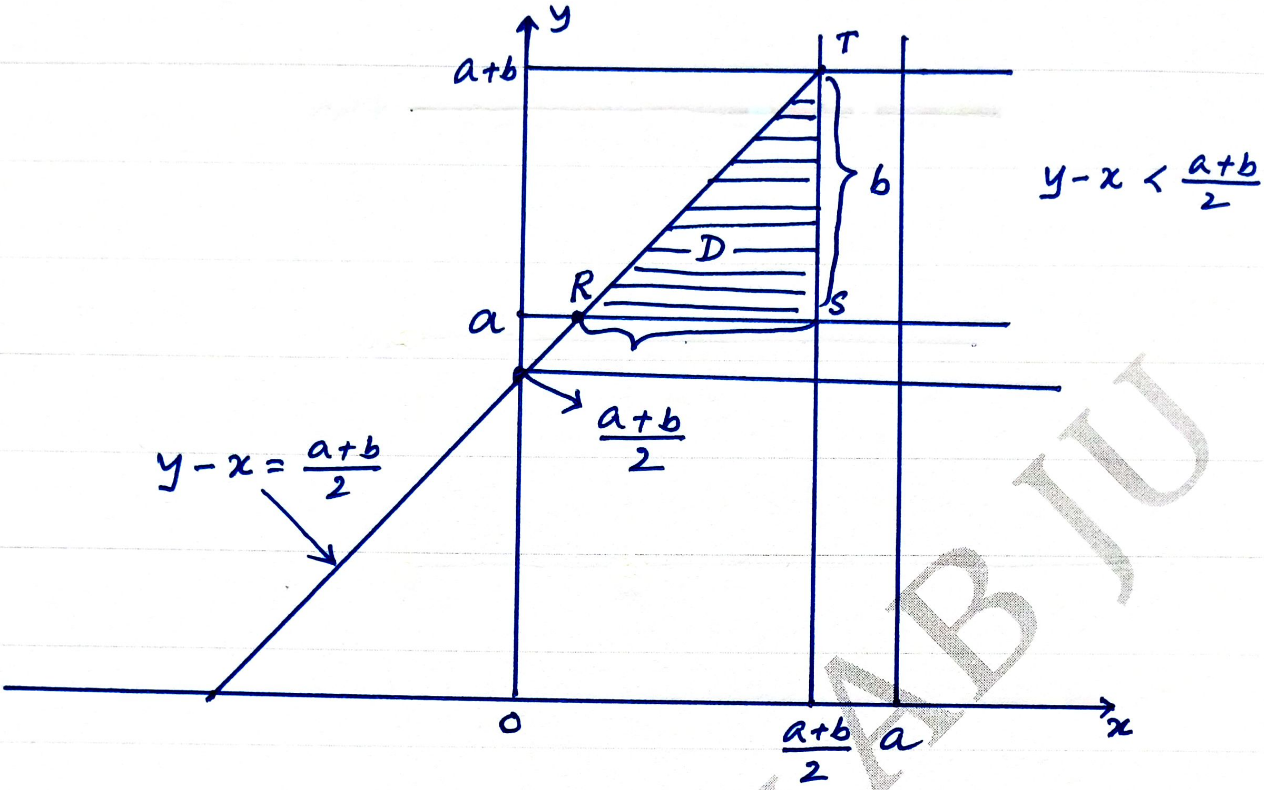
$= P((x,Y) \in D)$

[where $D = \left\{(x,y) : x < \frac{a+b}{2},\ y > \frac{a+b}{2},\ y-x < \frac{a+b}{2}\right\}$]

$$= \iint_{(x,y)\in D} f(x,y)\, dx\, dy$$

$$= \iint_{(x,y)\in D} \frac{1}{ab}\, dx\, dy = \frac{1}{ab} \iint_{(x,y)\in D} dx\, dy = \frac{1}{ab}|D|$$

where $|D|$ = area of D.



coordinate of $R = (h,k) \Rightarrow k=a,\quad k-h=\frac{a+b}{2}$

$$\Downarrow$$

$$a-h=\frac{a+b}{2}$$

$$\Downarrow$$

$$h=a-\frac{a+b}{2}=\frac{a-b}{2}$$

$RS=\frac{a+b}{2}-\frac{a-b}{2}=b$ ✓

$ST=b$

$|D|=\frac{1}{2}b\cdot b=\frac{b^2}{2}$

$\therefore$ Required probability $=\frac{|D|}{ab}=\frac{1}{ab}\cdot\frac{1}{2}b^2=\frac{b}{2a}$

Problem-5: In the quadratic equation

$x^2 + 2ax + b = 0$,

a, b are independently chosen at random from the interval $(-1, 1)$. Find the probability that the roots of the quadratic equation are real.

Answer: Let A and B be the random variables corresponding to a and b respectively.

Therefore, we have

$$f_A(a) = \begin{cases} \frac{1}{1-(-1)}, & -1 < a < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{1}{2}, & -1 < a < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_B(b) = \begin{cases} \frac{1}{2}, & -1 < b < 1 \\ 0, & \text{elsewhere} \end{cases}$$

As A and B are independent, the joint probability density function is given by

$$f(a,b) = f_A(a) f_B(b) = \begin{cases} \frac{1}{4}, & -1 < a < 1, -1 < b < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{1}{4}, & -1 < a, b < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Now the roots of the equation

$x^2 + 2ax + b = 0$

are real iff $(2a)^2 - 4 \cdot 1 \cdot b \geq 0$

i.e., iff $4a^2 - 4b \geq 0$

i.e., iff $a^2 - b \geq 0$

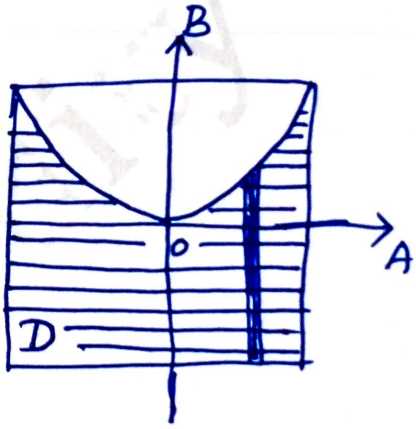
i.e., iff $a^2 \geq b$

i.e., iff $b \leq a^2$

$\therefore$ The required probability

$= P(B \leq A^2)$

$= \iint_D f(a, b)\, da\, db$



$= \iint_D \frac{1}{4}\, da\, db$

$= \frac{1}{4} \iint_D da\, db$

$D = \{(a, b) : -1 < a, b < 1, b \leq a^2\}$

$= \frac{1}{4} |D|$

$|D| =$ area of D

$= \frac{1}{4} \int_{-1}^{1} \int_{-1}^{b} da\, db = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{a^2} da\, db$

$= \frac{1}{4} \int_{-1}^{1} \left\{ \int_{-1}^{a^2} db \right\} da = \frac{1}{4} \int_{-1}^{1} \left(b \Big|_{-1}^{a^2} \right) da$

$= \frac{1}{4} \int_{-1}^{1} (a^2 + 1)\, da = \frac{1}{4} \cdot 2 \int_{0}^{1} (a^2 + 1)\, da = \frac{1}{2} \left[\frac{1}{3} + 1 \right] = \frac{2}{3}$

Problem-6: Two numbers are chosen independently at random from the interval $(0,1)$. Find the probability that the sum of the numbers is greater than 1 but the sum of their square is less than 1.

Answer: Let X and Y be the random variables corresponding to two chosen numbers x and y from the interval $(0,1)$. Therefore, if $f_X(x)$, $f_Y(y)$, $f(x,y)$ are the density function of X, density function of Y and the joint density function respectively, then we have

$$f_X(x) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} 1, & 0<x,y<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 1, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

The required probability

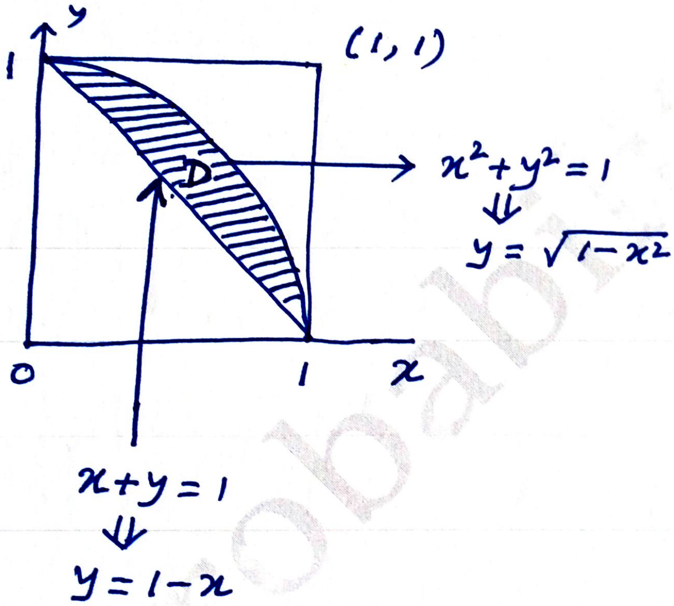
$= P(X+Y>1 \ \& \ X^2+Y^2<1)$

$= P((X,Y) \in D)$, where

$D = \{(x,y) : 0<x<1, 0<y<1, x+y>1 \ \& \ x^2+y^2<1\}$

$= \iint_D f(x,y)\, dx\, dy$

$= \iint_D 1 \cdot dx\, dy = \iint_D dx\, dy = |D|$



$|D| = \dfrac{\pi \cdot 1^2}{4} - \dfrac{1}{2} \cdot 1 \cdot 1$

$= \dfrac{\pi}{4} - \dfrac{1}{2}$

$= \dfrac{1}{4}(\pi - 2)$

The required probability

$= \dfrac{1}{4}(\pi - 2)$

$$|D| = \int_0^1 \left\{ \int_{1-x}^{\sqrt{1-x^2}} dy \right\} dx$$

$$= \int_0^1 \left(y \Big|_{1-x}^{\sqrt{1-x^2}} \right) dx$$

$$= \int_0^1 \{\sqrt{1-x^2} - (1-x)\} dx$$

$$= \int_0^1 \sqrt{1-x^2}\, dx - \int_0^1 (1-x)\, dx$$

$$= \int_0^1 \sqrt{1-x^2}\, dx - \left(x - \frac{x^2}{2}\right)\Big|_0^1$$

$$= \int_0^1 \sqrt{1-x^2}\, dx - \left(1 - \frac{1}{2}\right)$$

$$= \int_0^1 \sqrt{1-x^2}\, dx - \frac{1}{2}$$

$$= \int_0^{\pi/2} \cos\theta \cos\theta\, d\theta - \frac{1}{2}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} = \frac{1}{4}(\pi - 2)$$

Problem-7: Two numbers are chosen independently at random between 0 and 1. such that their product is less than k, where $0 < k < 1$

Problem-8: If X and Y are independent Binomial (n_1, p) and (n_2, p) variates then show that their sum $X+Y$ is a Binomial (n_1+n_2, p) variate.

Answer: Let $U = X+Y$

Now $X \simeq B(n_1, p)$

$$\Downarrow$$

$$P(X=i) = \binom{n_1}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, 2, \ldots, n_1$$

Again $Y = B(n_2, p)$

$$\Downarrow$$

$$P(Y=j) = \binom{n_2}{j} p^j (1-p)^{n-j}, \quad j = 0, 1, 2, \ldots, n_2$$

Therefore, the spectrum of U is $\{0, 1, 2, \ldots, n_1+n_2\}$

$$P(U=k) = P(X+Y=k)$$

$$= \sum_{i+j=k} P(X=i, Y=j)$$

$$= \sum_{i+j=k} P(X=i)\,P(Y=j) \quad \text{[As X and Y are independent]}$$

$$P(U=k) = \sum_{i+j=k} P(X=i)\, P(Y=j)$$

$$= \sum_{i+j=k} \binom{n_1}{i} p^i (1-p)^{n_1-i} \binom{n_2}{j} p^j (1-p)^{n_2-j}$$

$$= \sum_{i+j=k} \binom{n_1}{i}\binom{n_2}{j} p^{i+j} (1-p)^{n_1+n_2-(i+j)}$$

$$= \sum_{i+j=k} \binom{n_1}{i}\binom{n_2}{j} p^{k} (1-p)^{n_1+n_2-k}$$

$$= p^k (1-p)^{n_1+n_2-k} \sum_{i+j=k} \binom{n_1}{i}\binom{n_2}{j} \quad \text{---} (1)$$

Consider the following identity

$$(1+x)^{n_1+n_2} = (1+x)^{n_1}(1+x)^{n_2}$$

$$\Rightarrow \sum_{p=0}^{n_1+n_2} \binom{n_1+n_2}{p} x^p = \sum_{i=0}^{n_1} \binom{n_1}{i} x^i \sum_{j=0}^{n_2} \binom{n_2}{j} x^j$$

$$\Rightarrow \sum_{p=0}^{n_1+n_2} \binom{n_1+n_2}{p} x^p = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i}\binom{n_2}{j} x^{i+j}$$

Equating the coefficient of x^k on both sides, we get

$$\binom{n_1+n_2}{k} = \sum_{i+j=k} \binom{n_1}{i}\binom{n_2}{j} \quad \text{------ } (2)$$

From (1) & (2), we get

$$P(U=k) = p^k (1-p)^{n_1+n_2-k} \binom{n_1+n_2}{k}$$

$$\Rightarrow P(U=k) = \binom{n_1+n_2}{k} p^k (1-p)^{n_1+n_2-k}$$

$$\Rightarrow U = B(n_1+n_2, p)$$

Problem-9: If X and Y are two independent Poisson variates with parameters μ and λ respectively, then show that their sum $U = X+Y$ is also a Poisson variate with parameter $\lambda+\mu$.

Answer: According to the problem,

$$P(X=i) = \frac{e^{-\mu}\mu^i}{i!}, \quad i = 0, 1, 2, \ldots$$

$$P(Y=j) = \frac{e^{-\lambda}\lambda^j}{j!}, \quad j = 0, 1, 2, \ldots$$

Therefore the spectrum of U is $\{0, 1, 2, \ldots\}$

$$P(U=k) = P(X+Y=k)$$

$$= \sum_{i+j=k} P(X=i,\ Y=j)$$

$$= \sum_{i+j=k} P(X=i)\,P(Y=j)$$

$$= \sum_{i+j=k} \frac{e^{-\mu}\mu^{i}}{i!}\,\frac{e^{-\lambda}\lambda^{j}}{j!}$$

$$= e^{-\mu-\lambda} \sum_{i+j=k} \frac{\mu^{i}\lambda^{j}}{i!\,j!}$$

$$= e^{-\mu-\lambda} \sum_{i=0}^{k} \frac{\mu^{i}\lambda^{k-i}}{i!\,(k-i)!}$$

$$= e^{-(\mu+\lambda)} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!}\cdot\frac{1}{k!}\,\mu^{i}\lambda^{k-i}$$

$$= e^{-(\mu+\lambda)}\cdot\frac{1}{k!}\cdot\sum_{i=0}^{k}\binom{k}{i}\mu^{i}\lambda^{k-i}$$

$$= \frac{e^{-(\mu+\lambda)}}{k!}(\mu+\lambda)^{k}$$

$$= \frac{(\mu+\lambda)^{k}e^{-(\mu+\lambda)}}{k!}$$

$\Rightarrow$ U is a Poisson variate with parameter $\underline{\mu+\lambda}$

Problem-10: If X and Y are two independent $N(m_x, \sigma_x)$ and $N(m_y, \sigma_y)$ variates then their sum is a normal (m, σ) variate where $m = m_x + m_y$ and $\sigma = \sqrt{\sigma_x^2 + \sigma_y^2}$

Answer:

$$X = N(m_x, \sigma_x) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left(\frac{x-m_x}{\sigma_x}\right)^2}, -\infty < x < \infty$$

$$Y = N(m_y, \sigma_y) \Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{1}{2}\left(\frac{y-m_y}{\sigma_y}\right)^2}, -\infty < y < \infty$$

Therefore, the joint density of X and Y is given by

$f(x,y) = f_X(x) f_Y(y)$ [As X and Y are independent]

$$= \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-m_x}{\sigma_x}\right)^2 + \left(\frac{y-m_y}{\sigma_y}\right)^2\right]}, -\infty < x < \infty, -\infty < y < \infty$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left\{\left(\frac{x-m_x}{\sigma_x}\right)^2 + \left(\frac{y-m_y}{\sigma_y}\right)^2\right\}\right], -\infty < x < \infty, -\infty < y < \infty$$

Let $U = X + Y$, $V = X \Rightarrow u = x + y$, $v = x$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0 \;\forall x, y$$

If $g(u,v)$ is the joint pdf of U and V, then

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$g(u,v) = f(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = f(x,y)\,|-1| = f(x,y)$$

$$\Rightarrow g(u,v) = f(x,y)$$

$$= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left\{\left(\frac{x-m_x}{\sigma_x}\right)^2+\left(\frac{y-m_y}{\sigma_y}\right)^2\right\}\right]$$

$$= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left\{\left(\frac{v-m_x}{\sigma_x}\right)^2+\left(\frac{u-v-m_y}{\sigma_y}\right)^2\right\}\right] \quad \text{(1)}$$

Now $\left(\frac{v-m_x}{\sigma_x}\right)^2+\left(\frac{u-v-m_y}{\sigma_y}\right)^2$

$$= \left(\frac{v-m_x}{\sigma_x}\right)^2+\left\{\frac{u-m_y-(v-m_x+m_x)}{\sigma_y}\right\}^2$$

$$= \left(\frac{v-m_x}{\sigma_x}\right)^2+\left\{\frac{u-m_y-m_x-(v-m_x)}{\sigma_y}\right\}^2$$

$$= \left(\frac{v-m_x}{\sigma_x}\right)^2+\left\{\frac{u-m-(v-m_x)}{\sigma_y}\right\}^2, \quad [m = m_x+m_y]$$

$$= \left(\frac{v-m_x}{\sigma_x}\right)^2+\left(\frac{u-m}{\sigma_y}\right)^2+\left(\frac{v-m_x}{\sigma_y}\right)^2-2\cdot\frac{u-m}{\sigma_y}\cdot\frac{v-m_x}{\sigma_y}$$

$$= (v-m_x)^2\left\{\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}\right\}+\left(\frac{u-m}{\sigma_y}\right)^2-2\cdot\frac{u-m}{\sigma_y}\cdot\frac{v-m_x}{\sigma_y}$$

$$= (v-m_x)^2\frac{\sigma^2}{\sigma_x^2\sigma_y^2}+\left(\frac{u-m}{\sigma_y}\right)^2-2\cdot\frac{u-m}{\sigma_y}\cdot\frac{v-m_x}{\sigma_y}$$

$$\boxed{\sigma^2 = \sigma_x^2+\sigma_y^2}$$

$$= \frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left[(v-m_x)^2+\frac{\sigma_x^2}{\sigma^2}(u-m)^2 - 2\cdot\frac{\sigma_x^2}{\sigma^2}(u-m)(v-m_x)\right]$$

Anup Bandyopadhyay, Department of Mathematics, Jadavpur University

$$\left(\frac{v-m_x}{\sigma_x}\right)^2 + \left(\frac{u-v-m_y}{\sigma_y}\right)^2$$

$$= \frac{\sigma^2}{\sigma_x^2\sigma_y^2}\Big[(v-m_x)^2 - 2\cdot(v-m_x)\cdot\left\{\frac{\sigma_x^2}{\sigma^2}(u-m)\right\}$$

$$+\left\{\frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2 - \left\{\frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2 + \frac{\sigma_x^2}{\sigma^2}(u-m)^2\Big]$$

$$= \frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left[\left\{v-m_x - \frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2 + \frac{\sigma_x^2}{\sigma^2}(u-m)^2\left(1-\frac{\sigma_x^2}{\sigma^2}\right)\right]$$

$$= \frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left[\left\{v-m_x - \frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2 + \frac{\sigma_x^2}{\sigma^2}(u-m)^2\frac{\sigma_y^2}{\sigma^2}\right]$$

$$= \frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left\{v-m_x - \frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2 + \left(\frac{u-m}{\sigma}\right)^2$$

$$= \frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left\{v-m_x - \frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2 + \left(\frac{u-m}{\sigma}\right)^2$$

The density function of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} g(u,v)\,dv$$

$$= \int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left\{v-m_x-\frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2 - \frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]dv$$

$$= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]\times I$$

$$I = \int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left\{v-m_x-\frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2\right]dv$$

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$$I = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\frac{\sigma^2}{\sigma_x^2\sigma_y^2}\left\{v - m_x - \frac{\sigma_x^2}{\sigma^2}(u-m)\right\}^2\right] dv$$

Put $p = v - m_x - \frac{\sigma_x^2}{\sigma^2}(u-m)$

$$= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\frac{\sigma^2}{\sigma_x^2\sigma_y^2} p^2\right] dp$$

$$= 2\int_{0}^{\infty} \exp\left[-\frac{1}{2}\frac{\sigma^2}{\sigma_x^2\sigma_y^2} p^2\right] dp$$

Put $\frac{1}{2}\frac{\sigma^2}{\sigma_x^2\sigma_y^2} p^2 = q$

$$\Downarrow$$

$$p^2 = \frac{2q\,\sigma_x^2\sigma_y^2}{\sigma^2}$$

$$\Downarrow$$

$$p = \frac{\sqrt{2}\,\sigma_x\sigma_y}{\sigma}\sqrt{q}$$

$$\Downarrow$$

$$dp = \frac{\sqrt{2}\,\sigma_x\sigma_y}{\sigma}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{q}}dq$$

$$= 2\int_{0}^{\infty} \exp(-q)\cdot\frac{\sqrt{2}\,\sigma_x\sigma_y}{2\sigma}\cdot\frac{1}{\sqrt{q}}\, dq$$

$$= \frac{\sqrt{2}\,\sigma_x\sigma_y}{\sigma}\int_{0}^{\infty} \exp(-q)\, q^{\frac{1}{2}-1}\, dq$$

$$= \frac{\sqrt{2}\,\sigma_x\sigma_y}{\sigma}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{2}\,\sigma_x\sigma_y}{\sigma}\cdot\sqrt{\pi}$$

$$= \frac{\sqrt{2\pi}\,\sigma_x\sigma_y}{\sigma}.$$

$$\therefore\ f_U(u) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2\right]\times\frac{\sqrt{2\pi}\,\sigma_x\sigma_y}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2}\left(\frac{u-m}{\sigma}\right)^2\right],\ -\infty < u < \infty$$

$$\Rightarrow \boxed{U = N(m, \sigma)}$$

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Reproductive property of random variables:

1. Sum of two independent normal variates is also a normal variate

2. Sum of two independent Binomial variates $B(n_1, p)$ & $B(n_2, p)$ is also a Binomial (n_1+n_2, p) variate

3. Sum of two independent Poisson variate is also a Poisson variate

Problem – 11 : The joint probability density function of two random variables X and Y is given by

$$f(x,y) = \begin{cases} kxy, & 0<x<4, \; 1<y<5 \\ 0, & \text{elsewhere} \end{cases}$$

1. Find the value of k
2. Find the probability density of $X+2Y$
3. Find the joint density of $U = XY^2$ & $V = X^2Y$
4. Find the density of $U = XY^2$
5. Find the density of $V = X^2Y$
6. Find $P(X+Y<3)$
7. Find $P(X^2+Y^2<4)$