Physics 22: Homework 8 Hints

1. Since the speeds of the particles are much less than the speed of light, then the magnetic-field of a moving point charge is quite applicable in this case:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}.$$

Here, $\hat{r}$ points from the moving charge to the observation point and q can either be positive or negative depending on the sign of the charge. We take $\odot$ to be the $+z$-direction and $v_e = v_p = v$.

(a) The answer is:

$$\vec{B}(0,0) = \vec{B}_e(0,0) + \vec{B}_p(0,0) = \frac{\mu_0}{4\pi}\left(-\hat{z}\frac{ev_e}{b^2} - \hat{z}\frac{ev_p}{a^2}\right) = -\hat{z}\frac{\mu_0}{4\pi}ev\left(\frac{1}{b^2}+\frac{1}{a^2}\right).$$

(b) The answer is:

$$\vec{B}_e(a,0) = -\hat{z}\frac{\mu_0}{4\pi}\frac{evb}{\left(a^2+b^2\right)^{3/2}}.$$

(c) The electric field of the electron at the location of the proton at this instant is:

$$\vec{E}_e(a,0) = \frac{e}{4\pi\varepsilon_0}\frac{1}{\left(a^2+b^2\right)^{3/2}}\left(-a\hat{x}+b\hat{y}\right).$$

Thus, the electric force on the proton is

$$\vec{F}_{E_{pe}} = e\vec{E}_e(a,0) = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\left(a^2+b^2\right)^{3/2}}\left(-a\hat{x}+b\hat{y}\right).$$

The magnetic force on the proton is:

$$\vec{F}_{B_{pe}} = q_p\vec{v}_p\times\vec{B}_e(a,0) = +\hat{x}\frac{\mu_0}{4\pi}\frac{e^2v^2b}{\left(a^2+b^2\right)^{3/2}}.$$

Thus, the Lorentz force becomes:

$$\vec{F}_{L_{pe}} = \vec{F}_{E_{pe}} + \vec{F}_{B_{pe}}.$$

(d) The electric force on the electron is

$$\vec{F}_{E_{ep}} = -e\vec{E}_p(0,b) = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\left(a^2+b^2\right)^{3/2}}\left(a\hat{x}-b\hat{y}\right).$$

The magnetic force on the electron is

$$\vec{F}_{B_{ep}} = q_e\vec{v}_e\times\vec{B}_p(0,b) = +\hat{y}\frac{\mu_0}{4\pi}\frac{e^2v^2a}{\left(a^2+b^2\right)^{3/2}}.$$

One can then see that neither are the strengths of the magnetic forces on either charge due to the other equal, nor are they in opposite directions. The resolution to this paradox is the fact that the fields also carry momentum. If the field momentum density of the system is integrated over all of space, then the contribution this field momentum has can account for the missing information that seemingly violates the 3$^{\text{rd}}$ Law as it currently stands.

2. For this wire, radial distances lie in planes parallel to the yz-plane, so that an arbitrary radial distance, r, is given by

$$r = \sqrt{y^2+z^2},$$

while the magnitude of the field produced by this long, straight wire has the form

$$B(y,z) = \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{y^2 + z^2}}.$$

At the location of this electron, the field is directed in the $-z$-direction has the form

$$\vec{B}(y_e, z_e) = -\hat{z} \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{y_e^2 + z_e^2}} = -\hat{z} \frac{\mu_0 I}{2\pi} \frac{1}{y_e}.$$

Taking $v_x = 50.0$ km/s and $v_y = 30.0$ km/s, the force on this electron, with charge $q_e = -e$, is

$$\vec{F}_B = -e\vec{v}_e \times \vec{B}(y_e, z_e) = -\frac{\mu_0 e I}{2\pi y_e} (v_x \hat{y} + v_y \hat{x}).$$

3. The main point of this problem is to note that the field profile for any one of these wires has the form

$$B = \frac{\mu_0 I}{2\pi} \frac{1}{r},$$

with the form of the field lines being concentric circles with the center of the circle being the wire in question. The circle radius is determined by exactly where the field is to be determined by each wire. Finally, the total field will be the superposition of the fields.

(a) In this case, draw four circles centered on each of the wires, with the circles intersecting exactly at the center with each circle having a radius of $\ell/\sqrt{2}$, since this is the length from each corner of the square to its center. The magnetic field vector at the center will point tangent to the circle at that point, amounting to each field making a 45° angle relative to the horizontal (or vertical). Use the Right-Hand Rule (RHR) for determining the sense of the circulating magnetic field. Wires 1 and 3 contribute the exact same field, while Wires 2 and 4 also contribute the same field. Decompose the vectors into horizontal and vertical components to find the net field at the center. The net field will be leftward.

(b) In this case, for Wires 1, 2, and 3 draw circles with each one's center as each respective wire. Note that Wires 1 and 3 are closer to Wire 4 than Wire 2. Wire 1's field will be leftward, Wire 2's will have an upward and a leftward component, while Wire 3's will be downward.

(c) Upon finding the net field at the location of Wire 4 from Part (b), note that this field will, indeed, be perpendicular to Wire 4's current itself. Moreover, the field is exactly the same (in both magnitude and direction) along the whole length of Wire 4, given the translational symmetry of the field profile longitudinally along this wire. Since this is the case, consider a segment, of length L of Wire 4. If the net field due to the other wires is labeled $\vec{B}$, then the magnitude of the force on this segment will be

$$F_4 = \left| \int I_4 d\vec{L}_4 \times \vec{B} \right| = I_4 B \int dL_4 = IBL.$$

Therefore, the force per unit length on this wire will be

$$f_4 = \frac{F_4}{L} = IB,$$

with the direction given by the RHR for cross products.

(d) The field at the center of the configuration will zero when the currents running through Wires 2 and 3 is flipped. Parts (b) and (c) follow the exact same procedure, except with a modified field profile due to the flip.

4. Since we treat the fixed conducting wire at the very bottom to be very long, its field magnitude will have the form

$$B = \frac{\mu_0 I}{2\pi} \frac{1}{r},$$

where r is the radial distance from this wire. The net magnetic field felt by the levitating rod on the sliders will be the value of the field above at $r = h$, which is to be determined. Note that the force on the sliding rod due to the bottom fixed rod will be upward based on the Right-Hand Rule (RHR) for cross products applied to the magnetic force on a current-carrying wire. Indeed, we have seen that parallel wires supporting opposing currents will repel each other. This upward magnetic force will be countered by the downward gravitational force on the sliding rod. Since the magnetic force has a magnitude

$$F_B = IB\ell = \frac{\mu_0 I^2}{2\pi}\frac{\ell}{h},$$

and the gravitational force on the sliding rod has a magnitude

$$w = mg,$$

then a force balance between F_B and w will output the desired height. The answer is 3.84 mm.

5. Pick a point that is a distance s radially away from the wire segment, and a distance d, longitudinally away from an edge of the segment (e.g., the left edge). Upon taking $+z$ into the page, the profile becomes (upon proper application of the Biot-Savart Law)

$$\vec{B} = \hat{z}\frac{\mu_0 I}{4\pi}\frac{1}{s}\left[\frac{(\ell - d)}{\sqrt{s^2 + (\ell - d)^2}} + \frac{d}{\sqrt{s^2 + d^2}}\right].$$

   (a) Using the result above one can determine the field at point A by setting $d = \ell/2$, and the field at point B by setting $d = 0$. Again, it is instructive to derive the general form above to then see how these results are obtained from a first-principles application of the Biot-Savart Law.

   (b) At point A, the field profile becomes

$$\vec{B}_{\text{A}} = \hat{z}\frac{\mu_0 I}{2\pi}\frac{1}{s}\frac{\ell/2}{\sqrt{s^2 + \ell^2/4}} = \hat{z}\frac{\mu_0 I}{2\pi}\frac{1}{s}\frac{\ell/2}{\ell/2}\frac{1}{\sqrt{1 + 4s^2/\ell^2}} = \hat{z}\frac{\mu_0 I}{2\pi}\frac{1}{s}\left(1 + \frac{4s^2}{\ell^2}\right)^{-1/2}.$$

   Using the Binomial Approximation to zeroth order will give you the lowest-order, non-zero result for this field, which becomes the well-understood field profile of an infinite, current-carrying straight wire:

$$\vec{B}_{\text{A}} \approx \hat{z}\frac{\mu_0 I}{2\pi}\frac{1}{s}.$$

   Indeed, when one is infinitesimally close to the wire, the edges cannot be seen, so that the wire behaves as if it's infinite in this scheme.

6. Even though there is no battery physically making the charge move within the conducting material in this problem, the act of rotating the ring makes the glued charges move relative to the laboratory reference frame. As a result, this rotation amounts to the same effect as having a battery drive the charge carriers!

   (a) Since a given reference charge on the ring performs a full revolution in a time

$$\mathfrak{T} = \frac{2\pi}{\omega},$$

   which, of course, is known as the period, then the current is simply how much charge cross the reference line in a given amount of time. If we take this time to be $\mathfrak{T}$, then the total amount of charge that would cross a fixed reference line will be Q. Thus

$$I = \frac{Q}{\mathfrak{T}} = \frac{\omega Q}{2\pi}.$$

   (b) Since the loop is planar and the current is uniform throughout, then the dipole moment's magnitude will simply be the product of the current and the loop's area:

$$\mu = IA = \frac{\omega Q R^2}{2}.$$

7. In this problem, each segment of the rectangular loop of wire must be treated piece-by-piece in the analysis of the magnetic force. Recall that the field of this long straight wire is given by

$$B_1(r) = \frac{\mu_0 I_1}{2\pi}\frac{1}{r},$$

where $r$is measured radially from the wire. The direction of this wire's field at any location on the rectangular loop is out of the page based on the Right-Hand Rule (RHR) for determining the direction of the magnetic field of a long, straight, current-carrying wire.

   (a) Note that the bottom and top segments have the magnetic field due to the long straight wire vary spatially along the lengths of these segments (with the field getting weaker as one moves farther away from the wire along each of these segments), whereas the left and right segments have no variation in the field profile along their respective lengths since they are a fixed radius away from the wire along their lengths; however, the field strength for the right part of the rectangular loop is higher than for the left part. Although the top and bottom segments have a varying magnetic field across their lengths, note that for every infinitesimal segment chosen for the top wire, there is an identical infinitesimal segment for the bottom wire with the force contributions completely canceling (since they point in opposite directions). Thus, the net magnetic force contribution from the top and bottom segments completely cancel one another. However, the left and right segments of the rectangular loop will not completely cancel each other due to the asymmetry in the field strength when comparing the two segments. In particular, note that the direction of the force on the right segment is toward the long wire, while the direction of the force on the left segment is away from this wire. The net magnetic force is, therefore, rightward, and its magnitude is

$$F = I_2 \ell \left[B_1(a) - B_1(a+w)\right].$$

   (b) The force will flip if either I_1 or I_2 flips direction. This does not change the magnitude of the force, but it does change the force's direction based on the magnetic force law.

8. Here, one must be careful to use the Right-Hand Rule (RHR) for determining the direction of the magnetic field due to a long-straight wire. Note that the field of I_1 points out of the page to its left, but into the page to its right; whereas the field of I_2 points out of the page above it, but into the page below it.

   (a) Take the wires themselves to represent x- and y-axes, with $+x$ along the direction of I_2 and $+y$ along the direction of I_1, with the point of intersection between the wires as the origin of coordinates. Again, the fields of each of these wires will have a strength that varies inversely with the radius from each wire. To be very firm in the meaning of signs, take "out of the page" to be positive (i.e., $\odot$ means $+z$). In particular, the magnitude of the field of Wire 1 in the xy-plane may be written as

$$B_1(x) = -\frac{\mu_0 I_1}{2\pi}\frac{1}{x},$$

with $x > 0$ indicating that the field points into the page (since $B_1 < 0$ here), while $x < 0$ indicating that the field point out of the page (since $B_1 > 0$ here). Moreover, the magnitude of the field of Wire 2 in the xy-plane may be written as

$$B_2(y) = \frac{\mu_0 I_2}{2\pi}\frac{1}{y},$$

with $y > 0$ indicating that the field points out of the page (since $B_2 > 0$ here), while $y < 0$ indicating that the field points into the page (since $B_2 < 0$ here). Thus, the total field will be

$$B(x,y) = B_1(x) + B_2(y) = \frac{\mu_0}{2\pi}\left(-\frac{I_1}{x} + \frac{I_2}{y}\right) = \frac{\mu_0 I}{2\pi}\left(\frac{1}{y} - \frac{1}{x}\right),$$

with $B(x,y) > 0$ transparently indicating a net field that points out of the page (in line with our definition of the $+z$-direction above) and with $B(x,y) < 0$ indicating a net field that points into the page. Based on the analysis above, note that the fields point in the same direction in Quadrants II and IV, while the fields point in opposite direction in Quadrants I and III. Because of this, the fields along the line $y = -x$ will be double the field of one of the wires at a point on that line, while the fields along the line $y = x$ will be totally zero.

(b) In this case, $B(x,y)$ from Part (a) becomes

$$B(x,y) = B_1(x) + B_2(y) = \frac{\mu_0}{2\pi}\left(-\frac{I_1}{x} + \frac{I_2}{y}\right) = \frac{\mu_0 I}{2\pi}\left(\frac{2}{y} - \frac{1}{x}\right),$$

so that the fields along the line $y = -2x$ will be triple the field of Wire 1 at each of the points, while the fields along the line $y = 2x$ will completely cancel.

9. Consider breaking up the ribbon into a stack of very thin wires laterally along the ribbon, each supporting a current dI_2 running upward. Because this is a ribbon of current in which the current is uniformly distributed, the notion of current density may be used; however, the current density is not that of an areal density, but rather of a linear density, taken laterally along the width of the wire. Because the density of the current is uniform, then let us define the (linear) current density, $\vec{K}$, as a flux condition for the current dI_2:

$$dI_2 = \vec{K} \cdot \hat{n} dw,$$

where the direction of $\vec{K}$ is in the direction of the drift velocity of the charge carriers, $\hat{n}$ is the unit vector perpendicular to the reference "finish" line that we draw to count the charge passing the reference line in a given time, and dw is the infinitesimal width of one of these very thin wires laterally along the ribbon. Since $\vec{K}$ and $\hat{n}$ are parallel, then

$$dI_2 = K dw.$$

Finally, since the current is uniformly distributed, then we may write

$$K = \frac{I_2}{w}.$$

(a) The field of the straight wire will be into the page where the ribbon is location and will have the magnitude

$$B_1 = \frac{\mu_0 I_1}{2\pi} \frac{1}{r},$$

where r is measured radially from the straight wire. Consider one of the very thin wires—that is part of the stack that makes up the ribbon itself—that is a distance r away from the long, straight wire. Using these coordinates, the width of this piece of the ribbon is $dw = dr$. Taking right to be the $+x$-direction, for an infinitesimal (longitudinal) length of this infinitely long piece of the ribbon, $d\ell$, the magnetic force on it due to the wire's field is

$$d^2\vec{F}_{21} = dI_2 d\vec{\ell} \times \vec{B}_1(r) = -\hat{x} K dr d\ell B_1(r),$$

so that the force per unit length of this wire becomes

$$d\vec{f}_{21} = \frac{d^2\vec{F}_{21}}{d\ell} = -\hat{x}\frac{I_2}{w} dr \frac{\mu_0 I_1}{2\pi}\frac{1}{r},$$

with $r \in [a, a+w]$. We may, thus, integrate the force per unit length over the width of the ribbon to obtain the total force per unit length.

$$\vec{f}_{21} = \int d\vec{f}_{21} = -\hat{x}\frac{\mu_0 I_1 I_2}{2\pi w}\int_a^{a+w}\frac{dr}{r} = -\hat{x}\frac{\mu_0 I_1 I_2}{2\pi w}\ln\left(1 + \frac{w}{a}\right).$$

(b) The force per unit length on such a wire will, in general, be

$$\vec{f}_{21} = -\hat{x}\frac{\mu_0 I_1 I_2}{2\pi\xi}.$$

To make the results match, we must have

$$\xi = \frac{w}{\ln(1 + a/w)}.$$

10. The field of a single circular loop, of radius R and of current I, with its center at the origin of a z-axis that serves as its axis of symmetry was found in class to be

$$\vec{B} = \hat{z}\frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}},$$

which is valid for all z. Suppose we define the xy-plane to be parallel to the plane of either loop and cutting through the midpoint of the two loops. As a result, the center of the left loop will be at $z = -d/2$, while the center of the right loop will be at $z = +d/2$.

(a) If the currents flow in the same direction, then the above result can be used for both loops with no sign alterations, just with translations of the functions to account for each loop's respective new zero. In particular, we have for the Left (L) loop

$$\vec{B}_{\mathrm{L}} = \hat{z}\frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+(z+d/2)^2\right)^{3/2}},$$

while for the Right (R) loop

$$\vec{B}_{\mathrm{R}} = \hat{z}\frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+(z-d/2)^2\right)^{3/2}},$$

so that

$$\vec{B} = \hat{z}\frac{\mu_0 I R^2}{2}\left(\frac{1}{\left(R^2+(z+d/2)^2\right)^{3/2}} + \frac{1}{\left(R^2+(z-d/2)^2\right)^{3/2}}\right).$$

(b) If the L preserves its current's orientation but R switches, then the field of R will just pick up a negative sign. Thus, the net field along the z-axis will be

$$\vec{B} = \hat{z}\frac{\mu_0 I R^2}{2}\left(\frac{1}{\left(R^2+(z+d/2)^2\right)^{3/2}} - \frac{1}{\left(R^2+(z-d/2)^2\right)^{3/2}}\right).$$

11. Recall that from the Biot-Savart Law, any time the infinitesimal displacement along the (conventional) current direction, $d\vec{\ell}$, is along a line that is parallel to the unit vector, $\hat{r}$, that points from the current element to the observation point, then by the rules of the cross product the contribution to the magnetic field of such current elements is zero. Thus, for this reason, straight segments of wire on the very right, on the very left, and on top of point P make absolutely no contribution to the field at this point. Thus, the only contributions arise from the quarter circles. However, also note that for any current segment on either of the quarter circles, $d\vec{\ell} \times \hat{r}$ points out of the page and always has $d\vec{\ell}$ and $\hat{r}$ orthogonal to one another. Moreover, for each quarter circle, the distance from each current element is fixed because P is the center of either quarter circle. Taking $\odot$ to be the $+z$ direction, then the answer is:

$$\vec{B}_{\mathrm{P}} = \hat{z}\frac{\mu_0 I}{8}\left(\frac{a+b}{ab}\right).$$

12. With the same logic as in Problem 11, the straight segments on the left and right sides of point P do not contribute. The only contributions will come from the semicircles. However, because the semicircle of radius a is closer than the semicircle of radius b, the field from the smaller semicircle will overwhelm the field of the larger one, so that the net field will point out of the page, which we will define as the $+z$ direction. Indeed, the fields from these semicircles will be in competition. The answer is:

$$\vec{B}_{\mathrm{P}} = \hat{z}\frac{\mu_0 I}{4}\left(\frac{b-a}{ba}\right).$$

13. The point of this problem is to note that the cylinder's uniformly-distributed, circumferential current may be broken up into infinitesimally thin rings, of thickness $d\ell$, that are stacked laterally to form the cylinder when the stack is taken as a complete unit. Note again that the xy-plane cuts the length of the cylinder in half.

(a) Without loss of generality, take $z > \ell/2$. Based on the Right-Hand Rule (RHR) for determining magnetic fields, the field will point along the $-z$-direction for this particular placement of the observation point. Consider an infinitesimal ring, supporting a current dI, that is at a position $0 < z' < \ell/2$. Since z is measured from the xy-plane, this means that the distance of this particular ring from the observation point is $z - z'$. However, for this ring, we actually know what its magnetic field at this observation point is explicitly

$$d\vec{B} = -\hat{z}\frac{\mu_0 dI}{2}\frac{R^2}{\left(R^2 + (z - z')^2\right)^{3/2}}.$$

Similar to the formalism of Problem 9, we may define a (linear) current density, $\vec{K}$, which is directed along the circumferential current that is defined by

$$dI = Kd\ell = Kdz',$$

where since the circumferential current is presumed to be uniform across the length of the cylindrical surface,

$$K = \frac{I}{\ell}.$$

Thus, the magnetic field due to this one ring becomes

$$d\vec{B} = -\hat{z}\frac{\mu_0 IR^2}{2\ell}\frac{dz'}{\left(R^2 + (z - z')^2\right)^{3/2}} = -\hat{z}\frac{\mu_0 I}{2\ell R}\frac{dz'}{\left((z' - z)^2/R^2 + 1\right)^{3/2}}.$$

The total magnetic field of all of the rings may then be obtained by integrating the above expression over $z' \in [-\ell/2, +\ell/2]$:

$$\vec{B} = -\hat{z}\frac{\mu_0 I}{2\ell R}\int_{-\ell/2}^{+\ell/2}\frac{dz'}{\left((z' - z)^2/R^2 + 1\right)^{3/2}}$$

Since z is fixed (since it represents the observation point), then notice that

$$dz' = d\left(z' - z\right) = Rd\left[\left(z' - z\right)/R\right] \equiv Rdu,$$

so that

$$\vec{B} = -\hat{z}\frac{\mu_0 I}{2\ell}\int_{u_1}^{u_2}\frac{du}{\left(1 + u^2\right)^{3/2}},$$

with $u_1 = -\left(z + \ell/2\right)/R$ and $u_2 = -\left(z - \ell/2\right)/R$. Integrating, we find

$$\vec{B} = -\hat{z}\frac{\mu_0 I}{2\ell}\left[\frac{u}{\sqrt{u^2 + 1}}\right]_{u_1}^{u_2},$$

which could certainly be simplified.

(b) In this case, the procedure is fairly similar. One way to approach it is to break the integration up into two parts. Take the case for where the observation point is chosen such that $0 < z < \ell/2$. Then, the rings can be broken up into two intervals: one set which is at values $z' > z$ (with ring positions given by $z' - z$), and another set which is at values $z > z'$ (with ring positions $z - z'$). Respective integrals may then be set up—which will have practically identical forms to the one done in Part (a)—but where the limits will be slightly different.

14. Because the disk is insulating, the charges will remain stationary, and the movement of these charges will result in some current. However, it is worthwhile to take the charges along concentric, infinitesimally thin rings, similar to Problem 6; however, in this case, the current in each ring is infinitesimal and given by

$$dI = \frac{\omega dq}{2\pi} = \frac{\omega\sigma dA}{2\pi},$$

where dA is the area bounded by one of these infinitesimally thin rings—of radius r and thickness dr—which is given by

$$dA = 2\pi r dr.$$

However, we already know what the magnetic field of a current-carrying ring is along its axis of symmetry, and we may adapt the result for this infinitesimal case and write

$$d\vec{B} = \hat{z}\frac{\mu_0 dI}{2}\frac{r^2}{(r^2+z^2)^{3/2}},$$

for a point that is a distance z from the plane of this ring with some specific directionality of the current along the ring. Putting all of this together, we have:

$$d\vec{B} = \hat{z}\frac{\mu_0\omega\sigma}{2}\frac{r^3}{(r^2+z^2)^{3/2}}dr.$$

(a) To determine the result at the disk's center, simply take $z=0$ for the above expression and integrate over $r \in [0, R]$. In particular, we find

$$\vec{B}_0 = \hat{z}\frac{\mu_0\omega\sigma}{2}\int_0^R dr = \hat{z}\frac{\mu_0\omega\sigma R}{2}.$$

(b) For this problem, $z \neq 0$. So, we thus integrate

$$\vec{B} = \hat{z}\frac{\mu_0\omega\sigma}{2}\int_0^R \frac{r^3 dr}{(r^2+z^2)^{3/2}}.$$

Looking up the integral (or using a trigonometric substitution), the result is

$$\vec{B}(z) = \hat{z}\frac{\mu_0\omega\sigma}{2}\left(\frac{2z^2+R^2}{\sqrt{R^2+z^2}} - 2z\right).$$

(c) For this part, one has to find the magnetic dipole moments of each of the rings. Using the Right-Hand Rule (RHR) for determining magnetic dipole moments, we can see that given the direction of the current, the dipole moment will point along the axis of symmetry and, without loss of generality, suppose that the way that the current is flowing makes it so that the direction is along the $+z$-direction. Then, for an infinitesimal ring of radius r and thickness dr, the dipole-moment contribution from this specific ring is

$$d\vec{\mu} = \hat{z}AdI = \hat{z}\pi r^2\frac{\sigma\omega}{2}rdr = \hat{z}\frac{\pi}{2}\sigma\omega r^3 dr.$$

Integrating over the whole radius of the disk, the total dipole moment is then

$$\vec{\mu} = \hat{z}\frac{\pi}{2}\sigma\omega\int_0^R r^3 dr = \hat{z}\frac{\pi}{8}\sigma\omega R^4.$$

15. The important thing to note is that radial paths from the wire result in zero circulation, since the field is circumferential and, therefore, at right angles to the radial line. However, traversing circumferentially has an invariance in terms of the size of the circle, because just as much as the field of a long, straight wire decays inversely with the radial distance, the arc length increases linearly with this distance. Finally, when the current does not pierce the surface bounded by some loop (i.e., a loop does not "enclose" the current) then there is no net circulation of the field.

16. Consider a Anti-Clock-Wise (ACW) circulation to be taken as the positive sense of circulation. This goes hand-in-hand with saying that, relative to the figure, a current that pierces out of the page is taken to be positive. In turn, this implies that the normal vector of all relevant surfaces is taken to be out of the page, so that when current is interpreted as flux, the flux contribution of currents coming out of the page is a positive one. With this in mind, with an ACW circulation about A, the net circulation is $\Gamma_\mathrm{A} = +3\mu_0 I$. With an ACW circulation about B, the net circulation is $\Gamma_\mathrm{B} = 0$. With an ACW circulation about C, the net circulation is $\Gamma_\mathrm{C} = -\mu_0 I$. With an ACW circulation about D, the net circulation is $\Gamma_\mathrm{D} = +\mu_0 I$.

17. Consider a Clock-Wise (CW) circulation taken about the imaginary rectangular loop shown. Because the field points purely in the $+x$-direction, notice that neither of the vertical segments will contribute to the circulation, since $\vec{B} \perp d\vec{\ell}$ for either of those piece-wise smooth segments. As for the top and bottom horizontal segments of the loop, since the field strength increases as one goes to larger values of y, then the field strength for the top segment at $y = h$ (where $\vec{B}_\mathrm{top} = (\alpha + \beta h)\,\hat{x}$) is going to be larger than the field strength for the bottom segment at $y = 0$ (where $\vec{B}_\mathrm{bot} = \alpha\hat{x}$). However, notice that the sense of movement on the top segment is in the direction of the field (i.e., $d\vec{\ell}_\mathrm{top} \parallel \vec{B}_\mathrm{top}$), while the sense of movement on the bottom segment is against the direction of the field (i.e., $d\vec{\ell}_\mathrm{bot} \nparallel \vec{B}_\mathrm{bot}$). Thus, the top segment will contribute positively to the overall circulation, while the bottom segment will contribute negatively to this circulation. So, the Left-Hand Side (LHS) of Ampere's Law becomes

$$\oint_{\partial\mathcal{S}} \vec{B} \cdot d\vec{\ell} = (\alpha + \beta h)\,\ell - \alpha\ell = \beta h\ell,$$

where the first term to the right of the first equality comes from the circulation contribution of the top segment, and the second term to the right of that equality comes from the bottom segment. By Ampere's Law, there must be a current piercing the surface, $\mathcal{S}$, bounded by $\partial\mathcal{S}$ given by the ratio of the circulation to the permeability of free space. In particular,

$$I_\mathrm{thru} = \frac{\beta h\ell}{\mu_0}.$$

Since the value of the circulation is positive when the circulation direction is taken to be CW, then this implies that the net piercing current must be going into the page.

18. We have done this in class. We argued that if the solenoid is fairly long with tightly wound coils, then the field will be practically uniform and axial within the solenoid, and the field outside will be practically zero. Taking the field direction to be along the $+z$-direction, which would be the axis that defines the symmetry axis of the solenoid itself, the result is

$$\vec{B} = \begin{cases} \hat{z}\mu_0 nI & \text{for } r < a \\ \vec{0} & \text{for } r > a \end{cases},$$

where r measures the radial distance from the solenoid's symmetry axis.

19. A toroid is just a solenoid with its two caps fused together, with the body of the coils forming a circle. Think of it as a donut with a circular cross-section (i.e., a donut that when placed on a horizontal table that a vertical cut across the dough will show you a circular cross-section for the dough).

    (a) In some sense, we may use the arguments put forth for the solenoid to say that since the field outside the solenoid is practically zero, then fusing the two ends of the solenoid will still preserve this effect (perhaps even more so) simply because you force the field lines that potentially exited the solenoid on one end to now physically connect to the field lines entering the other end. However, a more useful way of showing this is to consider any Amperian loop that does not get pierced by a coil of the toroid. Since the current piercing this loop will be zero, then the circulation of the magnetic field will also be zero. However, since the choice of this loop is arbitrary in shape and orientation, then the only way to ensure that the circulation is always zero under any arbitrary circumstance is to have the field be identically zero as well.

    (b) Although the resulting field profile for a toroid is only approximate especially when the toroid's cross-section is not rectangular, the result is nonetheless pretty accurate. To use Ampere's Law to determine the field within the toroid, consider an Amperian loop that loops within and around the toroid at some fixed radius r from the toroid's axis of symmetry (i.e., if a donut is placed horizontally, this radius would be measured from the vertical axis running through the center of the donut's hole radially toward a point within the dough of the donut). [Who knew that Ampere's Law could make someone so hungry?] By approximate symmetry, the circulation of the magnetic field (i.e., the Left-Hand Side (LHS) of Ampere's Law) may be written as

$$\oint_{\partial\mathcal{S}_{\mathrm{A}}} \vec{B}\cdot d\vec{\ell} = B\,(2\pi r)\,.$$

Note that because the Amperian loop runs through all N turns of the solenoid, then the total current piercing the disk, $\mathcal{S}_{\mathrm{A}}$, bounded by the Amperian loop $\partial\mathcal{S}_{\mathrm{A}}$ is $I_{\mathrm{thru}} = NI$. Setting the two sides of Ampere's Law equal to one another outputs a field profile that decays as the inverse of this radial distance from the axis through the hole of the donut. As such, the largest field value occurs when r is the smallest it can be (i.e., the inner radius of the donut) and the smallest value of the field occurs when r is the largest it can be (i.e., the outer radius of the donut).

20. Note that because this system consists of currents distributed within cross-sections of conducting regions, then one will inevitably have to investigate the current density, particularly in regions where only a fraction of the current is piercing the surface of some Amperian loop that is used to determine the field by symmetry principles. As far as Part (a) is concerned, the field profile for $r < a$ and for $a < r < b$ has been worked out in class by finding the magnetic field of some long cylinder of uniformly distributed current. Indeed, Amperian loops drawn concentric to such a cylinder within and outside the region defined by the conducting cylinder only depend on the current piercing the loop. For this reason, even though there is a current present on the outer conducting shell, it will not effect the circulation calculation of Ampere's Law, or, for that matter, the calculation of the field, within the bounds of that shell. Thus, the field will circulate Anti-Clock-Wise (ACW) in these regions with

$$B\,(r < a) = \frac{\mu_0 I_1}{2\pi a^2} r = \frac{\mu_0 I}{2\pi a^2} r,$$

and

$$B\,(a < r < b) = \frac{\mu_0 I_1}{2\pi}\frac{1}{r} = \frac{\mu_0 I}{2\pi}\frac{1}{r}.$$

Upon entering the region with $b < r < c$, notice that a counter-propagating current begins to reduce the ACW circulation of the inner conductors current. We draw a concentric Amperian loop for $b < r < c$. For the same symmetry principles as before, the Left-Hand Side (LHS) of Ampere's Law still reads

$$\oint_{\partial\mathcal{S}_{\mathrm{A}}} \vec{B}\cdot d\vec{\ell} = B\,(2\pi r)\,.$$

As for the Right-Hand Side (RHS), we have to be a bit more careful regarding the current piercing the surface. On the one hand, I_1 is piercing the surface within radius a—which is contained within this Amperian loop and which we call $\mathcal{S}_{\mathrm{inner}}$—going out of the page (which we take as positive); whereas I_2 is piercing the surface going into the page (which we take as negative) in the region defined by $r > b$, which we call $\mathcal{S}_{\mathrm{outer}}$. There is no current present in the region between radii a and b, which we call $\mathcal{S}_{\mathrm{empty}}$. Thus, the total Amperian surface, $\mathcal{S}_{\mathrm{A}}$, bounded by the loop $\partial\mathcal{S}_{\mathrm{A}}$ in this region is the union of the sub-regions mentioned above:

$$\mathcal{S}_{\mathrm{A}} = \mathcal{S}_{\mathrm{inner}} \bigcup \mathcal{S}_{\mathrm{empty}} \bigcup \mathcal{S}_{\mathrm{outer}}.$$

As a result,

$$I_{\mathrm{thru}} = \int_{\mathcal{S}_{\mathrm{A}}} \vec{J}\cdot d\vec{A} = \int_{\mathcal{S}_{\mathrm{inner}}} \vec{J}\cdot d\vec{A} + \int_{\mathcal{S}_{\mathrm{empty}}} \vec{J}\cdot d\vec{A} + \int_{\mathcal{S}_{\mathrm{outer}}} \vec{J}\cdot d\vec{A} = I_1 + 0 - J_2 \int_{\mathcal{S}_{\mathrm{outer}}} dA = I_1 - J_2\pi\left(r^2 - b^2\right).$$

However, because the current is uniformly distributed within the outer conductor, then

$$J_2 = \frac{I_2}{\pi\left(c^2 - b^2\right)} = \frac{I}{\pi\left(c^2 - b^2\right)}.$$

Thus, the total current piercing this Amperian loop for $b < r < c$ is

$$I_{\text{thru}} = I_1 - I_2 \left(\frac{r^2 - b^2}{c^2 - b^2} \right) = I \left[1 - \left(\frac{r^2 - b^2}{c^2 - b^2} \right) \right] = I \left(\frac{c^2 - r^2}{c^2 - b^2} \right).$$

As far as for $r > c$, the LHS for Ampere's Law takes on the same form, while the piercing current becomes $I_{\text{thru}} = I_1 - I_2 = 0$. Thus, the field outside the assembly is identically zero because the net piercing current is zero for any arbitrary Amperian loop drawn with its perimeter strictly outside the conducting assembly. The sketched for Part (b) immediately follow from the functional forms derived in Part (a). In particular, note that the field is everywhere continuous, but happens to not be differentiable at $r = a$, $r = b$, and $r = c$. As for Part (c), one of the key differences is that the field outside the entire assembly will no longer be zero. Instead, there will be a circulating field that is in the direction dictated by the Right-Hand Rule (RHR) applied to the inner-conductor's current direction.

21. In this case, we are faced with a current density that happens to be distributed radially. So, when calculating the current piercing some circular Amperian loop within the beam, we cannot take the current density out of the surface integral directly, since it has a piece dependent on the radial position.

    (a) To determine this constant, use the fact that the total current must obey a flux condition using the current density. Take an infinitesimally thin ring of radius r and thickness dr that is concentric with the axis of the beam. The area of this ring is $dA = 2\pi r dr$. Since the "membrane" taken as the reference for counting the charge carriers piercing the membrane is the disk bounded by this ring, its normal vector is parallel to the current density. Thus, $\vec{J} \cdot d\vec{A} = J dA = J(r) 2\pi r dr$. Thus, if we call the disk bounded by the edge of the beam line to be $\mathcal{S}_{\text{beam}}$, then we mandate

    $$I = \int_{\mathcal{S}_{\text{beam}}} \vec{J} \cdot d\vec{A} = 2\pi \frac{J_0}{R} \int_0^R r^2 dr,$$

    from which J_0 may be determined in terms of given quantities.

    (b) For $r \geq R$, the field profile is that of a long, straight cylinder carrying a current I:

    $$B_{\text{out}} = \frac{\mu_0 I}{2\pi} \frac{1}{r}.$$

    For $r < R$ and upon taking a circular Amperian loop of such a radius, the Left-Hand Side (LHS) of Ampere's Law has the usual symmetry arguments applying to it to allow the extraction of the magnetic field magnitude out of the circulation integral:

    $$\oint_{\partial \mathcal{S}_{\text{A}}} \vec{B} \cdot d\vec{\ell} = B (2\pi r).$$

    The Right-Hand Side (RHS) will just have an adaptation of the procedure employed to determine the total current, except now the piercing current is only a fraction of the total current

    $$I_{\text{thru}} = 2\pi \frac{J_0}{R} \int_0^r (r')^2 dr',$$

    with J_0 given in Part (a) in terms of the other given quantities. One will then find that the field profile is quadratic in the radial position within the beam.

    (c) The graph follows from the results obtained in Part (b). In particular, one will notice that the field is continuous across the whole radial spectrum, but is not differentiable at $r = R$.

22. The formalism of Ampere's Law is quite useful in tackling this problem. Note that height-wise, the slab is cut in half via the xy-plane. In order to determine the field inside, consider a rectangular Amperian loop arranged symmetrically about the xy-plane, with length ℓ parallel to the xy-plane, height $2z$ parallel to the z-axis, and bounded area with normal vector along the $+x$-axis (i.e., in the direction of the current density $\vec{J}$). Taking an Anti-Clock-Wise (ACW) orientation of the Amperian loop, symmetry forbids the height-wise

segments of the loop to contribute anything to the circulation of the magnetic field (i.e., the Right-Hand Side (RHS) of Ampere's Law). The contributions from the length-wise segments will both be positive for the chosen circulation direction, so that

$$\oint_{\partial \mathcal{S}_{\mathrm{A}}} \vec{B} \cdot d\vec{\ell} = 2B\ell.$$

For the Left-Hand Side (LHS) of Ampere's Law, the current piercing the area bounded by the rectangular Amperian loop must be determined. In particular,

$$I_{\text{thru}} = \int_{\mathcal{S}_{\mathrm{A}}} \vec{J} \cdot d\vec{A} = JA = J2z\ell.$$

Setting the two sides equal, we have

$$2B\ell = \mu_0 J2z\ell,$$

so that

$$B_{\text{in}}(z) = \mu_0 Jz,$$

with the field pointing along the $-y$-direction for $z > 0$ and pointing along the $+y$-direction for $z < 0$. In other words,

$$\vec{B}_{\text{in}}(z) = -\mu_0 Jz\hat{y}.$$

As far as the outside is concerned, take an Amperian rectangular loop with exactly the same orientation and length, but with height $2z > h$. The LHS of Ampere's Law will take an identical form to the inside, however the outside's RHS will be simpler, as there is no current density beyond the slab's height. For this reason,

$$I_{\text{thru}} = J\ell h.$$

Thus, putting the two sides together, we find

$$\vec{B}_{\text{out}}(z) = -\hat{y}\frac{\mu_0 Jh}{2},$$

which is actually identical to the result of of an infinite sheet of uniformly flowing current, so long as one can see that the surface current density, $\vec{K}$, satisfies the condition

$$K = Jh,$$

where J is the magnitude of the (volume) current density, $\vec{J}$.