Physics 1A
Spring 2015

Challenge Problem 23

A uniform, solid sphere of mass M and radius R and a uniform, solid cylinder of the same mass and radius are released simultaneously at rest from the top of an inclined plane of length ℓ that makes an angle θ with the horizontal.

Which reaches the bottom first if they both roll without slipping?
Solution.

Using conservation of kinetic plus potential energy from the top to the bottom of the inclined plane (there is no net non-conservative work done by the force of static friction), we find that

$$Mg(\ell \sin\theta + R\cos\theta) = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + MgR\cos\theta \tag{1}$$

where we have assigned the bottom of the inclined plane to be $y = 0$, V is the speed of the cylinder at the bottom of the incline, and ω is its angular speed at the bottom of the incline.

Notice also that we have included an $MgR\cos\theta$ term on either side of the conservation equation due to the fact that the center of mass of the cylinder is actually a distance $\ell\cos\theta$ above the plane at any given point, but this term cancels on both sides, so it could have been omitted from the beginning. We also note that since the cylinder is rolling without slipping, we have the constraint

$$V = R\omega \tag{2}$$

Plugging this into the conservation of energy equation and solving for V gives

$$v = R\sqrt{\frac{2Mgd\sin\theta}{MR^2 + I}} \tag{3}$$

to determine the velocity of the cylinder at the bottom, we simply plug in the moment of inertial of the cylinder which is

$$I = \frac{1}{2}MR^2, \tag{4}$$

and we get

$$\boxed{v_{\text{cyl}} = \sqrt{\frac{4}{3}g\ell\sin\theta}}. \tag{5}$$

Doing the same computation with the moment of inertia of the sphere gives

$$\boxed{v_{\text{sph}} = \sqrt{\frac{10}{7}g\ell\sin\theta}}. \tag{6}$$

The acceleration of the center of mass of each object is constant along the inclined plane, so the one that has the higher final speed is the one that gets to the bottom first. We determine which is larger by taking their ratio;

$$\frac{v_{\text{sph}}}{v_{\text{cyl}}} = \sqrt{\frac{10}{7}\frac{3}{4}} = \sqrt{\frac{15}{14}} > 1. \tag{7}$$

Hence the sphere gets to the bottom faster.