Physics 22: Homework 6 Hints

1. Here are some basic thoughts to these conceptual problems.

   (a) The source battery must necessarily maintain some potential difference across the conductor in order to allow charges to flow continually. This can only be done by sustaining an electric field to promote this current in powering the circuit.

   (b) The issue here is that we would ideally like to consider that the battery outputs electrons to then move physically through the entire circuit like a person running along a racetrack. However, the source voltage, as mentioned in Part (a), sets up an electric field that sets all of the conduction electrons within the wire in motion. This occurs because electrical forces are long-range forces that require no physical contact. Thus, the presence of this electric field is propagated along as a reconfiguration of the fabric of space, communicated at the speed of light. For this reason, the light turns on instantaneously as these charge carriers are instructed to all drift simultaneously within the wire, allowing the ones in the resistive region to heat up the filament of the bulb and, thus, allow it to glow by heating up.

   (c) Here, the charges accelerate between collisions according to Newton's 2^{nd} Law, but they are constantly colliding with the lattice ions (which consist of the metal nuclei surrounded by core electrons that do not participate in the conduction band). So, on average, these charge carriers move with some constant drift velocity, which takes into account the acceleration of these charges in between colliding targets, as well as the catastrophic collisions with the ion cores that tend to, on average, result in the electrons starting consistently from rest after the conclusion of a collision. These effects are summarized in the Drude model, which is a model that makes use of these collisions to derive a result for the conductivity of an Ohmic material (e.g., copper, aluminum).

2. Similar arguments were employed in class with a similar example in order to understand how current is defined not only as the rate at which charge passes a given reference membrane, but also in terms of the flux of current density. Review the notes and try to piece together the logic via some dimensional analysis and brute-force computation.

3. Another aspect of copper that's great is that it can be quite easily soldered. Albeit, when different pieces of copper wire are connected in commercial or residential applications, the joining is done mechanically using a threaded wire connector that twists two exposed copper wires together. Here, the concentration provides us with the number of conduction electrons per unit volume (i.e., $n = 1.1 \times 10^{29}$ electrons/m^3).

   (a) The point here is to convert the density and atomic mass information to see if we can get to a number that's pretty close to the concentration provided. Specifically,

   $$8.9\,\frac{\text{g}}{\text{cm}^3}\left(\frac{1}{63.55\text{ g/mol}}\right)\left(\frac{6.022\times 10^{23}\text{ atoms}}{1\text{ mol}}\right)\left(\frac{10^2\text{ cm}}{1\text{ m}}\right)^3 = 0.84\times 10^{29}\text{ atoms/m}^3,$$

   which is fairly close to the value for the concentration of conduction electrons.

   (b) Taking the concentration of conduction electrons and dividing by the concentration of copper atoms gives us roughly 1.3 electrons/atom.

4. Based on the provided information, one can simply determine the resistance of each of the pieces. In particular

   $$R_{\text{B}} = \rho_{\text{B}}\frac{\ell_{\text{B}}}{A_{\text{B}}} = \rho\frac{\ell}{A} \equiv R,$$

   while

   $$R_{\text{A}} = \rho_{\text{A}}\frac{\ell_{\text{A}}}{A_{\text{A}}} = 3\rho\frac{2\ell}{A} = 6\rho\frac{\ell}{A} = 6R.$$

   (a) The total resistance of the assembly becomes

   $$R_{\text{AB}} = R_{\text{A}} + R_{\text{B}} = 7R.$$

i. The answer is: 1. Indeed, the current outputted by the battery has no choice but to run through both conductors. As such, this ratio is unity.

ii. The answer is: 6.

iii. By Ohm's Law,

$$\frac{\Delta V_{\mathrm{A}}}{\Delta V_{\mathrm{B}}} = \frac{I_{\mathrm{A}} R_{\mathrm{A}}}{I_{\mathrm{B}} R_{\mathrm{B}}} = \frac{R_{\mathrm{A}}}{R_{\mathrm{B}}},$$

so that the answer is: 6.

iv. The presumption is that the field through each of the conductors is uniform, but the field cannot be the same in each by virtue of how the nature of the collisions is different in each. Since we may write that $E = \Delta V/\ell$, then the answer is: 3.

v. This ratio may be written as,

$$\frac{P_{\mathrm{A}}}{P_{\mathrm{B}}} = \frac{I_{\mathrm{A}} \Delta V_{\mathrm{A}}}{I_{\mathrm{B}} \Delta V_{\mathrm{B}}} = 6.$$

(b) The answers are: $\Delta V_{\mathrm{A}} = 6V/7$ and $\Delta V_{\mathrm{B}} = V/7$.

(c) If ground is, for example, presumed to be where the current flows out of resistor B, then at the junction it must be $V_{\mathrm{jxn}} = V/7$.

(d) As seen in Part (aiv), the electric field is different in the conductors, and the change from A to B is an abrupt one. For this reason, it should be clear that the electric field undergoes a jump discontinuity at the spatial location of the junction. From past experience, this occurs any time there is an accumulation of surface charge (i.e., the presence of an areal charge density). Since the conventional current flows from left to right in the diagram, then the field must be directed from left to right. Take a Gaussian pillbox that has the same cross-section as the wires and that is concentric with the wires, centered on the junction. The only contribution for this Gaussian pillbox will result from the end-caps of the pillbox. The normal on the end-cap within A will be directed leftward (i.e., opposite to the field), while the normal on the end-cap within B will be directed rightward (i.e., in the direction of the field). Then, the Left-Hand Side (LHS) of Gauss's Law would say:

$$\oint_{\partial \mathrm{V_G}} \vec{E} \cdot d\vec{A} = -E_{\mathrm{A}} A + E_{\mathrm{B}} A = \left(E_{\mathrm{B}} - E_{\mathrm{A}}\right) A = E_{\mathrm{B}} \left(1 - \frac{E_{\mathrm{A}}}{E_{\mathrm{B}}}\right) A = -2 E_{\mathrm{B}} A.$$

The Right-Hand Side (RHS) would say:

$$Q_{\mathrm{encl}}/\varepsilon_0 \equiv Q/\varepsilon_0.$$

Setting the LHS and RHS equal, we find:

$$\sigma = -2\varepsilon_0 E_{\mathrm{B}} = -2\varepsilon_0 \frac{\Delta V_{\mathrm{B}}}{\ell_B} = -\frac{2}{7}\varepsilon_0 \frac{V}{\ell}.$$

(e) This was already answered in Part (d).

5. The power running through a resistor may be determined via the use of the electric-power formula in conjunction with Ohm's Law:

$$P_R = I_R \Delta V_R = \frac{\left(\Delta V_R\right)^2}{R} = I_R^2 R.$$

(a) Upon neglecting the voltage drop across the wires, the answer is: 0.522 A.

(b) One will find that the resistance of the cord is $R_{\mathrm{cord}} = 64.2\ \mathrm{m\Omega}$, so that the power is $P_{\mathrm{cord}} = 17.5\ \mathrm{mW}$.

(c) Indeed, noting that the resistance of the 60-W bulb is $R_{60} = V^2/P_{60} = 220\ \Omega$, then clearly $R_{\mathrm{cord}} \ll R_{60}$, warranting the negligence of the resistive effects of such wires.

(d) From Part (c), note that for a given voltage, $R \sim P^{-1}$. Thus, the higher-wattage bulb actually has the lower resistance.

(e) The current running through both bulbs with be the same since they're connected in series. Thus, the bulb with the higher resistance will consume more power, which is the bulb with the lower power rating. As a result, the lower-wattage bulb will glow brighter.

6. Here are some basic thoughts to these conceptual problems.

   (a) If all of the decorative lights were in series, then if one of the lights breaks, then there is a break in the circuit, not allowing any current to flow. So, not only would none of the lights be on, it would also be very difficult to troubleshoot in terms of identifying the bad bulb since all of the bulbs would not be lit.

   (b) This is really just a matter of choice for the flow of current. When resistors are in parallel, the charges have more of a choice through which resistor to move. With more options comes a lower resistance. However, with a series configuration, the charges have to flow through more of a tortuous path, resulting in additional collisions. With the absence of choice in this case, the resistance is higher.

   (c) The Junction Rule comes from the principle of charge conservation, since this rule (albeit looking at the rate of charge transfer) looks into the fact that if there are no additional pathways through which charge may transit, whatever charge comes into a junction must necessarily come out of that junction. The Loop Rule comes from the principle of the conservation of energy. Although the Loop Rule is technically a statement of energy per unit charge (i.e., potential), it nonetheless comes from the fact that in a closed system, there is no way for a charge to gain additional energy by starting and ending at the same endpoint. If this was the case, then energy may be harvested from nothing, which, of course, violates this principle.

7. The fact that all of the light bulbs are identical effectively means that the bulbs all have the same resistance. These bulbs are treated as Ohmic, since they are all incandescent. Now, these bulbs are no longer ones that we see in stores, as practically all bulbs are either LED or Compact-Fluorescent (CFL) bulbs. The bulbs known as "halogen" bulbs actually still are incandescent (i.e., they light up by virtue of heat that causes the filament to glow), but they have slightly better efficiencies relative to incandescent bulbs because of the halogen gas that is found within these bulbs. Nonetheless, the Wattage of a bulb is still represented in terms of the old-school incandescent wattage (e.g., a 40-Watt equivalent LED bulb could just consume a few Watts of power). The equivalence is made mostly to describe bulb brightness so that one an LED bulb that is equivalent to an incandescent one in terms of Wattage will necessarily be able to output the same brightness of light at a much, much lower power consumption. This is primarily why LED bulbs are not only more "green," but also much safer (because they don't get anywhere near as hot as incandescent bulbs).

   (a) An ideal battery is one with no internal resistance. Of course, in reality, batteries will have some internal resistance (which is a consequence of how rapidly it can maintain the potential difference between the electrodes via the quickness with which the chemical reactions can retain their charge separation), but a battery will have idyllic features if its internal resistance is kept to a minimum.

   (b) The relative brightness may equivalently be measured via the current running through the bulbs. This is primarily because one can empirically show that for equivalent light bulbs, more current equates to more consumed power, which, in turn, results in higher luminosity at a given distance from the bulb. Note that bulbs C and D have the largest current running through them (i.e., the current outputted by the battery), while the bulbs A and B have half this current. Thus, C and D are equally bright, while B and A are equally bright but less so that C and D.

   (c) If A is unscrewed, the current that once was split between B and A now will no longer be split but will, instead, run completely through B. However, the battery no longer outputs the same current, because the battery "sees" a different equivalent resistance in the circuit. In particular, the overall resistance of the circuit increases on account of unscrewing A.

      i. The output current will be less than when A was present, so that C and D would be less bright than before. However, even with the lessened outputted current, B now is brighter than before.

      ii. All bulbs will be equally bright in this scheme.

8. Again, all batteries and resistors are identical in this problem. The hints below will only comprise a partial discussion of each figure. To determine the resistor with the largest power, since all resistors are the same, then the one with the largest current running through it (or, equivalently, with the largest potential difference across it) will be the most powerful one.

(a) For Figure 13a, 1 and 2 are in parallel, while the 12 combination is in series with 3. For Figure 13b, 2 and 3 are in series, 23 is in parallel with 4, and 234 is in series with 1. For Figure 13c, even though both batteries are identical, none of the resistors fit the bill of being in series or parallel with another. For Figure 13d, the conducting wire that serves as the bridge between the two branches forces the potential between 2 and 3 to be the same as the potential between 1 and 4. For this reason, 1 and 2 are in parallel, 3 and 4 are in parallel, while 12 and 34 are in series. For Figure 13e, 2 and 3 are in series, 1 and 4 are in series, while 23 and 14 are in parallel. For Figure 13f, one can quite easily convince oneself that the potential difference between the two ends of 5 is zero. For this reason, 2 and 3 may be treated in series, 1 and 4 may be treated in series, and 23 and 14 may be treated in parallel. (Technically, for Figures 13d, 13e, and 13f, all resistors are in parallel and in series simultaneously, which is a consequence of the symmetry in the circuits with all resistances being identical.)

(b) Here, all resistors are the same except for R_4. For Figure 13d, 1 and 2 are in parallel, 3 and 4 are also in parallel, while 12 and 34 are in series. For Figure 13e, 2 and 3 are in series, 1 and 4 are in series as well, while 23 and 14 are in parallel. For Figure 13f, the symmetry is completely broken, so that the potential difference across R_5 will no longer be zero (as in Part (a)). For this reason, the resistors are neither in parallel nor in series. Thus, Kirchhoff's Rules must be used to analyze the circuit in Figure 13f under these broken-symmetry circumstances.

9. A potentiometer is a variable resistor. The device works by having a fixed electrode at one end of a resistive wire (a design of which is a coiled one) with another electrode sliding across the resistive wire, making contact with it for proper electrical conduction. As the slider electrode is moved, the wire through which the current moves has a different length. Thus, the resistance may be changed in this way. The equivalent resistance in the circuit is $R + r$, by virtue of the internal resistance being in series with the external resistance.

(a) The answer is:

$$P = I^2 R = \left(\frac{R}{r+R}\right)\frac{V^2}{r+R}.$$

(b) This will be determined via the extremum condition

$$\left.\frac{dP}{dR}\right|_{R_{\max}} = 0.$$

This condition will be a maximum, which can be seen from the functional form of the power delivered to the external resistor.

(c) As discovered in Part (b), when $R = R_{\max}$, the source is delivering the maximum amount of power to the external resistor. However, it will turn out that this also means that a good chunk of power is also being dissipated within the battery itself.

(d) When $R > r$, more power is delivered to the external resistor than to the internal one, but there is less overall power generated. When $R < r$, more power is generated by the source, but most of it is wasted as dissipated power within the battery itself (i.e., through its internal resistance).

10. When the middle branch carries no current, this means that when the switch is closed, the potential difference across the entire middle branch is zero. However, this would also be the case when the switch is open based on the criteria mentioned in the problem. As such, the current running through the circuit may be found:

$$I = \frac{V_1 + V_2}{R_1 + R_2}.$$

(a) Using the Loop Rule for the right loop in the circuit, we have

$$V_3 = V_2 - IR_2,$$

or equivalently using the left loop in the circuit, we have

$$V_3 = -V_1 + IR_1,$$

where I is the current running through the circuit.

(b) As discussed above, there is no potential difference across the switch either in its open or closed state. However, there usually is a potential difference across a switch, which is really the reason why such a switch would turn things on or off; however, this problem looks at a special case in which the switch serves no real function.

11. Here, Kirchhoff's Loop and Junction Rules must be employed. Although there is no concrete way to know exactly how the current flows through each of the branches in general, one may choose a general sense of current movement in each of the branches. Then, upon solving the circuit equations for the currents in the branches, if one happens to find a negative value for one of the currents, then this just means that the current is actually flowing opposite to what was originally presumed. If the value turns out to be positive, then the chosen direction is, indeed, correct.

    (a) The answer is: $I_2 = 4.77$ mA, moving downward through the battery V_2.

    (b) With the same equations used in Part (a), state that $I_2 > 0$ moving up. Working through the circuit equations, the answer is: $V_2 > 5.49$ V.

12. Ammeters and voltmeters have internal resistances. As such, for a given circuit that is to be analyzed, the act of placing such a device into the circuit in order to measure the current and/or the voltage naturally alters the circuit itself, so that the measurements being made will, in principle, not reflect the actual values of the current and voltage at different parts of the circuit. However, these alterations may be minimized by making the internal resistance of the ammeter zero and by making the internal resistance of the voltmeter infinite.

    (a) In each case, the circuit gets altered to one with a parallel resistor attached to R_1, with the resistance of the parallel branch being the internal resistance of the voltmeter under question.

    (b) The voltmeter with the highest internal resistance will produce the most accurate (i.e., most unaltered) result of the voltage across R_1 relative to the original circuit.

    (c) This would cause most of the current to bypass R_1, since proportionately more current will go through a parallel branch with less resistance than the other branch. Thus, if R_1 was supposed to represent the load resistance of, for example, some device, then the device would likely not even turn on because most such devices operate above some minimum threshold current. Moreover, the current outputted by the battery would be a lot larger in this case, resulting in an overall larger current running through R_2 and R_3. Again, this may pose problems for those resistors as well, particularly if they needed to operate at some maximum threshold current. As another important point, if the resistors, R_2 and R_3, were tiny in their value, putting an ammeter across R_1 would inevitably result in a current so high to be drawn from the battery, that it may cause the wires to overheat and potentially cause the ammeter to burn out.

13. Again, remember that an ideal ammeter has zero resistance, while a voltmeter has infinite resistance.

    (a) For the circuit with the ammeter bridge, since the ammeter has zero resistance, then it serves as a perfectly conducting wire. For this reason, $V_{\mathrm{B}\to\mathrm{A}} = 0$. For the circuit with the voltmeter bridge, since the voltmeter has infinite resistance, then it serves somewhat like there is no wire connecting the points A and B. Because the resistors in either branch "above" the bridge are different, then there will be a potential difference between these points, but there will be practically no current running through the voltmeter due to its gigantic resistance. For this reason, we effectively have two parallel branches, each with series resistors R and $2R$. Thus, one finds $V_{\mathrm{B}\to\mathrm{A}} = +4$ V. For the circuit with the resistor bridge, none of the resistors are in parallel or in series with another. Thus, the formalism of Kirchhoff's Rules must be utilized to relate the voltages, currents, and resistances together. The system does have some symmetry, in the sense that the currents through the resistors $2R$ will be the same, and that the non-bridge resistors R will also have the same current running through them. In any case, using those rules, one will find that $V_{\mathrm{B}\to\mathrm{A}} = 1.72$ V.

    (b) This was already answered in Part (a).

    (c) For the voltmeter and ammeter bridges, the problem is fairly straightforward since one can use series- and parallel-resistor formulations to solve the problem. For the one with the resistor bridge, you must use the full formalism of Kirchhoff's Rules to get the result. With the absence of too much use of symmetry, the procedure will, nonetheless, be identical to the one employed for this bridge in the previous parts.

14. The point here is to essentially solve the effective circuit with the shunt resistance that is placed in conjunction with the galvanometer's internal resistance for the desired outcome.

    (a) The answer is: 0.113 Ω.

    (b) The answer is: 6592 Ω.

15. The general solution to this problem was explicitly worked out in class. In particular, the charge on the capacitor has the form

$$Q(t) = CV\left(1 - e^{-t/\tau}\right)$$

and

$$I(t) = \frac{V}{R}e^{-t/\tau},$$

where $\tau = RC$.

(a) This can be seen quite clearly if we take $Q(T) = 0.99CV$ and find that $T \approx 5\tau$.

(b) Recall that the capacitor voltage is given by the rule $V_C = Q/C$, so that

$$V_C(t) = \frac{1}{C}Q(t) = V\left(1 - e^{-t/\tau}\right).$$

i. Taking a time derivative of the above expression, one finds that

$$\frac{dV_C}{dt} = \frac{V}{RC}\left(1 - e^{-t/\tau}\right),$$

so that

$$\frac{dV_C}{dt}(t=0) = \frac{V}{RC}.$$

ii. Since $V_C = Q/C$, then this means that if the rate of voltage increase on the capacitor is constant, then

$$\frac{dQ}{dt} = \frac{V}{RC}.$$

Integrating this expression, we have

$$\begin{aligned}\int \frac{dQ}{dt}dt &= \int \frac{V}{RC}dt \\ \int_0^Q dQ' &= \frac{V}{RC}\int_0^\tau dt \\ Q &= CV.\end{aligned}$$

However, this charge is what one obtains when $t \to \infty$ for the actual RC problem. In other words, it would take exactly one time constant for the capacitor to fully charge if the rate at which it voltage increased remained the same as its rate at $t = 0$. Of course, that doesn't happen because the current decays as a function of time due to the capacitor building up charge and reducing the rate at which charge accumulates via repulsive effects.

(c) This was also done in class exploiting the fact that the battery power may be integrated over time over the entire duration to determine the result. Indeed,

$$\Delta U_{\text{batt}} = \int_0^\infty P_{\text{batt}}dt = \int_0^\infty I_{\text{batt}}V_{\text{batt}}dt = \frac{V^2}{R}\int_0^\infty e^{-t/\tau}dt = CV^2.$$

Of course, since the initial charge on the capacitor is zero and the final charge is $Q_\infty = CV$, then the energy to charge the capacitor fully becomes

$$\Delta U_C = \frac{1}{2}CV^2.$$

(d) This can be done either by exploiting energy conservation, or by directly integrating the resistor power through the entire duration of the circuit:

$$\Delta U_R = \int_0^\infty I^2 R dt = \frac{V^2}{R} \int_0^\infty e^{-2t/\tau} dt = \frac{1}{2} C V^2.$$

16. Assume that the bottom plate of C_2 is charged, and suppose that the initial charge of C_2 is denoted Q_2 and given by $Q_2 = C_2 V = 150\ \mu$C. Of course, $Q_1 = 0$. The point here is that when the switch is closed, positive charges will migrate from the bottom plate of C_2 to the bottom plate of C_1. In doing so, a Clock-Wise (CW) current will be set up in the circuit.

(a) By Charge Conservation, we must have

$$Q_1 + Q_2 = Q_1' + Q_2'$$

or

$$Q_2 = Q_1' + Q_2',$$

where Q_1' and Q_2' denote the charges on these two capacitors at any arbitrary time $t > 0$. However, once charges stop migrating, then there will no longer be a circulating current in the circuit, resulting in no voltage drop across the resistor. As a result, the voltages across each capacitor would be exactly the same, and this will presumably happen as $t \to \infty$. Mathematically, we may write

$$V_{1\infty}' = V_{2\infty}',$$

or

$$\frac{Q_{1\infty}'}{C_1} = \frac{Q_{2\infty}'}{C_2}.$$

Along with charge conservation

$$Q_2 = Q_{1\infty}' + Q_{2\infty}',$$

there are two equations with two unknowns ($Q_{1\infty}'$ and $Q_{2\infty}'$) which may then be determined to then find the specific value of these voltages.

(b) The initial total energy in the system is just the energy in the capacitor C_2. The final energy in each capacitor is

$$U_{j\infty}' = \frac{1}{2} C_j \left(V_j'\right)^2$$

for $j = 1, 2$. The simplest way to determine this is to use energy conservation and to say that the energy difference between the capacitors before the switch is closed and after the switch is closed must have resulted in the energy loss through the resistor:

$$\Delta U_R = U_2 - \left(U_{2\infty}' + U_{1\infty}'\right).$$

The other way is to integrate the power of the resistor with respect to time; however, this requires first-hand knowledge of what the current through (or voltage across) the resistor is at any time after the switch is closed. This endeavor is a bit more involved, but will still output the same result as above.

17. Suppose that the top plate is charged positively and the bottom plate is charged negatively.

(a) Since the dielectric has a small but finite resistivity, then the conventional current will flow from the top plate to the bottom plate, with the dielectric acting as a resistor. As such, the capacitor would discharge while, say, heating up the dielectric as the charges conduct through the medium. For this reason, the system may be conceived as such a circuit.

(b) The time constant of this circuit will be the product of the resistance of the dielectric and the capacitance of the capacitor. Recall that the presence of the dielectric makes the capacitance of the capacitor modified by the dielectric constant, so that for this parallel-plate capacitor system,

$$C = \kappa \frac{\varepsilon_0 A}{d},$$

where A is the plate area and d is the plate separation. As for the resistance of the dielectric, note that the current is moving along the distance separating the plates, so that the cross-sectional area of this resistor must simply be the plate area. Thus,

$$R = \rho \frac{d}{A}.$$

Because this is a discharging RC series circuit, then the time constant is

$$\tau = RC = \varepsilon_0 \kappa \rho.$$

(c) The form of the charge of the capacitor as a function of the time is

$$Q(t) = Q_0 e^{-t/\tau},$$

so that the energy is

$$U_C(t) = \frac{1}{2C} Q^2(t) = \frac{Q_0^2}{2C} e^{-2t/\tau}.$$

So, at $t = T$, we require $U(T) = U_0/2 = Q_0^2/(4C)$. Thus,

$$\frac{1}{2} = e^{-2T/\tau},$$

which ultimately means that

$$T = \frac{\tau}{2} \ln(2) = \frac{\varepsilon_0 \kappa \rho}{2} \ln(2) \approx 8.0 \times 10^4 \text{ s} \approx 22 \text{ hours}.$$

18. An important qualitative feature to use in these qualitative analyses involving capacitors is that a branch with a bare, uncharged capacitor essentially acts like a short when moving charges are exposed to it. In general, if a capacitor is allowed to gradually charge, then the capacitor acts somewhat like a variable resistor, reducing the current in its branch due to the repulsive effect borne out of the charges that have initially populated the capacitor.

   (a) Note that when the switch is closed, R_1 will limit the maximum current outputted by the battery. However, as discussed above, because there is no resistor in the branch with capacitor C_1, then this capacitor will suck up this entire current at the instant the switch is closed.

      i. Because C_1 sucks up all of the current since its branch has absolutely no resistance, then the current running through R_2 will be zero immediately after closing the switch.

      ii. Although C_2 will not admit a current immediately after the switch is closed (since it has a series resistor in its branch and since C_1 has no such resistance and, thus, sucks up all of the current), C_2 will get a boost in current a short time after the switch is closed. However, it and C_1 will eventually charge up fully. For this reason, a long time later, these branches with capacitors will no longer admit any current. Thus, all of the current will eventually just run through R_2 and the circuit as $t \to \infty$ will effectively be one where the current runs just through resistors R_1 and R_2, making these resistors in series with one another at these long times after the switch has been closed.

   (b) As discussed above, there will initially be no current running through R_3 immediately after the closing of the switch; however, the current will see an initial rise practically right after the shorting effect by C_1 is done. That being said, the capacitor in R_3's branch will charge up and, thus, continually begin reducing the current in this branch after the initial rise. Eventually, the current through this branch will become zero as C_2 fully charges.

19. The presence of r is paramount in this circuit, as it prevents the capacitor from driving an infinite current from the battery.

   (a) As in the same spirit as Problem 18, the series resistor to the battery, r, will limit the outputted current by the battery; however, because the capacitor branch has absolutely no resistance and the capacitor is initially uncharged, the current will be totally sucked up by the capacitor branch at the instant the switch is closed. Thus, immediately after the switch is closed is when the battery outputs the largest current (since the battery sees the least resistance at the instant the switch is closed). Slightly afterwards, the capacitor has already built up some charge, and, as a result, begins limiting the current that runs through its branch. As a result of the junction rule, the resistor R progressively gets a larger and larger current running through it as the capacitor gets progressively charged. As $t \to \infty$, the capacitor admits absolutely no current into its branch, so that the resulting circuit is effectively one where the outputted current runs strictly through r and R. As such, one can then see that the current running into the capacitor branch monotonically decreases, the current running through R monotonically increases, while the current running through r monotonically decreases because the battery progressively sees a larger overall resistance as the charge on the capacitor monotonically increases.

(b) The current running through r will be an exponential decay from its initial (largest) value of V/r to its final value of $V/(r+R)$ that it approaches asymptotically. The current running through R will be a bounded exponential increase, starting from zero at $t=0$ and increasing towards the horizontal asymptote of $V/(r+R)$. Finally, the rate at which the charge increases on the capacitor (i.e., the current in the capacitor branch) will initially start at its largest value of V/r, and then exponentially decay and taper off at zero as $t \to \infty$.

(c) Note that the (conventional) current will run upward where the battery is, will run to the right across r and will run down along the branches of R and C. If we look at some time $t > 0$ and we label the rate at which the capacitor accumulates charge as $I_C = dq/dt$, then the junction rule would state

$$i = I + I_C,$$

where i is the current running through r (i.e., the current outputted by the battery) and I is the current running through R. Now, take a loop (Loop A) that runs Clock-Wise (CW) across the battery, r, and R. This loop rule would state

$$+V - ir - IR = 0.$$

Finally, take a loop (Loop B) that runs CW through C and against the conventional current through R. This loop rule states

$$-\frac{q}{C} + IR = 0.$$

Using the junction rule in Loop A gives

$$V - (I + I_C)\, r - IR = 0,$$

or

$$V - I\,(r+R) - r\frac{dq}{dt} = 0.$$

Solving for I in Loop B and plugging into the expression above gives

$$V - \frac{q}{RC}\,(r+R) = r\frac{dq}{dt}.$$

Upon dividing both sides by r, we get

$$\frac{dq}{dt} = -\frac{q}{C}\left(\frac{r+R}{rR}\right) + \frac{V}{r}.$$

Since the Left-Hand Side (LHS) has units of current, the first term on the Right-Hand Side (RHS) must also have the same units. This implies that the factor multiplying q on the RHS must have units of inverse time. Indeed, this factor will be the inverse of the time constant:

$$\frac{1}{\tau} = \frac{1}{C}\left(\frac{r+R}{rR}\right),$$

so that

$$\tau = C\left(\frac{rR}{r+R}\right),$$

as desired.

(d) Beginning with the explicit form of the rate of the capacitor charge from Part (c), we may rearrange to write

$$\frac{dq}{dt} = -\frac{q}{\tau} + \frac{V}{r} = -\frac{1}{\tau}\left(q - \frac{V\tau}{r}\right),$$

so that

$$\frac{1}{(q - V\tau/r)}\frac{dq}{dt} = -\frac{1}{\tau}.$$

Integrating both sides of the expression above with respect to time and taking $q\,(t=0) = 0$ (since the capacitor is initially uncharged) and $q\,(t) = Q$, we have

$$\begin{aligned}
\int \frac{1}{(q - V\tau/r)}\frac{dq}{dt}dt &= -\int \frac{dt}{\tau} \\
\int_0^Q \frac{dq'}{(q' - V\tau/r)} &= -\frac{1}{\tau}\int_0^t dt' \\
\left[\ln|q' - V\tau/r|\right]_0^Q &= -\frac{t}{\tau} \\
\ln\left(1 - \frac{Qr}{V\tau}\right) &= -\frac{t}{\tau}.
\end{aligned}$$

Exponentiating both sides, and rearranging, we ultimately find

$$Q\,(t) = \frac{V\tau}{r}\left(1 - e^{-t/\tau}\right),$$

whereupon recalling the form for $\tau$from Part (c), we get

$$Q\,(t) = \frac{V}{r}C\left(\frac{rR}{r+R}\right)\left(1 - e^{-t/\tau}\right) = CV\frac{R}{R}\left(\frac{1}{r/R+1}\right)\left(1 - e^{-t/\tau}\right) = \frac{CV}{1+r/R}\left(1 - e^{-t/\tau}\right),$$

which is the desired result.

(e) Since the time constant may be rewritten as

$$\tau = \left(\frac{rR}{r+R}\right)C = \frac{R}{R}\left(\frac{r}{1+r/R}\right)C = \left(\frac{r}{1+r/R}\right)C,$$

then note that as $R \to \infty$, then $\tau \to rC$. Keeping this in mind and noting that

$$\lim_{R\to\infty}\frac{1}{1+r/R} = 1,$$

then

$$\lim_{R\to\infty} Q\,(t) = CV\left(1 - e^{-t/(rC)}\right),$$

which says that the circuit can be treated as if the resistor R is not there, in which case the circuit because a charging RC series circuit with battery potential V, resistor r, and capacitor C.

(f) Note that

$$\lim_{r\to 0}\tau = \lim_{r\to 0}\left(\frac{rR}{r+R}\right)C = 0,$$

which means that for an infinitesimal value of t,

$$\lim_{r\to 0} e^{-t/\tau} = 0.$$

Noting also that

$$\lim_{r\to 0}\frac{1}{1+r/R} = 1,$$

then we find that for an infinitesimal time, $t = \epsilon$, after the switch is closed

$$Q\,(\epsilon) \to CV,$$

noting that ϵ can be made vanishingly small. Thus, the capacitor acquires the voltage across the battery in practically no time, so that the capacitor, indeed, charged instantaneously.