Physics 1A
Spring 2015

Challenge Problem 5

Alice and Bob are having a competition to see who can throw a baseball the farthest from the top of a hill which slopes downward uniformly at an angle ϕ. Alice plays softball, and Bob doesn't have a particularly good arm, so he knows that he's probably gonna lose. Fortunately, Bob has taken mechanics, and he realizes that there is an optimal angle θ at which he should throw the ball so that it has the greatest range; what is that optimal angle?

Solution. Our strategy will be as follows:

1. Determine the range of the ball as a function of θ, the launch angle.
   (a) Determine the the trajectory of the ball $y_b(x)$.
   (b) Determine an equation for the surface of the hill $y_h(x)$
   (c) Determine the value of x where they intersect (this is the range).
2. Maximize the range by determining at what value of θ its derivative with respect to θ vanishes.

To Determine the trajectory of the ball, we assume that the ball is launched from the top of the hill which we choose to be the origin. For a given initial speed v_0 The position as a function of time of the ball is given by

$$x(t) = v_0 \cos\theta t \tag{1}$$

$$y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2 \tag{2}$$

Solving for t in the x-equation, and plugging the result into the y-equation gives the equation for y as a function of x for the ball:

$$y_b(x) = \tan\theta\, x - \frac{1}{2}g\frac{x^2}{\cos^2\theta} \tag{3}$$

On the other hand, the equation for y as a function of x for the hill is

$$y_h(x) = -\tan\phi\, x \tag{4}$$

The point at which the ball hits the hill has an x-position that satisfies $y_b(x) = y_h(x)$, which means that

$$\tan\theta\, x - \frac{1}{2} g \frac{x^2}{\cos^2\theta} = -\tan\phi\, x \tag{5}$$

Solving for x in this equation gives

$$x = \frac{2}{g}\cos^2\theta(\tan\theta + \tan\phi) \tag{6}$$

To maximize this, we take the derivative with respect to θ, set it to zero, and solve for the θ that satisfies the resulting equation. First, note that we can write the expression on the right as

$$\frac{2}{g}\left(\frac{\sin(2\theta)}{2} + \cos^2\theta\tan\phi\right) \tag{7}$$

so taking its derivative with respect to theta, and setting the result to zero gives

$$\cos(2\theta) - 2\sin\theta\cos\theta\tan\phi = 0 \tag{8}$$

and therefore

$$\cot(2\theta) = \tan\phi \tag{9}$$

It follows that

$$\phi = \tan^{-1}\cot(2\theta) = \frac{\pi}{2} - 2\theta, \tag{10}$$

and thus

$$\boxed{\theta = \frac{\pi}{4} - \frac{\phi}{2}}. \tag{11}$$

Notice that when $\phi = 0$, this reduces to $\theta = \pi/4$. This makes sense because, as one can verify more easily, range on flat ground is maximized when one throws at a 45° angle.