Physics 22: Homework 4 Hints

1. Note that if we consider the work done by the conservative electric force, then we may write

$$W_E^{(\mathrm{i}\to\mathrm{f})} = -\Delta U_{\mathrm{i}\to\mathrm{f}} = U_\mathrm{i} - U_\mathrm{f}.$$

This will allow one to determine all of the desired values, particularly noting that

$$W_E^{(\mathrm{i}\to\mathrm{f})} = \vec{F}_E \cdot \Delta\vec{\ell} = q\vec{E} \cdot \Delta\vec{\ell}.$$

Moreover, we may incorporate the potential into the mix using the fact that

$$\Delta V = \frac{\Delta U}{q},$$

so that

$$W_E^{(\mathrm{i}\to\mathrm{f})} = -q\Delta V.$$

(a) Filling out the table, we have:

$\Delta U_{\mathrm{A}\to\mathrm{B}}$ (μJ)	$\Delta U_{\mathrm{B}\to\mathrm{C}}$ (μJ)	$\Delta U_{\mathrm{A}\to\mathrm{C}}$ (μJ)	$W_E^{(\mathrm{A}\to\mathrm{B})}$ (μJ)	$W_E^{(\mathrm{B}\to\mathrm{C})}$ (μJ)	$W_E^{(\mathrm{A}\to\mathrm{C})}$ (μJ)
0	+16	+16	0	−16	−16

(b) Filling out the table again, we have:

$\Delta U_{\mathrm{A}\to\mathrm{B}}$ (μJ)	$\Delta U_{\mathrm{B}\to\mathrm{C}}$ (μJ)	$\Delta U_{\mathrm{A}\to\mathrm{C}}$ (μJ)	$W_E^{(\mathrm{A}\to\mathrm{B})}$ (μJ)	$W_E^{(\mathrm{B}\to\mathrm{C})}$ (μJ)	$W_E^{(\mathrm{A}\to\mathrm{C})}$ (μJ)
0	−24	−24	0	+24	+24

(c) In this case, we adopt the convention that

$$V_\mathrm{fi} \equiv V_{\mathrm{i}\to\mathrm{f}} \equiv V_\mathrm{f} - V_\mathrm{i}.$$

Thus, we have:

$$\begin{aligned} V_\mathrm{AB} &= 0 \\ V_\mathrm{BC} &= -8\ \mathrm{V} \\ V_\mathrm{AC} &= -8\ \mathrm{V} \end{aligned}$$

Because the potential is a source-centric quantity, it only depends on the source. Moreover, because there are two different types of charge, there are all sorts of variations in sign that can arise when one considers different types of charge moving within an electric field. However, quite plainly and simply, if one moves along an electric-field line, then the potential does decrease. As such, one can then incorporate the sign of the charge being moved from one spatial location to another within a field. Similar to the electric field, the electric potential describes the energy properties of space itself.

2. The grounded plate is at the lower potential, so that the field lines will point toward this plate.

(a) Note that the equipotential surfaces are drawn equidistant from one another and from the conducting plates. This has to follow because of the uniformity of the field between the parallel plates. In particular, since the strength of the field is described via the density of field lines, this will also translate to the density of equipotential surfaces when adjacent surfaces are held at the same potential difference. These planes are equipotential surfaces because the field will point perpendicular to the plates, which will also result in the field lines pointing perpendicular to the equipotential surfaces. Indeed, because the definition of the potential difference is

$$\Delta V = -\int_c \vec{E} \cdot d\vec{\ell},$$

then the locus of points satisfying the condition for which this integral is zero nontrivially satisfy the condition that

$$\vec{E} \cdot d\vec{\ell} = 0$$

for all $d\vec{\ell}$'s defined in this locus of points. As such, this set comprises a surface, as in a three-dimensional space, a two-dimensional surface will satisfy this orthogonality condition for such (electric-field) lines. Because the plates are themselves conducting and in electrostatic equilibrium, then they will satisfy $\Delta V = 0$, whether one considers points within the conductor (where $\vec{E} = \vec{0}$ and thus makes the condition trivially satisfied) or points on the surface of the conductor (where $\vec{E}$ must be perpendicular to the metallic surfaces and thus makes the condition satisfied in a less trivial manner).

(b) These equipotential surfaces satisfy the condition (defining $+y$ to point from plate A to plate B)

$$\Delta V = E\Delta y.$$

In other words, the potential increases when one moves against the field, while the potential decreases when one moves with the field. Since $V_{\mathrm{A}} = 0$ and $V_{\mathrm{B}} = 36$ V, then

$$\{V_1, V_2, V_3, V_4, V_5\} = \{6\ \mathrm{V}, 12\ \mathrm{V}, 18\ \mathrm{V}, 24\ \mathrm{V}, 30\ \mathrm{V}\}.$$

(c) Some of the results, among many others, are:

$$\begin{aligned} V_{12} &\equiv V_{2\to1} = -6\ \mathrm{V} \\ V_{53} &\equiv V_{3\to5} = +12\ \mathrm{V} \\ V_{2\mathrm{B}} &\equiv V_{\mathrm{B}\to2} = -24\ \mathrm{V} \end{aligned}$$

(d) The answers are:

i. $+18\ \mu$J

ii. $-6\ \mu$J

iii. 1.92×10^{-18} J

iv. Here, one must consider the work done by the electrostatic force on the moved charge by the source charges that are present on the source plates.

A. $-36\ \mu$J. One requires an external force to move this charge, as the charge would rather move towards the grounded plate if left to interact with the field generated by the source plates.

B. $+36\ \mu$J. In this case, the negative charge will naturally move away from the grounded plate.

(e) The field will point towards the grounded plate with a magnitude of

$$E = \frac{V_{\mathrm{BA}}}{d} = 6\ \mathrm{V/cm}.$$

(f) Since this is a parallel-plate capacitor configuration with very large plate areas, then the field magnitude will be given by

$$E = \frac{\sigma}{\varepsilon_0},$$

so that the areal charge density is

$$\sigma = \varepsilon_0 E = 5.31\ \mathrm{nV/m} = 53.1\ \mu\mathrm{V/cm}.$$

(g) The potential will be 0 V for $y < 0$, it will be 36 V for $y > d$, while

$$V(y) = \frac{V_{\mathrm{BA}}}{d} y$$

for $y \in (0, d)$. The result for the field will follow by simply differentiating the expression above.

(h) Any result involving changes in energy, changes in potential, or work will remain the same. However, the results involving the values of of potential energy or potential at a particular location will be different, because a different reference is being used for the zero of the potential energy or potential.

3. The (uniform) field in each of the regions will be determined via the use of the relationship

$$E = E_x = -\frac{dV}{dx}.$$

   (a) It is unclear what the state of the field is for $x < 0$ and $x > 75$ cm as there is not enough information provided (i.e., the potential profile in those regions is not provided) in order to determine the specific form of the field. However, because the system consists of very large parallel plates, we can, at least, be confident that the field will be uniform in those regions. Wherever the potential is linear, the field will be constant but nonzero. In particular, the field will be -2 V/cm for $x \in (0, 50 \text{ cm})$, with the minus sign indicating that the field points towards plate A (i.e., along the $-x$-direction). For $x \in (50 \text{ cm}, 75 \text{ cm})$, the field will be $+12$ V/cm, with the plus sign indicating that the field points towards plate C (i.e., along the $+x$-direction).

   (b) This graph will follow immediately from the results of Part (a).

   (c) Because it was uncovered in Part (a) that the field between plates A and B points towards plate A, then this would mean that the force an electron would feel in between these plates must be towards plate B. One may use Newton's 2nd Law to uncover the electron's acceleration towards this plate, and use either constant-acceleration kinematics or energy conservation to determine the speed with which the electron would reach plate B.

   (d) Because it was uncovered in Part (a) that the field between plates B and C points towards plate C, then this would mean that the force an electron would feel in between these plates must be towards plate B. However, since the electron would have an initial velocity pointing away from plate B when exiting through the bored hole, then the electron will inevitably slow down, because that's what an object does when its velocity points opposite to its acceleration. One of the points of this part is to see if there is substantial-enough acceleration to prevent the electron from reaching plate C given the initial speed with which it enters the region between plates B and C.

4. This potential has a linear variation in x for a fixed value of y, and has a linear variation in y for a fixed value in x.

   (a) Because x and y presumably have SI units of meters, then in order to ensure that the potential is written in units of Volts, then the units of α must be

$$[\alpha] = \text{V/m}^2.$$

   (b) The answer is

$$\vec{E} = E_x\hat{x} + E_y\hat{y} = -\alpha\left(y\hat{x} + x\hat{y}\right),$$

   so that the field component along x, $\vec{E}_x$, varies purely as a linear function of y and points towards the $-x$-direction, while the field component along y, $\vec{E}_y$, varies purely as a linear function of x and points toward the $-y$-direction.

   (c) Note that for

$$V_n \equiv n\alpha$$

   (for $n = 0, \pm 1, \pm 2, \ldots$), the equipotential surface will have the form

$$y = \frac{n}{x}.$$

   Note that if one enforces that adjacent equipotential surfaces have the same potential difference (which is the customary rule to follow, and is guaranteed if we take $|\Delta n| = 1$), then these surfaces will be quite closely spaced particularly when x and y acquire large values simultaneously, while they will be more sparsely spaced when x or y is quite comparatively much smaller than y or x, respectively. This is to be expected, as the equipotential surfaces do have an inverse relationship between y and x.

(d) Using the result of Part (b), note that, for example, the field lines will point purely along the $-x$-direction along the $+y$-axis (i.e., $x = 0,\ y > 0$) and along the $+x$-direction along the $-y$-axis (i.e., $x = 0,\ y < 0$). Moreover, the field lines will point purely along the $-y$-direction along the $+x$-axis (i.e., $y = 0,\ x > 0$) and along the $+y$-direction along the $-x$-axis (i.e., $y = 0,\ x < 0$). Furthermore, the field lines will point at 45° angles along points where $y = \pm x$, pointing towards the origin in Quadrants I and III, and pointing away from the origin in Quadrants II and IV. The field strength is particularly large when both x and y are large, while it is small when both x and y are small. The field will point more dominantly along the x-direction when $|y| \gg |x|$, while it will point more dominantly along the y-direction when $|x| \gg |y|$. A lot of these general trends can be seen quite clearly by recalling that field lines have to point locally perpendicular to the equipotential surfaces, and that they must point towards lower potential.

5. Both the potential of a point source varies inversely with the distance from the source, while the potential energy between point charges varies inversely with the distance between the charges. Taking $e \equiv 1.60 \times 10^{-19}$ C, note that the charge of the hydrogen's nucleus is e, while the charge of the electron orbiting the hydrogen nucleus is $-e$.

   (a) The answer is: $U_{\mathrm{g}} \equiv U_{\mathrm{ground}} = -ke^2/a_0$ and $U_{\mathrm{e}} \equiv U_{\mathrm{excited}} = -ke^2/\left(4a_0\right)$. Recall that $k = \left(4\pi\varepsilon_0\right)^{-1}$, $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C}^2/\left(\mathrm{N}\cdot\mathrm{m}^2\right)$, and that 1 eV $= 1.60 \times 10^{-19}$ J.

   (b) The answer is: $V_{\mathrm{g}} \equiv V_{\mathrm{ground}} = ke/a_0$ and $V_{\mathrm{e}} \equiv V_{\mathrm{excited}} = ke/\left(4a_0\right)$.

   (c) Use the fact that $W_E^{(\mathrm{g}\to\mathrm{e})} = U_{\mathrm{g}} - U_{\mathrm{e}}$.

   (d) Here, $\Delta U = U_{\mathrm{e}} - U_{\mathrm{g}}$.

   (e) Since the potential energy is negative (since the interaction between the electron and nucleus is attractive), this is a $1/r$ dependence that diverges to $-\infty$ as $r \to 0$, and converges to 0 as $r \to \infty$. Since the nucleus (i.e., the source charge) is positive, then the potential undergoes a $1/r$ dependence that diverges to $+\infty$ as $r \to 0$, and converges to 0 as $r \to \infty$.

   (f) Nothing changes in this case regarding the quantities of potential energy, change in potential energy, or work. However, since the source charge is now negative instead of positive, then the potential, as well as the potential difference, will be different, as the potential will be inherently negative, while potential differences will increase for larger radial distances.

6. Recall that the work required to assemble a configuration of charge (i.e., bring each charge from ∞ and place them at specific finite locations from one another in some predetermined geometrical configuration) is the same as determining the potential energy of the configuration of charge. The first charge in the assembly costs nothing energetically, whereas assembling the second charge into its location must consider its interaction with the first charge, assembling the third charge into its location must consider its interaction with the first and second charges, and so on and so forth.

   (a) Using proper book-keeping, and bringing in the charges starting with q_4 and ending with q_1, we may write

$$
\begin{aligned}
U &= U_4 + U_3 + U_2 + U_1 \\
&= 0 + U_{34} + \left(U_{24} + U_{23}\right) + \left(U_{14} + U_{13} + U_{12}\right) \\
&= k\left[q_3\left(\frac{q_4}{r_{34}}\right) + q_2\left(\frac{q_4}{r_{24}} + \frac{q_3}{r_{23}}\right) + q_1\left(\frac{q_4}{r_{14}} + \frac{q_3}{r_{13}} + \frac{q_2}{r_{12}}\right)\right].
\end{aligned}
$$

   (b) It can be seen that the net force on 1 will aim to move the charge away from the configuration. That initial push will eventually result in a weaker and weaker interaction force with the remaining charges of the assembly as the charge moves farther away, resulting in the charge eventually trekking infinitely far away from the configuration if one waits long enough. One may use the conservation of mechanical energy to determine the kinetic energy very far away. Since the kinetic energy initially is zero, then

$$\begin{aligned} K_1 + U_1 &= K_\infty + U_\infty \\ 0 + U_1 &= K_\infty + 0 \\ kq_1\left(\frac{q_4}{r_{14}} + \frac{q_3}{r_{13}} + \frac{q_2}{r_{12}}\right) &= \frac{1}{2}m_1 v_\infty^2 \\ 2\frac{kq^2}{\ell}\left(1 - \frac{1}{\sqrt{2}}\right) &= \frac{1}{2}m_1 v_\infty^2. \end{aligned}$$

Note that because the Left-Hand Side above is positive, then it is indeed possible for the charge to reach a position infinitely far-away with a well-defined speed, v_∞.

7. There are a couple of ways to determine this result. One may use the "brute-force" method and calculate the amount of energy necessary to bring in infinitesimal amount of charges from ∞ bit-by-bit to then determine the total energy. Another, possibly more straightforward, means by which this may be calculated would be to integrate over all of the volume occupied by the electric field. This method will be further illustrated when considering, for example, the energy to charge up a capacitor, which will be an upcoming problem of interest in the class. It will, indeed, turn out that both answers will be the same.

   (a) Using the latter of the two methods, if we consider that some charge, q, has already been placed on the conducting sphere, then the configuration will behave as a point charge. The additional energy required to place another charge, dq, onto this sphere would be the potential of a point charge, q, at position R, multiplied by the charge, dq:

   $$dU = dqV = dq\left(k\frac{q}{R}\right),$$

   so that the total energy would be

   $$U = \int dU = \frac{k}{R}\int_0^Q q dq = \frac{1}{2}k\frac{Q^2}{R}.$$

   (b) Using the former of the two methods, recall that the field for such a sphere has the form

   $$\vec{E} = \begin{cases} k\frac{Q}{R^3}r\hat{r} & \text{for } 0 \le r \le R \\ k\frac{Q}{r^2}\hat{r} & \text{for } r > R \end{cases}.$$

   To obtain the total energy, one may simply integrate the energy density,

   $$u_E = \frac{1}{2}\varepsilon_0 E^2,$$

   over the entire volume over which the sphere exists (i.e., over all of three-dimensional space in this case):

   $$U = \int_{\mathrm{V}} u_E d\mathcal{V}.$$

   Note that we may write that $d\mathcal{V} = 4\pi r^2 dr$.

8. The charge on the inner sphere will reside fully on its surface; however, because the shell is neutral and conducting, the charge on the inner sphere will polarize the shell. In particularly, because the field must be zero within any conducting volume that is in electrostatic equilibrium, the field will be zero within the inner sphere, as well as within the conducting shell.

   (a) The conducting sphere as well as the conducting shell will act as equipotential volumes since they are both conductors in electrostatic equilibrium. Indeed, since $\vec{E} = \vec{0}$, they will trivially satisfy the equipotential criterion that $\Delta V = 0$. Field lines will exist within the cavity between the inner sphere and the inner portion of the shell (i.e., for $a < r < b$), and they will also exist for $r > c$, pointing radially away from the center of the configuration in these regions. In these regions, equipotential surfaces will be defined by the surfaces that satisfy the criterion that $\vec{E} \perp d\vec{\ell}$, which will happen to be spherical surfaces concentric with the center of the configuration. Although it is not known what the specific relationship between the radii a, b, and c is, one can nonetheless conclude that the equipotential surfaces in the regions with nonzero field will be more closely spaced when closer to the inner sphere within the cavity, and will get more sparsely spaced the farther away one gets to the center of the configuration.

(b) This is easily determined using Gauss's Law for the various regions of interest. In particular, the field will be identically that of a point charge, Q, at the center of the configuration, with regions of zero field where a conducting body exists.

(c) Since the potential is referenced from ∞ (where it is zero), one way to methodically determine it is to pick radial positions in the four distinct regions where it is desired, starting with the outermost region and working inward.

i. For $r > c$,

$$V(r) = V(r) - V(r \to \infty) = -\int_{\infty}^{r} E(r' > c)\, dr'.$$

ii. For $b < r < c$,

$$V(r) = -\int_{\infty}^{c} E(r > c)\, dr - \int_{c}^{r} E(b < r' < c)\, dr' = -\int_{\infty}^{c} E(r > c)\, dr.$$

iii. For $a < r < b$,

$$\begin{aligned} V(r) &= -\int_{\infty}^{c} E(r > c)\, dr - \int_{c}^{b} E(b < r < c)\, dr - \int_{b}^{r} E(a < r' < b)\, dr' \\ &= -\int_{\infty}^{c} E(r > c)\, dr - \int_{b}^{r} E(a < r' < b)\, dr'. \end{aligned}$$

iv. For $r < a$,

$$\begin{aligned} V(r) &= -\int_{\infty}^{c} E(r > c)\, dr - \int_{c}^{b} E(b < r < c)\, dr - \int_{b}^{a} E(a < r < b)\, dr - \int_{a}^{r} E(r' < a)\, dr' \\ &= -\int_{\infty}^{c} E(r > c)\, dr - \int_{b}^{a} E(a < r < b)\, dr. \end{aligned}$$

(d) This follows by graphing the results of Parts (b) and (c), particularly noting that (as seen in Part (c)), the potential is continuous while the field is discontinuous. This discontinuity in the field is a tell-tale sign of the existence of only surface charges.

9. Parts (a) - (c) follow standard protocol, but be mindful of the fact that the field within the sphere is different than outside it. For this reason, determining the potential—particularly within the sphere—must be treated carefully. Since the sphere acts like a point charge, the for $r > R$,

$$V(r) = k\frac{Q}{r},$$

so that

$$V(R) = k\frac{Q}{R}.$$

Thus, for $r < R$,

$$V(r) = V(R) - \int_{R}^{r} E(r' < R)\, dr'.$$

For Part (d), one must use a similar formalism to find the energy required. For example, one may first find the potential difference between $5R$ and $R/2$:

$$\begin{aligned} \Delta V &\equiv \Delta V_{5R \to R} + \Delta V_{R \to R/2} \\ &= \left[\frac{kQ}{r}\right]_{5R}^{R} - \int_{R}^{R/2} E(r < R)\, dr \\ &= \frac{4}{5}\frac{kQ}{R} - \int_{R}^{R/2} \left(\frac{kQ}{R^3} r\right) dr. \end{aligned}$$

Then, $\Delta U = q\Delta V$.

10. Recall that the potential across a conducting path is constant when in electrostatic equilibrium. This will be especially useful for when the spheres are connected.

    (a) Since both spheres have the same charge, the sphere with the smaller radius will have the higher potential. The reasoning can be understood by considering, for example, which one takes more energy to assemble.

    (b) Because the potentials are different, then connecting the conducting wire will naturally cause charges to flow from the higher-potential sphere to the lower-potential one. After a very short time, the system will reach electrostatic equilibrium, resulting in the potential difference between the spheres being identically zero. Since the conducting wire is quite thin, the amount of charge it will hold is negligible compared to the spheres themselves. For this reason, the sphere with the larger radius will hold proportionately more charge than the one with the smaller radius to ensure a minimum-energy configuration, towards which systems naturally tend.

    (c) The answer is: a/b. This is in line with the analysis discussed in Part (b).

    (d) The answer is: b/a. Note that this shows that the final surface charge densities are not equal.

    (e) This follows from the fact that the smaller sphere has the larger surface charge density, as seen in Part (d). As such, this will also result in the smaller sphere having the larger field strength when near it. Thus, it is much easier to generate the breakdown of the field through air and, thus, create a lightning-bolt through the ionized air from the sharp protuberance to another conducting surface.

11. Note that q_1 has a larger charge (in magnitude) than q_2.

    (a) Because the potential of a point charge is inversely proportional to the observation point's distance away from it, then to compensate for the fact that $q_1 = 4\,|q_2|$, the distance from the observation point to each of the charges must satisfy the condition $r_1 = 4r_2$. This will only happen for $0 < x < d$, and for $x > d$.

    (b) So long as the condition for the relative positions of the charges from the observation point in Part (a) are satisfied, then one will clearly see that there are, indeed, other points that satisfy this condition. The locus of such points will form an equipotential line of zero voltage in the xy-plane.

    (c) For the field to be zero, one also has to factor direction, because, unlike the potential, the field is a vector. Thus, the vector sum of electric-fields at a specific location must give zero in order for the field to be zero there. Also, the field has a different spatial profile for these point charges. In particular, it's inversely proportional to the square of the distance of the observation point from the source charges. To be more rigorous, because the field and the potential have the relationship,

    $$\vec{E} = -\vec{\nabla} V,$$

    which along the x-axis would reduce to

    $$\vec{E} = -\hat{x}\frac{dV}{dx},$$

    note that just because a function has a zero value doesn't necessarily imply that its derivative at that zero point is also zero. Indeed, the field changes in value as one moves through the point at which $\vec{E} = \vec{0}$.

12. A similar problem was done in class as one of the first examples of determining the potential due to a discrete charge configuration.

    (a) The answer is:

    $$V(x) = \frac{2kq}{x^2 - a^2}\begin{cases} a \cdot \operatorname{sgn}(x) & \text{for } |x| > a \\ -x & \text{for } |x| < a \end{cases},$$

    where

    $$\operatorname{sgn}(x) \equiv \frac{|x|}{x} = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}.$$

    (b) The answer is zero. The y-axis serves as an equipotential line in the xy-plane.

(c) Since the negative charge is to the left of the y-axis (i.e., in the half-plane with $x < 0$), then that half-plane will be the direction in which the potential is going lower from the y-axis. In other words, the field at any point along the y-axis will point in the $-x$-direction. For this reason, the force on an electron placed on that axis will point in the $+x$-direction.

(d) This can be stitched together quite simply using smooth lines that pass through perpendicular to the field lines. Of course, you also made measurements of this in the equipotential-surfaces lab.

(e) This was done in class, but was exclusively determined via the polar coordinates r and θ. One would simply need to convert back to the Cartesian coordinate x and y. The answer is:

$$V(x,y) = kq\left[\frac{1}{\sqrt{(x-a)^2+y^2}} - \frac{1}{\sqrt{(x+a)^2+y^2}}\right].$$

(f) It is useful to do the following with the denominator terms in the result for Part (e):

$$\begin{aligned}\sqrt{(x\pm a)^2+y^2} &= \sqrt{x^2 \pm 2ax + a^2 + y^2} \\ &= \sqrt{(x^2+y^2+a^2) \pm 2ax} \\ &\approx \sqrt{x^2+y^2 \pm 2ax} \\ &\approx \sqrt{(x^2+y^2)\left(1 \pm \frac{2ax}{(x^2+y^2)}\right)} \\ &\approx \sqrt{x^2+y^2}\left(1 \pm \frac{2ax}{(x^2+y^2)}\right)^{1/2}.\end{aligned}$$

Then, we have:

$$\begin{aligned}\frac{1}{\sqrt{(x\pm a)^2+y^2}} &\approx \frac{1}{\sqrt{x^2+y^2}}\left(1 \pm \frac{2ax}{(x^2+y^2)}\right)^{-1/2} \\ &\approx \frac{1}{\sqrt{x^2+y^2}}\left(1 \mp \frac{ax}{(x^2+y^2)}\right).\end{aligned}$$

The result quoted then follows.

(g) To get this result, derivatives must be evaluated with great care. In particular, the partial derivatives of the potential must be taken to get the field components

$$E_x = -\frac{\partial V}{\partial x} \quad \& \quad E_y = -\frac{\partial V}{\partial y}.$$

Recognizing the transformation between polar and Cartesian coordinates,

$$r = \sqrt{x^2+y^2}$$

and

$$\tan\theta = \frac{y}{x},$$

the result will follow.

(h) Even if no success was achieved in deriving the result in Part (g), note that the field along the $+x$-axis would have $\theta = 0$ and $r = x$; whereas, the field along the $+y$-axis would have $\theta = \pi/2$ and $r = y$.

13. Note that, in this case, there will be no finite point in space for which the potential will be zero!

(a) The answer is:

$$V(x) = \frac{2kq}{x^2-a^2}\begin{cases}|x| & \text{for } |x| > a \\ -a & \text{for } |x| < a\end{cases}.$$

(b) The answer is:

$$V(y) = \frac{2kq}{\sqrt{y^2 + a^2}}.$$

(c) These results follow by using the zeroth-order binomial approximation for each respective potential. The result, as expected, will simply vary inversely with the coordinate.

14. Since the rod is deemed to be very long, then we can comfortably approximate the rod's length to be practically infinite for all intents and purposes.

(a) Since the conductor will be in electrostatic equilibrium, the net charge will reside on the cylindrical surface. However, since there is charge on the cylindrical surface, field lines must exist exterior to the cylinder. However, since the field lines must be created by these charges on the surface, these field lines must necessarily point perpendicular to the surface to ensure that $\Delta V = 0$ on the surface.

(b) Within the conductor, the field will be zero, trivially resulting in a potential difference of zero within the conductor.

(c) Since the rod is presumed to be infinitely long, then there will be charge residing at ∞. As such, the potential will not tend to zero as $r \to \infty$. For this reason, one will find that using

$$V = k \int \frac{dq}{r}$$

will result in a divergence of the potential. Indeed, the formula above results from agreeing that $V \to 0$ as $r \to \infty$, but this only works if the charge configuration is finite in its spatial extent.

(d) Equipotential surfaces will be cylindrical shells that are concentric with the conducting rod's axis of symmetry. The field will be given by Gauss's Law to be

$$\vec{E} = \begin{cases} \frac{2k\lambda}{r}\hat{r} & \text{for } r > a \\ \vec{0} & \text{for } r < a \end{cases}.$$

i. The answer is:

$$V_{\text{AB}} \equiv V_{\text{A}} - V_{\text{B}} = 2k\lambda \ln\left(\frac{b}{a}\right).$$

ii. The result will become

$$V(r) = \begin{cases} 2k\lambda \ln(b/r) & \text{for } r > a \\ 2k\lambda \ln(b/a) & \text{for } r \leq a \end{cases}.$$

Notice that $V > 0$ for $a < r < b$, $V < 0$ for $r > b$, and that the potential is a positive constant for $r \leq a$ (as it should be for a conductor in electrostatic equilibrium).

(e) The potential inside this the shell is no different than in Part (dii), but the potential outside the shell must be zero for all points outside the shell. Indeed, the field outside the shell will be zero since, by Gauss's Law, the charge enclosed will be zero for any Gaussian surface. As such, the potential difference must be zero everywhere outside the shell. However, since the potential must be continuous, it must have the same value outside the shell as at the shell. Thus, it must be zero outside the shell.

15. This problem is one involving a continuous collection of charges on a straight rod of finite length. Since it is a finite configuration of charge, then the potential may be determined using

$$V = \int dV = k \int \frac{dq}{r}.$$

(a) Since the linear density is constant in this case, then $\lambda = Q/\ell$.

i. In this part, be careful to integrate over the source and get an answer that depends on x. In other words, use, for example, a parameter x' to denote points on the rod, with $x' \in [-\ell/2, \ell/2]$. Then, the integral will be taken over the source (i.e., over x' with an integration measure dx'). The answer is:

$$V(x) = k\lambda \ln\left(\frac{x + \ell/2}{x - \ell/2}\right).$$

ii. This was done in class. Note that the integral may be taken over positions x, since now there is no longer a conflict between the observation point and the source domain, as in Part (ai) above. The answer is:

$$V(y) = 2k\lambda \ln\left[\sqrt{1+\left(\frac{\ell}{2y}\right)^2}+\frac{\ell}{2y}\right].$$

(b) We can no longer assume that the linear density is uniform. Indeed, $\lambda \neq Q/\ell$, but, rather, is given by the function provided. Note from the functional form of the density that the density is the largest at either edge of the rod (taking on a value of λ_0 at those edges), and has its smallest value at the center (where it is zero).

i. This will be found by integrating the charge density over the entire length of the rod and solving for λ_0:

$$Q = \int_{-\ell/2}^{\ell/2} \lambda(x)\, dx.$$

ii. The procedure is the same as in Part (ai), except that the density cannot be pulled out of the integral.

iii. The procedure is the same as in Part (aii), except that the density cannot be pulled out of the integral.

16. The ring has a uniform charge density, with linear density $\lambda = q/(2\pi a)$.

(a) This was done in class. The answer is:

$$V(z) = \frac{kq}{\sqrt{a^2+z^2}}.$$

(b) The field may be derived from the potential via the relationship:

$$E_z = -\frac{dV}{dz},$$

where regular (instead of partial) derivatives are being used since the variation in the potential is only along z. Indeed, note that the equipotential surfaces must be perpendicular to the z-axis since, as we have found before, the field along the symmetry axis points along this axis.

(c) Although the maximum of the potential may be found by taking its derivative and setting it equal to zero, the result may be seen quite intuitively to be where the distance from the source charges to the observation point is a minimum. This occurs exactly at the center of the ring. The maximum of the field definitely does not occur at the ring's center, because the field there actually happens to be a local minimum (since all contributions from the point charges vectorially cancel there). Instead, the field is maximal at a finite location along the symmetry axis, which occurs when there is a reasonable compromise between the size of the on-axis components of the field due to these point charges and the distance from the sources themselves.

(d) The properties described in Part (c) above will show through in the plots.

17. Note that the disk (as was done in class for determining the field of a disk) may be determined by breaking the disk up into concentric rings of infinitesimal thickness. In this way, the result for the potential of the ring may be used to now determine the potential for the disk.

(a) From Problem 16a, the contribution of an infinitesimally thin ring of radius r and charge dq is:

$$dV = \frac{kdq}{\sqrt{r^2+z^2}}.$$

Here, $dq = \sigma dA = \sigma 2\pi r dr$, with $\sigma = Q/\left(\pi R^2\right)$ .The result for the potential of this disk along the symmetry axis will follow upon proper integration.

(b) Again, the equipotential surfaces of this disk will be orthogonal to the z-axis, so that the field may be determined via taking the derivative of the disk's potential along this axis relative to the coordinate z:

$$E_z = -\frac{dV}{dz}.$$

18. Note that the field will point radially inward within the cavity of the sphere and the shell, while the field will point radially outward from the outside of the shell. In particular,

$$\vec{E} = \hat{r}\frac{kQ}{r^2}\begin{cases} 0 & \text{for } r < R \\ -1 & \text{for } R < r < 2R \\ +1 & \text{for } r > 2R \end{cases}.$$

(a) With the knowledge of the field between the spherical surface and the shell, the potential difference may be determined using the definition of potential in terms of the field:

$$\Delta V = -\int_c \vec{E} \cdot d\vec{r}.$$

The answer is:

$$V_{R\to 2R} = \frac{kQ}{2R}.$$

(b) Although the field is clearly discontinuous at the boundaries between the various regions, the potential will be continuous. Using the methods of Problem 8c, one may determine the potential everywhere. The answer is:

$$V(r) = kQ\begin{cases} \frac{1}{r} & \text{for } r > 2R \\ \left(\frac{1}{R} - \frac{1}{r}\right) & \text{for } R < r \leq 2R \\ 0 & \text{for } r \leq R \end{cases}.$$

(c) The graphs will clearly indicate the discontinuity in the field at $r = 2R$ and at $r = R$, but the potential will be continuous throughout. Moreover, the potential is not only zero throughout the volume of the inner conducting shell, but it also tends to zero as $r \to \infty$.

19. Because the slab is an insulating one, then the field will not be zero within the confines of the slab. Indeed, recall that the field for such a slab for $z > |d/2|$ will be uniform, but that the field within the slab will vary linearly with the coordinate, z, and will be zero at $z = 0$. Since this slab is considered to be infinite in its extent, it will no longer be the case that $V \to 0$ as $r \to \infty$. This is why the problem is requiring an appropriate—albeit, arbitrary—choice of the zero of the potential.

(a) Let us choose the potential to be zero at $z = 0$. Since we know the field (e.g., upon appropriate uses of Gauss's Law with a Gaussian pillbox) has the form

$$E(z) = 4\pi k\rho\begin{cases} (d/2)\cdot \operatorname{sgn}(z) & \text{for } z > |d/2| \\ z & \text{for } z \leq |d/2| \end{cases},$$

where

$$\operatorname{sgn}(z) \equiv \frac{|z|}{z} = \begin{cases} +1 & \text{for } z > 0 \\ -1 & \text{for } z < 0 \end{cases},$$

in order to pick out the fact that the field points upward (relative to the figure) when above the slab (i.e., for $z > d/2 > 0$) and points downward below the slab (i.e., for $z < -d/2 < 0$). Then the potential difference between $z = 0$ and $z = d/2$ will have the form

$$V_{d/2\to 0} = \frac{1}{2}\pi k\rho d^2.$$

(b) With the chosen reference in Part (a), the answer becomes:

$$V(z) = 2\pi k\rho\begin{cases} z^2 & \text{for } z \leq |d/2| \\ \left[\left(\frac{d}{2}\right)^2 - |z|\,d\right] & \text{for } z > |d/2| \end{cases}.$$

The "$|z|$" in the expression for $z > |d/2|$ is to ensure that the expression is symmetric on both sides. Indeed, one can see this to be the case by invoking the definition of the absolute-value function here:

$$|z| = \begin{cases} +z & \text{for } z \geq 0 \\ -z & \text{for } z < 0 \end{cases}.$$

Furthermore, note that this would also reproduce the correct result for the constant field in the regions outside the slab in Part (a).

20. By energy conservation, the speed of the electron upon exiting the cathode-anode assembly will be given by

$$eV = \frac{1}{2}mu_x^2.$$

(a) Note that the deflecting capacitor assembly generates a uniform field that causes the electrons to undergo constant acceleration while moving through the length L of this deflector. From Newton's 2nd Law, the acceleration will point upward with the magnitude

$$a = \frac{e}{m}\frac{V'}{d},$$

where the field in the deflector is $E' = V'/d$. After exiting the deflector, the electrons will move in a straight line towards the fluorescent screen. Thus, the main part of the problem focuses on figuring out exactly how much the electrons have deflected vertically upon exiting the deflector. Once this is known, then the striking position may be immediately determined by following that beam line straight from the exiting position of the deflector all the way to the screen. Qualitatively, it should be evident that the size of the field within the deflector will determine the extent to which the beam gets deflected, so that a stronger field within those deflector plates will undoubtedly make the beam deflect even more and, thus, hit the fluorescent screen at a larger value of y. To make the electrons hit the screen below the x-axis, the polarity of the deflector plates must be switched to allow deflection in the other direction. The answer is:

$$y = \left(\frac{eV'L}{mu_x^2 d}\right)\left(\frac{L}{2} + D\right) = \frac{1}{2}\frac{V'}{V}\frac{L}{d}\left(\frac{L}{2} + D\right),$$

where the last equal sign above follows by using energy conservation with

$$\frac{1}{2}mu_x^2 = eV.$$

(b) An oscilloscope effectively reads the points at which the electrons hit the fluorescent screen. Since we see from Part (a) that the location that the electrons hit the screen as a deflection from the original collimated beam-line, must be proportional to the deflecting voltage, V'. As such, one can measure potential differences by, for example, placing an unknown potential difference across the deflecting plates, and then figuring out exactly where the beam hits the fluorescent screen. Upon working backwards (i.e., solving for V' in the result of Part (a) in terms of the deflecting position y), one can determine the unknown potential difference with a knowledge of all of the other parameters.

(c) This question imagines that there is another orthogonal axis, z, that points perpendicular to the page of the drawing at the screen. Just as the vertically oriented deflecting capacitor in the figure allows for the beam to sweep in the y-direction, one can place another deflecting capacitor assembly superposed with the one shown in the figure, except that this other plate system has the plates placed in front and in back of the beam line. Thus, collectively, the deflecting capacitor would look like the front, back, top, and bottom sides of a box. The front and back plates will then be responsible for deflecting the beam to cover (or "sweep") all points along the z-axis of the screen. (As an interesting side-note, the same mechanism is used in the old-school Cathode-Ray Tube (CRT) televisions, where pixels are made to light up by an electron beam that hits the picture tube of the television. To form an image, electrons must be made to scan through all points in the yz-plane of the fluorescent screen rapidly enough to give the viewer a smooth video of a televised event. Of course, these televisions have now become obsolete in favor of Liquid-Crystal Display televisions which work off of a different operating principle.)

21. This problem will be (or was), for the most part, done in class. The answers are:
   (a) $C = \frac{1}{4\pi k}\frac{A}{d}$.
   (b) $C = \frac{R}{k}$.
   (c) $C = \frac{1}{k}\frac{ba}{(b-a)}$.
   (d) $C = \frac{1}{2k}\frac{\ell}{\ln(b/a)}$.
   (e) Since $1/k \equiv 4\pi\varepsilon_0$—where ε_0 is the permittivity of free space—the dielectric aims to make $\varepsilon_0 \to \kappa\varepsilon_0$, where κ is the dielectric constant of the dielectric. The dielectric constant can be unambiguously defined as the ratio of the capacitance when the dielectric completely fills the space between the electrodes, C', to the bare capacitance (i.e., the capacitance in the absence of the dielectric), C:

$$\kappa \equiv \frac{C'}{C}.$$

22. This was (or will be) discussed in class. Recall that the dielectric between the capacitor plates gets polarized via the alignment of its dipole moments with the externally applied field from the plates. This dipole-moment alignment in turn generates a field within the dielectric that opposes the externally applied field; however, the field strength of the dipoles within the dielectric is always smaller than the applied field, so that the dielectric aims to reduce the net field between the capacitor plates. The "free" charge refers to the charge on the conducting plates, which, via the discussion of conduction electrons, is free to mobilize. The "bound" charge refers to the induced charge that exists within the dielectric (caused as a result of the polarization of the dielectric from the source plates). These charges are bound to specific lattice sites defined by the molecular make-up of the dielectric.

23. Because we have analyzed such a system multiple times, we are already quite aware of the field profile and voltage implications of such a system. The field only exists between the plates of the capacitor, since the spatial uniformity of the field of the individual plates aims to cancel the field outside of the plates via superposition.
   (a) As in Problem 7b, a nice way to determine the energy of assembly is to integrate the energy density over the volume in which the field exists. As stated, the field only exists between the plates of the capacitor, the volume of which is $\mathcal{V} = Ad$. Because the field is uniform, the necessary energy is

$$U = u_E\mathcal{V} = \frac{1}{2}\varepsilon_0 E^2\mathcal{V} = \frac{\sigma^2}{2\varepsilon_0}Ad = \frac{Q^2}{2\varepsilon_0}\frac{d}{A}.$$

   Notice that this is no different than just saying

$$U = \frac{Q^2}{2C},$$

   where since

$$C = \frac{\varepsilon_0 A}{d}$$

   for a parallel-plate capacitor, then we obtain the exact same result as in the approach using the energy density.
   (b) In this case, consider trying to do work to separate the plates by an amount z. The energy that is necessary in this case may be written out by substituting $d \to z$ in the result of Part (a):

$$U = \frac{Q^2}{2\varepsilon_0 A}z.$$

   Thus, the force becomes

$$F = -\frac{Q^2}{2\varepsilon_0 A},$$

   where the minus sign indicates the attractive nature of the force.

(c) Since the field is

$$E = \frac{Q}{\varepsilon_0 A},$$

then, in magnitude,

$$F = \frac{1}{2} Q E.$$

(d) This is the case because each plate only experiences the field from the other plate and not the field it generates. (This is the same as saying that an object cannot exert a force on itself, but can be the victim of a force exerted on it by some external agent.) For this reason, the factor of 1/2 arises from the result of Part (c), because we must only consider the field acting on each plate due to the other plate, which happens to be half the field between the plates.