Physics 22: Homework 1

The following problems encompass the topics of charge, as well as electrostatic forces, torques, and fields.

1. What is the total charge of all the electrons in 1.2 mol of diatomic hydrogen gas (H_2)?

2. It is well known that protons and neutrons are not truly fundamental, as they are made up of constituents known as quarks. There are six known quarks, and the proton and the neutron are composed of different combinations of the two most common quarks: the "up" quark, which is denoted by "u" and has a charge $q_{\mathrm{u}} = +2e/3$; and the "down" quark, which is denoted by "d" and has a charge of $q_{\mathrm{d}} = -e/3$. (Here, $e \equiv +1.60 \times 10^{-19}$ C is the magnitude of the charge of the electron.) Determine the combination of three of these quarks that would make up the proton and the combination of three of these quarks that would make up the neutron. (Although quarks are more fundamental, they cannot be isolated. So, although they have a charge that is a fraction of the charge of the electron, we still consider the charge of the electron to be the fundamental charge since we may easily isolate electrons without any issue. The problem of isolating quarks is known in the physics literature as "quark confinement," which is still a topic being researched theoretically and experimentally by particle physicists.)

3. The distance between the electron and the proton in elemental (neutral) hydrogen (H) is the so-called Bohr radius, which is $a_0 \approx 0.5 \times 10^{-10}$ m $= 5 \times 10^{-2}$ nm.

   (a) Determine the electrostatic force of attraction between a proton and an electron separated by this distance.

   (b) To determine the gravitational attraction between these particles separated by a Bohr radius, one may use Newton's Law of Universal Gravitation, which states that the gravitational force magnitude between two point masses, m_1 and m_2, separated by a distance, r , is directly proportional to the product of the masses and inversely proportional to the square of the distance between the point masses. The constant of proportionality is known as Newton's Gravitational Constant, $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$, and is necessary to both make the aforementioned proportionalities agree unit-wise with that of a force, but to also correspondingly calibrate the value of the force with the definition of a Newton. Mathematically, the force magnitude in this case is

   $$F_G = G\frac{m_1 m_2}{r^2}.$$

   Use this law to determine the value of the gravitational force between this proton and electron.

   (c) Based on comparing your results to Parts (a) and (b), is it reasonable to ignore gravitational forces when considering physics at the nanoscale?

4. Consider three charged spheres—A, B, and C—that are small enough so as to be treated as point charges. The charges on the spheres are $q_{\mathrm{A}} = 3q$, $q_{\mathrm{B}} = -2q$, and $q_{\mathrm{C}} = -q$, with $q > 0$. As in Figure 1, the charges are arranged along a straight line, with the distance between charges A and B being $r_{\mathrm{AB}} = 2d$, and the distance between charges B and C being $r_{\mathrm{BC}} = d$.

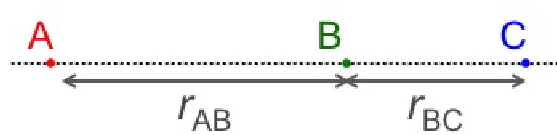

Figure 1: A linear arrangement of three small, charged spheres.

   (a) Draw Free-Body Force Diagrams (FBFDs) for each charge.

   (b) Determine the magnitude and direction of the net electrostatic force on each of the charges.

   (c) Suppose that charges A and C are fixed in space (e.g., glued to a neutral wooden table).

      i. Along the line defined by these charges, determine the location(s) (besides infinity) where charge B may be placed in order for it to be in equilibrium.

      ii. Is (Are) the location(s) determined in Part (ci) stable or unstable?

   (d) Determine the electric field at the location of each charge.

(e) If one of the charges is removed (e.g., B), how is the electric field affected at that point? Explain your reasoning.

(f) Now, suppose that the charges are, instead, arranged in the configuration shown in Figure 2. Here, $r_{\mathrm{AC}} = 3d$ (as it was before) and $r_{\mathrm{BC}} = 4d$. The line connecting A with C and C with B are mutually perpendicular.

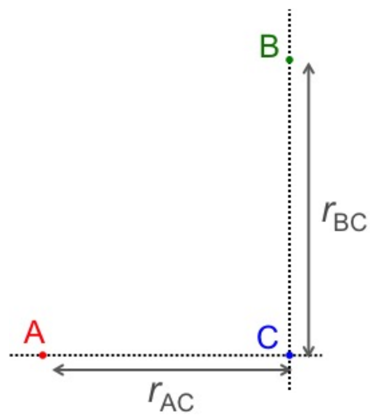

Figure 2: A right-triangular arrangement of three small, charged spheres

i. Determine the net electrostatic force on each of the spheres.

ii. Being allowed to alter the charge on A, what charge must A have (denoted q'_{A}) in order to make the net electrostatic force on B purely horizontal?

5. When conducting objects (such as metals) are connected for a brief time, charge can be made to flow quite easily from one to the other, even if the objects are in contact for an extremely brief amount of time. To that end, consider two identical metallic ball-bearings (i.e., two very tiny metallic solid spheres) having *unknown* charges q_1 and q_2. It is found that when they are placed 1 m apart, they experience a 25-N *attractive* force.

(a) What can you conclude about the charges on these metal marbles from the information given?

(b) The ball-bearings are made to touch very briefly and then are separated by 1 m again. What can you conclude about the charges now? Explain.

(c) Suppose that when the charges are 1 m apart, after briefly touching, they again feel a force of 25 N. Determine the charges q_1 and q_2.

6. As in Figure 3, two identically-charged, small spheres—each of mass m—are hanging from massless strings of length ℓ. In this equilibrium configuration, the strings make a symmetric angle ϕ relative to the vertical.

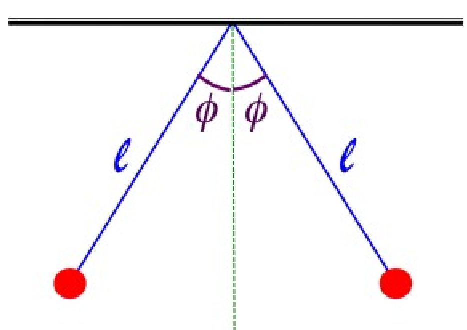

Figure 3: Two identical, equally charged metal spheres attached with identical strings to a common anchor on a rigid ceiling.

(a) Why are the angles that the strings make relative to the vertical symmetric?

(b) Obtain an expression for the charge, q, on either sphere in terms of the given quantities.

(c) *Qualitatively* describe, in detail, exactly what happens to the angles made by each string (and why) in the following cases.

i. The mass of one of the spheres is increased.

ii. The charge on one of the spheres is increased.

iii. The length of one of the strings is shortened.

(d) Now, suppose that we return back to the original configuration: two identically charged spheres (of charge q as determined in Part 6b) with equal mass, m, and with equal string lengths, ℓ. However, these tiny spheres are now immersed in a uniform electric field due to some external source. As shown in Figure 4, this external field is directed horizontally rightward and is seen to make one of the charged spheres hang perfectly vertical and the other one to hang at an angle α relative to the vertical.

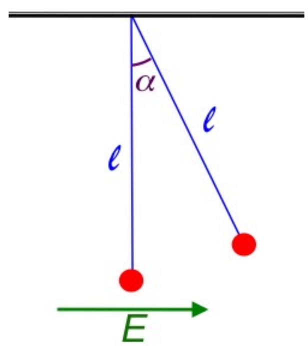

Figure 4: The charge arrangement in Figure 3 is immersed in a uniform electric field. The left-hand string is vertically oriented, while the right-hand string makes an angle, α, relative to the vertical.

i. For the equilibrium configuration shown in Figure 4, why must $q > 0$?

ii. Determine the value of the electric field that would allow for such an equilibrium configuration.

7. Many molecules have a bond structure that lends itself to some form of charge separation that is dipolar. As such, (neutral) electric dipoles are quite important structures to study to get a better understanding of the nature of such molecules. Thus, consider such a (neutral) electric dipole, consisting of a pair of equally, but oppositely, charged points separated by some distance. Suppose that the magnitude of the charge on each of the poles is q, and that the distance separating the charges is d. Let us place this dipole on a two-dimensional coordinate system as in Figure 5, where the charges lie on the y axis and are placed symmetrically on either side of the x axis, which serves as a perpendicular bisector of the dipole configuration. Thus, in this configuration, each pole is a distance $d/2$ away from the x axis.

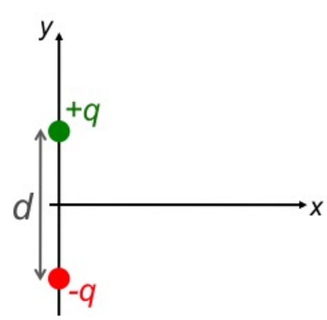

Figure 5: A neutral dipole placed on the y-axis and oriented symmetrically about the x-axis.

(a) Derive a general formula for the electric field (magnitude and direction) of this dipole at any point, y, along the y-axis.

(b) Where along the y-axis is the electric field a maximum?

(c) Derive a general formula for the electric field (magnitude and direction) of this dipole at any point, x, along the x-axis.

(d) Where along the x-axis is the electric field a maximum?

(e) Sketch graphs of the electric-field functions determined in Parts 7a and 7c.

(f) Since the bond length of typical molecules is in the sub-nanometer length scale, if one is at an observation point that is quite macroscopic, then one could impose the condition that the observation point is much greater than the dipole separation.

  i. Assuming that $y \gg d$, take appropriate approximations in Part 7a to obtain an expression for which $E \sim y^{-3}$.

  ii. Assuming that $x \gg d$, take appropriate approximations in Part 7c to obtain an expression for which $E \sim x^{-3}$.

8. As shown in Figure 6, consider a similar arrangement to Problem 7, except that both poles are charged identically (i.e., each pole has the same charge q in both sign and magnitude).

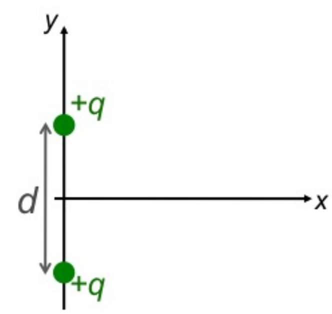

Figure 6: A dipole with positive poles placed on the y-axis and oriented symmetrically about the x-axis.

(a) Repeat Parts (a) - (e) of Problem 7 for this configuration.

(b) Unlike the neutral dipole in Problem 7—which, when viewed from very far, seems to have no charge since the poles practically overlap at such macroscopic length scales—this charged dipole will behave differently. In particular, show that when $y \gg d$ for the field along the y axis, and that when $x \gg d$ for the field along the x axis, the field resembles that of a monopole (i.e., a point charge) of charge $2q$.

9. As in Figure 7, another useful charge configuration to consider is that of the quadrupole, which, in this simplest case, consists of two equal charges q (separated by distance d from one another) and another opposite charge, $-2q$, midway between them. In other words, the charges are configured collinearly along the x axis, with the charges q arranged symmetrically about the y axis and the charge $-2q$ at the origin.

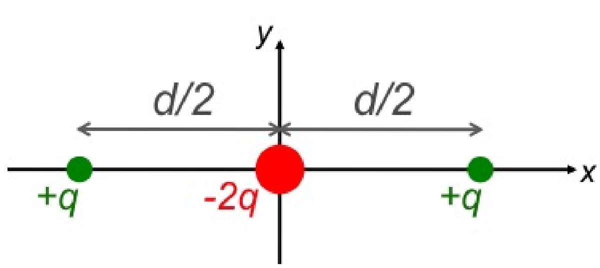

Figure 7: A linear neutral quadrupole placed on the x-axis and oriented symmetrically about the y-axis.

(a) Assuming that $x \gg d$, take appropriate approximations to show that the electric field of this quadrupole has the form that $E \sim x^{-4}$.

(b) In chemistry, it is common to learn about the fact that the forces between molecules are deemed as "short range." Given the analysis done in Problems 7, 8, and 9, discuss the validity of such an assertion.

10. Consider a (neutral) dipole, consisting of charges q and $-q$ that are separated by a distance d. The dipole is placed in a uniform electric field, $\vec{E}$, so that its dipole moment, $\vec{p} \equiv q\vec{d}$, makes an angle, θ, relative to the field, as shown in Figure 8.

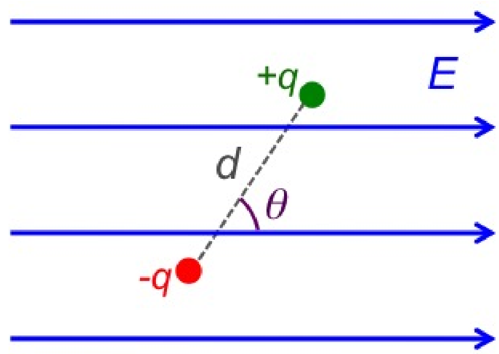

Figure 8: A neutral dipole immersed in a uniform electric field, $\vec{E}$.

(a) Show that the net force on this dipole is zero regardless of the dipole's orientation relative to the field.

(b) Determine the torque on the dipole.

(c) How much work does it take to rotate the dipole from an initial angular position, θ_1, to a final angular position, θ_2?

(d) Determine the angular positions at which the dipole is in stable and unstable equilibrium.

(e) If the dipole is displaced from its stable equilibrium position by an angle $\theta \ll 1$ (measured, for example, in radians), the dipole will oscillate in Simple Harmonic Motion (SHM). If the Moment of Inertia (MI) of the dipole is I, show that the frequency of this small-angle oscillation is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{Eqd}{I}}.$$

11. Consider an xy-coordinate system with a point charge, $Q > 0$, placed at the origin of coordinates. The x-axis of this coordinate system also serves as a perpendicularly bisecting line for a (neutral) dipole located at a distance x along the x-axis, as shown in Figure 9. The dipole has charges q and $-q$ that are separated by a distance, d.

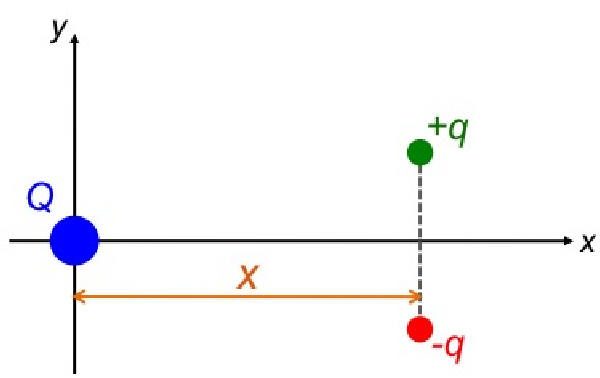

Figure 9: A point charge interacting with a neutral dipole placed parallel to the y-axis and oriented symmetrically about the x-axis, at a perpendicular distance, x, away from the y-axis.

(a) In terms of the given quantities, determine the net force on the dipole.

(b) In terms of the given quantities, determine the net torque about the center of the dipole.

(c) In terms of the given quantities, determine the net force on the point charge, in turn showing that Newton's 3^{rd} Law is satisfied.

(d) Repeat Parts (a) - (c) assuming that $d \ll x$.

(e) Is there a torque on the point charge? Justify your answer.

12. Consider a negatively charged metallic plate that generates a practically constant electric field. At the instant shown in Figure 10, a proton (of mass m_p and charge $q_p = +e$) is near the edge of the plate, traveling at a speed v parallel to its surface, a perpendicular distance D away from the surface of the plate, and a length L from the opposite edge of the plate.

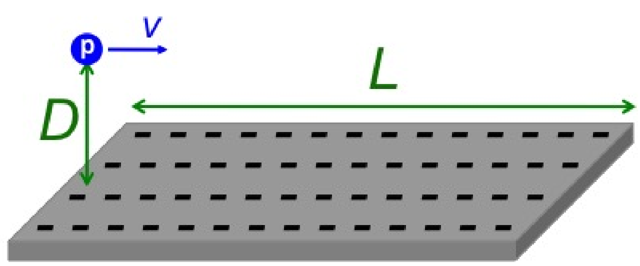

Figure 10: A proton entering an edge of a negatively charged metallic plate, moving parallel to the plate towards its opposite edge.

(a) What is the direction of the electric field generated by the plate?

(b) What is the direction of the electric force on the proton due to the plate?

(c) What must be the electric-field strength of the plate if the proton is to barely graze the opposite edge of the plate in its trajectory?

(d) Determine the angle by which the proton would have deflected from its initially parallel movement to the plate upon reaching the opposite edge.

(e) How would the circumstances be different if the particle had been an electron?

13. In a similar setup to Problem 12, an electron (of charge $q_e = -e$ and mass m_e) is launched from the surface of an infinitely large metallic plate which generates a uniform electric field, E_0. The electron is launched at a speed v, and at an angle ϕ relative to the surface of the plate, as shown in Figure 11.

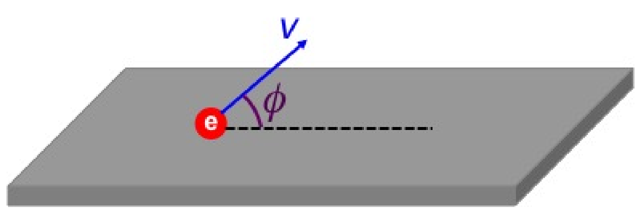

Figure 11: An electron is launched at a speed, v, from an infinite, charged plate at an angle, ϕ, relative to the plate's surface.

(a) What must be the sign of the charges on the plate in order to ensure that the electron eventually returns back to the surface of the plate?

(b) How does the direction of the electric field of the plate compare to the direction of the electric force on the electron due to the plate? Discuss the comparison both qualitatively and quantitatively.

(c) Determine the amount of time it takes for the electron to return to the surface of the plate.

(d) Determine the final speed of the electron.

(e) Determine the maximum height above the plate to which the electron reaches in its trajectory.

14. Consider a uniformly charged rod, of length ℓ and total charge Q, that is placed on an xy-coordinate system in such a way that its body lies completely on the x-axis and is symmetric about the origin of coordinates (i.e., the y-axis serves as a perpendicularly bisecting line to the rod).

    (a) Determine the electric field of the rod at any point x along the x-axis, such that $|x| > \ell/2$.

    (b) Determine the electric field of the rod at any point $y \neq 0$ along the y-axis.

    (c) Determine the electric field of the rod at any point (x, y) on the xy-plane such that $|x| \leq \ell/2$ and $y \neq 0$.

    (d) Keeping the (linear) charge density of the rod fixed but taking the length of the rod to infinity, determine the electric field of the rod a perpendicular distance $y \neq 0$ away from the rod.

    (e) Show that the solution to Part 14b reduces to the solution to Part 14d when one is very, very close to the rod.

15. A rod of length L lies along the y-axis with its center at the origin. The rod has a *nonuniform* linear charge density given by

$$\lambda = \alpha\,|y|\,,$$

    where α is some constant that has units of (Charge) / (Length)^2.

    (a) Draw a graph of $\lambda\,(y)$.

    (b) In terms of the given quantities, determine the total charge of this rod.

    (c) Find the electric field of this rod at any point x along the x-axis (which, of course, serves as a perpendicularly bisecting line of this rod).

16. Consider a circular arc, of angle 2θ, with its center of curvature located at the origin of an xy-plane, as shown in Figure 12. The radius of the arc is R and its constant charge density is $\lambda < 0$.

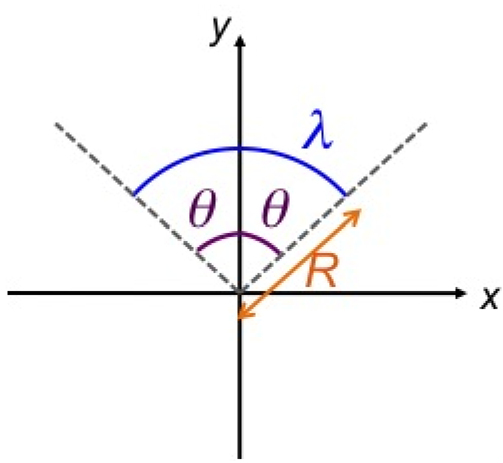

Figure 12: A circular arc, of radius R and charge density λ, oriented symmetrically about the y-axis with its center of curvature at the origin of the xy-plane. The arc subtends an angle 2θ.

    (a) Determine the electric field of this arc at the center of curvature.

    (b) Using the expression in Part (a), determine the electric field at the center of curvature if $\theta = \pi/2$.

    (c) Using the expression in Part (a), determine the electric field at the center of curvature if $\theta = \pi$. Qualitatively justify the outcome of your answer.

17. A uniformly charged ring has total charge $Q > 0$ and radius a. The ring lies in the xy-plane with its center at the origin of this plane. The z-axis serves as this ring's axis of symmetry, as shown in Figure 13.

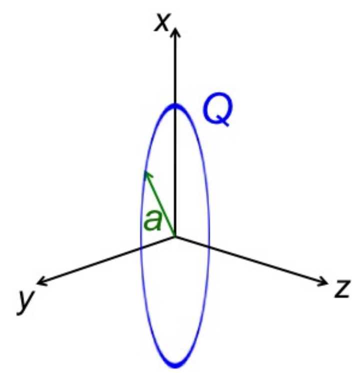

Figure 13: A uniformly charged ring placed on the xy-plane with its center of curvature at the origin of this plane. The z-axis serves as the ring's axis of symmetry.

(a) Show that the electric field of this ring along the z-axis is purely directed along this axis.

(b) Determine the expression for the electric field of this ring at any point z along its axis of symmetry.

(c) At what point(s) along its axis of symmetry is the electric field a maximum? At what point(s) is the electric field a minimum?

(d) A small point charge, $-q$ (with $q > 0$), is placed a small distance z (i.e., $z \ll a$) from the center of the ring. With appropriate approximations, show that the point charge will exhibit SHM and determine the period of oscillation.

18. Consider two uniformly charged rings, set up similarly to Problem 17. Specifically, one charged ring has total charge $Q > 0$, radius a, and has its center of curvature at $z = -b$; while the other charged ring has total charge $-Q$, radius a, and has its center of curvature at $z = +b$. Determine the electric field due to this two-ring configuration at all distinct regions along the z-axis.

19. Consider a semicircular arc of a ring that is uniformly charged—with total charge $Q < 0$—and radius a. This semicircle lies in the xy-plane with its center of curvature at the origin, and is described by the equation

$$x^2 + y^2 = a^2,$$

with $x \in [0, a]$ and $y \in [-a, a]$, as shown in Figure 14. The z-axis is perpendicular to the plane of the semicircle and runs through the arc's center of curvature. Determine the electric field at any point z along the z-axis.

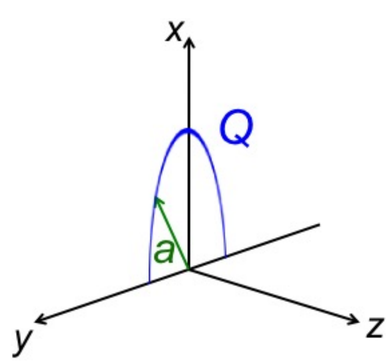

Figure 14: A charged semicircle placed on the xy-plane with its center of curvature at the origin of this plane. The z-axis runs through the semicircle's center of curvature and is perpendicular to the plane of the semicircle.

20. Consider a uniform disk of charge, of total charge $\mathcal{Q}$ and radius R. As shown in Figure 15, the disk lies in the xy plane with its center at the origin. The z-axis serves as its axis of symmetry.

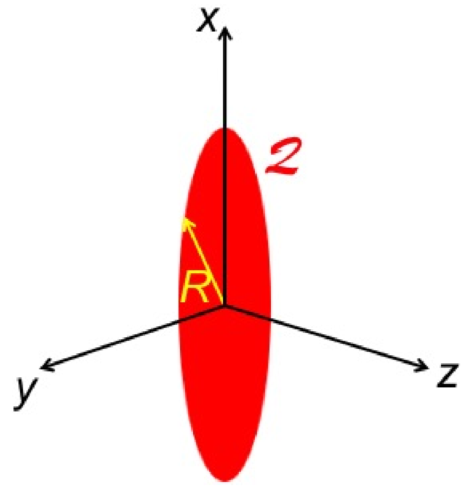

Figure 15: A uniformly charged disk on the xy-plane with its center at the origin of this plane. The z-axis serves as the disk's axis of symmetry.

(a) Using the result of Problem 17b and the "Washer Method" of integration, determine the electric field of this disk at any point z along its symmetry axis.

(b) Show that when the observation point on the symmetry axis is very close to the disk (i.e., $z \ll R$), the expression for the field reduces to a uniform value that only depends on the disk's charge density and is independent of z.

(c) Show that when the observation point on the symmetry axis is very far away from the disk (i.e., $z \gg R$), the expression for the field reduces to that of a point charge.

21. Consider an infinitely long, uniformly charged rod—of charge density $\lambda > 0$—situated along the y-axis. As in Figure 16, a finite, uniformly charged rod—of total charge $Q < 0$ and length ℓ—is situated along the x-axis, with the end closest to the infinite rod being a distance a away from the origin. Determine the net electric force on the finite rod.

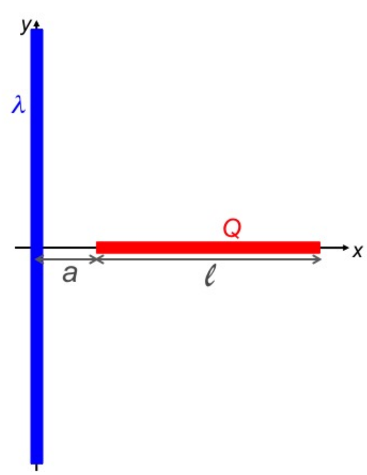

Figure 16: A thin, infinitely-long, uniformly charged rod (placed on the y-axis) is interacting with a thin, finitely-long, uniformly charged rod (placed on the x-axis) with its closest end a distance, a, away from the origin.