Physics 22: Homework 9 Hints

1. The current in the outer loop will move in an Anti-Clock-Wise (ACW) fashion when the switch is closed. When the switch is initially closed, the current cannot rise to its maximal value instantaneously. Thus, the current will require some time to increase up to its largest value. In this increase, the flux through the inner loop will increase, so that Lenz's Rule says that the induced field of the inner loop will aim to try to reduce the flux in this case. Thus, the induced current will flow Clock-Wise (CW). When the current is steady in the outer loop, there will be no induced current in the inner loop. When the switch is suddenly opened, the current in the outer loop will take some time to go to zero. In this process, the inner loop will see a reduction in the flux through it, so that it will apply an induced field to reinforce the external field's flux. As such, the induced current will flow ACW.

2. According to Faraday's Law, the current induced in the loop and, thus, the power dissipation in the loop (with $P \sim I^2$) will be largest when the flux change is the most drastic in time.

   (a) The flux change will be minimal when the bar magnet is close to and within the loop, even though the strength of the field will be among the largest values under these circumstances. The flux will change most abruptly when the magnet is some intermediate distance away from the loop (say on the order of a loop diameter), as in these distances one can see a progression of actual field lines begin to pierce (or cease to pierce) the loop given some generic drawing of the field-lines of a typical bar magnet. In the entering portion, the flux will increase (with the field pointing rightward relative to the figure), and the induced current will move upward on the near side of the loop to reduce the flux to the lower value it had just a short while ago. In the exiting portion, the flux will decrease (with the field, again, pointing rightward relative to the figure), and the induced current will move downward on the near side of the loop to increase the flux to the higher value it had just a short time before.

   (b) Based on how the induced current is circulating, the loop has its dipole moment (and, thus, its North pole) to the left when the bar magnet is approaching it. So, the loop will repel the bar magnet, so that a force needs to be applied opposite to this repulsive interaction to make the bar magnet move steadily through the loop. Thus, the work done by the external agent must be positive. An attractive interaction will instead result when the bar magnet begins exiting the loop, again resulting in positive work to be done against this repulsive force to again keep displacing the bar magnet to the right steadily.

3. The voltage rating of the light bulb suggests that the induced emf, for optimum brightness, must match the voltage rating. For maintaining consistent brightness, the induced emf must be constant, meaning that the rate at which the magnetic flux changes must also be constant. In other words, the field must be reduced linearly in time to ensure no brightness fluctuations. Because the inward-pointing field is being reduced in its strength, the induced magnetic field of the loop must point inward as well, so that the induced current must be circulating Clock-Wise (CW).

4. This solenoid can be presumed to be very long, given that is transverse dimension is much smaller than its longitudinal dimension. This solenoid has turns per unit length

   $$n \equiv \frac{N}{\ell} = 1000 \text{ m}^{-1} = 1 \text{ mm}^{-1},$$

   which seemingly represents a pretty tightly-packed solenoidal coil. The field of the solenoid under these circumstances is

   $$B = \begin{cases} 0 & \text{for } r > D/2 \\ \mu_0 n I & \text{for } r < D/2 \end{cases},$$

   where r is the radial distance away from the solenoid's symmetry axis.

   (a) In this case, only a fraction of the solenoid's field is piercing the loop:

   $$\frac{d\Phi_B}{dt} = \frac{d}{dt}\left(\mu_0 n I \frac{\pi d^2}{4}\right) = \frac{\pi}{4}\mu_0 n d^2 \gamma.$$

(b) In this case, all of the solenoid's field is piercing the loop, and this is the only contribution of flux since the solenoid's field is taken as zero when outside the solenoid's radius:

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\left(\mu_0 n I \frac{\pi D^2}{4}\right) = \frac{\pi}{4}\mu_0 n D^2 \gamma.$$

5. When a current runs through the solenoid, the coil will feel its magnetic field through its respective turns.

(a) Note that even though it is not explicitly drawn in the diagram of the figure, the current in the solenoid is taken to be running in such a way that the solenoid's field is pointing downward.

i. Since the current through the solenoid is decaying and since the solenoid field is downward, then the coil will generate an induced field that will help reinforce the original field in order to preserve the flux through the coil to the originally larger value. Based on the geometry of the figure, this will involve an induced current flowing upward through the resistor.

ii. Because the field of the solenoid is uniform on its inside and zero on its outside, and because the coil is practically at the same radius as the solenoid, then the flux through a single turn of the coil will be

$$\Phi_{\mathrm{c}}^{(1)} = B_{\mathrm{s}} A_{\mathrm{c}\perp} = \mu_0 n I_{\mathrm{s}} \frac{\pi D^2}{4} = \mu_0 \frac{N}{\ell} I_{\mathrm{s}} \frac{\pi D^2}{4},$$

so that the flux through all of the turns of the coil will be exactly $\mathcal{N}$ times the flux through one of the turns:

$$\Phi_{\mathrm{c}} = \mathcal{N}\Phi_{\mathrm{c}}^{(1)} = \left(\mu_0 \frac{\mathcal{N}N}{\ell}\frac{\pi D^2}{4}\right) I_{\mathrm{s}}.$$

Note that

$$\frac{dI_{\mathrm{s}}}{dt} = -\alpha I_0 e^{-\alpha t},$$

with

$$\frac{d\Phi_{\mathrm{c}}}{dt} = \left(\mu_0 \frac{\mathcal{N}N}{\ell}\frac{\pi D^2}{4}\right)\frac{dI_{\mathrm{s}}}{dt}.$$

Because the rate of flux change will be explicitly time-dependent, then one can determine the values of the induced current running through the coil's resistor at a variety of times. Note that because the decay gets flatter in its time profile as $t \to \infty$, then the induced current through the resistor will get smaller as time moves forward, becoming vanishingly small at very large values of the time, t.

iii. It can be seen that the induced current will have a profile in time that is an exponential decay, for which a sketch is quite trivial.

(b) Note that the profile is effectively piece-wise linear in the time, so that the field strength will also have such a profile.

i. As such, the rate of flux change will be flat, horizontal segments that undergo a jump discontinuity at the kinks of the current profile. Note that because the rise in the current is less gradual than the decay in the current in the profile of $I(t)$, the induced current will be a smaller constant value for the intervals for which the solenoid current is increasing, but a larger constant value for the intervals for which the solenoid current is decreasing. However, the direction of the induced current will also change between these rising and decaying intervals. In particular, if one takes the current through the coil to have the same positive sense as the current through the solenoid, when the solenoid current is increasing, the induced current in the coil will generate a magnetic field that opposes the solenoid's. As such, the induced current will flow in the opposite sense through the coils relative to the solenoid, so that we may define this to be a negative value of the induced current. Whereas, when the solenoid's current is decaying, then the coil's induced current will move in the same sense as that of the solenoid's, so that this will be a positive induced current. Thus, the profile for the induced current as a function of the time will be a rectangular wave, with regions that are negative constant values for time intervals where the solenoid current is increasing, and regions that are positive constant values for time intervals where the solenoid current is decreasing.

ii. The basic relationship is that of essentially a derivative plot. In particular, $I_{\text{ind}}(t)$ is the negative of the derivative of $I(t)$ with, of course, some additional scaling constants: the cross-sectional area, permeability of free space, the turn density of the solenoid, as well as the resistance of the coil.

(c) In line with Part (b) (particularly, Part (bii)), the induced current will be related to the negative of the derivative of the solenoid current.

i. The answer is:

$$I_{\text{ind}}(t) = -\frac{1}{4}\pi D^2 \frac{\mu_0 \mathcal{N} N}{R\ell} \omega I_0 \cos(\omega t).$$

ii. The peak current is the magnitude of the pre-factor to the $\cos(\omega t)$ in Part (di).

iii. Note from the result of Part (ci), in comparison to the solenoid's current profile, that when the induced current is a maximum, the solenoid current is zero. This follows from the fact that $\cos(\omega t)$ and $\sin(\omega t)$ are out of phase by $\pi/2$.

6. Note that based on the initial condition of the problem, there will be an induced current running through the loop since the magnetic flux goes from being initially zero to being nonzero. However, recall that any time when the magnetic flux is fixed, there will be no induced emf based on Faraday's Law.

(a) This is a simple kinematics problem for an object moving at constant velocity. Since the distance traveled by the right edge of the loop must be $\ell + w$ in order to completely clear the field region, then the time will be

$$T = \frac{\ell + w}{v}.$$

(b) The field pierces the loop consistently into the page. Since the magnetic flux is increasing when the loop begins entering the field region, then there will be an induced magnetic field that will point out of the page to restore the flux to the originally smaller value that it had when the loop was less immersed in the field region. Thus, there will be an Anti-Clock-Wise (ACW), and therefore negative, current flowing through the loop during this initial entry. Because $\ell < w$, then there will be a range of points for which the loop will be fully immersed within the field region, during which no induced current will be generated. However, as soon as the loop begins exiting the field region, the flux will begin to reduce, resulting in a Clock-Wise (CW), and therefore positive, induced current to flow through the loop. Because the geometry of the loop is rectangular with its front edge oriented parallel to the geometry of the field boundaries, and since the movement of the loop is at a steady speed, then when there is a flux change, it is a constant value. Thus, the profile of the induced current will be a negative constant value for $t \in [0, \ell/v]$, will be zero for $t \in [\ell/v, w/v]$, and will be a positive constant value for $t \in [w/v, (\ell + w)/v] = [w/v, T]$. The constant positive and negative values will have the same size.

(c) This involves taking the square of the plot in Part (b) and multiplying by the resistance, R, since a way to characterize the dissipated power through the loop running the induced current is

$$P_{\text{ind}} = I_{\text{ind}}^2 R.$$

(d) The novelty here is that there is no range of times for which the loop is fully immersed in the field region, on account of $\ell' > w'$. Thus, the flux will increase linearly in time for $t \in [0, w'/v]$, will remain constant for $t \in [w'/v, \ell'/v]$ (where in this region the loop will be moving but the field will nonetheless be restricted to the same sliver of area across the loop, therefore administering no change in flux), and will decrease linearly in time for $t \in [\ell'/v, (\ell' + w')/v] \equiv [\ell'/v, T']$.

(e) Under these circumstances, note that since the height of the triangle gets bigger as one moves from the right vertex to the left leg, then the flux profile will not be linear in the time, but will be quadratic in the time. This will result in a slightly different induced current profile on the premise of the slightly different geometry in comparison to the square loop before. However, similar to the earlier parts, upon entry the induced current runs to oppose the increasing magnetic flux using an induced magnetic field that opposes the applied field, whereas upon exit the induced current runs to oppose the decreasing magnetic flux using an induced magnetic field that reinforces the applied field.

7. This problem was explicitly done in class for the case of a circular loop spun about its diameter at a constant frequency, with the loop having N turns. Adapting the result from class to this result gives a time-dependent flux of

$$\Phi_B = NB\ell w \cos(2\pi f t + \phi),$$

for some phase angle, ϕ, that is placed to allow for the adjustment of when the stop watch is started and, at that point in time, how the field is oriented relative to the normal vector of the loop's plane. Thus, the induced emf has the form

$$\mathcal{E}_{\text{ind}} = -2\pi f NB\ell w \sin(2\pi f t + \phi).$$

(a) Since the peak value of the emf is provided, and since $|\sin(x)| \leq 1$, then

$$\mathcal{E}_{\max} = 2\pi f NB\ell w,$$

from which the number of turns may be determined.

(b) From the result above for the flux, the maximum flux is just the pre-factor to the cosine function in the flux equation. Thus

$$\Phi_{B_{\max}} = NB\ell w.$$

(c) The current profile will itself be a sinusoidal function, while the power will be the square of a sinusoid since $P = I^2 R$ for the power dissipated by a resistor with a current running through it. The phase angle is not an important parameter here, so that one can calibrate the system, for example, to take $\phi = 0$.

8. Note that as the slider bar is moved to the right, the effective area of the circuit consisting of the resistor, conducting rails, and the conducting bar gets bigger.

(a) This increase in the area of the circuit is what leads to an increase in the magnetic flux piercing the conducting rectangular loop. Based on Faraday's Law, this will lead to the generation of an induced current that would aim to oppose the flux change. By Lenz's Rule, this induced current will flow in a way that generates an induced magnetic field that opposes the applied field, in hopes of getting the flux to retain its initially lower value. As such, the induced current will flow Anti-Clock-Wise (ACW) (i.e., downward through the resistor in the figure).

(b) Note that if an induced current is present, one has moving charges in the presence of a magnetic field. This particular induced current is flowing upward through the sliding bar, which leads to a magnetic force leftward. Thus, if the slider felt to rightward force, it would inevitably begin slowing down; however, since it maintains its rightward velocity, it must feel an applied force to the right.

(c) Because the rate at which work is done is the power, and since the applied force must be equal and opposite to the magnetic force felt by the moving conducting bar, this means that the power generated by the person pushing the bar must be the same magnitude as the power dissipated by the resistor. The induced current magnitude is

$$I_{\text{ind}} = \frac{1}{R}\left|\frac{d\Phi_B}{dt}\right| = \frac{B\ell v}{R},$$

so that the power dissipated by the resistor will be

$$P_{\text{ind}} = I_{\text{ind}}^2 R = \frac{B^2\ell^2 v^2}{R}.$$

(d) An ideal voltmeter ideally has an infinite internal resistance; however, this does not change the fact that the induced current in the circuit—albeit small—will be running ACW in the circuit.

i. In particular, since the induced current runs down along the resistor, and since we know that the current running through the resistor must do so from high to low potential, then a point above the resistor must be at a higher potential than a point below the resistor. Thus, one would put the positive lead of the voltmeter at a point above the resistor, and the negative lead at a point below.

ii. If this voltmeter truly is ideal, then note from the result of Part (c), that

$$\lim_{R\to\infty} P_{\text{ind}} = \lim_{R\to\infty} \frac{B^2\ell^2 v^2}{R} = 0.$$

In other words, the work done on the bar due to the person pushing it must be independent of the time in order for the power to be zero.

9. Because the bar is initially at rest, the application of the force, $\vec{F}$, to the right will be the net force at $t = 0$, so that the bar will accelerate to the right and, thus, will begin moving to the right.

(a) As the bar develops some velocity, then a magnetic force will result to the left (whether one wishes to derive this magnetic force from the notions of Faraday's Law or motional emf). Because the applied force, $\vec{F}$, is kept fixed, then the leftward-pointing magnetic force will aim to make the net force less than it was a little time ago. However, since the magnetic force is proportional to the speed of the moving bar, a very short time after $t = 0$ (i.e., at $t \gtrsim 0$) this magnetic force will most definitely be less than the applied force because the speed would not have become very much in this short amount of time. Thus, by Newton's 2nd Law, the acceleration of the bar will still be rightward, it just won't be as large as it once was. Thus, the speed of the bar will still keep increasing (albeit at a slower rate), making the magnetic force increase in size, leading to a further decrease in the net force and, therefore, the acceleration of the bar. In this process, the magnetic force will eventually get large enough so as to balance the applied force. At this point, the acceleration will be zero, so that, thereafter, the bar will cease to change its velocity. For this reason, we say that the bar reached a terminal velocity, so that if the system maintains its parameters, then the bar will be moving at the same velocity as it had when the forces balanced. Ultimately, this implies that the velocity of the bar as a function of the time is a monotonically increasing function of the time with an ever-diminishing slope as time progresses forward.

(b) Suppose the bar has a speed, $v = v(t)$, at some point in time, t. As in Problem 8, the leftward-pointing magnetic force magnitude on the bar at this point in time is

$$F_B = I_{\text{ind}}\ell B = \frac{B^2\ell^2}{R}v.$$

Taking right as positive, the net force will be

$$F_{\text{net}} = F - F_B = F - \frac{B^2\ell^2}{R}v.$$

Since $a = dv/dt$, then Newton's 2nd Law states

$$m\frac{dv}{dt} = F - \frac{B^2\ell^2}{R}v.$$

This actually happens to be an identical differential equation to the charging DC RC series circuit. Mimicking the solution to that differential equation via the method of the separation of variables, the result will be

$$v(t) = v_\infty\left(1 - e^{-t/\tau}\right),$$

with

$$v_\infty = \frac{FR}{B^2\ell^2},$$

and

$$\tau = \frac{mR}{B^2\ell^2}.$$

Of course, this has the proper qualitative form described in Part (a) of the velocity of the bar as a function of the time.

(c) We found from Part (b) that

$$a = \frac{F}{m} - \frac{B^2\ell^2}{mR}v,$$

so that when $a \to 0$,

$$v \to \frac{FR}{B^2\ell^2} \equiv v_\infty.$$

Thus, v_∞ is the terminal speed, which occurs as $t \to \infty$.

10. This problem is identical to Problem 9, except that the force, F, pushing the bar is exactly supplied via the steady current supplied to the bar via the battery:

$$F = I\ell B = \frac{\mathcal{E}\ell B}{R}.$$

Note that this form for the force means that, actually, the terminal speed is independent of the resistance. Indeed, using one of the results of Problem 9,

$$v_\infty = \frac{FR}{B^2\ell^2} = \frac{\mathcal{E}}{\ell B}.$$

Thus, the resistance in the circuit really only controls how quickly one approaches the terminal speed (or, how quickly one approaches some fixed fraction of the terminal speed). Indeed, since the time constant has the form

$$\tau = \frac{mR}{B^2\ell^2},$$

then the larger the circuit's resistance, the longer the time constant, implying a longer time to reach a desired speed for the bar.

11. Consider a point at some arbitrary radius, r, away from the axis of rotation of this wheel. Since the wheel is rotating at a fixed frequency, the linear speed of a point at this radius will be

$$v(r) = 2\pi f r.$$

If a point charge happens to be at this radius, it will feel a magnetic force. Note that since the charge would be moving in the plane of the spinning disk and that the magnetic field is perpendicular to the disk, then there will be a nonzero magnetic force on this charge. If the charge is positive, one can show, based on the figure, that the force is radially outward and given by the magnitude

$$F_B(r) = qv(r)B = 2\pi f q B r.$$

Thus, point charges farther away from the axis of rotation will feel a larger force. Since the magnetic force is responsible for the separation of charge (specifically, positive charges being pushed radially outward and negative charges being pushed radially in), then this magnetic interaction will generate an emf between the axle and the rim purely based on the movement of the charges on the disk relative to the perpendicularly oriented magnetic field. Because the force has a spatial dependence, this motional emf must be integrated:

$$\mathcal{E}_{\text{mot}} = \int \frac{\vec{F}_B}{q} \cdot d\vec{\ell},$$

where $d\vec{\ell}$ is an infinitesimal displacement in the direction of the separating charges (i.e., $d\vec{\ell} = \hat{r}dr$). Thus,

$$\mathcal{E}_{\text{mot}} = \int_0^{D/2} 2\pi f q B r dr = \frac{\pi}{4} f q B D^2.$$

This will be the steady-state voltage reading that the voltmeter will make. Now, since the positive charges will be at the rim and the negative charges will be at the axle, then this implies that a current will flow upward (relative to the figure) through the voltmeter. As such, the point below the voltmeter will be at a higher potential, which is where the positive lead of the voltmeter should be placed to get a positive reading of the steady-state emf above.

12. In this problem, there is no indication of how B_2 compares to B_1. However, this detail really only matters if one were interested in how the current specifically flows in the conducting loop. Since we are more interested in how much charge flows in the time that the field changes in its strength, the direction of the flow is immaterial. The magnetic flux of this system at any point in time is given by

$$\Phi_B = B\pi a^2,$$

while its rate of change is

$$\frac{d\Phi_B}{dt} = \pi a^2 \frac{dB}{dt}.$$

By Faraday's and Ohm's Law, the magnitude of the induced current flowing in the loop is thus

$$I = \frac{1}{R}\frac{d\Phi_B}{dt} = \frac{\pi a^2}{R}\frac{dB}{dt}.$$

Since $I = dq/dt$, then

$$\frac{dq}{dt} = \frac{\pi a^2}{R}\frac{dB}{dt}.$$

Integrating both sides with respect to time, we have

$$\int \frac{dq}{dt} dt = \int \frac{\pi a^2}{R}\frac{dB}{dt} dt,$$

so that, upon using the chain rule

$$\int_{Q_1}^{Q_2} dq = \frac{\pi a^2}{R} \int_{B_1}^{B_2} dB.$$

Thus,

$$\Delta Q \equiv Q_2 - Q_1 = \frac{\pi a^2}{R} \left(B_2 - B_1\right),$$

which takes care of both parts.