Physics 1A
Spring 2015

Midterm Exam 2

May 18, 2015
4:00 PM - 4:50 PM
PAB 1-425

READ THIS BEFORE YOU BEGIN

- You are allowed to use only yourself and a writing instrument on the exam.
- Print your name on the top right of your exam.
- If you finish more than 5 minutes before the end of the exam period, then please raise your hand and a proctor will collect your exam. Otherwise, please stay in your seat until the end of time is called.
- When the exam is finished, please remain in your seat, and the proctor(s) will come around and collect your exam. Once your exam is collected, you may leave the room.
- **Show all work.** The purpose of this exam is primarily to test how you think; you will get more partial credit for a logical, well-thought-out, descriptive response.
- Please **box all of your final answers** to computational problems.

ADVICE

- Do not just attempt to blindly calculate answers to computational questions.
- **Use your physical and geometric intuition** first to try and determine as much about the answer as you can before you launch into computation!
- **Check units and limits** of your answers whenever possible!

Problem 1. (5 points)

Suppose you take a rigid cube (like a die), and attempt to balance it on its corner on a frictionless surface. It is extremely difficult to do so, and it is overwhelmingly likely that if you let go of the cube, it will fall down.

Suppose you let go of the unbalanced cube, and suppose that it is completely stationary just before you let go. As the cube slides and falls, describe and draw the path followed by its center of mass from the time it's released, to the time when one of its edges or faces first hits the surface.

Solution. Recall that if $\mathbf{F}_{\text{ext}}$ is the net external force on a system, if M is the total mass of the system, and if $\mathbf{R}$ is the center of mass position of the system, then

$$\mathbf{F}_{\text{ext}} = M\ddot{\mathbf{R}}. \tag{1}$$

Let the cube be the system under consideration. The only external forces on the system are the normal force, and the gravitational force, neither of which is in the horizontal direction. If we let the x-y plane be the horizontal direction, this means that the x- and y- components of the net external force are zero. If we let X, Y, Z denote the components of the center of mass position, then breaking the equation above into components gives

$$0 = M\ddot{X}, \qquad 0 = M\ddot{Y}, \qquad N - Mg = M\ddot{Z}. \tag{2}$$

In particular, the first two equations indicate that the acceleration of the center of mass in the horizontal directions is zero. This means that

The center of mass of the cube falls straight down.

Problem 2. (10 points)

A roller coaster car (car 1) of mass M is free to slide on a frictionless track and initially sits at rest at the bottom of a circular loop of radius R. At some time later, another roller coaster car (car 2) of the same mass crashes into it, and they both begin to move up the loop such that they remain in contact with one another.

What is the minimum speed car 2 needs to have before the collision so that both cars make it to the top of the loop without ever losing contact with the track?

Hint: What is the normal force of the track on the cars if they lose contact with the track?

Solution. The roller coaster cars will reach the top of the track and remain in contact with the track provided the normal force with the track is nonzero. In order to just make it to the top, the worst case scenario is for the normal force to be zero exactly when the two cars get to the top of the track – if this happens, then the cars are just barely keeping contact with the track. If the normal force at the top of the track is zero, then Newton's Second Law for the two cars (which we approximate as being particles in this analysis) reads as follows at that point in the vertical direction:

$$-(2M)g = -(2M)\frac{v_{\text{top}}^2}{R}. \tag{3}$$

Notice that if the cars are moving around the circle, then they will have centripetal acceleration of magnitude v^2/R downward toward the center of the loop, hence the right hand side of this equation. It follows that the speed of the cars at the top of the track must be

$$v_{\text{top}}^2 = gR. \tag{4}$$

We can use conservation of energy to now determine what the speed of the pair of cars at the bottom of the track needs to be right after the collision:

$$\frac{1}{2}(2M)v_{\text{bottom}}^2 = \frac{1}{2}(2M)v_{\text{top}}^2 + (2M)g(2R). \tag{5}$$

where we have taken the zero of gravitational potential energy at the bottom of the loop. Plugging the expression above for v_{top}^2 into this conservation of energy equation allows us to solve for v_{bottom}:

$$v_{\text{bottom}} = \sqrt{5gR} \tag{6}$$

Finally, we use momentum conservation in the horizontal direction to determine the necessary velocity for cart 2 before the collision. We take the direction in which cart 2 was moving before the collision to be the positive horizontal direction and get

$$Mv_2 = (2M)v_{\text{bottom}} = (2M)\sqrt{5gr} \tag{7}$$

so that

$$\boxed{v_2 = 2\sqrt{5gR}}. \tag{8}$$

Problem 3. (20 points)

Lonestar is a galactic bounty hunter in search of a fugitive. The fugitive's last known location is the planet Alzar, a planet whose entire surface is perpetually covered by clouds. To hide her spaceship while she's looking for the fugitive, and to make a quiet descent to the surface, Lonestar plans to hover her ship above the clouds, parachute to the surface of the planet, and then use a jetpack to get her and the fugitive back to her ship.

The combined mass of Lonestar, her empty jetpack, and the fugitive is m. Lonestar fills her jetpack with fuel having a total mass $(e-1)m$ (here $e = 2.718..$ is Euler's number). The acceleration due to gravity on Alzar is one third of g, and Lonestar's jetpack is programmed to transport her and the fugitive back to the spaceship with an acceleration of two thirds of g in the vertical direction. The jetpack fuel has an exhaust velocity u relative to the jetpack.

Neglecting air resistance, at what maximum height above Alzar's surface can Lonestar park her spacecraft and still make it back once she has apprehended the fugitive?

Solution. The crux of this problem is simply to determine how high lonestar can go given the amount of fuel she has in her jetpack, and given the other parameters in this problem.

One tricky part of the problem is to realize that even once her pack runs out of fuel, Lonestar will continue to fly upwards (like in the challenge problem where the ball was dropped through the floor of the elevator), so to determine how high she can go, one needs to determine not only how high she goes during the rocket state when her jetpack is firing, but also how high she goes after that. The total height h she goes is the sum of these two. If t_{burn} represents the amount of time for which her jetpack is on – the "burn" stage, then using the fact that she accelerates at two thirds of g during the burn stage, kinematics tells us that the height she goes during the burn stage is

$$h_{\text{burn}} = \frac{1}{2}\left(\frac{2}{3}g\right)t_{\text{burn}}^2 = \frac{1}{3}gt_{\text{burn}}^2 \tag{9}$$

Now let v denote the upward speed she has right when her jetpack loses fuel, then the kinematics equation $v^2 = v_0^2 + 2a(y - y_0)$ tells us that the extra height she rises to during free fall is

$$h_{\text{free}} = \frac{v^2}{2\left(\frac{1}{3}g\right)} = \frac{3}{2}\frac{v^2}{g} = \frac{3}{2}\frac{1}{g}\left(\frac{2}{3}gt_{\text{burn}}\right)^2 = \frac{2}{3}gt_{\text{burn}}^2 \tag{10}$$

The total height she gets to is therefore

$$h = h_{\text{burn}} + h_{\text{free}} = gt_{\text{burn}}^2 \tag{11}$$

To finish the problem, we need to determine t_{burn}. To do so, we turn to the mass flow equation in the vertical direction where we take the positive y-direction to be "up."

$$F_{\text{ext},y} = M\frac{dv}{dt} - \frac{dM}{dt}u. \tag{12}$$

Noting that the net external force is just gravity, and noting that Lonestar's jetpack is programmed to accelerate where upward at two-thirds of g

$$-\frac{1}{3}Mg = \frac{2}{3}Mg - \frac{dM}{dt}u \tag{13}$$

A little simplification gives

$$\frac{dM}{dt} = \frac{g}{u}M \tag{14}$$

The solution to this equation is

$$M(t) = M(0)e^{gt/u} \tag{15}$$

This is the mass as a function of time of Lonestar + the fugutive + the jetpack plus its remaining fuel. Since u is a velocity, and since the velocity of the fuel points downward, u is a negative number, so this equation says that during the burn stage, the mass of the system decreases exponentially with time in order to sustain the constant acceleration two-thirds of g upward. If we take $t = 0$ just as Lonestar takes off, then we get

$$M(0) = m + (e-1)m = em, \qquad M(t_{\text{burn}}) = m \tag{16}$$

and therefore

$$e^{gt_{\text{burn}}/u} = \frac{M(t_{\text{burn}})}{M(0)} = \frac{1}{e} \tag{17}$$

so that taking the natural log of both sides and solving for t_{burn} gives

$$t_{\text{burn}} = -\frac{u}{g}. \tag{18}$$

Finally, we plug this into the expression above for the height h to get

$$\boxed{h = \frac{u^2}{g}}. \tag{19}$$

Problem	Score
Problem 1	
Problem 2	
Problem 3	
Total	