Physics 22: Homework 3

The following problems encompass the theory and applications of Gauss's Law.

1. Four point charges—$q_1 = +q$, $q_2 = -2q$, and $q_3 = +3q$, and $q_4 = -4q$—exist in the space shown in Figure 1. Two (possibly complex) closed surfaces, A and B, are shown, in cross-section. All charges that happen to be within the boundaries of the cross-sections of these surfaces happen to be within the closed volumes defined by these surfaces. Determine the net flux through each of the surfaces.

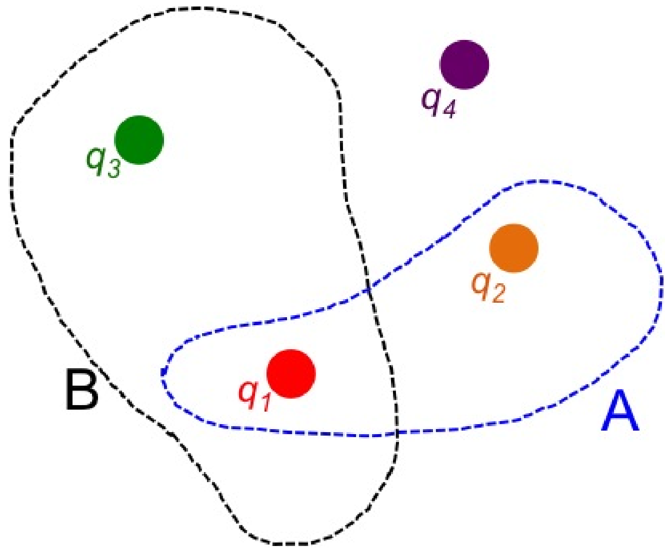

Figure 1: Closed surfaces A and B in the presence of point charges. Where the point charges overlap with the cross-sectional outlines of the surfaces indicates that such point charges are within the volume of said surfaces.

2. Gauss's Law is a statement on what the net electric flux through a closed surface, ∂V, is solely based on the net charge present within the volume, V, bounded by that surface (i.e., the net enclosed charge):

$$\oint_{\partial V} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0},$$

where

$$Q_{\text{encl}} = \sum_{i \in V} q_i$$

in the case when the charges enclosed are discrete, or

$$Q_{\text{encl}} = \int_V dq = \int_V \rho d\mathcal{V}$$

in the case when the charges enclosed are continuous, and where the infinitesimal areal vector, $d\vec{A} = \hat{n} dA$, is flat and has its normal vector, $\hat{n}$, pointing locally perpendicular and away from the bounded volume by the Gaussian surface. With this in mind, if a charge configuration administers high amounts of symmetry, one may simply pick an imaginary surface to enclose this charge configuration, carefully chosen so that this so-called "Gaussian surface" contains the same symmetries as the configuration itself. For a small subset of charge configurations, one could then reason that the electric field on the highly symmetric Gaussian surface is not only constant in magnitude at any point on the surface, but is also in a constant direction relative to the normal vector at any point on the surface. Under these circumstances, one may then take the "E" out of the surface integral. Since determining the enclosed charge is quite straightforward, then one sees that Gauss's Law may thus be used to determine the electric field of such highly symmetric charge configurations.

(a) Consider three different Gaussian surfaces: a sphere, a cylinder, and a rectangular box. Which of these surfaces could one use to determine the electric field generated by: (i) a point charge; (ii) a uniformly charged sphere; (iii) a uniformly charged, long rod; and (iv) a uniformly charge, large plate? Explain.

(b) Explain why Gauss's Law is not useful in determining the electric-field profile of: (i) a (neutral) dipole; (ii) a ring of charge; (iii) near the ends of a short rod of charge; and (iv) near the edges of a sheet of charge.

(c) Consider a thin spherical shell of charge. Imagine that the points on this shell are described via the use of latitudinal and longitudinal lines, in exactly the same way as the way to which locations on the surface of the Earth are referred. Suppose this shell had a distribution of charge that was dependent on the latitude and/or longitude.

   i. Would Gauss's Law be useful in determining the electric-field profile at observation points within the shell? Explain.

   ii. Would Gauss's Law be useful in determining the electric-field profile at observation points outside of the shell? Explain.

3. Use Gauss's Law to argue in favor of the following proposals.

   (a) A person is safe inside a metal car in a lightning storm.

   (b) A charge completely surrounded by a conductive shell induces an equal, but opposite, amount of charge on the surface of the conductor.

4. Consider a conducting solid sphere, of charge Q and radius R.

   (a) Determine the electric field for radial positions $r < R$.

   (b) Use symmetry arguments to argue that the electric field of this sphere must only have a radial component for $r > R$.

   (c) Obtain an expression for the electric field at any radial position $r > R$.

   (d) What happens to the field at a point outside of the sphere in the limit as $R \to 0$?

5. An insulating solid sphere, of total charge Q and radius R, is uniformly charged throughout its volume.

   (a) Use symmetry arguments to argue that the electric field of this sphere must only have a radial component for $r < R$.

   (b) Use symmetry arguments to argue that the electric field of this sphere must only depend on its radial position from the center for $r < R$.

   (c) Determine the electric field at any radial position $r < R$.

   (d) Determine the electric field at any radial position $r \geq R$.

   (e) Graph $E(r)$ for $r \in [0, \infty)$.

   (f) Suppose that modifications are done on this sphere in such a way as to preserve the total charge Q, but to now make the charges distributed non-uniformly. In particular, consider a charge profile such that the charge density is linear in the radial position,

$$\rho(r) = \alpha r,$$

   where $\alpha > 0$ is a constant to be determined, and $r \in [0, R]$.

      i. Use symmetry arguments to argue that the electric field of this sphere must, again, only have a radial component for $r < R$ and for $r \geq R$.

      ii. Use symmetry arguments to argue that the magnitude of the electric field must only depend on its radial position from the center for $r < R$ and for $r \geq R$.

      iii. Determine α in terms of the given quantities.

      iv. Determine the electric field at any radial position $r < R$ and $r \geq R$.

      v. Graph $E(r)$ for $r \in [0, \infty)$.

6. Consider a conducting, neutral spherical shell, of inner radius a and outer radius b. At the center of this shell exists a point charge, $-q$ (with $q > 0$), as shown in Figure 2.

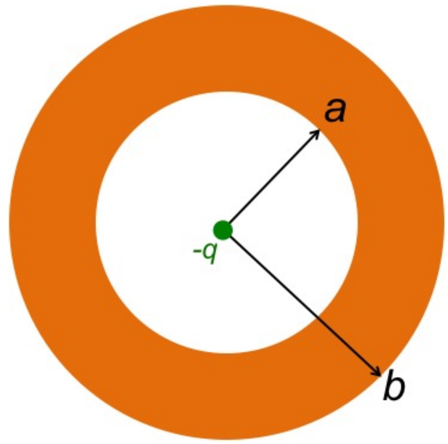

Figure 2: A neutral conducting shell with a negative point charge placed at the shell's center, which is clearly within the cavity of the sphere.

   (a) Determine the electric field at all radial positions (i.e., for $0 \leq r < a$, for $a < r < b$, and for $r > b$).
   (b) Determine the surface charge densities on the inner and outer surfaces of the conducting shell.
   (c) Suppose that the conducting shell is now charged by a total charge $Q = 2q$. Repeat Parts (a) and (b) in this case.

7. Consider an insulating, very long cylinder of charge where the charge is distributed uniformly throughout its volume. The charge density of the cylinder is ρ and the cylinder has radius R.

   (a) Use symmetry arguments to argue that the electric field of this cylinder must only have a radial component for $r < R$ and for $r \geq R$.
   (b) Use symmetry arguments to argue that the magnitude of the electric field of this cylinder must only depend on its radial position from the center for $r < R$ and for $r \geq R$, and that the field does not depend on the position along the cylinder.
   (c) Since Gaussian surfaces must be closed, choosing a cylindrical Gaussian surface must necessarily have end-caps attached. In other words, if one rolls a sheet of paper into a cylinder, the ends of the cylinder are technically open, so that one would have to close those ends off with some type of end-cap (disks, hemispheres, etc.).
      i. Show that the end-caps do not contribute to the electric flux.
      ii. Determine the electric field at any radial position from the center (i.e., for $0 \leq r < R$ and for $r \geq R$).

8. Consider a very large, *insulating* sheet of charge, with total charge Q, area A, and negligible thickness.

   (a) Use symmetry arguments to argue that the electric field of this sheet must only point along the line that is perpendicular to the sheet.
   (b) Use symmetry arguments to argue that the electric field of this sheet must be independent of the position along the sheet.
   (c) Use symmetry arguments to argue that the electric field of this sheet must also be independent of how far away the observation point is from the sheet.
   (d) Using Gauss's Law, determine the electric field at any point in space (excluding points on the sheet itself).
   (e) Rewrite the result in Part 8d in terms of the surface charge density, σ.

9. As in Figure 3, consider a very large, *conducting* slab of charge, with total charge Q, cross-sectional area A, and thickness d. The slab is placed in a three-dimensional coordinate system with x-, y-, and z-axes. Specifically, the origin of coordinates is at the physical center of the slab, with the z-axis taken along the thickness, while the xy-plane is parallel to the aforementioned cross-section.

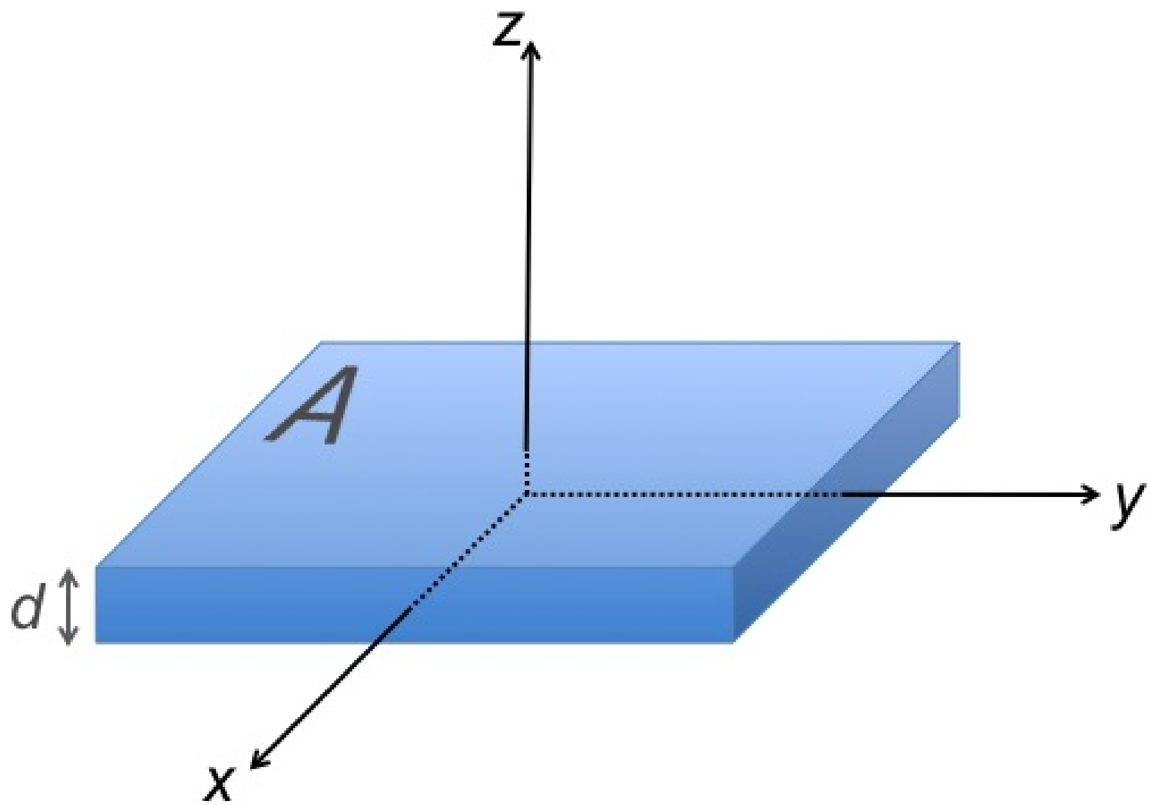

Figure 3: A conducting slab of charge with total charge Q, cross-sectional area A, and thickness d. The origin of coordinates is placed at the center of the slab, with the z-axis perpendicular to its cross-sectional face.

(a) Determine the electric field for $|z| < d/2$.

(b) Using the results of Problems 8a, 8b, and 8c, determine the electric field for $|z| > d/2$.

(c) Rewrite the results of Part 9b in terms of the surface charge density, σ'.

(d) Compare the result of Part 9c with the result of Problem 8e. In particular, explain why the expressions differ by a factor of two, even though they technically represent the same value of the field outside of the sheet/slab.

10. Consider a very large, *insulating* slab of charge, with total charge Q, cross-sectional area A, and thickness d. The charge in this slab is uniformly distributed throughout the volume of the slab. As in Problem 9, a similar coordinate system is adopted for this insulating slab.

(a) Determine the electric field for $|z| < d/2$ and for $|z| \geq d/2$.

(b) Rewrite the results of Part (a) in terms of the slab's volume charge density, ρ.

(c) Unlike for the conducting slab of Problem 9, this insulating slab does not have a discontinuity in the field upon transitioning from within the slab to outside of it. Nonetheless, define an overall charge density for this insulating slab, $\sigma \equiv Q/A$. Using this definition, show that the result for this insulating slab is identical to the one for the insulating sheet of Problem 8.

11. Consider three parallel, very large, insulating sheets, that have negligible thickness and are denoted as Sheets 1, 2, and 3. All sheets have the same area, A, and have their centers along a single line represented by the z-axis, as shown in Figure 4. Sheet 1 is located at $z = -a$, Sheet 2 is located at $z = 0$, and Sheet 3 is located at $z = +a$. However, $Q_1 = +Q$, $Q_2 = +Q$, and $Q_3 = -Q$, with $Q > 0$.

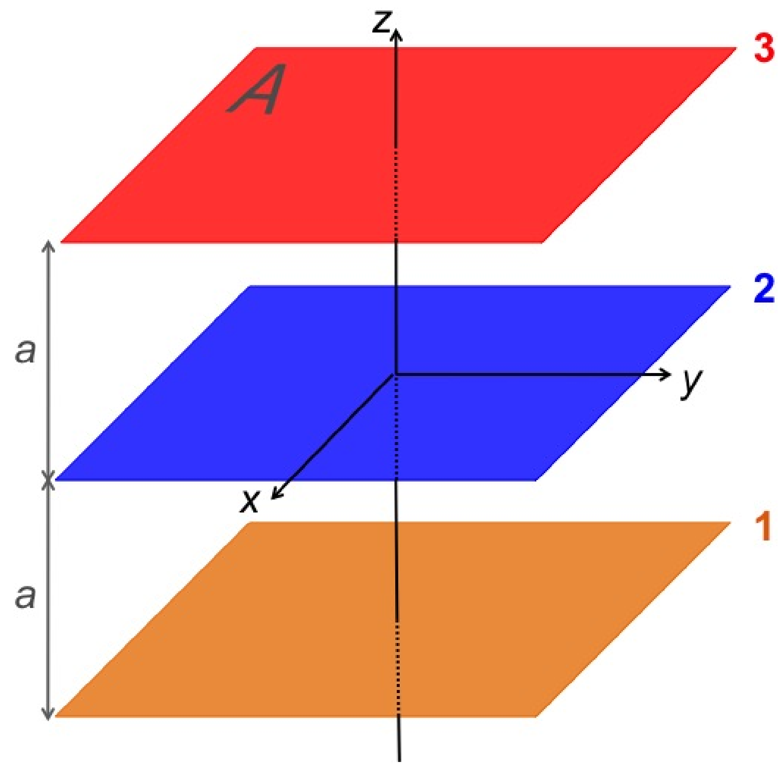

Figure 4: Three equally spaced, equally sized, charged, very large sheets arranged parallel to one another. The z-axis serves as the line connecting all of the centers of the sheets, and Sheet 2 is on the xy-plane.

(a) Determine the electric field in all of the relevant regions along the z-axis (i.e., for $z < -a$, $-a < z < 0$, $0 < z < a$, and $z > a$).

(b) Determine the surface charge densities on the surfaces of all of the plates.

(c) Note that the fields on the outside of this arrangement (i.e., for $z < -a$ and $z > a$) are equal in magnitude and opposite in direction. In other words, they are mirror images of one another. Provide a qualitative argument as to why this is the case.

12. Gauss's Law can also be applied to problems dealing with gravity via Newton's Law of Universal Gravitation, which states that the force of attraction between two point masses, m_1 and m_2, is proportional to the product of these masses and is inversely proportional to square of the separation, r, between the masses. Mathematically, the magnitude of this force on m_2 due to m_1 is

$$F_{G_{21}} = G\frac{m_1 m_2}{r^2},$$

where $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ is Newton's Gravitational Constant.

(a) Show that the gravitational acceleration, $\vec{g}$, is really the gravitational field at a point.

(b) Define gravitational flux, Φ_G, and use it to express Gauss's Law for Gravitational Fields.

(c) Use Gauss's Law to prove that the gravitational field *outside* a uniform, spherical planet is the same as if the mass was concentrated at the center.

13. After J. J. Thomson discovered that atoms contained negative particulate matter (called electrons), he postulated the first atomic model that consisted of electrons embedded in a ball of positively charged matter. He described the model as plums (electrons) in a pudding (positive matter), which led to the coining of this model as the "plum-pudding" model. In this model (roughly displayed in Figure 5), the electrons were believed to not orbit, but were rather believed to be located at equilibrium positions within the positive matter of the atom. In the case of hydrogen, there would only be one negative "plum" in the positive "pudding."

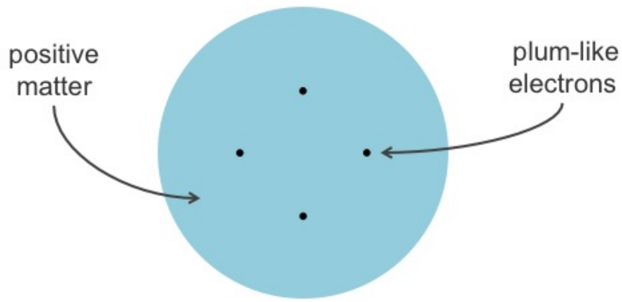

Figure 5: Thomson's "plum-pudding" model, which consists of a ball of positively charged matter (i.e., the pudding) with a collection of electrons (i.e, the plums) placed in equilibrium within the ball. This was one of the earliest models of the atom that was based on legitimate scientific observations, specifically Thomson's discovery of negatively charged particulate matter from experiments performed on cathode rays in a partially evacuated tube.

(a) In the case of the hydrogen atom, explain why the equilibrium position of the single electron in this model must necessarily be located at the center of the ball of positively charged matter.

(b) Show that if the electron is displaced by a small amount from its equilibrium position at the center, then the electron will undergo Simple Harmonic Motion (SHM).

(c) At the time that Thomson solidified this model (around 1897), it was well known that light waves can be generated by jiggling charges at the frequencies corresponding to those of visible light. Thomson postulated that this was caused by the electron vibrating back and forth.

i. What would have to be the size of the hydrogen atom in order to produce light of frequency 4.5×10^{14} Hz (which most of us would claim to be essentially red in its color)?

ii. Compare this size to the known diameter of the hydrogen atom, which is approximately 10^{-10} m $\equiv$ 1 Å.

(d) If the electron was displaced from its equilibrium position by a distance greater than the radius of the positively charged ball of matter, would the resulting motion still be SHM? Explain.

(e) Consider a Thomson model of the helium atom, which consists of two electrons and a positive spherical ball of charge $+2e$ (where $e \equiv 1.60 \times 10^{-19}$ C) and radius R. How far from the center must the two electrons have to be in order for this atomic configuration to be stable?

14. Thomson's model of the atom was proven to be inconsistent with the results of the so-called "gold-foil" experiment (performed in 1912), which was performed by Geiger and Marsden under the supervision of Rutherford (who happened to be Thomson's former student). This experiment showed that the positive ball of matter—discussed in Problem 13—was actually concentrated inside a tiny region of the atom (about 10^{-14} m) in diameter which consisted (to their knowledge at that point in time) with positively charged particles called protons. This finding was the birth of the planetary model. However, more advances were made, most notably by Niels Bohr and other pioneers of quantum mechanics, to then finally describe the model of the atom as a positively charged nucleus at the center of a so-called "cloud" of electrons. In the case of hydrogen, the nucleus consists of a charge $+q_e$ with a surrounding, spherically symmetric, electron cloud of charge $-q_e$, as shown in Figure 6. From quantum mechanics, the charge density of this electron cloud is given by

$$\rho(r) = -\left(\frac{q_e}{\pi a_0^3}\right) e^{-2r/a_0},$$

where $e = 2.71828$ is sometimes called Euler's number (i.e., the base of the natural logarithm), $a_0 = 5.3 \times 10^{-11}$ m $= 0.53$ Å is the so-called Bohr radius, and $q_e = +1.60 \times 10^{-19}$ C is the (fundamental) electron charge magnitude.

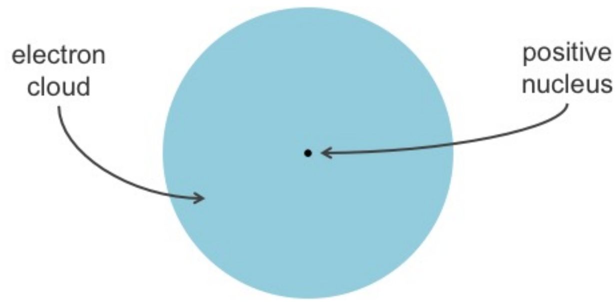

Figure 6: The electron-cloud model representation of a hydrogen atom. The proton in the center of the cloud is surrounded by a cloud containing the charge of a single electron. Technically, this cloud represents a non-uniform, radial distribution of charge which extends out into infinity. The spherical nature of the model is shown for the sake of showcasing the spherically symmetric nature of the charge distribution of this cloud.

(a) Find the *total* amount of charge within a volume of radius r from the center of the atom. (Note that the proton's charge, which is presumed to be a point-like charge at the center of the cloud, must also be considered in this calculation of the total charge.)

(b) Show that as $r \to \infty$, the net enclosed charge goes to zero. Explain this result. (Note that the functional form of $\rho(r)$ suggests that there is really no "edge" for the electron cloud, unlike what is shown in Figure 6. Instead, the charge density extends to ∞, so that it makes sense to talk about such a limit.)

(c) Find the electric field of this hydrogen atom as a function of the radial position, r.

(d) Roughly sketch a graph of $E(r)$ for this hydrogen atom.