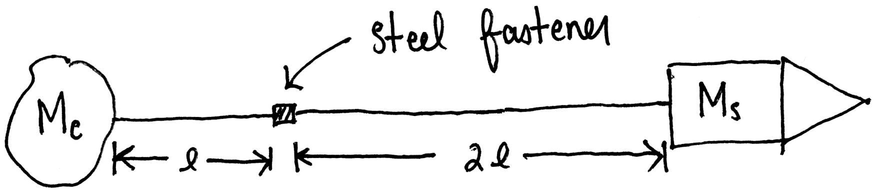
Physics 1A
Spring 2015

Challenge Problem 11

As shown in the figure below, a spaceship of mass M_s tows cargo of mass M_c with two pieces of adamantium cable joined by a small steel fastener whose length and mass are negligible. One piece is of length ℓ, the other is of length 2ℓ, and they both have mass per unit length λ. The steel fastener can be under a maximum tension $T_{\max}$ before it breaks.



(a) What is the maximum allowable thrust (force) that the spaceship's engine can exert on the spaceship before the fastener breaks?

(b) At this maximum thrust, what is the tension in the entire cable (both pieces) as a function of x, where x is the coordinate along the cable and its point of contact with the cargo is $x = 0$?

Solution.

(a) I'll demonstrate 2 methods of doing this part of the problem. First, I'll do it the systematic way by drawing FBDs for all objects in the diagram and using Newton's Laws etc. Second, I'll do it a rather faster way by picking certain clever subsystems of the whole system and applying Newton's Laws only to those.

<u>**Method 1: Systematic FBDs**</u>

We can ignore the y-direction for this entire problem, so we draw FBDs for all bodies including only the forces in the x-direction. There are four bodies: The cargo, the cable of length ℓ, the cable of length 2ℓ, and the spaceship. They all have the same acceleration a because they are connected by the taught cable. Newton's Second and Third Laws give the following equations:

$$T_c = M_c a, \qquad T - T_c = \lambda \ell a, \qquad T_s - T = 2\lambda \ell a, \qquad F - T_s = M_s a. \quad (1)$$

where T_c is the tension in the cable where it is fastened to the cargo, T is the tension at the location of the fastener, T_s is the tension in the cable where it is fastened to the ship, and F is the force of the thrusters on the ship.. This is four equations in five unknowns T_c, T, T_s, a, F. We can solve this set of equations for F in terms of T, and then plugging in $T = T_{\max}$ gives the max thrust $F = F_{\max}$. Adding the first two equations allows us to solve for a in terms of T;

$$a = \frac{T}{M_c + \lambda \ell}. \quad (2)$$

The second two equations then give us F in terms of T;

$$\begin{aligned} F &= T_s + M_s a & (3) \\ &= T + 2\lambda \ell a + M_s a & (4) \\ &= \left(\frac{1}{M_c + \lambda \ell} + 2\lambda \ell + M_s \right) T & (5) \\ &= \left(\frac{M_c + M_s + 3\lambda \ell}{M_c + \lambda \ell} \right) T. & (6) \end{aligned}$$

so setting $T = T_{\max}$ gives the desired result;

$$\boxed{F_{\max} = \left(\frac{M_c + M_s + 3\lambda \ell}{M_c + \lambda \ell} \right) T_{\max}} \quad (7)$$

Method 2: Clever subsystems.

Drawing a free body diagram for the cargo plus the section of adamantium of length ℓ yields the following equation via Newton's Second Law for the tension at the steel fastener:

$$T_{\max} = (M_c + \lambda \ell) a_{\max} \quad (8)$$

where $a_{\max}$ is the acceleration of the whole system above which the fastener will break. Considering the whole system (all cables + spaceship + cargo) and noting that the thrust F is the only external force gives the following relation between the max thrust and max acceleration of the system

$$F_{\max} = (M_c + M_s + 3\lambda\ell)a_{\max} \tag{9}$$

Combining these facts gives

$$\boxed{F_{\max} = \left(\frac{M_c + M_s + 3\lambda\ell}{M_c + \lambda\ell}\right) T_{\max}} \tag{10}$$

(b) There are two ways to find the tension in the cable. One is to use differential equations and analyze the forces on a small segment of rope along the cable as we did in class, the other is to just focus on the free body diagram of the subsystem consisting of the cargo plus a length x of the cable. In either case, we ignore the steel fastener since its length and mass are negligible.

Method 1: Differential equation method.

This method is exactly what we did in class. If you look at a small segment of the adamantium cable of length Δx, you find that

$$T(x + \Delta x) - T(x) = \lambda \Delta x a_{\max}. \tag{11}$$

Dividing both sides by Δx and taking the limit $\Delta x \to 0$ gives the following differential equation for $T(x)$:

$$\frac{dT}{dx} = \lambda a_{\max}. \tag{12}$$

Integrating both sides from ℓ to x, using the fact that $T(\ell) = T_{\max}$, and using the expression for $a_{\max}$ from part (a) gives

$$T(x) = T_{\max} + \lambda \left(\frac{T_{\max}}{M_c + \lambda\ell}\right)(x - \ell) \tag{13}$$

which can be simplified to

$$\boxed{T(x) = \left(\frac{M_c + \lambda x}{M_c + \lambda\ell}\right) T_{\max}}. \tag{14}$$

Method 2: Clever subsystem again.

Let's do the second method. Drawing a free body diagram for the cargo plus a length x of the cable measured from the cargo immediately gives

$$T(x) = (M_c + \lambda x) a_{\text{max}}. \tag{15}$$

Plugging in the expression for a_{max} from part (a) then gives the desired answer:

$$\boxed{T(x) = \left(\frac{M_c + \lambda x}{M_c + \lambda \ell} \right) T_{\text{max}}}. \tag{16}$$

Notice that when $x = \ell$, this reduces to T_{max} which makes sense since that's the location of the steel fastener.