Physics 1A
Spring 2015

Midterm Exam 1

April 27, 2015
4:00 PM - 4:50 PM
PAB 1-425

READ THIS BEFORE YOU BEGIN

- You are allowed to use only yourself and a writing instrument on the exam.
- Print your name on the top right of your exam.
- If you finish more than 5 minutes before the end of the exam period, then please raise your hand and a proctor will collect your exam. Otherwise, please stay in your seat until the end of time is called.
- When the exam is finished, please remain in your seat, and the proctor(s) will come around and collect your exam. Once your exam is collected, you may leave the room.
- **Show all work.** The purpose of this exam is primarily to test how you think; you will get more partial credit for a logical, well-thought-out response.
- Please **box all of your final answers** to computational problems.

ADVICE

Do not just attempt to blindly calculate answers to computational questions. Use your physical and geometric intuition first to try and determine as much about the answer as you can before you launch into computation!

Problem 1. (5 points)

A mass m is suspended from the ceiling of an office building elevator by a Hooke's Law spring of spring constant k and natural length ℓ_0. Let the positive y-direction be the direction in which the floor numbers in the building increase.

Assuming that the y-component of the acceleration of the elevator is a, which of the following is the length of the spring as the elevator accelerates?

Circle one answer, then explain the reasoning behind your answer. Reasoning may include calculation, but it doesn't have to.

(a) $\boxed{\ell_0 + \dfrac{m}{k}(a+g)}$

(b) $\ell_0 - \frac{m}{k}(g-a)$

(c) $\ell_0 + \frac{m}{k}(g-a)$

(d) $\ell_0 - \frac{m}{k}(a+g)$

Solution. This problem can be solved in more than one way.

Method 1. - Limiting Cases
One can solve it by considering the limits $a \to 0$ and $a \to -g$. When $a \to 0$, one expects that the spring is longer than its natural length by an amount mg/k because of the force of gravity. This leaves only (a) and (c) as possibilities. In the limit $a \to -g$, the elevator is in free fall, so one expects that the spring's length is its natural length. This leaves only (a).

Method 2. - Newton's Second Law
Drawing a free body diagram of the spring in the elevator, and being sure to analyze the situation from the ground outside of the elevator since the elevator frame in non-inertial, Newton's Second Law tells us that

$$k(\ell - \ell_0) - mg = ma \tag{1}$$

Notice that the sign in from of the spring force term is such that when $\ell < \ell_0$, namely when the spring is stretched, it exerts a positive force (upward), on the mass. Solving for ℓ gives (a).

Problem 2. (15 points)

Suppose that at time $t = 0$, Novak Djokovic drops a tennis ball from rest onto a tennis court. Let h_0 be the initial height from which the ball is dropped. For $i = 1, 2, 3, \ldots$, let v_i denote the speed of the ball just after its i^{th} bounce, and let h_i denote its maximum height after its i^{th} bounce. Assume that each bounce is instantaneous.

(a) After each bounce, the subsequent maximum height reached by the ball is reduced by a factor of $9/16$. Write this as a mathematical condition relating h_{i+1} and h_i.

(b) What is the ratio of v_{i+1} to v_i? Justify mathematically.

(c) Draw graphs of the y-component of position, velocity, and acceleration of the tennis ball as functions of time. Draw your graphs for all times up to and including the third bounce. Write a short explanation in words of how you generated your graphs.

(d) **Extra Credit. (5 points)** Let Δt_i be the time between bounce i and bounce $i + 1$. Determine Δt_i in terms of h_0.

Solution.

(a) The max height reached after bounce $i + 1$ is $9/16$ times the max height reached after the previous bounce, namely bounce i. Translating this into equation form gives

$$\boxed{h_{i+1} = \frac{9}{16} h_i} \tag{2}$$

(b) Using the kinematics equation

$$v_y^2 = v_{y,0}^2 - 2g(y - y_0), \tag{3}$$

with the initial time being just after bounce i, and the final time being the apex reached immediately after bounce i, we obtain

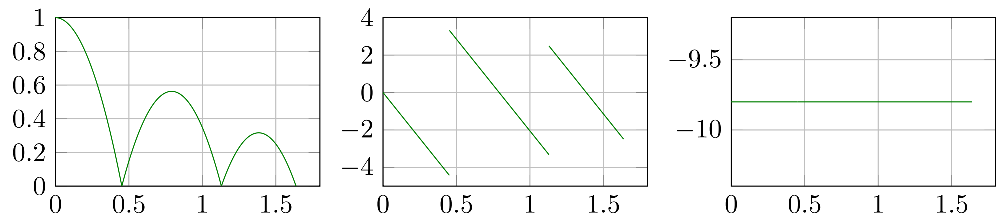
$$0^2 = v_i^2 - 2g(h_i - 0) \tag{4}$$

which implies that

$$v_i = \sqrt{2gh_i}. \tag{5}$$

It follows that

$$\frac{v_{i+1}}{v_i} = \frac{\sqrt{2gh_{i+1}}}{\sqrt{2gh_i}} = \sqrt{\frac{h_{i+1}}{h_i}} = \sqrt{\frac{9}{16}} = \boxed{\frac{3}{4}} \tag{6}$$



(c) Recall that velocity is the derivative of position, and acceleration is the derivative of velocity. The plots are as follows. I generated the plots using Python code – they are numerically accurate in the sense that they were generated from kinematics formulas using the value $h_0 = 1\,\mathrm{m}$. I would be more than happy to provide the IPython notebook with the code upon request.

(d) Using the kinematics equation for $y(t)$ in free fall for the interval of time between bounce i and bounce $i+1$ gives

$$0 = v_i \Delta t_i - \frac{1}{2} g \Delta t_i^2. \tag{7}$$

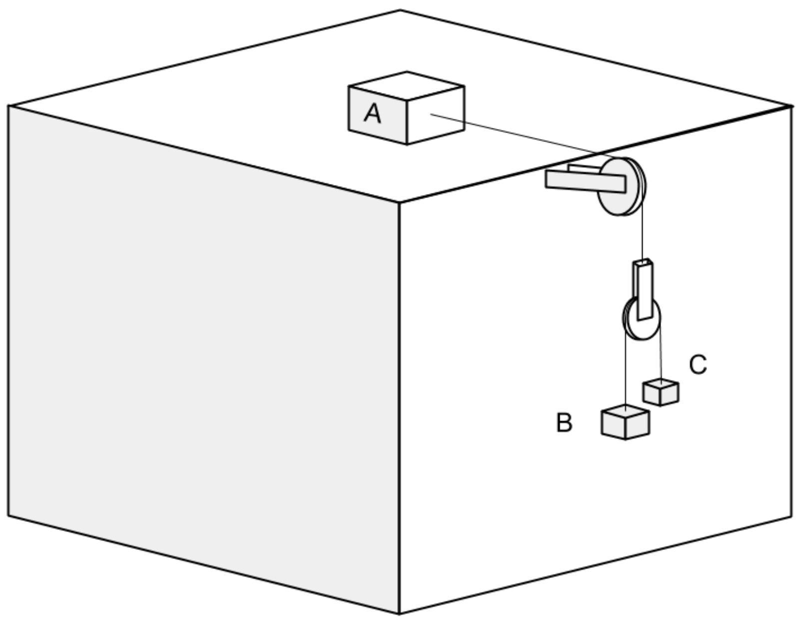
The nonzero Δt_i solution to this equation is the time interval we're after:

$$\Delta t_i = \frac{2v_i}{g} = \frac{2}{g}\sqrt{2gh_i} = \sqrt{\frac{8}{g}}\sqrt{\left(\frac{9}{16}\right)^i h_0} = \boxed{\left(\frac{3}{4}\right)^i \sqrt{\frac{8h_0}{g}}} \tag{8}$$

Problem 3. (22 points)

Consider the following system. All ropes and pulleys are massless, and all surfaces are frictionless. The mass m_A slides on an *immovable* table and is pulled by two masses m_B and m_C dangling over the edge of the table and connected by a rope threaded over a pulley.

Take the positive x-direction to be the direction of the resulting motion of mass A, and take the positive y-direction to be the direction opposite the motion of the pulley holding masses B and C.



(a) What do you expect the x-acceleration of mass A to be in the following limits? Explain your answers in words.

- **(i)** $m_A \to 0$
- **(ii)** $m_A \to \infty$
- **(iii)** $m_A \to m$, $m_B \to m/2$ and $m_C \to m/2$. In this case, apply all of these limits simultaneously.

(b) Determine an expression for the x-acceleration of mass m_A in terms of the masses m_A, m_B, m_C and g.

Solution. We make the abbreviations $a_A = a_{A,x}$, $a_B = a_{B,y}$, $a_C = a_{C,y}$ during the entirety of this solution.

(a) **(i)** In the limit $m_A \to 0$, the pulley holding masses B and C and those masses themselves will be in free fall, so m_A will be moving with acceleration g towards the pulley. Since that is the positive direction of motion as specified in the problem, we get

$$\boxed{a_A \to g}. \tag{9}$$

(ii) In the limit $m_A \to \infty$, block A will not accelerate at all, so we expect

$$\boxed{a_A \to 0}. \tag{10}$$

(iii) In the limit $m_A \to m$, $m_B \to m/2$ and $m_C \to m/2$, the situation becomes equivalent to a block of mass m being pulley by another block of mass m over the side of the table, and in this simpler case, one can show that the acceleration will be $g/2$ towards the pulley:

$$\boxed{a_A \to \frac{g}{2}}. \tag{11}$$

(b) Let T_1 be the tension in the rope attached to block A, and let T_2 be the tension in the rope connection blocks B and C. Newton's Second Law on the masses A, B, C and hanging pulley (which is massless!) respectively gives

$$T_1 = m_A a_A, \qquad T_2 - m_B g = m_B a_B, \qquad T_2 - m_C g = m_C a_C, \qquad T_1 - 2T_2 = 0. \tag{12}$$

This is four equations in five unknowns T_1, T_2, a_A, a_B, a_C. We are missing some information. Newton's Third Law doesn't help us in this regard here, but we can write the following constraints:

$$a_A = -a_p, \qquad 2a_p - a_B - a_C = 0. \tag{13}$$

where in the first equation, a_p represents the acceleration of the hanging pulley, and we have been careful to note that when block A moves toward the pulley, the hanging pulley moves down, so these accelerations have opposite sign. We thus have six equations in six unknowns $T_1, T_2, a_A, a_B, a_C, a_p$, and we can now solve for a_A (I leave the algebra to you) to obtain

$$\boxed{a_A = \frac{1}{1 + \frac{1}{4}\left(\frac{m_A}{m_B} + \frac{m_A}{m_C}\right)} g}. \tag{14}$$

One readily checks that this expression yields results that are consistent with the limits from part (a) of the question.

Problem	Score
Problem 1	
Problem 2	
Problem 3	
Total	
Extra Credit	