Physics 1A
Spring 2015

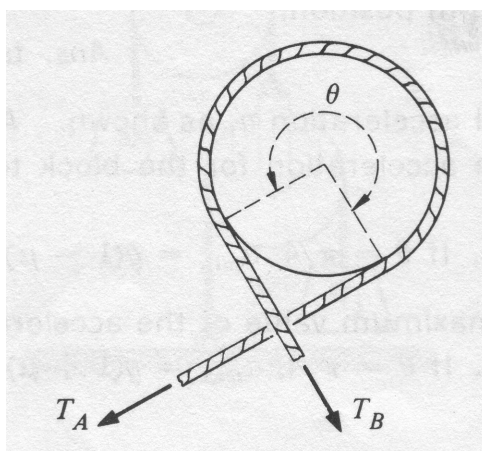
Challenge Problem 9

A device called a capstan is used aboard ships to control a rope which is under great tension. The rope is rapped around a fixed drum, usually for several turns (the drawing shows about three-fourths of a turn). The load on the rope pulls it with a force T_A, and the sailor holds it with a much smaller force T_B.

(a) Show that

$$T_B = T_A e^{-\mu\theta}$$

where μ us the coefficient of static friction of the drum's surface, and θ is the total angle subtended by the rope on the drum.



(b) Alice can pull a rope with a force of magnitude 500 N. If the drum has a diameter of 1/2 m, the surface has coefficient of static friction 1/2, and Alice has a 5 m length of rope, then by using the capstan, approximately what is the maximum load she can hold? Assume that all but 1 m of the rope will wrapped around the drum.

(a) A free body diagram on a small piece of the rope subtending an angle $\Delta\theta$ shows that Newton's Second Law in the x-direction is

$$T(\theta + \Delta\theta)\cos\left(\frac{\Delta\theta}{2}\right) - T(\theta)\cos\left(\frac{\Delta\theta}{2}\right) + f_s = 0 \tag{1}$$

and in the y-direction is

$$N - T(\theta)\sin\left(\frac{\Delta\theta}{2}\right) - T(\theta + \Delta\theta)\sin\left(\frac{\Delta\theta}{2}\right) = 0. \tag{2}$$

where f_s is the magnitude of the force of static friction. We assume that the static friction force achieves its maximum value μN to help us hold the load on the capstan the greatest amount possible;

$$f_s = \mu N \tag{3}$$

and we use the small angle approximations

$$\cos\left(\frac{\Delta\theta}{2}\right) \approx 1, \qquad \sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2}, \tag{4}$$

since we will be taking the limit $\Delta\theta \to 0$ in the end, and therefore this approximation will become exact. So the Newton's Second Law equations become

$$T(\theta + \Delta\theta) - T(\theta) = -\mu N \tag{5}$$

$$(T(\theta + \Delta\theta) + T(\theta))\frac{\Delta\theta}{2} = N. \tag{6}$$

Plugging what N is according to the second equation into the first equation, dividing both sides by $\Delta\theta$, and taking the limit $\Delta\theta \to 0$ gives

$$\frac{dT}{d\theta} = -\mu T. \tag{7}$$

Separating variables and integrating from 0 to θ where the tension at 0 is T_A and the tension at θ is T_B gives

$$\ln\left(\frac{T_B}{T_A}\right) = -\mu\theta \tag{8}$$

so that

$$\boxed{T_B = T_A e^{-\mu\theta}}. \tag{9}$$

(**b**) 4 m of the rope will be wound around the drum. The total angle θ in radians that it will be wrapped around for is therefore this length divided by the radius of the drum which is 1/4 m;

$$\theta = \frac{4\,\mathrm{m}}{1/4\,\mathrm{m}} = 16. \tag{10}$$

If we plug T_B in the formula from part (a) as the force Alice exerts on the rope, then T_A is the force at the other end, namely the load that she can carry;

$$T_A = (500\,\mathrm{N})e^{(1/2)(16)} \approx \boxed{e^8(500)\,\mathrm{N}}. \tag{11}$$

Now it turns out that $e^8 \approx 3000$, so the capstan allows her to hold about 3000 times more weight!!! And that's only with about 2.5 turns around the capstan. HOLY CRAP WOW! Capstans are AMAZING!!!