Physics 1A: Mechanics

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Notes on Motion in a Straight Line

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(Alternatively, just e-mail me your questions.)

WHERE DO ALL THESE FORMULAS COME FROM?

Basically, all equations of kinematics come from discussion of changes happening in time. If we observe some object and this object is at time t_1 at a position $\vec{x}_1$, and later we observe same object at time t_2 to be at $\vec{x}_2$, then we know that in time $\Delta t = t_2 - t_1$ the object changed its location by $\Delta\vec{x} = \vec{x}_2 - \vec{x}_1$. We can form a notion of how fast these changes happen: intuitively, things happen faster if more changes occur in the same amount of time. A straightforwardly following example of a quantity that measures how fast things change in time is *average velocity*:

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{x}}{\Delta t} \ . \tag{1}$$

It is called average because the velocity could be in principle constantly changing from 0 to c (where c is the speed of light, which is approximately $300,000 \left[\frac{\text{km}}{\text{s}}\right]$ - nothing can be faster that! [You will have to wait until Physics 1C to discuss this in more detail.]) within the time interval Δt, but we know only the average result of the motion which is the displacement $\Delta\vec{x}$ that happened within time Δt. (To get a clear understanding of this point imagine the following situation: you run with a velocity of $10 \left[\frac{\text{m}}{\text{s}}\right]$ for 1 minute and then you walk with velocity $1 \left[\frac{\text{m}}{\text{s}}\right]$ for 9 minutes. What is your *average* velocity in this time interval?)

To get a more precise notion of the *instantaneous velocity*, i.e. the velocity at a given time t, we form a derivative:

$$\vec{v}(t) = \frac{d\vec{x}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{x}}{\Delta t} \ . \tag{2}$$

Basically, to get the velocity at time t we measure the change in position that happens at smaller and smaller intervals of time around t. Mathematically it is a very well defined notion. If you have problems understanding this, you should refer first to some basic calculus textbook (or try *Wikipedia* for an introduction). There are many texts for science majors - "practical approaches" - that should allow you to spend no more than a couple of afternoons grasping this subject (this of course depends on what calculus classes you have taken so far).

Now, knowing the velocity at any time, given by the function $v(t)$, how do we get the information on how far has the object moved in a given time interval $\Delta t = t_f - t_i$? By means of integration! I.e., we can write down the infinitesimal change in position and integrate that to obtain the total change in position:

$$d\vec{x} = \vec{v}(t)dt \qquad \Rightarrow \qquad \vec{x} = \int d\vec{x} = \int_{t_i}^{t_f} \vec{v}(t)\, dt \ , \tag{3}$$

where we have just integrated both sides of the equation. Again, if this looks unfamiliar, you might want to spend another afternoon or two on reviewing integration.

An especially simple case occurs if the velocity is constant over the discussed period of time, i.e. when $\vec{v}(t) = \vec{v}_0$. Then we have

$$\vec{x} = \int_{t_i}^{t_f} \vec{v}_0\, dt = \vec{v}_0 \int_{t_i}^{t_f} dt = \vec{v}_0(t_f - t_i) = \vec{v}_0 \Delta t \ . \tag{4}$$

Actually, to be totally specific, we have

$$\vec{x} = \vec{x}_0 + \vec{v}_0 \Delta t \ , \tag{5}$$

as every integral is specified only up to a constant, in this case $\vec{x}_0$ - the so-called *initial displacement*, i.e., the displacement of the object at time t_i.

Now, just as we can consider changes in location, we can also consider changes in velocity. In full analogy to the definition of velocity, we define *acceleration* as

$$\vec{a} = \frac{d\vec{v}}{dt} \ . \tag{6}$$

Using (2), we immediately get

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} \ . \tag{7}$$

First, let us see that knowing the acceleration in some time interval $\Delta t = t_f - t_i$ we can calculate velocity as a function of time:

$$d\vec{v} = \vec{a}\ dt \qquad \Rightarrow \qquad \vec{v} = \int d\vec{v} = \int_{t_i}^{t_f} \vec{a}\, dt \ . \tag{8}$$

In a special case when the acceleration is constant (think for example about gravity, which to a good approximation is constant close to the Earth's surface) we get

$$\vec{v} = \int_{t_i}^{t_f} \vec{a}\, dt = \vec{a} \int_{t_i}^{t_f} dt = \vec{a}\Delta t + \vec{v}_0 \ , \tag{9}$$

where yet again we note that the integral is definite only up to a constant. Usually, the problem will specify whether we have some initial velocity $\vec{v}_0$ or not. Also, usually we assume that the initial time is zero, $t_i = 0$, in which case $t_f = t$ and we can drop the specific Δt symbol and just use t instead. Then we have a general expression of the form

$$\vec{v}(t) = \vec{a}t + \vec{v}_0 \ . \tag{10}$$

Now, having calculated the velocity we can calculate the position according to

$$\vec{x} = \int \vec{v}\, dt = \int (\vec{a}t + \vec{v}_0)\, dt = \int \vec{a}t\, dt + \int \vec{v}_0\, dt = \int \vec{a}t\, dt + \vec{v}_0 \int dt = \int \vec{a}t\, dt + \vec{v}_0 t \ , \tag{11}$$

where we have immediately used the fact that $\vec{v}_0$ is constant from the very definition. If the acceleration happens to be constant as well, then we have

$$\vec{x} = \vec{v}_0 t + \vec{a} \int t\, dt = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 + \vec{x}_0 \ , \tag{12}$$

where yet again we note that integration is definite only up to a constant, this time the initial displacement $\vec{x}_0$.

This is the most general formula specifying the position in time of an object moving under constant acceleration, and you must have already seen it during the lecture. If one happens to be dealing with acceleration or initial velocity that are not constant this formula obviously does not apply. In those cases we are usually given how the acceleration or velocity are described as functions of time, which allows us to perform the relevant integrations to obtain the displacement $\vec{x}(t)$.