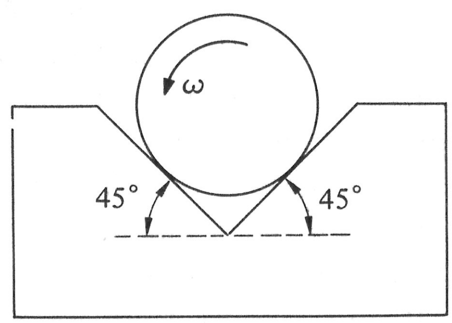
Physics 1A
Spirng 2015

Challenge Problem 22

A cylinder of mass M and radius R is rotated in a uniform V-shaped groove with constant angular velocity ω. The coefficient of kinetic friction between the cylinder and each surface is μ_k. What is the magnitude and direction of the torque that must be applied to the cylinder to keep it rotating at that angular velocity?



Solution.

We recall that for any system of particles, the net torque on the system about its center of mass equals the rate of change of its angular momentum about its center of mass

$$\boldsymbol{\tau}' = \frac{d\mathbf{L}'}{dt} \tag{1}$$

In this case we want the angular velocity of the cylinder to remain constant, and this implies that its rotational angular momentum about its center of mass doesn't change. It follows that the right hand side vanishes, and we find that the torques relative to the center of mass must sum to zero.

$$\boldsymbol{\tau}' = 0. \tag{2}$$

Excluding the applied forces that keep the cylinder rotating (the ones whose corresponding torques we are trying to solve for) there are five forces that can potentially contribute to the net torque about the center of mass: the normal force on the left contact point $\mathbf{N}_1$, the kinetic friction force on the left contact point $\mathbf{f}_1$, the normal and friction forces on the right contact point $\mathbf{N}_2$, $\mathbf{f}_2$, the force due to gravity $M\mathbf{g}$.

The torque due to gravity relative to the center of mass vanishes because it can be treated as acting at the center of mass. The torque due to the normal forces vanish as well because they point along the same lines as their position vectors $\mathbf{r}'$ relative to the center of mass. This leaves us with the torques due to the friction forces. The friction forces oppose the rotation and point tangent to the cylinder in the clockwise direction at a distance R from the center of mass, so assuming the x-y plane is perpendicular to the axis of rotation, their respective torques are

$$\boldsymbol{\tau}_1' = -f_1 R\mathbf{k}, \qquad \boldsymbol{\tau}_2' = -f_2 R\mathbf{k} \tag{3}$$

The sought-after applied torque must sum with these frictional torques to zero, so we get

$$\boldsymbol{\tau}_{\text{applied}} = (f_1 + f_2)R\mathbf{k}. \tag{4}$$

Now we just need to compute f_1 and f_2. We assume that that whatever agent is supplying the external torque supplies no net force, and supposing that the positive y-axis points up, Newton's Second Law in the x- and y-directions tells us that

$$\frac{N_1}{\sqrt{2}} - \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} - \frac{N_2}{\sqrt{2}} = 0 \tag{5}$$

$$\frac{f_1}{\sqrt{2}} + \frac{N_1}{\sqrt{2}} + \frac{N_2}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} - Mg = 0 \tag{6}$$

and the relationship between kinetic friction and normal force tells us that

$$f_1 = \mu_k N_1, \qquad f_2 = \mu_k N_2 \tag{7}$$

This constitutes four equations in four unknowns f_1, f_2, N_1, N_2. The results for f_1 and f_2 are

$$f_1 = \frac{Mg}{\sqrt{2}}\frac{\mu_k(1+\mu_k)}{1+\mu_k^2} \tag{8}$$

$$f_2 = \frac{Mg}{\sqrt{2}}\frac{\mu_k(1-\mu_k)}{1+\mu_k^2} \tag{9}$$

Adding these together and plugging into (4) gives

$$\boxed{\boldsymbol{\tau}_{\text{applied}} = \sqrt{2}Mg\left(\frac{\mu_k}{1+\mu_k^2}\right)R\,\mathbf{k}} \tag{10}$$