Physics 1A
Spring 2015

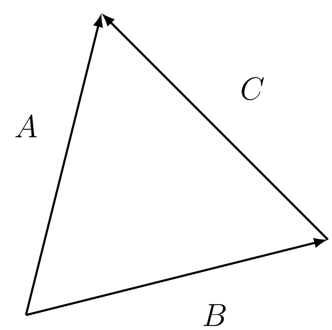
Challenge Problem 1

The Law of Cosines states that for any triangle with sides of length A, B, and C, the angle θ subtended by sides A and B satisfies

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

Prove this law using vector methods (e.g. perhaps things like vector addition, the dot product of two vectors, etc.)

Solution. A triangle can be formed using vectors as in the following diagram:



Then notice that $\mathbf{A}$ is the sum of $\mathbf{B}$ and $\mathbf{C}$, or equivalently

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \tag{1}$$

Taking the dot product of both sides with itself gives

$$\mathbf{C}\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} - 2\mathbf{A}\cdot\mathbf{B} \tag{2}$$

But notice that for any vector $\mathbf{v}$, the dot product of that vector with itself is just the squared magnitude of that vector.

$$\mathbf{v}\cdot\mathbf{v} = v_x^2 + v_y^2 + v_z^2 = |\mathbf{v}|^2. \tag{3}$$

Moreover, recall that the dot product of two vectors $\mathbf{v}$ and $\mathbf{w}$ can be written in terms of their magnitudes and the sign of the angle θ between them as

$$\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta. \tag{4}$$

Using these two facts in (2) gives

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta. \tag{5}$$

If we simply use the notation $A = |\mathbf{A}|$, $B = |\mathbf{B}|$, and $C = |\mathbf{C}|$, we obtain the desired identity.