Physics 1A: Mechanics
Winter 2016

Discussion: Week 2

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(Alternatively, just e-mail me your questions.)

Problem 1

a) A body travels in a straight line with a constant acceleration a. At $t = 0$, it is located at $x = x_0$ and has a velocity $v_x = v_0$. Show that its position and velocity at time $t > 0$ are given by

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \ ,$$
$$v(t) = v_0 + at \ .$$

b) Eliminate t from preceding equations and thus show that, at any time,

$$v_x^2 = v_0^2 + 2a(x - x_0) \ .$$

c) The formula you derived in part b) is a very useful one. Write down what were the assumptions made to derive it so that you know in what situations you can use it.

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a) We know that

$$\frac{dv}{dt} = a(t) \quad \Rightarrow \quad dv = a(t)\ dt \quad \Rightarrow \quad v(t) = \int dv = \int_0^t a(t')\ dt' = a\int_0^t dt' = at + C_1 \ , \tag{1}$$

where we have used the fact that the acceleration is constant, $a(t') = a$, and where C_1 is the integration constant. We know that at $t = 0$ we want to have $v(0) = v_0$, so that the integration constant is $C_1 = v_0$ and

$$v(t) = v_0 + at \ . \tag{2}$$

Similarly, we know that

$$\frac{dx}{dt} = v(t) \quad \Rightarrow \quad dx = v\ dt \quad \Rightarrow \quad x(t) = \int dx = \int_0^t v(t')\ dt' = \int_0^t \big(v_0 + at'\big)\ dt' = \tag{3}$$

$$= v_0 \int_0^t dt' + a\int_0^t t'\ dt' = v_0 t + \frac{1}{2}at^2 + C_2 \ , \tag{4}$$

where C_2 is another integration constant. Since at $t = 0$ we want $x(0)$, we can see that the integration constant must be $C_2 = x_0$, and thus we arrive at

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \ . \tag{5}$$

b) In part a) we have obtained

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \text{and} \qquad v(t) = v_0 + at \ . \tag{6}$$

Let us solve the second equation for t,

$$t = \frac{v(t) - v_0}{a} \ , \tag{7}$$

and plug that into the first equation:

$$\begin{aligned}
x(t) = x_0 + v_0\frac{v(t)-v_0}{a} + \frac{1}{2}a\left(\frac{v(t)-v_0}{a}\right)^2 \qquad &\Rightarrow \qquad x(t) - x_0 = \frac{v_0 v(t) - v_0^2}{a} + \frac{a}{2}\frac{v^2(t) - 2v_0 v(t) + v_0^2}{a^2} && (8)\\
&\Rightarrow \qquad x(t) - x_0 = \frac{v_0 v(t)}{a} - \frac{v_0^2}{a} + \frac{v^2(t)}{2a} - \frac{v_0 v(t)}{a} + \frac{v_0^2}{2a} && (9)\\
&\Rightarrow \qquad x(t) - x_0 = \frac{v^2(t)}{2a} - \frac{v_0^2}{2a} && (10)\\
&\Rightarrow \qquad v^2(t) = v_0^2 + 2a\big(x(t) - x_0\big) \ . && (11)
\end{aligned}$$

Very often one sees the above formula in the following form:

$$v = \sqrt{v_0^2 + 2ax} \ , \tag{12}$$

where it is assumed that x_0 and the time dependence is not explicitly written.

c) It is important to remember that one can only use (11) for *constant acceleration*. Additionally, as already has been noted, one can only use the special version (12) for $x_0 = 0$.

Problem 2

This is a homework problem. I will not include homework problems in discussions very often, but I want to make sure I get you started on this one!

A rocket of mass m blasts off vertically from the launch pad - its engines give it a constant upward acceleration of a and it feels no appreciable air resistance. When it has reached a height of h, its engines suddenly fail so that the only force acting on it is now gravity.
a) What is the maximum height this rocket will reach above the launch pad?
b) How much time after engine failure will elapse before the rocket comes crashing down to the launch pad, and how fast will it be moving just before it crashes?
c) Sketch a_y-t, v_y-t, and y-t graphs of the rocket's motion from the instant of the blast-off to the instant just before it strikes the launch pad. (Assume y-axis is in the direction of motion of the rocket, i.e. vertical.)

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a) The distance the rocket is going to travel upwards will consist of two parts: part one when the rockets engines were OK and part two when they already have failed, but the rocket's velocity hasn't changed direction yet.

Part one is given to be $\Delta x_1 = h$. We know that h is given by

$$h = \frac{1}{2}(a-g)\Delta t_1 \ , \tag{13}$$

where $a - g$ is the total acceleration the rocket is experiencing. Then we immediately can calculate that it takes the rocket

$$\Delta t_1 = \sqrt{\frac{2h}{a-g}} \tag{14}$$

to get to this point. By this time it has the velocity

$$v_h = (a-g)\Delta t_1 = (a-g)\sqrt{\frac{2h}{a-g}} = \sqrt{2h(a-g)} \ . \tag{15}$$

Now the rocket starts to decelerate (since the engines have failed and the only acceleration is due to gravity, which acts against the rockets motion at this point). First, we want to know how much time it will take for it to stop moving upwards, that is what is the Δt_2 that satisfies

$$v_h - g\Delta t_2 = 0 \qquad \Rightarrow \qquad \Delta t_2 = \frac{v_h}{g} = \frac{\sqrt{2h(a-g)}}{g} \ . \tag{16}$$

Now, given that, we can calculate that the distance traveled by the rocket in this time is

$$\Delta x_2 \ = \ v_h\Delta t_2 - \frac{1}{2}g(\Delta t_2)^2 = \sqrt{2h(a-g)}\frac{\sqrt{2h(a-g)}}{g} - \frac{1}{2}g\frac{2h(a-g)}{g^2} = \frac{2h(a-g)}{g} - \frac{h(a-g)}{g} = h\frac{a-g}{g} \ , \tag{17}$$

so that the height at which the rocket starts to fall is

$$\Delta x = \Delta x_1 + \Delta x_2 = h + h\frac{a-g}{g} = h\left(1 + \frac{a-g}{g}\right) = h\frac{a}{g} \ . \tag{18}$$

b) The time it will take the rocket to fall is

$$\Delta t_3 = \sqrt{\frac{2\Delta x}{g}} = \sqrt{\frac{2ha}{g^2}} = \frac{\sqrt{2ha}}{g} \ , \tag{19}$$

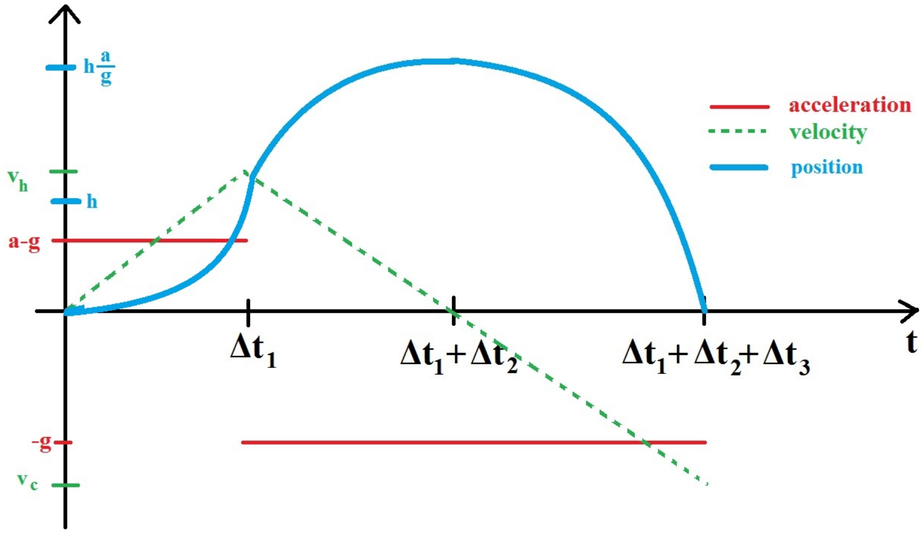
so that the total time from the moment the engines failed to the moment the rocket hits the ground is

$$\Delta t_{\text{total}} = \Delta t_2 + \Delta t_3 = \frac{\sqrt{2h(a-g)} + \sqrt{2ha}}{g} \ . \tag{20}$$

At the moment of crash, the rocket will be moving with the speed

$$v_{\text{crash}} = g\Delta t_3 = g\frac{\sqrt{2ha}}{g} = \sqrt{2ha} \ . \tag{21}$$

c)



It should be clear that I put three different graphs on one graph, since all of them were dependent on time. You should understand that in fact one should imagine three different axes where values for a, v and y are annotated. Also, it should be more than clear that this graph is *not to scale*.

Problem 3

This is on original Corbin midterm problem.

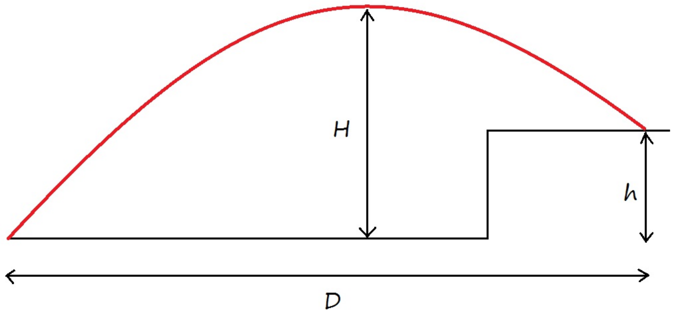
A projectile lands on a hill of height h, a horizontal distance D from where it was launched, after attaining a maximum height H. Find the following quantities in terms of h, H and D:

a) the time of the flight,

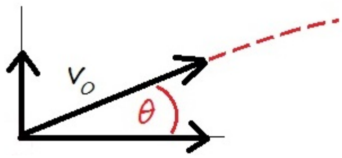
b) the horizontal and vertical components of the initial velocity vector,

c) the angle (above horizontal) that the projectile is fired at.

You may solve for these in any order that is convenient for you.



Let us assume that we start at $t_0 = 0$ the projectile starts its motion at $x_0 = 0$ and $y_0 = 0$. When the projectile starts to move, it has a velocity of magnitude v and a direction that we still have to find. This direction may be specified by an angle θ as shown in the picture below:



For a given angle θ we can calculate the horizontal and vertical components of velocity:

$$v_x = v\cos\theta \ , \qquad v_y = v\sin\theta \ . \tag{22}$$

We do not know θ yet, but we will find it in the course of this problem.

Let us write the kinematic equations describing the projectile motion:

$$x(t) = v\cos\theta \ t \ , \tag{23}$$

$$y(t) = v\sin\theta \ t + \frac{1}{2}\vec{g}t^2 = v\sin\theta \ t - \frac{1}{2}gt^2 \ , \tag{24}$$

where in the second equation we have used the fact that in our frame of reference the gravity acts downwards.

We proceed by applying this set of equations to special points in space and time (it might sound complicated, but it really is quite easy - just read on!). For example, if we denote the time at which the projectile lands at the hill - at height h - as t_h, then we know that the following equations has to be satisfied:

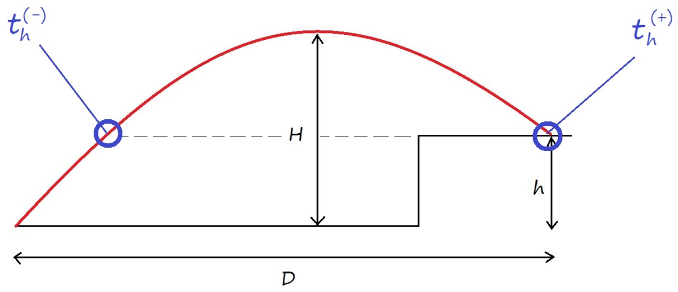
$$h = v\sin\theta \ t_h - \frac{1}{2}gt_h^2 \ . \tag{25}$$

This is nothing else than a quadratic equation (for t_h) that we all know and love (;)), and most of all which we know how to solve. We proceed in the usual way:

$$\frac{1}{2}gt_h^2 - v\sin\theta \ t_h + h = 0 \qquad \Rightarrow \qquad \Delta = b^2 - 4ac = v^2\sin^2\theta - 2gh \qquad \Rightarrow \qquad \sqrt{\Delta} = \sqrt{v^2\sin^2\theta - 2gh} \tag{26}$$

$$\Rightarrow \qquad t_h^{(-)} = \frac{v\sin\theta - \sqrt{v^2\sin^2\theta - 2gh}}{g} \ , \qquad t_h^{(+)} = \frac{v\sin\theta + \sqrt{v^2\sin^2\theta - 2gh}}{g} \ . \tag{27}$$

How do we know which of these solutions for t_h to choose? If you look at the graph, you will easily see that in fact there are two times at which the projectile is at the height h:



For obvious reasons we are interested in the later of the times, that is $t_h^{(+)}$, so that we set

$$t_h = t_h^{(+)} = \frac{v\sin\theta + \sqrt{v^2\sin^2\theta - 2gh}}{g} \,. \tag{28}$$

This in principle should answer part a) of the question, however, we do not know the angle θ nor the velocity v yet. Luckily, we still got some information we haven't used yet. For one, we know that the maximal height of the trajectory is H. At this height the vertical velocity of the projectile is zero, which plugged into our very useful formula derived in Problem 1b yields

$$0 = \sqrt{v^2\sin^2\theta - 2gH} \,, \tag{29}$$

where we have taken into account the fact that gravity acts against the initial velocity (hence the minus sign). This straightforwardly leads to

$$v\sin\theta = \sqrt{2gH} \,, \tag{30}$$

i.e., we have just solved for the vertical component of velocity.

Then, we know that at time t_h the projectile should travel a horizontal distance D, that is we know that

$$D = v\cos\theta \; t_h = v\cos\theta \, \frac{v\sin\theta + \sqrt{v^2\sin^2\theta - 2gh}}{g} = v\cos\theta \, \frac{\sqrt{2gH} + \sqrt{2g(H-h)}}{g} \,, \tag{31}$$

where we have plugged in (30). Thus we can also solve for the horizontal component of velocity,

$$v\cos\theta = \frac{Dg}{\sqrt{2gH} + \sqrt{2g(H-h)}} \,. \tag{32}$$

Using this we can solve for the magnitude of velocity,

$$v = \frac{Dg}{\cos\theta\big(\sqrt{2gH} + \sqrt{2g(H-h)}\big)} \,, \tag{33}$$

and by subsequently plugging this into (30) we arrive at

$$\frac{Dg}{\cos\theta\big(\sqrt{2gH} + \sqrt{2g(H-h)}\big)}\sin\theta = \sqrt{2gH} \quad \Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{\sqrt{2gH}\Big(\sqrt{2gH} + \sqrt{2g(H-h)}\Big)}{Dg} \tag{34}$$

$$= \frac{2gH + \sqrt{4g^2H(H-h)}}{Dg} \,. \tag{35}$$

Finally then

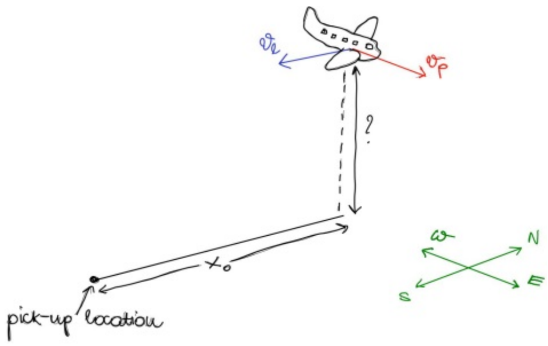
$$\theta = \tan^{-1}\left(\frac{2gH + \sqrt{4g^2H(H-h)}}{Dg}\right) \,. \tag{36}$$

Thus we have found all the quantities the problem asked for (we can now plug the solution for θ into the solutions for velocity as well as its horizontal and vertical components, and then plug the vertical component into the solution for the time of the flight).

Problem 4

A plane flies due East with velocity v_p. The plane is equipped with a launcher that is able to horizontally fire a package with velocity v_l, perpendicular to the direction of the motion of the plane. The pilot of the plane did very poorly in his mechanics class, and this is the reason the following situation takes place. He is supposed to fire the package in such a way that it ends up in a secret pick-up location. He fires when the plane is a distance x_0 (as measured on the ground) due north of the secret pick up location. Assuming that the operator chose the right distance to reach the package in the longitudinal direction,
a) what is the height h at which the plane flied at the moment the pilot launched the package?
b) by how much did the pilot overshoot in the latitudinal direction?

The situation described is the following:



We know that the package will not land in the pick-up location, because it will not only fall in a projectile motion south due to the velocity gained from the launcher, but also it will fall east due to the velocity of the plane itself.

a) If the longitudinal distance is correct, then

$$x_0 = v_l t \qquad \Rightarrow \qquad t = \frac{x_0}{v_l} \ . \tag{37}$$

This is the same time it takes the package to reach the ground, i.e.

$$h = \frac{1}{2} g t^2 = \frac{g x_0^2}{2 v_l^2} \ . \tag{38}$$

b) The package will overshoot the pick-up location by a distance d east, given by

$$d = v_p t = x_0 \frac{v_p}{v_l} \ . \tag{39}$$

Problem 5

A student is standing in an elevator that is on the ground. At time $t = 0$ the elevator begins to ascend from the ground with uniform speed (the value of which we do not know). At time t_d (the "drop time") the student drops a marble through the floor. The marble falls with uniform acceleration g and hits the ground an amount of time T later.

a) Find the height of the elevator at the drop time t_d.

b) On physical grounds, what would you expect the answer to be in the limit

$$\frac{T}{t_d} \to 0 \ ?$$

Why?

c) Does your expectation of what would happen from part b) match your mathematical answer from part a)?

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a) Before the student releases the marble, the equation of motion of the marble is given by

$$h(t) = vt \ . \tag{40}$$

At time t_d the marble is at as of yet unknown height H, and it satisfies the equation

$$H = vt_d \ . \tag{41}$$

After the marble is released its motion is described by

$$h(t) = H + v(t - t_d) - \frac{1}{2}g(t - t_d)^2 \ . \tag{42}$$

As stated in the problem, we do not know v, but we can solve for it using (41):

$$v = \frac{H}{t_d} \tag{43}$$

and plug that into (42):

$$h(t) = H + \frac{H}{t_d}(t - t_d) - \frac{1}{2}g(t - t_d)^2 \ . \tag{44}$$

From the description of the problem we know that when $t - t_d = T$ the marble hits the ground, i.e. that

$$0 = H + H\frac{T}{t_d} - \frac{1}{2}gT^2 = H\left(1 + \frac{T}{t_d}\right) - \frac{1}{2}gT^2 \ . \tag{45}$$

We can solve this for H:

$$H = \frac{g}{2}\frac{T^2}{1 + \frac{T}{t_d}} \ . \tag{46}$$

b) When $\frac{T}{t_d} \to 0$, it means that it takes the elevator much more time to get to a given height (t_d) then it takes the marble to fall down from that height (T). This basically means that the elevator is extremely slow and in the limit we can just consider the motion of the marble after t_d as that of a free falling body without an initial velocity (i.e., the velocity of the elevator is slow that we can neglect it).

c) We can see that in the limit $\frac{T}{t_d} \to 0$ equation (46) becomes

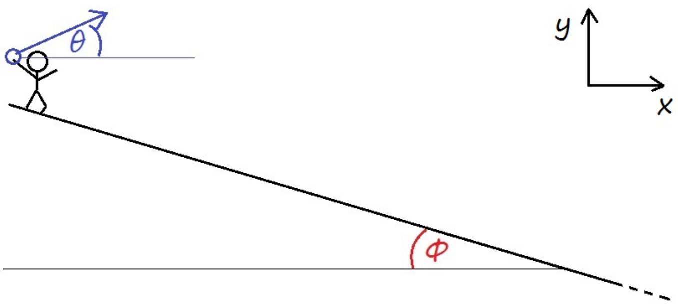
$$H = \frac{1}{2}gT^2 \ , \tag{47}$$

which is just the equation for the initial height of a body falling time in T under acceleration g.

Problem 6

You are standing on a hill of a uniform slope ϕ. You want to throw the ball down the hill in such a way that its range is maximal. At what angle θ should you throw it? Assume the speed with which you throw the ball is v.
Hint: Assume for ease that the hill in infinitely long, i.e., there is no way you could overthrow the slope.

The situation in question is the following:



We know that the maximal range for the projectile motion on a level ground is associated with $\theta = \frac{\pi}{4}$ (you should be able to show it yourself!). Here, because of the sloped hill, the situation will change. The usual equations of kinematics will still apply,

$$y(t) = v \sin\theta\ t - \frac{1}{2}gt^2\ , \tag{48}$$

$$x(t) = v \cos\theta\ t\ . \tag{49}$$

Usually, we would now proceed by setting $y(t_{gr})$ to zero (ground level), solving for t_{gr} and plugging that into the equation for $x(t)$ to obtain the range. Here, the ground level y_{gr} is not zero, but is a function of x:

$$\tan\phi = \frac{-y_{gr}}{x} \qquad \Rightarrow \qquad y_{gr} = -x\tan\phi\ . \tag{50}$$

Obviously, x is itself dependent on time as in (49), so that we really have

$$y_{gr}(t) = -x(t)\tan\phi = -v\cos\theta\ \tan\phi\ t\ . \tag{51}$$

This we plug into (48) at t_{gr} to obtain

$$-v\cos\theta\ \tan\phi\ t_{gr} = v\sin\theta\ t_{gr} - \frac{1}{2}gt_{gr}^2 \qquad \Rightarrow \qquad \frac{1}{2}gt_{gr}^2 - v\big(\sin\theta + \cos\theta\tan\phi\big)t_{gr} = 0\ , \tag{52}$$

$$\Rightarrow \qquad t_{gr}\left(\frac{1}{2}gt_{gr} - v\big(\sin\theta + \cos\theta\tan\phi\big)\right) = 0\ , \tag{53}$$

$$\Rightarrow \qquad t_{gr} = \frac{2v\big(\sin\theta + \cos\theta\tan\phi\big)}{g}\ . \tag{54}$$

This we can again plug into (49) to obtain the range:

$$x(t_{gr}) = \frac{2v^2\cos\theta\big(\sin\theta + \cos\theta\tan\phi\big)}{g}\ . \tag{55}$$

Now we want to maximize this range - to find the maximum we take derivative with respect to θ and set the result to zero:

$$\frac{dx(t_{gr})}{d\theta} = \frac{2v^2}{g}\Big[-\sin\theta\big(\sin\theta + \cos\theta\tan\phi\big) + \cos\theta\big(\cos\theta - \sin\theta\tan\phi\big)\Big] = \tag{56}$$

$$= \frac{2v^2}{g}\Big[-\sin^2\theta - \sin\theta\cos\theta\tan\phi + \cos^2\theta - \sin\theta\cos\theta\tan\phi\Big] = \tag{57}$$

$$= \frac{2v^2}{g}\Big[\cos 2\theta - \sin 2\theta\tan\phi\Big] = 0\ , \tag{58}$$

where in the last equality we have used $2\sin\theta\cos\theta = \sin 2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$. Then we can solve the last equation for θ:

$$\cos 2\theta - \sin 2\theta \tan\phi = 0 \qquad \Rightarrow \qquad \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta = \tan\phi \; . \tag{59}$$

Now, one can find that $\tan^{-1}\cot x = \frac{\pi}{2} - x$, so that we can calculate

$$\phi = \tan^{-1}\cot 2\theta = \frac{\pi}{2} - 2\theta \qquad \Rightarrow \qquad \theta = \frac{\pi}{4} - \frac{\phi}{2} \; . \tag{60}$$

Note that when $\phi = 0$ the angle θ reduces to $\frac{\pi}{2}$, as it should!

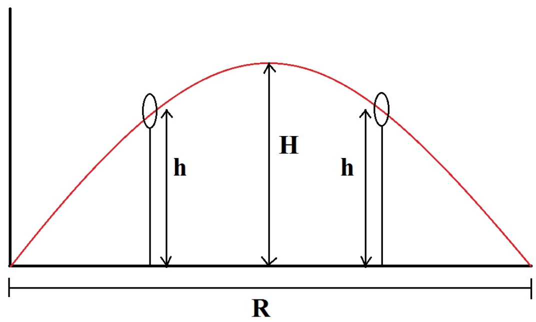
Problem 7

This is an original Corbin midterm problem. This is the kind of a crazy problem you should expect to see on your midterm in the 4th week. Corbin likes unusually sounding problems as they force you to really use your knowledge instead of just remembering solutions for standard setups.

Because there just really isn't anything better to do, a group of students studying for a Physics 1A Midterm decides to play the following game: they put together a catapult designed to throw squirrels through a parabolic arc of height H and range R. To make things a little more interesting, flaming hoops, each centered on a height h, have been mounted on movable poles and placed strategically so that they intersect the flight path of the thrown squirrels.

a) Find the horizontal and vertical components of the squirrels' initial velocity.

b) How much horizontal separation should there be between the poles?



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a) We know that the vertical and horizontal components of the velocity have to satisfy

$$v_{0x}t = \frac{R}{2} \qquad \text{and} \qquad v_{0y}t - \frac{1}{2}gt^2 = H \ . \tag{61}$$

We also know that at the highest point we have

$$v_0y - gt = - \qquad \Rightarrow \qquad t = \frac{v_{0y}}{g} \ . \tag{62}$$

We plug this into the second equation from (61) and get

$$v_{0y}\frac{v_{0y}}{g} - \frac{1}{2}g\frac{v_{0y}^2}{g^2} = \frac{1}{2}\frac{v_{0y}^2}{g} = H \qquad \Rightarrow \qquad v_{0y} = \sqrt{2gH} \ . \tag{63}$$

Similarly, the first equation from (61) becomes

$$v_{0x}\frac{v_{0y}}{g} = v_{0x}\sqrt{\frac{2H}{g}} = \frac{R}{2} \qquad \Rightarrow \qquad v_{0x} = R\sqrt{\frac{g}{8H}} \ . \tag{64}$$

b) The time that it takes for the squirrel to reach the hoop is given by

$$h = v_{0y}t_h - \frac{1}{2}gt_h^2 \qquad \Rightarrow \qquad \frac{1}{2}gt_h^2 - v_{0y}t_h + h = 0 \ , \qquad \Rightarrow \qquad \Delta = v_{0y}^2 - 2gh \ , \ \sqrt{\Delta} = \sqrt{v_{0y}^2 - 2gh} \ , \tag{65}$$

$$\Rightarrow \qquad t_h^- = \frac{v_{0y} - \sqrt{v_{0y}^2 - 2gh}}{g} \ , \qquad t_h^+ = \frac{v_{0y} + \sqrt{v_{0y}^2 - 2gh}}{g} \ , \tag{66}$$

where the smaller of the times describes the time when the squirrel is going through the *first* hoop and the larger of the times describes the time when the squirrel is going through the *second* hoop. The difference between those times is

$$\Delta t = t_h^+ - t_h^- = \frac{2\sqrt{v_{0y}^2 - 2gh}}{g} = \sqrt{\frac{8(H-h)}{g}} \ . \tag{67}$$

Then the horizontal distance the squirrel goes in this time is

$$\Delta x = v_{0x}\Delta t = R\sqrt{\frac{g}{8H}}\sqrt{\frac{8(H-h)}{g}} = R\sqrt{\frac{H-h}{H}}\ . \tag{68}$$

Problem 8

This is another homework problem. I think it's extremely interesting and thus I wanted to make sure you do it.

When tossing an object - say a tennis ball - straight up, one usually observes that the object in question seems to "hang" for a bit at the highest point of its trajectory. This, of course, is a false observation (if any object hovered in air for no apparent reason, it would contradict our basic physics intuition, wouldn't it?). Is there, however, some explanation as to why it *seems* to us as if there is some hovering after all?

Let's see whether we can calculate this. Let $y_{\max}$ be the maximum height above the point of release. To explain why the object seems to hang in the air, calculate the ratio of the time it is between $y_{\max}/2$ and $y_{\max}$ to the time it takes it to go from the point of release to $y_{\max}/2$. You may ignore air resistance.

We start from the basic notion that it takes just as much time to get from 0 to $y_{\max}$ as it takes to go from $y_{\max}$ (starting at zero velocity) to 0. Similarly, it takes the same time to get from $y_{\max}/2$ to $y_{\max}$ as it takes to get from $y_{\max}$ to $y_{\max}/2$. It is somewhat easier to consider the motion of the object as it's falling from $y_{\max}$, and we have just proven that this consideration will also answer our question.

Then let's denote the time it takes to go from $y_{\max}$ to $y_{\max}/2$ as t_{half} and the time it takes to fall from $y_{\max}$ to the ground as t_{tot}. Clearly, we have

$$\frac{y_{\max}}{2} = \frac{1}{2}gt_{\text{half}}^2 \qquad \text{and} \qquad y_{\max} = \frac{1}{2}gt_{\text{tot}}^2 \ , \tag{69}$$

so that (we just multiply the left equation by 2)

$$y_{\max} = gt_{\text{half}}^2 \qquad \Rightarrow \qquad gt_{\text{half}}^2 = \frac{1}{2}gt_{\text{tot}}^2 \qquad \Rightarrow \qquad t_{\text{half}} = \sqrt{\frac{t_{\text{tot}}^2}{2}} = \frac{t_{\text{tot}}}{\sqrt{2}} \ . \tag{70}$$

Naturally, the time it takes to fall from $y_{\max}/2$ to the ground, t_{rest}, is given by

$$t_{\text{rest}} = t_{\text{tot}} - t_{\text{half}} \ . \tag{71}$$

Then easily,

$$\frac{t_{\text{half}}}{t_{\text{rest}}} = \frac{\frac{t_{\text{tot}}}{\sqrt{2}}}{t_{\text{tot}} - \frac{t_{\text{tot}}}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1} = \frac{1}{1.41-1} = 2.44 \ . \tag{72}$$

We see that the ball spends nearly 2.5 more time in the upper half of its trajectory than in the lower part, and this is the very reason why it seems as if the ball is slightly hovering in the air!

Problem 9

This homework problem used to appear on Corbin's midterms. I would not expect it again, but it's important you go over this.

An elevator starts from rest and maintains a constant upward acceleration of a. A bolt in the elevator ceiling h above the elevator floor works loose and falls out the instant the elevator begins to move.
a) How long does it take for the bolt to reach the floor of the elevator?
b) Just as it reaches the floor, how fast is the bolt moving according to an observer
i) in the elevator?
ii) Standing on the floor landings of the building?
c) According to each observer in part b), how far has the bolt traveled between the ceiling and floor of the elevator?

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a) The bolt is subject to two accelerations: that of gravity g and that of the elevator a. In this situation they will add up (the floor "moving" towards the bolt!) to give the bolt time

$$t = \sqrt{\frac{2h}{g+a}} \tag{73}$$

to reach the floor.

b) i) In the elevator, the accelerations add up to give

$$v = (g+a)t \; . \tag{74}$$

ii) Outside the elevator we see only the gravity acting:

$$v = gt \; . \tag{75}$$

c) i) Intuitively, the bolt has traveled just as much as we can see in the elevator, that is h. This result is also given by

$$s = \frac{1}{2}(g+a)t^2 = h \; , \tag{76}$$

which should not surprise us, as we used this relation to obtain t in the first place.

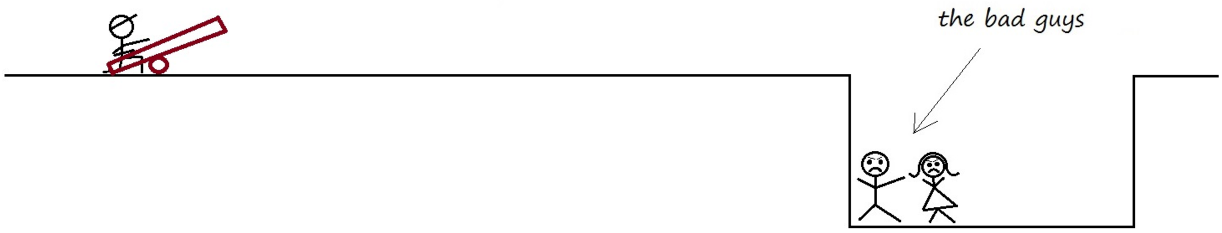
ii) Outside the elevator the bolt has traveled

$$s = \frac{1}{2}gt^2 = \frac{g}{g+a}h \; , \tag{77}$$

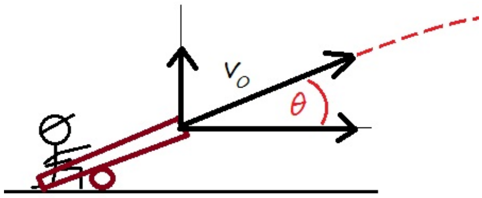
which is *shorter than h!*

Problem 10

A group of terrorists is hiding in a sort of large, rectangular ditch of depth d. A SWAT team member stands a distance L from the edge of the ditch manning a mortar of a negligible height (that is, the depth d of the ditch is so big that it does not *really* matter that the mortar itself has some height - just assume that the projectiles are fired from the ground level). The terrorists may expect the attack and thus may cover down really close to the bottom edge of the ditch, as pictured. How close to the edge of the ditch can rockets reach if the muzzle speed of the projectile is v?



Let us assume that we start at $t_0 = 0$ and that the mortar is positioned at $x_0 = 0$ and $y_0 = 0$. This means that the edge of the ditch is at $x = L$, and the bottom of the ditch is at $y = -d$. When the projectile leaves the muzzle, it has a velocity of magnitude v and a direction that we still have to find. This direction may be specified by an angle θ as shown in the picture below:



For a given angle θ we can calculate the horizontal and vertical components of velocity:

$$v_x = v\cos\theta \ , \qquad v_y = v\sin\theta \ . \tag{78}$$

We do not know θ yet, but we expect to find it in the course of this problem.

We can write down the kinematic equations describing the motion of the projectile:

$$x(t) = x_0 + v\cos\theta\ t = v\cos\theta\ t \ , \tag{79}$$

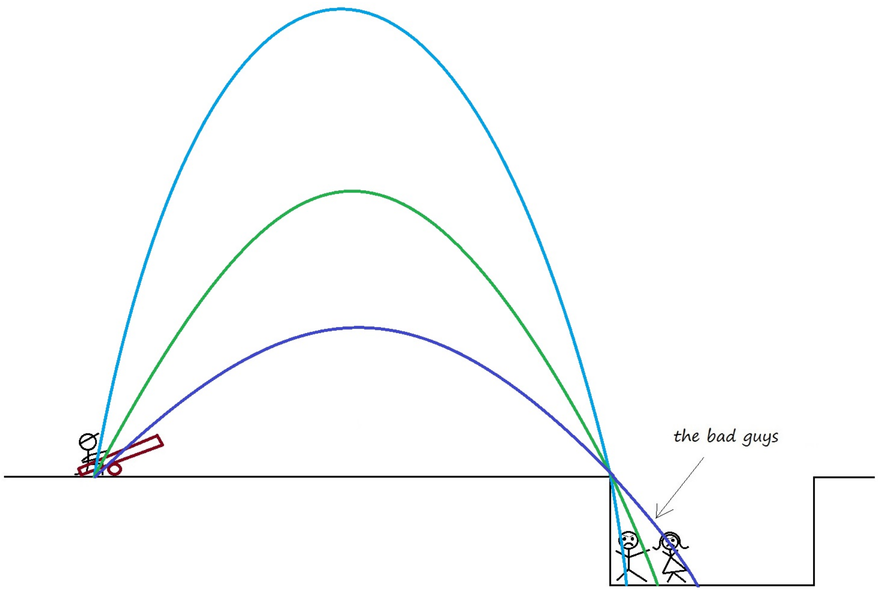
$$y(t) = y_0 + v\sin\theta\ t + \frac{1}{2}at^2 = v\sin\theta\ t - \frac{1}{2}gt^2 \ , \tag{80}$$

where in the second equation we have taken into the account the fact that gravity acts in the negative y direction (in our chosen frame of reference).

Now: we want the range to be minimal, but still larger than L. We also want the minimal range trajectory to be such that the projectile does not hit the edge of the ditch. That is, if we denote the time it takes the projectile to get back to the ground level as t_{gr}, for this time we want

$$x(t_{gr}) \geq L \ . \tag{81}$$

In the limit of a point-like projectile we can just denote this condition as $x(t_{gr}) = L$. There are many trajectories which satisfy this condition, as pictured below:



We can see from this picture that the trajectory with the biggest maximal height will result in the shortest range. The question is whether we can obtain different maximal heights given the initial speed and the condition $x(t_{gr}) = L$?

The mathematical problem we are facing is clear: for the given velocity v we must find the angle (or angles) θ that produce a trajectory satisfying $x(t_{gr}) = L$. Once we're done with that, we must calculate what is the range (including the ditch) of a projectile with such velocity. Let's do it!

The time t_{gr} can be calculated by inserting it into (80) and demanding that the left hand side of the equation is zero (the ground level), i.e. by writing

$$v \sin\theta \; t_{gr} - \frac{1}{2} g t_{gr}^2 = 0 \; . \tag{82}$$

This, of course, is just a quadratic equation, easily solved by rewriting it as

$$t_{gr} \left(v \sin\theta - \frac{1}{2} g t_{gr} \right) = 0 \; . \tag{83}$$

This equation has two solutions: we can immediately discard the trivial solution $t_{gr} = 0$ because we set our initial time to be zero and t_{gr} *must* satisfy $t_{gr} > t_0$. Thus we are left with the other solution:

$$t_{gr} = \frac{2v \sin\theta}{g} \; . \tag{84}$$

Now we plug t_{gr} into (79) and demand that the horizontal distance traveled at this time is L, i.e. we write

$$v \cos\theta \; t_{gr} = \frac{2v^2 \sin\theta \cos\theta}{g} = L \; . \tag{85}$$

Since we can use the trigonometrical identity $2 \sin\theta \cos\theta = \sin 2\theta$, the above equation becomes a condition on θ:

$$\sin 2\theta = \frac{Lg}{v^2} \qquad \Rightarrow \qquad \theta = \frac{1}{2} \sin^{-1} \left(\frac{Lg}{v^2} \right) \; . \tag{86}$$

This is the angle at which we should fire our projectile!

To proceed and calculate the range of this projectile we need to find the time it takes it to get to the bottom of the ditch, let's call it t_d. At this time we will have $y(t_d) = -d$. That is we can write

$$-d = v \sin\theta \; t_d - \frac{1}{2} g t_d^2 \qquad \Rightarrow \qquad \frac{1}{2} g t_d^2 - v \sin\theta \; t_d - d = 0 \; . \tag{87}$$

This, again, is just a quadratic equation that we perfectly know how to solve:

$$\Delta = b^2 - 4ac = v^2 \sin^2\theta + 2gd \qquad \Rightarrow \qquad \sqrt{\Delta} = \sqrt{v^2 \sin^2\theta + 2gd} \; , \tag{88}$$

$$t_d^- = \frac{v \sin\theta - \sqrt{v^2 \sin^2\theta + 2gd}}{g} \; , \qquad t_d^+ = \frac{v \sin\theta + \sqrt{v^2 \sin^2\theta + 2gd}}{g} \; . \tag{89}$$

We can immediately see that t_d^- does not constitute a viable solution, as it's less than zero and our problem starts at $t_0 = 0$. Therefore we set for

$$t_d = t_d^+ = \frac{v\sin\theta + \sqrt{v^2\sin^2\theta + 2gd}}{g} \ . \tag{90}$$

Having calculated the time it takes the projectile to reach the bottom of the ditch we can find the range of the projectile using

$$x(t_d) \ = \ v\cos\theta \ t_d = v\cos\theta \ \frac{v\sin\theta + \sqrt{v^2\sin^2\theta + 2gd}}{g} = \frac{v^2\sin\theta\cos\theta + v\cos\theta\sqrt{v^2\sin^2\theta + 2gd}}{g} \ . \tag{91}$$

Then the distance closest to the edge that the projectile can reach is given by

$$l = x(t_d) - L = \frac{v^2\sin\theta\cos\theta + v\cos\theta\sqrt{v^2\sin^2\theta + 2gd}}{g} - L \ . \tag{92}$$

Now, let us actually use some numbers (it won't happen often, but *sometimes* it is useful to do so). This problem comes from *The Feynman Lectures on Physics*, where the data given was

$$L = 27000 \text{ ft} \ , \qquad d = 350 \text{ ft}, \qquad v = 1000 \ \frac{\text{ft}}{\text{s}} \ , \qquad g = 32.2 \ \frac{\text{ft}}{\text{s}^2} \ . \tag{93}$$

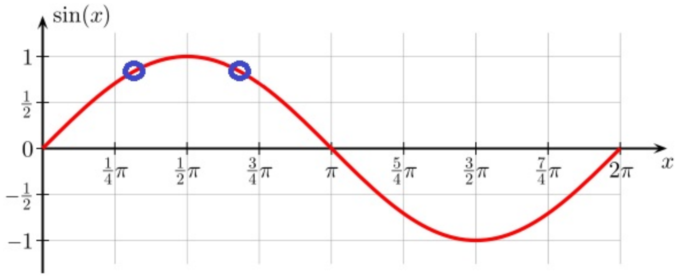
Using this data, please solve for θ using (86) and then plug it into (92) to solve for the minimal range. *Only then* turn the page.

Is your minimal range 202 ft (which is the correct answer)? No? Then I bet then it's about 589 ft - that is way too much!

Here is the explanation to this weird discrepancy: there exist *two* angles $\theta \in (0, \pi)$ satisfying (86), i.e. satisfying the equation

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{Lg}{v^2}\right) = \frac{1}{2}\sin^{-1}(0.8694) \ . \tag{94}$$

It is easily seen on the graph below:



What a typical calculator or mathematical program yields is the first of the shown angles,

$$\phi_1 = \sin^{-1}(0.8694) = 1.054 = 60.39^\circ \ . \tag{95}$$

The other solution is connected to the first one via

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \sin\left(\frac{\pi}{2} - \alpha\right) \ , \tag{96}$$

that is

$$\phi_2 = \frac{\pi}{2} + \left(\frac{\pi}{2} - \phi_1\right) = \pi - \phi_1 = 2.0876 = 119.61^\circ \ . \tag{97}$$

The corresponding projectile angles θ_1 and θ_2 are then

$$\theta_1 = 0.527 = 30.21^\circ \qquad \text{and} \qquad \theta_2 = 1.0438 = 59.81^\circ \ . \tag{98}$$

These two angles both satisfy the condition that $x(t_{gr}) = L$, but they lead to different ranges at $x(t_d)$! It turns out it is the larger of the angles, θ_2, that leads to the minimal range of 202 ft.

Note: some of you tried to solve for the shortest distance by taking a derivative of (91) with respect to θ and setting it to zero. Mind you: this method only gives you the *extremum* of the function and one is still to find whether this extremum is a minimum or a maximum. In this case what one obtains is a maximum! The absolutely minimum range is obtained for $\theta = \frac{\pi}{2}$ - then obviously the range is zero. This is entirely not what we were looking for!