Physics 1A: Mechanics

Winter 2016

Discussion: Week 3

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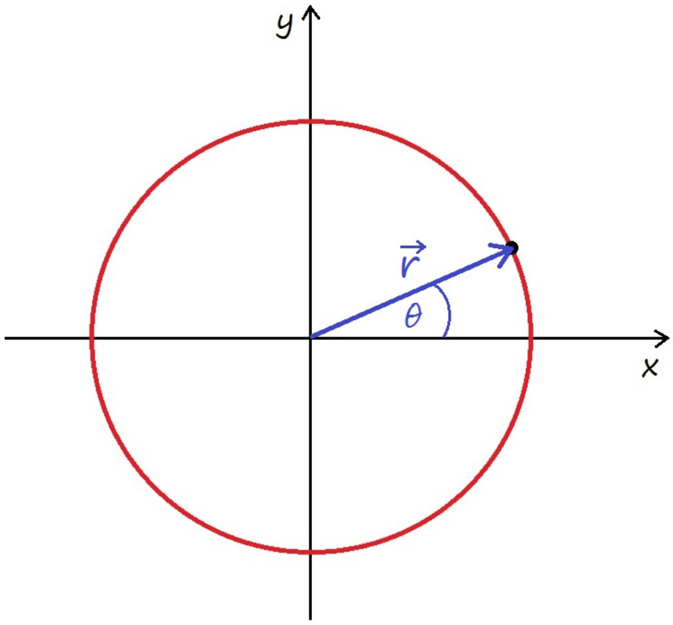
Office hours: Mondays, Tuesdays, Wednesdays 4-7 p.m. *by appointment!*

(Alternatively, just e-mail me your questions.)

Problem 1

a) We are considering an object moving along a circle of radius r, as depicted below:



Note that the center of the circle has been aligned with the center of the frame of reference.

The position of an object is described by the position vector $\vec{\boldsymbol{r}}$. Note that $|\vec{\boldsymbol{r}}| = r$, the radius of the circle. For the purpose of derivation it is better to express this position vector by means of other variables. Specifically, express $\vec{\boldsymbol{r}}$ in terms of r, the angle θ, and unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.

Hint: If you're at a loss, read about the so-called polar coordinates.

We can write $\vec{\boldsymbol{r}}$ in the following way:

$$\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{x}} + r_y\hat{\boldsymbol{y}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} = r\cos\theta\ \hat{\boldsymbol{x}} + r\sin\theta\ \hat{\boldsymbol{y}}\ . \tag{1}$$

b) That $\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt}$ is *always* true, also for circular motion. Using your result from a), obtain an expression for $\vec{\boldsymbol{v}}$ in circular motion. Note: for circular motion the angle θ describing the position of the object changes in time, i.e., $\theta = \theta(t)$.
Hint: Use the chain rule when taking the derivative! Remember: $\vec{\boldsymbol{r}}$ is a function of θ, and then θ is a function of t.
Hint 2: To introduce some consistency into our derivations we usually consider the object to be moving counterclockwise along the circle. It is not very important for this problem, but I thought it may make it easier for you to imagine the motion.

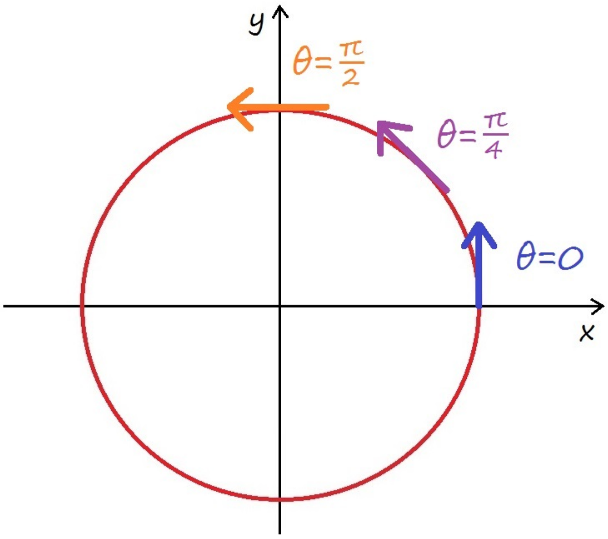
Straightforwardly,

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = \frac{d}{dt}\big(r\cos\theta\ \hat{\boldsymbol{x}} + r\sin\theta\ \hat{\boldsymbol{y}}\big) = r\left(\frac{d\cos\theta}{dt}\ \hat{\boldsymbol{x}} + \frac{d\sin\theta}{dt}\ \hat{\boldsymbol{y}}\right) = r\left(-\sin\theta\frac{d\theta}{dt}\ \hat{\boldsymbol{x}} + \cos\theta\frac{d\theta}{dt}\ \hat{\boldsymbol{y}}\right) = \tag{2}$$

$$= r\big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)\frac{d\theta}{dt}\ . \tag{3}$$

where we have used the fact that r is a constant.

c) If you did part b) correctly, you should have obtained an expression with a factor of $\big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)$. This factor is a unit vector (check this! One does that by checking whether the vector's magnitude is equal 1.) whose direction will change with θ. Please draw this vector for a few different θs on the diagram below. In what direction does it point? Is it along the radius, or maybe clockwise around the circle, or maybe otherwise? Note whether this direction follows your expectation.



We can see that the unit vector in question follows the direction along the circle. This is consistent with our expectations as the direction of the object's velocity clearly should point along the circle, and $\big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)$ specifies the direction of $\frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{v}}$. Moreover, since $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{v}}$, we can now figure out that the object's speed is given by $v = r\frac{d\theta}{dt}$.

d) Now we move further. As before, it is *always* true that acceleration is given by $\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt}$. Calculate acceleration in the circular motion.
Hint: the calculation will involve a product rule and a chain rule, and the final expression is a sum of two terms.

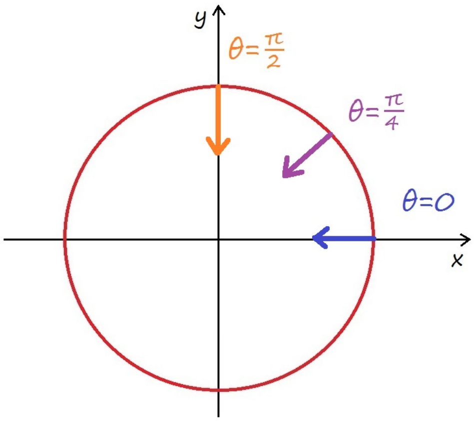
As asked, we differentiate the velocity:

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}\left[r\big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)\frac{d\theta}{dt}\right] = r\left[\left(-\frac{d\sin\theta}{dt}\ \hat{\boldsymbol{x}} + \frac{d\cos\theta}{dt}\ \hat{\boldsymbol{y}}\right)\frac{d\theta}{dt} + (-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}})\frac{d^2\theta}{dt^2}\right] = \tag{4}$$

$$= r\left[\left(-\cos\theta\frac{d\theta}{dt}\ \hat{\boldsymbol{x}} - \sin\theta\frac{d\theta}{dt}\ \hat{\boldsymbol{y}}\right)\frac{d\theta}{dt} + \big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)\frac{d^2\theta}{dt^2}\right] = \tag{5}$$

$$= r\left[\big(-\cos\theta\ \hat{\boldsymbol{x}} - \sin\theta\ \hat{\boldsymbol{y}}\big)\left(\frac{d\theta}{dt}\right)^2 + \big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)\frac{d^2\theta}{dt^2}\right] . \tag{6}$$

e) Once more, if you derived the expression above correctly, you should have obtained two terms. One of them has a factor of $\big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)$, which we already know from b) and c), and one has a factor of $\big(-\cos\theta\ \hat{\boldsymbol{x}} - \sin\theta\ \hat{\boldsymbol{y}}\big)$. Please draw this unit vector on the picture below. How can one interpret the two terms in $\vec{\boldsymbol{a}}$?



As we can see, the direction of $\big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)$ points along the radius, towards the center of the circle.

How to interpret the two terms? Let us first note that the term pointing along the circle is proportional to $r\frac{d^2\theta}{dt^2}$. It is reasonable to think that this tells us how fast the velocity of the object is changing *along* its trajectory (i.e., along the radius). The other term, proportional to $r\left(\frac{d\theta}{dt}\right)^2$, is a bit harder to figure out. To ease things up a bit, let us assume that $\frac{d^2\theta}{dt^2} = 0$ (this just means that the velocity along the circle is constant, which is perfectly viable). Then the total acceleration is still non-zero! It is because *the direction of the velocity has to constantly change in order for the object to move along a circle*. The term $r\big(-\cos\theta\ \hat{\boldsymbol{x}} - \sin\theta\ \hat{\boldsymbol{y}}\big)\left(\frac{d\theta}{dt}\right)^2$ describes this change in direction of the object's velocity. The presence of this acceleration is what makes the object move in a circle in the first place!

We give those two accelerations names: the acceleration along the circle is called the *translational acceleration* $\vec{a}_t$, while the radial acceleration is called the *centripetal acceleration* $\vec{\boldsymbol{a}}_r$. I.e.,

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}_t + \vec{\boldsymbol{a}}_r\ , \qquad \text{where} \qquad \vec{\boldsymbol{a}}_t = r\frac{d^2\theta}{dt^2}\big(-\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}\big)\ , \tag{7}$$

$$\vec{\boldsymbol{a}}_r = r\left(\frac{d\theta}{dt}\right)^2\big(-\cos\theta\ \hat{\boldsymbol{x}} - \sin\theta\ \hat{\boldsymbol{y}}\big) = r\left(\frac{d\theta}{dt}\right)^2\big(-\hat{\boldsymbol{r}}\big)\ . \tag{8}$$

Now, since (as we concluded in c))

$$v = r\frac{d\theta}{dt} \qquad \Rightarrow \qquad \left(\frac{d\theta}{dt}\right)^2 = \frac{v^2}{r^2}\ , \tag{9}$$

we can also write

$$\vec{\boldsymbol{a}}_r = \frac{v^2}{r}\big(-\hat{\boldsymbol{r}}\big)\ . \tag{10}$$

An important note: there really are no that many interesting circular motion problems. I will use some of the more interesting homework problems in this worksheet, but make sure you do the rest of the homework too because I will not dwell on circular motion! Honestly, it is just like the linear motion, only instead of meters one uses angles to measure it.

Also, all of the following problems involve the notion of force, so make sure you have reviewed last Friday's lecture!

Problem 2

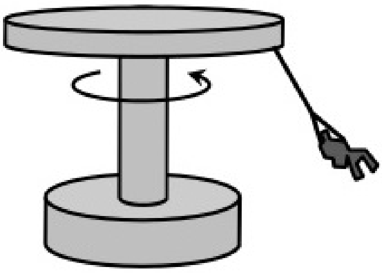
How is water removed from clothes during the spin cycle of a washer?
Hint: Your answer should involve physics-related terms such as "force", "acceleration" etc.

When the drum of the washer spins, the clothes (along with the water in the clothes) are dragged along due to friction. Due to the inertia both the clothes and the water particles tend to keep moving in a straight line. However, the clothes are acted upon by the walls of the drum, which effectively provide the centripetal force, thus making the clothes spin. Since there are holes in the drum, some of the water droplets move out through these holes and then fly off at a tangent to the drum's rotation.

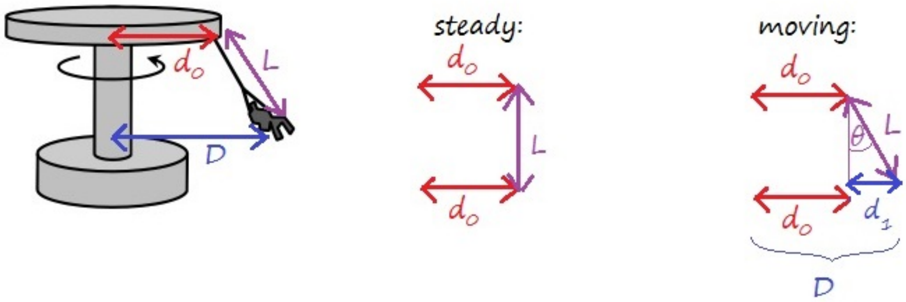
Of course once we start to *really* think deep about this problem we discover that there are more factors at play. For example, the water tends to cling to the clothes, so it takes some time for the water to make it through the layers of material. Then, since the water (at least some of it) doesn't just go flying out, it means it does experience some centripetal acceleration. While all that's going on, the clothes also get smashed up against each other on the inside of the drum, which additionally helps to squeeze the water out of the clothes. Nevertheless, the mechanism described in the first paragraph is the main working principle behind the spin cycle.

Problem 3

You take your little cousin Bill to a local amusement park. The kid could ride on a merry-go-round for hours, so you're stuck beside it waiting for him for quite some time. Bored, you're starting to consider Bill's motion around the center of the carousel. Bill's mass is m. The seats of the carousel are hung using lines of approximate length L. You note that when the carousel is not moving, the seats are approximately d_0 away from the center pole. In full motion, the seats are D away from the pole. What is Bill's total acceleration, and what is his linear velocity?



The lengths in the problem are connected in the following way:



Note that we do not know θ yet. However, we can relate it to other quantities known in the problem:

$$d_1 = D - d_0 \quad \text{and} \quad d_1 = L\sin\theta \quad \Rightarrow \quad D - d_0 = L\sin\theta \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{D-d_0}{L}\right) . \tag{11}$$

In the following we will just use θ since we already know it.

Now, there is tension T in the line holding the seat. This tension has the same direction as the line, but we can decompose it into its horizontal and vertical components:

$$T_h = T\sin\theta \quad \text{and} \quad T_v = T\cos\theta . \tag{12}$$

In equilibrium, the vertical component musty cancel gravity, so that

$$T\cos\theta - mg = 0 \quad \Rightarrow \quad T = \frac{mg}{\cos\theta} . \tag{13}$$

Then the horizontal component is

$$T_h = mg\tan\theta . \tag{14}$$

This horizontal component of tension plays the role of the centripetal force. Thus

$$a = \frac{T_h}{m} = g\tan\theta . \tag{15}$$

Then straightforwardly

$$a = \frac{v^2}{r} \quad \Rightarrow \quad v = \sqrt{ar} = \sqrt{gr\tan\theta} = \sqrt{gD\tan\theta} , \tag{16}$$

where in the last equation we have used the fact that radius of the motion is $r = D$.

Problem 4

Every human learns pretty early that if they hit something hard, e.g. a wall, how much it will hurt depends on how fast one moves (among other things). Same thing applies to falling: it's pretty safe to jump from a bench or a classroom table, while it's a rather stupid idea to jump from the roof of the Physics and Astronomy Building. Intuitively, after a few experiments (they can be so-called "thought experiments", you don't exactly need to jump off PAB to check this) we may infer that this has something to do both with the speed gained and with how fast this speed is diminished due to the collision (we will really elaborate on that soon). We also realize that when things fall, they gain in speed. Obviously, this is because when free falling, objects accelerate due to unrestricted action of gravity. That is why it hurts little to jump from a bench, while it hurts plenty to jump off of a building: if we jump from a larger height, there is more time to gain speed until we hit the ground.

Having concluded that, let us figure out the following thing. The floor of an average train is about as high as a classroom table. We know it's pretty safe to jump from this height. Why then it is not such a great idea to jump out of a fast moving train? Please try to use physical terms in your reasoning.

When you ride on a train - even when you are not moving with respect to the trains interior - you move with respect to the outside of the train (otherwise the whole concept of a train would be a failure, wouldn't it?). You can think of it this way: when the train starts moving, its floor acts with a friction on your feet and "drags" you along. This is the reason you move with the train at all!

Let us elaborate on this point: imagine you are standing on a platform with no walls which is pulled by such a train. If your feet were somehow levitating above this platform, you would just stay exactly where you were if the train started moving. But we all know that for all practical purposes, humans do not levitate. Your feet are planted firmly on the platforms floor. When the train starts moving, the floor of the platform also starts moving and acts with a friction on your feet, thus "dragging" you with it. This is why you acquire all this velocity (when friction acts on your feet, this means a force acts on your feet and we know that when a force acts on a body, this body moves with an acceleration due to this force!). Eventually, if the train stops accelerating and moves with a constant velocity, no force drags you anymore. But you have already acquired all this velocity and now you're moving with it through space...

It means that to the outside world, and for all practical purposes outside the train (such as jumping off or going through a collision), the passenger moves with the same velocity as the train. When this passenger jumps off of the train, he or she does not "shake off" this velocity due to the trains motion in any way. By jumping off, one just adds a component of velocity that is perpendicular to the direction of motion of the train. While in the air, the velocity of the passenger has a component parallel to the direction of motion of the train at the moment of the jump and equal to the velocity of the train at the moment of the jump, as well as a component perpendicular to the direction of motion of the train acquired due to the jump itself. When this passenger hits the ground, it's the component acquired from being inside a fast moving train that is most deadly. This is also why car accidents are dangerous - for all purposes, you are moving with the velocity of the car and when the car stops extremely abruptly due to a collision, you are still moving with this velocity to subsequently hit the window shield/side door/whatever you happen to fly against. Think about that next time you think it's cool not to put your seat belt on...

Problem 5

This is a homework problem.

A uniform cable of weight w hangs vertically downward, supported by an upward force of magnitude w at its top end. What is the tension in the cable

a) at its top end;
b) at its bottom end;
c) at its middle?

Your answer to each part must include a free-body diagram.

Hint: For each question choose the body to analyze to be a section of the cable or a point along the cable.

d) Graph the tension in the rope versus the distance from its top end.

The cable *is not moving*, therefore the net force (on every part of it!) must be zero.

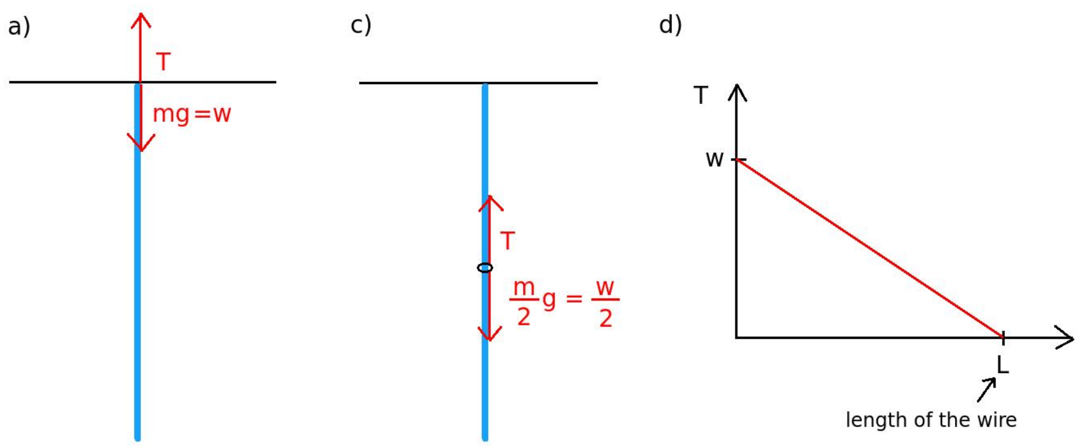
a) Since the net force is zero, the tension in this part must balance out all the weight. Hence $T(\text{top end}) = w$.

b) There is no weight anymore to support at the bottom, so the tension must be zero: $T(\text{bottom end}) = 0$. We may have also figured it out from the fact that the difference between the tension at the top and the bottom must be equal to the weight w, and that's all already in the top end tension.

Frankly, there isn't anything to draw.

c) The net force on the middle part should be zero. Now the middle part "holds" one half of the cable's weight $w/2$, and so the tension must be equal exactly that: $T = w/2$. Or $T = T(\text{top end}) - w/2 = w/2$.

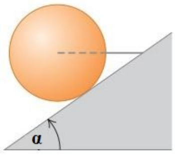
d)



Problem 6

This is another homework problem.

A horizontal wire holds a solid uniform ball of mass m in place on a tilted ramp that rises at an angle α above the horizontal. The surface of this ramp is perfectly smooth, and the wire is directed away from the center of the ball (see figure).
a) Draw a free-body diagram for the ball.
b) How hard does the surface of the ramp push on the ball?
c) What is the tension in the wire?



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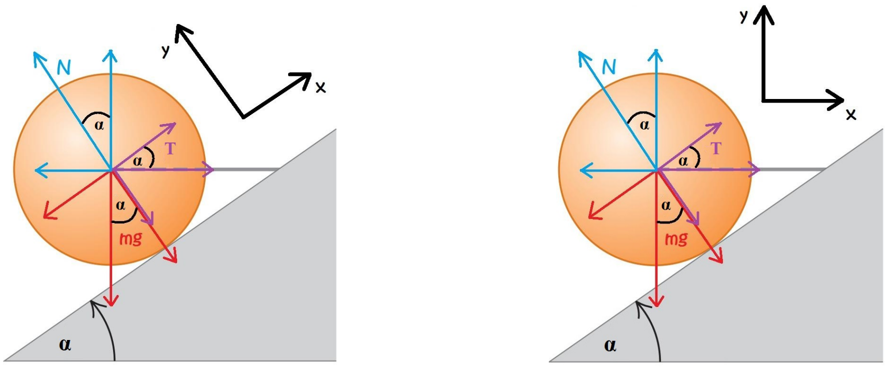
a) The pictures below are the same except for the axes, which refer to two ways of assigning the components of the forces described in part b).



b) There are two ways of solving this problem, connected to two ways in which we can assign the axes. Let us go over both of them. Both methods are correct - it depends on you which one you will choose!

i) First, we can think of the forces acting along the incline and perpendicular to it. Then clearly we have

$$mg\cos\alpha + T\sin\alpha = N \ , \tag{17}$$

$$T\cos\alpha - mg\sin\alpha = 0 \qquad \Rightarrow \qquad T = \frac{mg\sin\alpha}{\cos\alpha} \ , \tag{18}$$

which we combine into

$$mg\cos\alpha + \frac{mg\sin\alpha}{\cos\alpha}\sin\alpha = N \ . \tag{19}$$

We multiply both sides by $\cos\alpha$ to get

$$mg\cos^2\alpha + mg\sin^2\alpha = mg = N\cos\alpha \qquad \Rightarrow \qquad N = \frac{mg}{\cos\alpha} \ . \tag{20}$$

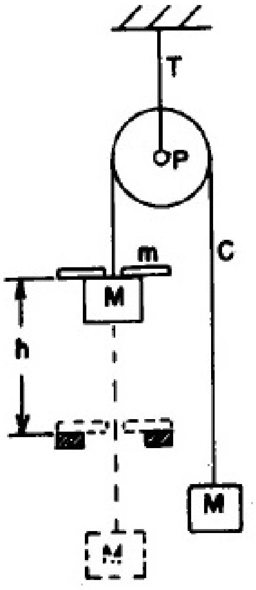
ii) Alternatively, we consider a vertical (with respect to the ground) axis y. The forces along this axis again have to be zero, and so we have

$$mg = N\cos\alpha \qquad \Rightarrow \qquad N = \frac{mg}{\cos\alpha} \ . \tag{21}$$

c) We've already solved for tension in (18).

Problem 7

An early arrangement for measuring the acceleration of gravity, called Atwood's machine, is shown in the figure. The pulley and cord have negligible mass and friction. The system is balanced with equal masses M on each side, and then a small C-shaped object of mass m is added to one side (as in the picture). After accelerating through a certain distance h the C-shaped object is caught on a ring and the two equal masses then move on with constant speed v. Find the value of g that corresponds to the measured values of m, M, h, and v.



With added mass m, the net force acting on the whole system is

$$F = mg \ , \tag{22}$$

where we assume g to be *unknown*. This force acts on a system whose total mass is $2M + m$, so that the resulting acceleration is

$$F = (2M + m)\, a \qquad \Rightarrow \qquad a = \frac{mg}{2M + m} \ . \tag{23}$$

Then, since the system starts from rest, we have

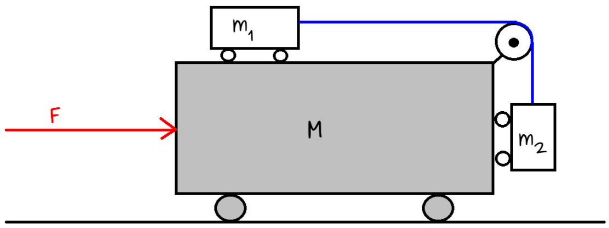
$$v = \sqrt{2ah} = \sqrt{\frac{2mgh}{2M + m}} \ . \tag{24}$$

We then easily solve for g:

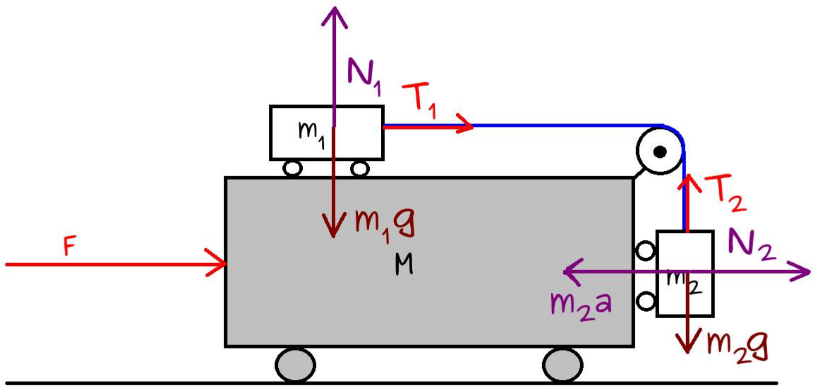
$$g = \frac{v^2 \big(2M + m\big)}{2mh} \ . \tag{25}$$

Problem 8

What horizontal force F must be applied to M so that m_1 and m_2 do not move relative to M? Neglect friction.



As usual, let us begin with drawing all the forces present in the system:



Both N_1 and N_2 are balanced, so we may disregard them. T_1 just comes from the fact that the pulley, which is attached to the cart, moves with same acceleration a as the cart. Therefore

$$T_1 = m_1 a \ . \tag{26}$$

On the other hand,

$$T_2 = m_2 g \ . \tag{27}$$

Of course, we also have $T_1 = T_2$, so that

$$m_1 a = m_2 g \ . \tag{28}$$

We still do not know what a is. We do know, however, that this is the acceleration that the whole system gains because of the force F, that is

$$F = (M + m_1 + m_2)a \qquad \Rightarrow \qquad a = \frac{F}{M + m_1 + m_2} \ . \tag{29}$$

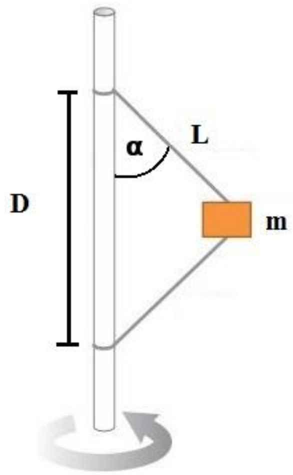
Easily now, we plug the above equation into (28) and get

$$F = g\frac{m_2}{m_1}\big(M + m_1 + m_2\big) \ . \tag{30}$$

Problem 9

And a yet another homework problem...

A block of mass m is attached to a vertical rod by means of two strings of same length L. When the system rotates about the axis of the rod, the strings are extended as shown in the diagram and the tension in the upper string is T_1.
a) What is the tension in the lower cord?
b) How many revolutions per minute does the system make?
c) Find the number of revolutions per minute at which the lower cord just goes slack.
d) Explain what happens if the number of revolutions per minute is less than in part c).



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a) The horizontal component of T_1 is given by $T_1 \sin\alpha$, and the vertical one by $T_1 \cos\alpha$. The horizontal component of the tension T_2 in the lower string is $T_2 \sin\alpha$, and the vertical is $T_2 \cos\alpha$. We have $\alpha = \cos^{-1}(D/2L)$ (from the geometry). Now then the forces acting in the vertical direction satisfy

$$T_1 \cos\alpha - T_2 \cos\alpha - mg = 0 \qquad \Rightarrow \qquad T_2 = T_1 - \frac{mg}{\cos\alpha} \ . \tag{31}$$

b) The radial acceleration is given by

$$a = \frac{1}{m}\big(T_2 \sin\alpha + T_1 \sin\alpha\big) = \frac{1}{m}\big(T_1 \sin\alpha - mg\tan\alpha + T_1 \sin\alpha\big) = \frac{2T_1 \sin\alpha - mg\tan\alpha}{m} \ . \tag{32}$$

Then easily

$$v = \sqrt{aR} \qquad \Rightarrow \qquad \omega = \sqrt{\frac{a}{R}} \qquad \Rightarrow \qquad f_{min} = \frac{60}{2\pi}\sqrt{\frac{a}{R}} = \frac{60}{2\pi}\sqrt{\frac{2T_1 \sin\alpha - mg\tan\alpha}{mR}} \ . \tag{33}$$

c) Let us take a limit here. If $T_2 \to 0$, we have

$$T_1' \cos\alpha = mg \qquad \Rightarrow \qquad T_1' = \frac{mg}{\cos\alpha} \ . \tag{34}$$

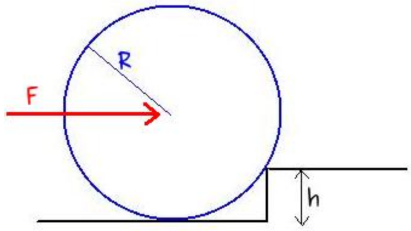
Then

$$T_1' \sin\alpha = m\frac{v^2}{R} \qquad \Rightarrow \qquad v = \sqrt{\frac{RT_1' \sin\alpha}{m}} \qquad \Rightarrow \qquad f_{min} = \frac{60}{2\pi}\sqrt{\frac{T_1' \sin\alpha}{Rm}} = \frac{60}{2\pi}\sqrt{\frac{ga}{R}}\tan\alpha \ . \tag{35}$$

d) If the block rotates at a slower rate than the limiting rate in part c), it means that the angle α becomes smaller.

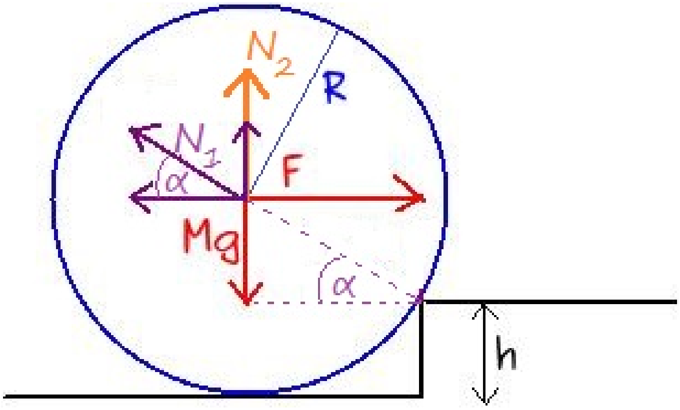
Problem 10

What horizontal force F is needed to push a wheel mass M and radius R over a block of height h?



Hint: as usual, start from making a drawing! Is the above drawing completely correct? Specifically, in kinematics, where do we apply forces?

In kinematics we usually assume that the applied forces act on the *center of mass* of considered object. That is, the situation described above looks the following way:



We have the force F pushing on the wheel from the left, we have the wheel's weight Mg, and we have the normal force from the tip of the block. You should we able to convince yourself that this force will always act towards the center of the wheel.

Now, from the geometry of the problem we can see that

$$\sin\alpha = \frac{R-h}{R} \qquad \text{and} \qquad \cos\alpha = \frac{\sqrt{R^2-(R-h)^2}}{R} = \frac{\sqrt{2hR-h^2}}{R} \; . \tag{36}$$

Now, when the wheel is stopped by the block, it means that the horizontal part of the normal force balances the force F:

$$F = N\cos\alpha = N\frac{\sqrt{2hR-h^2}}{R} \; . \tag{37}$$

On the other hand, for the wheel to "climb" the block we need the vertical part of the normal force to be equal or bigger than the force of gravity:

$$N\sin\alpha = N\frac{R-h}{R} \geq Mg \; . \tag{38}$$

We use (37) to solve for N,

$$N = \frac{FR}{\sqrt{2hR-h^2}} \; , \tag{39}$$

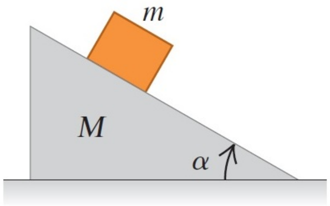
and plug that into (38):

$$\frac{FR}{\sqrt{2hR-h^2}}\frac{R-h}{R} = F\frac{R-h}{\sqrt{2hR-h^2}} \geq Mg \qquad \Rightarrow \qquad F \geq Mg\frac{\sqrt{2hR-h^2}}{R-h} \; . \tag{40}$$

Problem 11

A homework problem once more.

Moving Wedge. A wedge with mass M rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge (see figure). There is no friction between the block and the wedge. The system is released from rest.
a) Calculate the acceleration of the wedge and the horizontal and vertical components of the acceleration of the block.
b) Do your answers to part a) reduce to the correct results when M is very large?
c) As seen by a stationary observer, what is the shape of the trajectory of the block?



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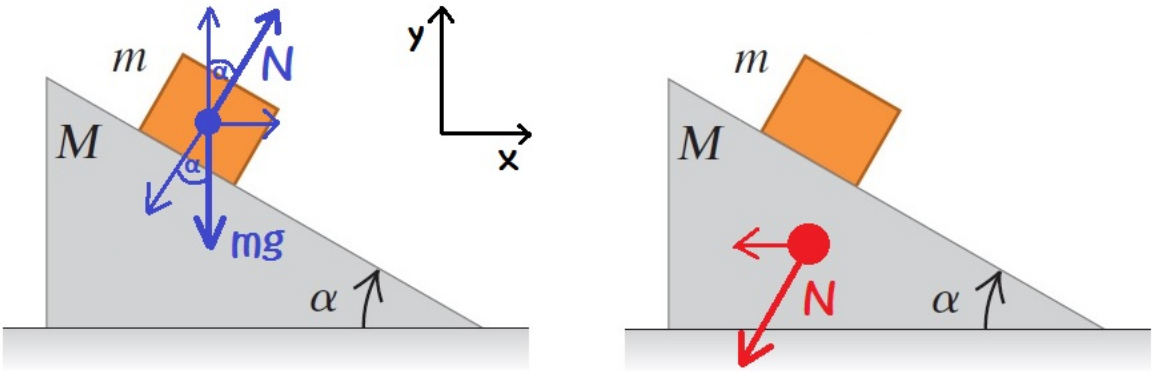
a) Let's first draw all the forces that act on both bodies:



Here you can see perfectly well one of the approximations we use, that is that the forces act on the center of mass.

First, let us write the equations of motion for m and M as described in the xy-plane. Let us note that we do not really know N, as the situation is not static!

$$N \sin\alpha = ma_x \ , \tag{41}$$
$$mg - N\cos\alpha = ma_y \ , \tag{42}$$
$$N \sin\alpha = -MA \ . \tag{43}$$

We have then four unknowns a_x, a_y, A, and N, and three equations, so we clearly need another one. We figure this one out by realizing what a person standing on the wedge would see. Clearly for this person we would have

$$a'_x = a_x - A \ , \qquad a'_y = a_y \ . \tag{44}$$

Still for this person, we have

$$\tan\alpha = \frac{a'_y}{a'_x} = \frac{a_y}{a_x - A} \ . \tag{45}$$

Now then we have all four equations we need and can proceed with solving them:

$$N \sin\alpha = ma_x \ , \tag{46}$$
$$mg - N\cos\alpha = ma_y \ , \tag{47}$$
$$N \sin\alpha = -MA \ , \tag{48}$$
$$\tan\alpha = \frac{a_y}{a_x - A} \ . \tag{49}$$

We can combine the first and the third to get

$$ma_x = -MA \qquad \Rightarrow \qquad a_x = -\frac{MA}{m} \ , \tag{50}$$

then we can plug it into the last one

$$\tan\alpha = \frac{a_y}{-\frac{MA}{m} - A} = -\frac{a_y}{A\left(\frac{M}{m}+1\right)} = -\frac{a_y}{A\left(\frac{M+m}{m}\right)} = -\frac{a_y}{A}\frac{m}{M+m} \qquad \Rightarrow \qquad a_y = -A\frac{M+m}{m}\tan\alpha \ , \tag{51}$$

and finally we plug this into the second:

$$mg - N\cos\alpha = -mA\frac{M+m}{m}\tan\alpha = -A(M+m)\tan\alpha \qquad \Rightarrow \tag{52}$$

$$\Rightarrow \qquad N = \frac{mg}{\cos\alpha} + A(M+m)\frac{\sin\alpha}{\cos^2\alpha} \ . \tag{53}$$

Clearly now the third becomes

$$mg\frac{\sin\alpha}{\cos\alpha} + A(M+m)\frac{\sin^2\alpha}{\cos^2\alpha} = -MA \qquad \Rightarrow \tag{54}$$

$$\Rightarrow \qquad A(M+m)\tan^2\alpha + MA = -mg\tan\alpha \tag{55}$$

$$\Rightarrow \qquad A\left((M+m)\tan\alpha + \frac{M}{\tan\alpha}\right) = -mg \tag{56}$$

$$\Rightarrow \qquad A = -\frac{mg}{(M+m)\tan\alpha + \frac{M}{\tan\alpha}} \ . \tag{57}$$

Once we have this, we can derive the other accelerations by plugging the above result into (50) and (51):

$$a_x = \frac{Mg}{(M+m)\tan\alpha + \frac{M}{\tan\alpha}} \qquad \text{and} \qquad a_y = \frac{(M+m)g\tan\alpha}{(M+m)\tan\alpha + \frac{M}{\tan\alpha}} \ . \tag{58}$$

(Note that we have defined a_y such that it points downward.)

b) When M becomes very large ($M \to \infty$), then A becomes arbitrarily close to zero, which indeed reduces all the other results to correct ones as well (think about (45)).

c) The trajectory will be given by

$$\vec{r} = \vec{r_0} + \frac{1}{2}a_x t^2\ \hat{\boldsymbol{x}} - \frac{1}{2}a_y t^2\ \hat{\boldsymbol{y}} = \vec{r_0} + \frac{1}{2}\frac{Mg\ \hat{\boldsymbol{x}} - (M+m)g\tan\alpha\ \hat{\boldsymbol{y}}}{(M+m)\tan\alpha + \frac{M}{\tan\alpha}}t^2 \ , \tag{59}$$

where $\vec{r_0}$ is the initial position of the block. Note that we can also write this as (taking the initial position to be zero)

$$x(t) = \frac{1}{2}a_x t^2 \ , \tag{60}$$

$$y(t) = -\frac{1}{2}a_y t^2 \ . \tag{61}$$

We can solve for time from the first equation and plug this into the second equation, so that we have y as a function of x:

$$y(t) = -x(t)\frac{a_y}{a_x} = -x(t)\frac{(M+m)\tan\alpha}{m} \ . \tag{62}$$

Clearly, $y(t)$ is a linear function of $x(t)$, and so we deduce that the trajectory of the block is a linear function with the slope $\frac{(M+m)\tan\alpha}{m}$.

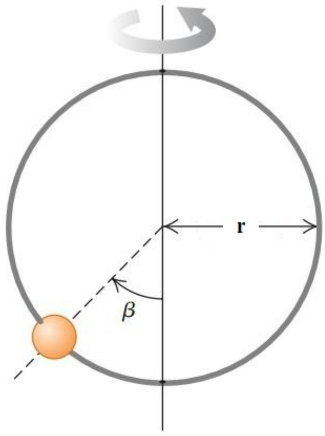
Problem 12

The last homework problem!

A small bead can slide without friction on a circular hoop that is in a vertical plane and has a radius r. The hoop rotates at a constant frequency f about a vertical diameter (see figure).

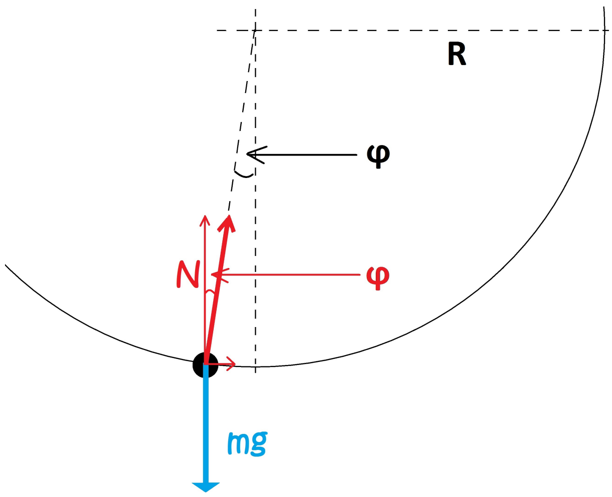
a) Find the angle β at which the bead is in vertical equilibrium. (Of course, it has a radial acceleration toward the axis.)

b) Is there some condition on the frequency f for the bead to move from the bottom of the loop at all?



First I'd like to go over the question why the bead is at an angle to the vertical at all, i.e. why is it not just sitting calmly at the bottom of the hoop. Well, we know for sure that if the hoop is at rest and then at some moment $t = 0$ it starts to rotate, then at this moment the bead is at the very bottom of the hoop and no horizontal force is acting on it. But will it stay this way? We know that nothing is perfect in this world and for some reason or other the bead can swerve a bit off its bottom-of-the-hoop-position (imagine that the machine that turns the hoop trembles a little or something of this sort - any minuscule action that could result in the bead moving a bit up the arc of the hoop is legit here). Let us find out, what happens next?

Let's start from picturing all the forces that will act on the bead in this new position, described by an infinitesimally small angle φ (which is not to scale on the picture, but drawing infinitesimally small things is not exactly possible). The normal force is directed towards the center of the hoop just because in general a line perpendicular to a fragment of a circle points towards the center.



We want to note that because of the rotating motion of the whole hoop the bead is now going around an infinitesimally small circle of radius $R\sin\varphi$, where R is the radius of the hoop. The hoop is rotating with a frequency f, which makes the bead's angular velocity $\omega = 2\pi f$, and the linear velocity $v = \omega R \sin\varphi = 2\pi f R \sin\varphi$. Thus there has to be a radial acceleration of magnitude

$$a_{\mathrm{rad}} = \frac{v^2}{R\sin\varphi} = 4\pi^2 f^2 R \sin\varphi \ . \tag{63}$$

Furthermore, we can also see that the horizontal component of the normal force is

$$N \sin\varphi \ , \tag{64}$$

and as clearly this must be the centripetal force (we cannot identify any other force around that would claim this role), we have

$$N\sin\varphi = m a_{\mathrm{rad}} = m 4\pi^2 f^2 R \sin\varphi \qquad \Rightarrow \qquad N = 4m\pi^2 f^2 R \ . \tag{65}$$

Now if we want this to be an equilibrium position, we need the sum of the vertical forces to be equal zero, that is:

$$N\cos\varphi = 4m\pi^2 f^2 R\cos\varphi = mg \qquad \Rightarrow \qquad 4\pi^2 f^2 R \cos\varphi = g \ . \tag{66}$$

Clearly, this is a condition on φ, additionally depending on the constant values of f and R. We know that the function cosine is such that it has the maximum value for $\varphi = 0$ and then decreases (until it reaches π, where it starts to increase again - in this example φ won't ever reach this limit, so we can as well take cosine to be just decreasing for increasing φ). If then

$$4\pi^2 f^2 R\cos\varphi < g \ , \tag{67}$$

for our infinitesimally small initial angle φ, then the normal force is too small to keep the bead in its off-the-bottom position and thus the bead will always remain in the equilibrium at the bottom, only deviating from it a bit every now and then due to some trembling etc. If, however, we have

$$4\pi^2 f^2 R\cos\varphi > g \ , \tag{68}$$

then we have a net upward force and consequently the bead will go up the hoop *until* φ is large enough for

$$4\pi^2 f^2 R \cos\varphi = g \tag{69}$$

to be satisfied, i.e. until the sum of forces in the vertical direction becomes zero.

Now let us see that in the limit $\varphi \to 0$ we have $\cos\varphi \to 1$. Thus the condition for the bead to go up the hoop for amount described by *any* angle $\varphi > 0$ can be written as

$$4\pi^2 f^2 R > g \qquad \Rightarrow \qquad f^2 R > \frac{g}{4\pi^2} \ . \tag{70}$$

The quantity on the right has a constant value and so this condition specifies a relation between f and R that has to be fulfilled in order for this phenomenon to occur.

Thus in result we can see that if (70) is satisfied, even the smallest deviation from the equilibrium (which in the real world will always have place for that reason or another) makes the bead move up the arc of the hoop until it reaches a position such that it finds an equilibrium given by (69).

Which leads us to the point a) of the problem:

a) Clearly, it must be that

$$g = 4\pi^2 f^2 R\cos\beta \qquad \Rightarrow \qquad \beta = \cos^{-1}\left(\frac{g}{4\pi^2 f^2 R}\right) \ . \tag{71}$$

(See derivation leading to (66).)

b) $\beta = \frac{\pi}{2}$ requires the argument of $\cos^{-1}$ to be zero. Given that g and R are constant, we would need f to be infinite. There are very large frequencies/velocities in the world, but there are *no* infinite frequencies/velocities and so this means that the bead can somehow approach $\beta = \frac{\pi}{2}$, but it can never reach it!

c) We have

$$\beta = \cos^{-1}\left(\frac{g}{4\pi^2 f^2 R}\right) \,. \tag{72}$$

Obviously, the quantity in the bracket should belong to the range of the cosine function, that is explicitly we should have

$$0 \leq \frac{g}{4\pi^2 f^2 R} \leq 1 \tag{73}$$

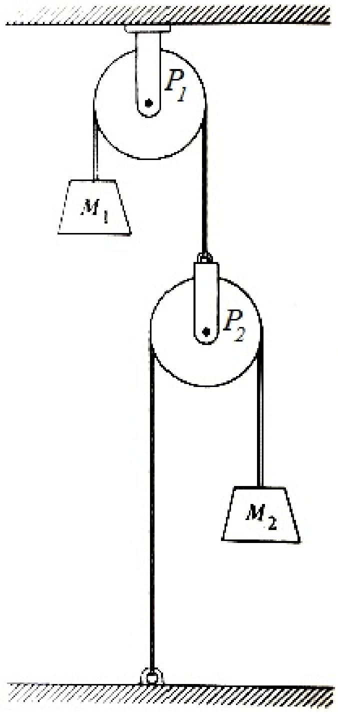
for an angle ranging from 0 to $\frac{\pi}{2}$. This means that we have the following condition on f:

$$f \geq \frac{1}{2\pi}\sqrt{\frac{g}{R}} \,. \tag{74}$$

For frequencies smaller than this the equilibrium at the bottom of the hoop will become stable, and thus the bead will never leave that position. (If you're not convinced, please consider (70) once more.)

Problem 13

Masses M_1 and M_2 are connected to a system of strings and pulleys as shown. The strings are massless and of constant length, and the pulleys are massless and frictionless.



a) What do you expect the acceleration of mass M_1 to be in the following limits? Explain your answers.

i) $M_2 \to 0$,

ii) $M_2 \to \infty$.

b) Determine the acceleration of mass M_1 and check that your answer agrees with your expectations from part a).

c) Suppose that the whole apparatus of this problem is in an elevator accelerating relative to the ground with acceleration A.

i) Determine the acceleration of mass M_1 relative to the ground,

ii) Determine the acceleration of mass M_1 relative to the elevator,

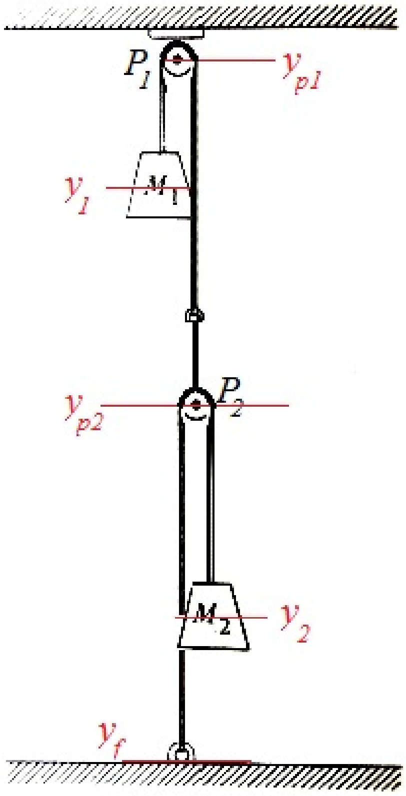
iii) Does your expression for the acceleration of mass M_1 relative to the elevator make sense in the limit $A \to 0$ and in the limit $A \to -g$?

Usually in problem solutions I assume that it is relatively easy to realize what the relative signs of accelerations and velocities are and thus I use pretty casual language when describing them. In this problem the situation is a bit more complicated than usual, and therefore I will pay more attention to be extremely explicit about the signs. For example, in part a), point i) I make it specific that the acceleration is $-g$ instead of just saying it's "g downwards".

a) i) In the limit $M_2 \to 0$, mass M_1 completely governs the motion - hence the acceleration of M_1 is $-g$.

ii) In the limit $M_2 \to \infty$ it's M_2 that governs the motion - clearly M_2 moves with acceleration $-g$. What is then the acceleration of M_1? Note: if mass M_2 moves down a certain amount, then the pulley from which it is hanging moves down half as much, meaning that the acceleration of this pulley is $-\frac{1}{2}g$. Consequently, the acceleration of mass M_1 is $+\frac{1}{2}g$.

b) Let us annotate on the picture the following variables: the (vertical) position of mass M_1, y_1, the position of mass M_2, y_2, the position of pulley P_1, y_{p1}, the position of pulley P_2, y_{p2}, and the floor level y_f:



Now this is what we do with these (it's a really nice trick for solving problems like these, so pay attention!): we note that the ropes that are connecting the objects of our system do not stretch, i.e., have constant lengths. Let denote the length of the rope connecting mass M_1 with pulley P_2 as L_1, and let us denote the length of the other rope as L_2. Then we can see that these lengths can be written as

$$L_1 = (y_{p1} - y_1) + (y_{p_1} - y_{p_2}) = 2y_{p1} - y_1 - y_{p2} \ , \tag{75}$$
$$L_2 = (y_{p2} - y_f) + (y_{p2} - y_2) = 2y_{p2} - y_f - y_2 \ . \tag{76}$$

We can use the second equation to solve for y_{p2},

$$y_{p2} = \frac{1}{2}L_2 + \frac{1}{2}y_f + \frac{1}{2}y_2 \ , \tag{77}$$

and plug it into the first equation,

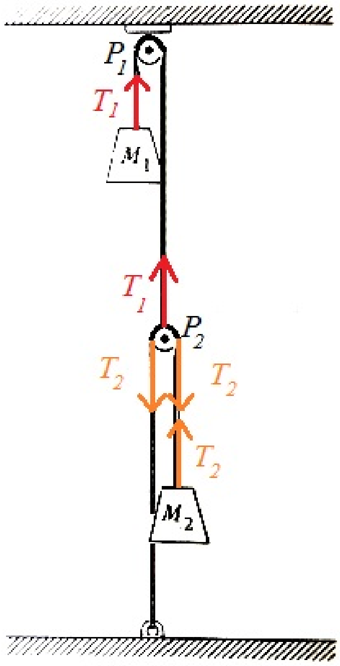
$$L_1 = 2y_{p1} - y_1 - \frac{1}{2}L_2 - \frac{1}{2}y_f - \frac{1}{2}y_2 \qquad \Rightarrow \qquad y_1 = 2y_{p1} - L_1 - \frac{1}{2}L_2 - \frac{1}{2}y_f - \frac{1}{2}y_2 \ . \tag{78}$$

This equation connects the positions of all the objects in our system. Now let us take a double derivative of both sides of this equation (obviously the constant terms drop out!):

$$\frac{d^2y_1}{dt^2} = -\frac{1}{2}\frac{d^2y_2}{dt^2} \qquad \Rightarrow \qquad a_1 = -\frac{1}{2}a_2 \ , \tag{79}$$

where in the last step we just acknowledged that second time derivative of position is acceleration.

This way we obtained a crucial, as you will soon see, relation between the accelerations of the two masses. Now let us try to find their values. We annotate the tensions at play on the figure below:



From this figure we can see that

$$T_1 - M_1 g = M_1 a_1 \ , \qquad T_2 - M_2 g = M_2 a_2 \ , \qquad T_1 = 2T_2 \ . \tag{80}$$

(Note that I haven't assigned any signs to a_1 and a_2 yet! We will know what they're signs are from the solution.) Using the last relation we can combine these equations into

$$2(M_2 g + M_2 a_2) - M_1 g = M_1 a_1 \ . \tag{81}$$

Finally, we use (79) to obtain

$$2M_2 g + 2M_2 a_2 - M_1 g = -\frac{1}{2} M_1 a_2 \qquad \Rightarrow \qquad a_2 = -g \frac{4M_2 - 2M_1}{4M_2 + M_1} \ , \tag{82}$$

$$\Rightarrow \qquad a_1 = g \frac{2M_2 - M_1}{4M_2 + M_1} \ . \tag{83}$$

Note that

$$\lim_{M_2 \to 0} a_1 = -g \qquad \text{and} \qquad \lim_{M_2 \to \infty} a_1 = \frac{1}{2} g \ . \tag{84}$$

c) We may relatively easily solve this part by using our approach from part b). That is, we express the lengths of the strings in terms of the positions of all the objects in our system. Here is what changes: now the positions of pulley P_1 and floor are also changing, specifically their acceleration is A. Then differentiating (78) leads to

$$\frac{d^2 y_1}{dt^2} = 2\frac{d^2 y_{p1}}{dt^2} - \frac{1}{2}\frac{d^2 y_f}{dt^2} - \frac{1}{2}\frac{d^2 y_2}{dt^2} \qquad \Rightarrow \qquad 2a_1 = 3A - a_2 \qquad \Rightarrow \qquad a_2 = 3A - 2a_1 \ . \tag{85}$$

We now plug that into (81) to obtain

$$2M_2 g + 2M_2\big(3A - 2a_1\big) - M_1 g = M_1 a_1 \qquad \Rightarrow \qquad g\big(2M_2 - M_1\big) + 6M_2 A = a_1\big(M_1 + 4M_2\big) \tag{86}$$

$$\Rightarrow \qquad a_1 = g\frac{2M_2 - M_1}{4M_2 + M_1} + A\frac{6M_2}{4M_2 + M_1} \ , \tag{87}$$

which is the answer to i).

ii) The acceleration of M_1 with respect to ground, a_1, is the acceleration of M_1 with respect to the elevator $a_{1,e}$ plus the acceleration of the elevator A,

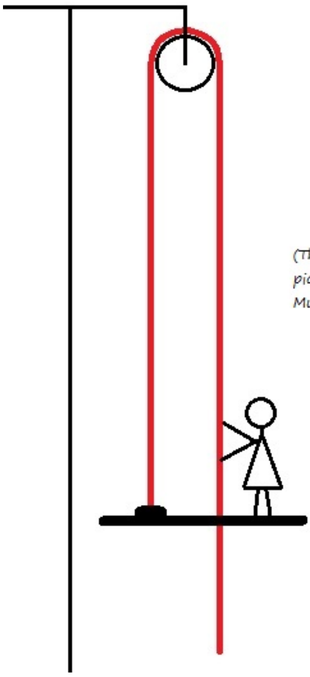
$$a_1 = a_{1,e} + A \qquad \Rightarrow \qquad a_{1,e} = a_1 - A = (g + A)\frac{2M_2 - M_1}{4M_2 + M_1} \ . \tag{88}$$

iii) In the limit $A \to 0$ we recover our expression from part b), which is what we should expect. In the limit $A \to -g$ the acceleration a_1, e becomes zero. This also makes sense, as that means that the whole elevator along with all its components is in free fall! They all fall in the same way and therefore $a_{1,e} = 0$ in that case.

Problem 14

This is an original Corbin midterm problem.

Eager to escape before she is detected, a jewel thief (mass M_1) lowers herself from the roof of the museum in the maintenance lift (mass M_2). The lift is operated manually, by pulling on a rope that is draped over a massless pulley, as shown. For the following questions, assume she is accelerating downward at a rate $|\vec{a}| = a$.
a) How much force is the jewel thief applying to the rope?
b) Strangely enough, someone has left a bathroom scale sitting on the bottom of the lift and the thief is standing on it. How much does it say she weighs?
c) How does the mass of the lift compare to the mass of the thief? How do we know that? Using your answers in part a) and b), explain what would happen if things were the other way around.
d) What is the fastest rate at which the jewel thief can accelerate downwards? Explain.



Let us first take a while to wonder how this thing works! The system is the following: we have a thief on a lift. The thief is holding the rope and pushing downwards. Through tension, it's like the thief is pulling the lift upwards. The whole system is moving downwards though. This seems counterintuitive at first sight, but think about it the following way: Just before the thief starts lowering herself, the lift stands on the edge of the roof. Then the lift leaves this edge, the thief is pushing downwards, and the lift is also going downwards with some acceleration a. *If the thief was not pushing down, the lift would fall with the acceleration of the free fall, g.* Which would contradict the sole purpose of the lift. So the thief is exerting some force on the rope, directed downwards, and through tension there is some upwards force on the lift *so that it doesn't just fall down.*

Now we're ready to solve this!

a) If we drew a free body diagram, we would have

$$(M_1 + M_2)g - 2T = (M_1 + M_2)a \; . \tag{89}$$

We also know that it must be that

$$T = F \; . \tag{90}$$

Then easily

$$F = T = \frac{(a-g)(M_1+M_2)}{2} \ . \tag{91}$$

b) For the thief alone, we clearly have

$$-N - T + M_1 g = M_1 a \ , \tag{92}$$

so that

$$\begin{aligned} N &= (g-a)M_1 + T = (g-a)M_1 + \frac{(a-g)(M_1+M_2)}{2} = (g-a)M_1 - \frac{(g-a)(M_1+M_2)}{2} = \\ &= \frac{1}{2}(g-a)(M_1-M_2) \ . \end{aligned} \tag{93}$$

c) We know that the normal force should only push and not pull (that would be weird). For it to pe pushing we need $N > 0$ and then it must be that $M_1 > M_2$.

d) We can again use the same equation to get that it must be that

$$g - a \geq 0 \qquad \Rightarrow \qquad a \leq g \ . \tag{94}$$

Clearly, the thief just cannot accelerate faster than the free fall.

Also, we know that from the intuition: for her to be accelerating faster than g something should accelerate her in that direction!

Problem 15

A small pebble is lodged in the tread of a tire of radius R. If this tire is rolling at speed V without slipping on a horizontal road, find the equations for the x and y coordinates of the pebble as a function of time. Let the pebble touch the road at $t = 0$. Find also the velocity and acceleration components as a function of the time.

Let us put our tire in a frame of reference:

We know from problem 1a) that the position vector of an object traveling counterclockwise on a circular orbit centered at $x = 0$, $y = 0$ is

$$\vec{r} = R\cos\theta\ \hat{\boldsymbol{x}} + R\sin\theta\ \hat{\boldsymbol{y}}\ . \tag{95}$$

In our case, our pebble traverses around a circle in a *clockwise* manner. Moreover, at $t = 0$ the pebble touches the ground, that is the initial phase of the motion is different than zero. Finally, the center of this circle is at $t = 0$ at $x = 0$, $y = R$, and travels to the right at velocity V. Let us apply these changes to (95).

We remember that the angular velocity is connected to the linear velocity by

$$\omega = \frac{V}{R}\ , \tag{96}$$

and the angle describing the position of the object as a function of time is

$$\theta = \omega t = \frac{Vt}{R}\ . \tag{97}$$

Now, for the *clockwise* motion the angle is negative, i.e.,

$$\vec{r}\ =\ R\cos(-\theta)\ \hat{\boldsymbol{x}} + R\sin(-\theta)\ \hat{\boldsymbol{y}} = R\cos\theta\ \hat{\boldsymbol{x}} - R\sin\theta\ \hat{\boldsymbol{y}}\ . \tag{98}$$

Our motion, by the condition that at $t = 0$ the pebble touches the ground, is also shifted by a phase of $\frac{\pi}{2}$, so that altogether we have

$$\vec{r} = R\cos\left(\frac{Vt}{R} + \frac{\pi}{2}\right)\ \hat{\boldsymbol{x}} - R\sin\left(\frac{Vt}{R} + \frac{\pi}{2}\right)\ \hat{\boldsymbol{y}} = -R\sin\left(\frac{Vt}{R}\right)\ \hat{\boldsymbol{x}} - R\cos\left(\frac{Vt}{R}\right)\ \hat{\boldsymbol{y}}\ . \tag{99}$$

Finally, we adjust for the offset of the center of the circle by adding $R\ \hat{\boldsymbol{y}}$,

$$\vec{r} = -R\sin\left(\frac{Vt}{R}\right)\ \hat{\boldsymbol{x}} + R\left[1 - \cos\left(\frac{Vt}{R}\right)\right]\ \hat{\boldsymbol{y}}\ , \tag{100}$$

and then adjust for the motion of the center of the circle in the horizontal direction, $x(t) = Vt$,

$$\vec{r} = \left[Vt - R\sin\left(\frac{Vt}{R}\right)\right]\ \hat{\boldsymbol{x}} + R\left[1 - \cos\left(\frac{Vt}{R}\right)\right]\ \hat{\boldsymbol{y}}\ . \tag{101}$$

From this we read off that

$$x(t) = Vt - R\sin\left(\frac{Vt}{R}\right) \qquad \text{and} \qquad y(t) = R\left[1 - \cos\left(\frac{Vt}{R}\right)\right] \ . \tag{102}$$

Now it's easy to calculate the components of velocity,

$$v_x(t) = \frac{dx}{dt} = V - R\omega\cos\frac{Vt}{R} \qquad \text{and} \qquad v_y(t) = \frac{dy}{dt} = R\omega\sin\frac{Vt}{R} \ , \tag{103}$$

and similarly the acceleration components are easily found to be

$$a_x(t) = \frac{dv_x}{dt} = R\omega^2\sin\frac{Vt}{R} \qquad \text{and} \qquad a_y(t) = \frac{dv_y}{dt} = R\omega^2\cos\frac{Vt}{R} \ . \tag{104}$$