Physics 22: Homework 6

The following exercises encompass problems dealing with resistance, current, and the use of circuit rules (e.g., Kirchhoff's Rules) to analyze and solve circuits. (In this assignment, the circuits to be solved involve resistors, voltmeters, ammeters, ideal and non-ideal batteries, as well as circuits that combine resistors with capacitors (i.e., RC circuits).)

1. Consider the following questions, which are to be explained qualitatively.

   (a) Although the electric field inside a conductor in electrostatic equilibrium is zero, why is this not the case inside a conductor that is carrying a current?

   (b) When a light switch is turned on, the light comes on right away by, for example, having electrons pass through the resistive filament causing the filament to glow. However, the electrons in such conductors typically move with a drift speed that is commonly in the range of 10^{-4} m/s = 0.1 mm/s. Explain this apparent disparity.

   (c) A constant electric field inside conducting wires produces a constant average velocity of the charge carriers which we call the drift velocity. How is this consistent with Newton's 2^{nd} Law of Motion?

2. Consider a marching band consisting of a uniform formation of identical band members (each of mass m) separated from nearest neighbors in some square lattice arrangement with some separation ℓ and moving at a uniform velocity, $\vec{v}$. As viewed from a helicopter, the band consists of a very large number of columns of band members (with a column, for example, being denoted as a line of band members perpendicular to the band's movement), but consists of η rows (with a row, for example, being denoted as a line of band members parallel to the band's movement). Consider a reference line, perpendicular to the band's velocity, through which the band members will cross.

   (a) Considering an arbitrary time over which the counting of band members is administered as they cross the reference line, obtain a formula for the rate at which band members cross this reference line.

   (b) Again, considering an arbitrary time over which the counting of band members is administered as they cross the reference line, obtain a formula for the rate of mass flow across this reference line.

   (c) For the value of $m = 60$ kg, $\ell = 4$ m, $v = 2$ m/s, and $\eta = 5$, verify the results of Parts (a) and (b) by brute-force counting.

   (d) How can this exercise be likened to the flow of charge in a conducting wire? Discuss.

   (e) Why can the concept of current be thought as a concept of flux, and of what is the current the flux?

3. Copper is the most common metal used in conducting wires (e.g., to run the electricity in residential and commercial buildings). It second only to silver in its conductivity, it is quite malleable, it is relatively cheap, and, if kept away from moisture, it would tend not to rust. The concentration of conduction electrons in copper is 1.1×10^{29} m^{-3}.

   (a) Show that the value for this concentration is quite consistent with the facts that copper has a mass density of about 8.9 g/cm^3 and that its atomic mass is 63.55 g/mol.

   (b) How many conduction electrons does each atom of copper contain?

4. As in Figure 1, consider two cylindrical conductors—A and B—connected end-to-end, thus effectively forming one cylindrical conductor. The other ends are connected to a battery, of voltage V. The conductors have the same cross-section, but A is twice as long and has three times the resistivity of B.

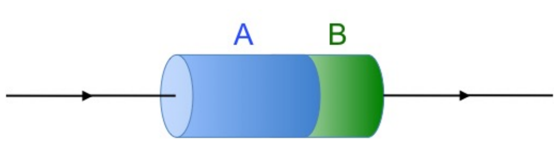

Figure 1: Two cylindrical conductors, A and B, fused end-to-end, with a current running through the cylinders supplied by a battery, of voltage V, to which the cylinders are connected on either of the non-fused ends.

   (a) Determine the following.

      i. The ratio of the current running through A to the current running through B: $I_{\text{A}}/I_{\text{B}}$.

ii. The ratio of the resistance of A to the resistance of B: R_A/R_B.

iii. The ratio of the potential difference across A to the potential difference across B: $\Delta V_A/\Delta V_B$.

iv. The ratio of the electric field in A to the electric field in B: E_A/E_B.

v. The ratio of the power dissipated through A to the power dissipated through B: P_A/P_B.

(b) Determine ΔV_A and ΔV_B in terms of the battery voltage, V.

(c) Determine the potential (relative to ground) at the junction between the two conductors.

(d) Use Gauss's Law to determine the net charge density that accumulates at the boundary between the two conductors.

(e) From the result in Part (d), how does the charge accumulation at the junction between the conductors explain the difference in electric field inside each conductor?

5. Electric devices are rated for the power they consume, assuming that they will be plugged into a standard American outlet delivering a Root-Mean-Square (or RMS) voltage of about 115 V. (The RMS value arises because the voltage being delivered to these outlets is actually alternating.)

   (a) How much current flows through a typical 60-W incandescent bulb when it is on?

   (b) If we assume that the cord delivering the current to the bulb is about 3 m long, with a 1-mm diameter, and made of copper ($\rho = 1.72 \times 10^{-8}\ \Omega \cdot \text{m}$), determine the power being consumed by the cord.

   (c) Based on the result to Part (b), is it a good approximation to ignore the connecting wires in a circuit, or is it quite relevant to consider the resistive effects of such wires? Explain.

   (d) Compare a 40-W bulb to a 60-W one. Which has more resistance?

   (e) If one was to connect a 40-W bulb in series with a 60-W bulb, and then made the free ends of those bulbs connect to the opposite terminals of an American wall outlet, which one would be brighter? Explain your reasoning.

6. Consider the following questions, which are to be explained qualitatively.

   (a) Consider decorative lights that one may put in the front yard of one's house to celebrate some holiday. Would it be wise to use wiring such that the lights are arranged in parallel strands or is it better to have all of the lights arranged in series? Explain.

   (b) Conceptually, discuss why resistors in series add while resistors in parallel add inversely.

   (c) From what conservation laws do Kirchhoff's Rules derive and why?

7. Four identical light bulbs—A, B, C, and D—are connected to an ideal battery, as shown in Figure 2.

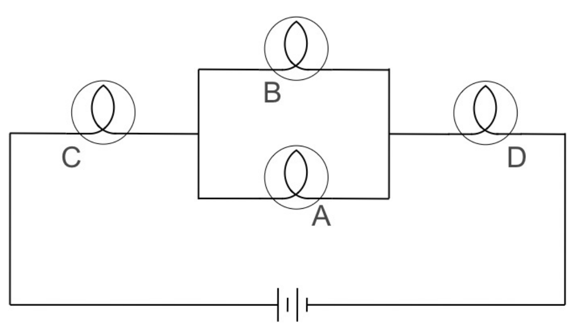

Figure 2: An arrangement of identical (incandescent) light bulbs placed in between the terminals of an ideal battery.

   (a) First of all, what is meant by an "ideal" battery?

   (b) Determine the relative brightness of each bulb.

   (c) Suppose A is unscrewed from its socket.

      i. How would the brightness of the remaining bulbs compare with their brightness when A was not unscrewed?

      ii. What would be the relative brightness of these remaining bulbs?

8. The circuits in Figure 3 consist of combinations of resistors, some of which also contain one battery, or multiple batteries. The circuits for which no battery is shown contain electrodes at points a and b, across which a battery may be connected. All resistors are the same, and so are all the batteries; the subscripts are merely shown to be able to distinguish one element from the other.

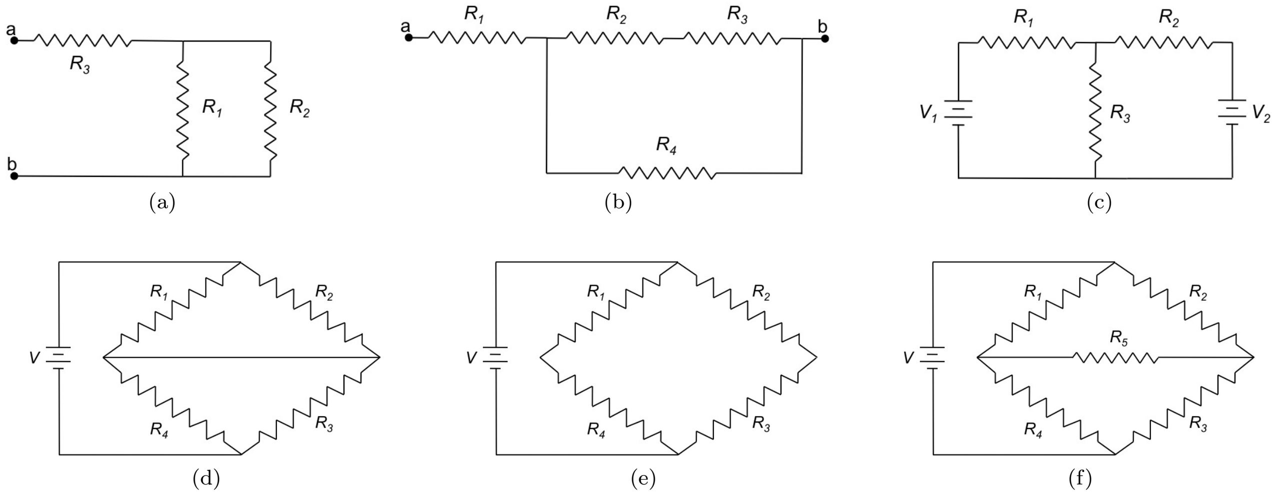

Figure 3: Resistor circuits with varying combinations. All resistors are identical ($R_j = R$) and all batteries are identical ($V_j = V$).

(a) For Figures 3a - 3f, determine (i) which resistors are in series, (ii) which resistors are in parallel, (iii) the equivalent resistance of the circuit, and (iv) the most powerful resistor, if any.

(b) Reconsider the answers to Part (a) specifically for Figures 3d, 3e, and 3f if *one* of the resistors (R_4 for instance) was different (i.e., $R_j = R$ for $j \neq 4$, and $R_4 = R'$).

9. A non-ideal battery, of voltage V and internal resistance r, is connected to a potentiometer, which has a variable resistance, R.

(a) Derive an expression for the power dissipated by each resistor in terms of the given quantities.

(b) Determine the value of the potentiometer, R_{max}, that maximizes the power output to the external resistor.

(c) When $R = R_{\text{max}}$, how much power is being dissipated by the battery? Explain why it would not be a practical endeavor to put a battery through such a situation. (Even though this is not a good predicament in which to put a battery, such a technique is often used to match loads in many electronic devices (e.g., amplifiers and speakers).)

(d) Discuss how the power generated by the battery's voltage output is divided between its internal resistance and the external resistance for the cases where $R > r$ and also for $R < r$.

10. The circuit shown in Figure 4 is designed in such a way so that it makes absolutely no difference if the switch is closed or open in terms of, for example, the current running through each of the resistors. In this circuit, $V_1 = V$, $V_2 = 5V$, $R_1 = R$, and $R_2 = 2R$.

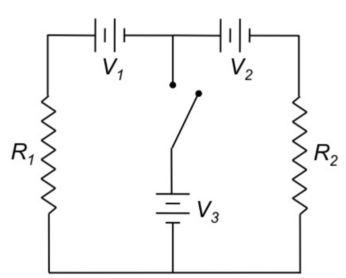

Figure 4: A resistor circuit with three separate branches, with each branch containing a battery. The voltage of battery V_3 is unknown, but it is known that leaving the switch open or closed has no bearing on the current running through the branches with resistors.

(a) What must be the voltage across V_3 in order to satisfy the conditions described in this circuit?

(b) Is there a potential difference across the switch when it is open? If there is, then what is it?

11. Consider the circuit in Figure 5, which consists of three batteries—$V_1 = 6.0$ V, $V_2 = 1.5$ V, and $V_3 = 4.5$ V—and four resistors—$R_1 = 270\ \Omega$, $R_2 = 560\ \Omega$, $R_3 = 820\ \Omega$, and $R_4 = 150\ \Omega$.

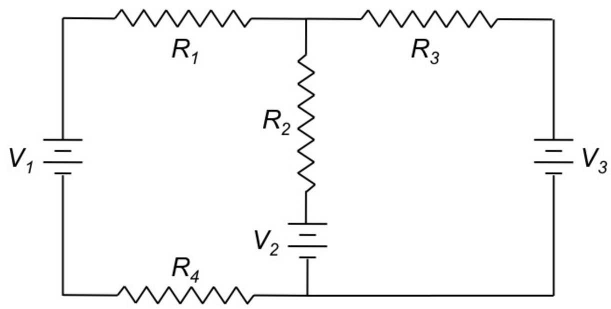

Figure 5: A resistor circuit with three separate branches, with each branch containing a battery.

(a) Find the current in R_2 and give its direction.

(b) Suppose that V_2 is replaced with another battery that makes the current in R_2 run upwards. Determine the minimum value of the replacement battery that would satisfy this requirement.

12. Devices that are used to measure voltage (voltmeters) and current (ammeters) have internal resistances themselves, as these devices happen to be made of resistive wires (sometimes on purpose) and they may also have some impedance mismatch when being connected to some arbitrary circuit that has not been well designed. To illustrate the importance of these internal resistances, consider the circuit in Figure 6, which consists of a battery, of voltage $V = 9$ V, and three resistors, of resistances $R_1 = 12$ kΩ and $R_2 = R_3 = 6$ kΩ. The voltage across R_1 is measured using three different voltmeters: V_1 with internal resistance $r_1 = 24$ kΩ, V_2 with internal resistance $r_2 = 120$ kΩ, and V_3 with internal resistance $r_3 = 10$ MΩ.

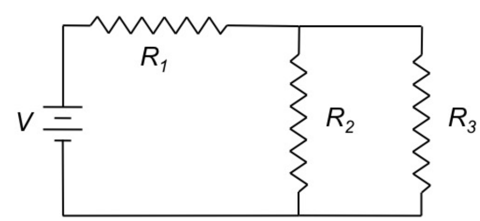

Figure 6: A resistor circuit with a battery powering the resistors. The resistor R_1 is to be attached to imperfect voltmeters and, mistakenly, to an ammeter.

(a) Determine the reading across each voltmeter when placed across R_1.

(b) Determine the percent errors in the voltmeter readings as compared to what the voltage should ideally read across R_1.

(c) Suppose a student places an ammeter, with internal resistance $r = 60\ \Omega$ across R_1 by mistake. What would the ammeter read and why would this be risky?

13. Figure 7 consists of a combination of resistors with a single battery and a bridge, with one of the circuits (Figure 7a) making use of an ideal ammeter as the bridge, another (Figure 7b) making use of an ideal voltmeter as the bridge, and yet another (Figure 7c) making use of a resistor as the bridge. In these circuits, $R = 3\ \Omega$ and $V = 12$ V.

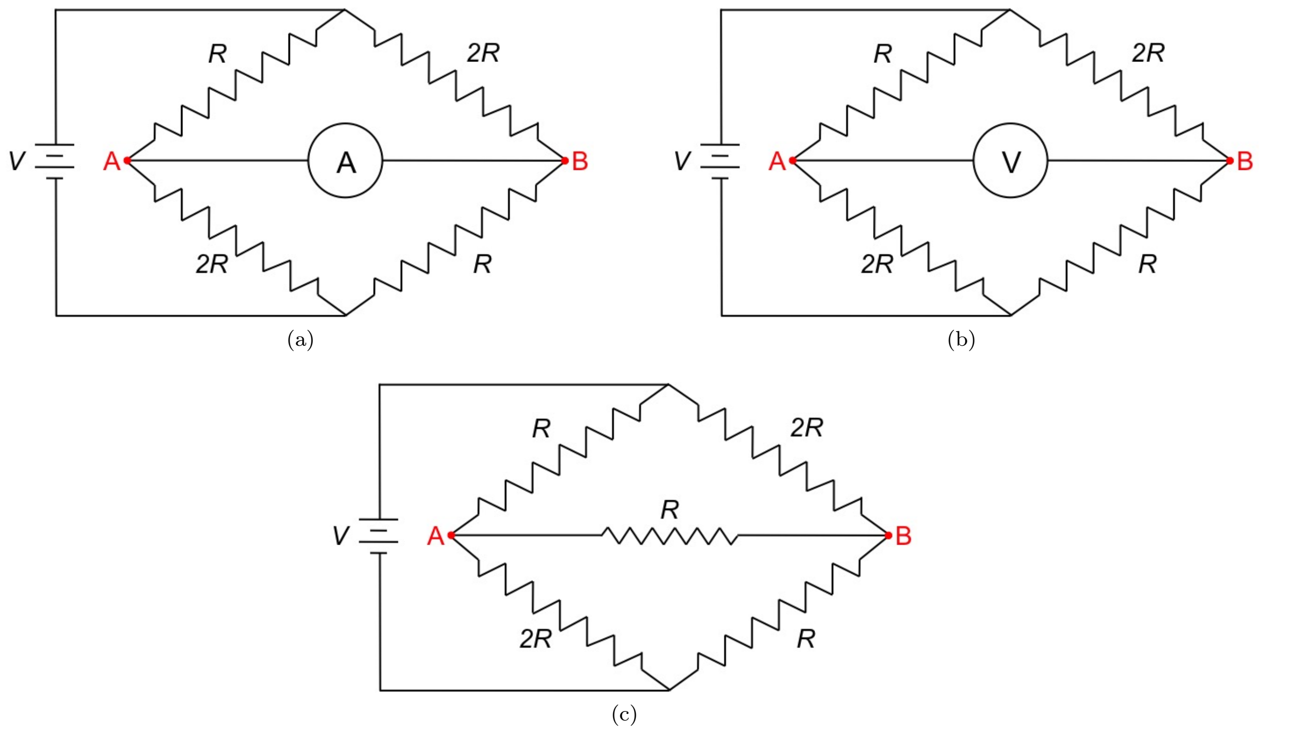

Figure 7: Identical resistor circuits with varying bridges.

(a) Determine the potential difference between points A and B.

(b) Determine which point (A or B) is at the higher potential.

(c) Determine how the results to Parts (a) and (b) would change by switching one of the $2R$ resistors into an R.

14. A galvanometer is a very sensitive ammeter used to measure very small amounts of current. A typical galvanometer has a resistance of 75 Ω and requires a current of 1.5 mA to deflect the needle of the galvanometer to its full scale. (A current higher than this value would damage the device.)

    (a) It is possible to use a galvanometer to measure larger currents by placing a relatively small resistance (called a "shunt") in parallel with it. The shunt diverts the bulk of the current to protect the galvanometer. What shunt resistance is necessary for the galvanometer to be able to measure 1 A of current?

    (b) It is also possible to use a galvanometer as a voltmeter by placing a relatively large resistance in series with it. The large resistance takes up the bulk of the voltage, keeping the current small and protecting the galvanometer. What resistance is necessary for the galvanometer to be able to measure a potential difference of 10 V?

15. Consider a resistor, of resistance R, attached in series to an uncharged capacitor, of capacitance C. This series combination is connected to an open switch, which is, in turn, attached to a battery, of voltage V. At $t = 0$, the switch is closed.

    (a) Show that the capacitor takes about 5 time constants to reach 99% of its final charge.

    (b) Consider the voltage of the capacitor as a function of the time, $t \geq 0$.

        i. Obtain an expression for the rate at which the capacitor voltage increases for $t \geq 0$, and evaluate the value of this rate at $t = 0$.

        ii. If the capacitor continued charging at the rate determined by the value of Part (bi) at $t = 0$, show that the capacitor would be fully charged in exactly one time constant.

    (c) Show that the total energy drawn from the battery in fully charging the capacitor is twice the energy stored in the capacitor when it is fully charged.

    (d) Show that half the energy drawn from the battery is dissipated as heat in the resistance of the circuit.

16. Capacitor $C_2 = 2.0\ \mu$F is initially charged to a voltage of $V = 150$ V. As shown in Figure 8, this capacitor is then connected through a switch to an uncharged capacitor, $C_1 = 1.0\ \mu$F, through a resistor, $R = 4.0$ kΩ. The switch is closed at $t = 0$.

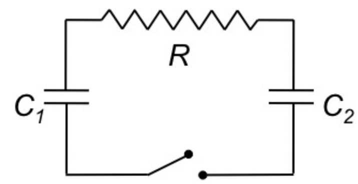

Figure 8: A charged capacitor, C_2, is to charge an uncharged capacitor, C_1, through a resistor, R.

(a) Determine the final voltage across each of the capacitors when the circuit comes to equilibrium.

(b) Determine the total energy dissipated through the resistor once the circuit comes to equilibrium.

17. A parallel-plate capacitor is insulated with a material of dielectric constant κ and resistivity ρ, as shown in Figure 9.

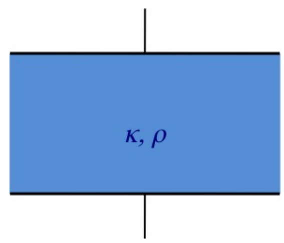

Figure 9: A parallel-plate capacitor with a dielectric that has a finite conductivity, making the capacitor discharge.

(a) Why can this system be modeled as a discharging RC series circuit?

(b) Assuming the parallel-plate capacitor is large enough to ignore edge effects (as is customarily the case), determine the time constant of this RC circuit.

(c) Assuming the material is polystyrene—with $\kappa = 2.6$ and $\rho = 10^{16}\ \Omega \cdot$ m, determine the time it takes for the capacitor's stored energy to reach half of its initial value.

18. The circuit in Figure 10 is a rather complicated RC circuit, consisting of a battery of voltage V, resistors of resistances $R_1 = R_2 = R_3 = R$, and uncharged capacitors of capacitances $C_1 = C_2 = C$. The switch in the circuit is initially open and prevents the capacitors from charging. The switch is closed at $t = 0$. (Do not attempt to solve this circuit quantitatively.)

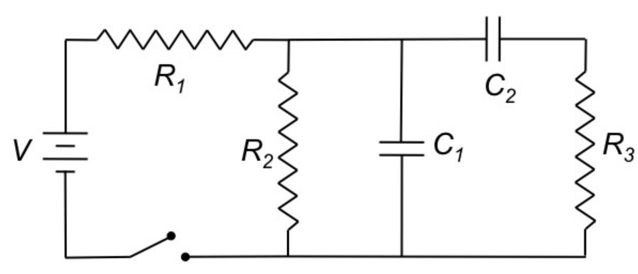

Figure 10: An RC circuit which will lead to the capacitors C_1 and C_2 to charge over time.

(a) Find an expression for the current in R_2

i. immediately after the switch is closed (i.e., at $t \gtrsim 0$).

ii. a very long time after the switch is closed (i.e., as $t \to \infty$).

(b) Discuss qualitatively how the current in R_3 changes over time after the switch is closed.

19. Consider the RC circuit shown in Figure 11 with an initially open switch. The capacitor, of capacitance C, is initially uncharged. The resistors have resistances r and R and the battery has voltage V. The switch is closed at $t = 0$.

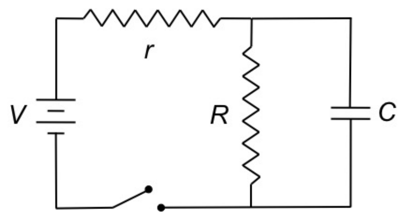

Figure 11: A parallel combination of a resistor, R, and capacitor, C, which is in series with a battery of voltage V and internal resistance r.

(a) Qualitatively describe the short-term and long-term behavior of the current in each branch.

(b) Based on your qualitative analysis in Part (a), sketch graphs of the current in each branch as a function of the time $t \geq 0$.

(c) Applying Kirchhoff's Loop and Junction Rules, show that the time constant of the circuit is given by

$$\tau = \left(\frac{rR}{r+R}\right) C.$$

(d) Attempt to solve the circuit to show that the charge on the capacitor for $t \geq 0$ is given by

$$Q\left(t\right) = \frac{CV}{1 + \left(r/R\right)} \left(1 - e^{-t/\tau}\right),$$

with τ given in Part (c).

(e) Show that the result in Part (d) condenses to the result expected when $R \to \infty$.

(f) Show that the capacitor charges instantaneously after the switch is closed when $r \to 0$.