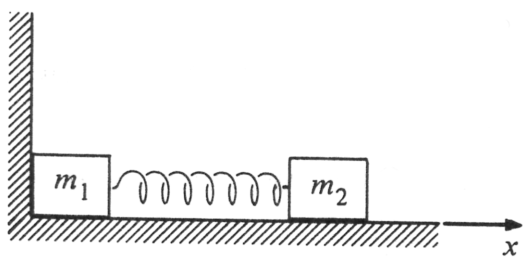
Physics 1A
Spring 2015

Challenge Problem 15

A system is composed of two blocks of mass m_1 and m_2 connected by a massless spring with spring constant k. The blocks slide on a frictionless plane. The unstretched length of the spring is ℓ_0. Initially, mass m_2 is held so that the spring is compressed to $\ell_0/2$ and m_1 is forced against a wall as shown. The mass m_2 is released at time $t = 0$.

Find the position $x(t)$ of the center of mass of the system as a function of time for all $t > 0$.



Solution. We ignore the vertical (y) direction in this problem. The key insights in this problem are as follows:

> While mass m_1 is in contact with the wall, there is a single external force on the system, namely the normal force of the wall on mass m_1 to the right. This means that as long as m_1 is in contact with the wall, the center of mass of the system accelerates to the right. However, m_2 moves to the right and eventually passes through the equilibrium position of the spring. A moment later, the spring is slightly stretched and mass m_1 is pulled off of the wall. For all times after this, the net external force on the system is zero, so the center of mass moves at a constant velocity. What is the value of this velocity? It's simply the velocity that the center of mass had right at the moment when mass m_2 passes through the equilibrium position.

All right, now that we know what happens on a conceptual level, let's do the math. We divide the motion of the system into two parts. The first part is from time 0 to time t_1 which is defined as when mass m_2 passes through the equilibrium position. The second part is all times after t_1.

Let the origin of our coordinates be at the left hand edge of mass m_1. For times 0 to t_1, free body diagrams show that the normal force on mass m_1 is the same as the tension in the spring which is given by

$$F_s = -k(x_2 - \ell). \tag{1}$$

So the equation of motion for that mass is

$$m\ddot{x}_2 = -k(x_2 - \ell). \tag{2}$$

This is almost the equation for simple harmonic motion. In order to get it into the form of the simple harmonic motion equation, we make a change of variables. We define $x = x_2 - \ell$ from which it follows that $\ddot{x} = \ddot{x}_2$ and the equation of motion becomes

$$m\ddot{x} = -k\ddot{x}. \tag{3}$$

The solution to this equation is

$$x(t) = A\cos\left(\sqrt{\frac{k}{m_2}}t + \phi\right) \tag{4}$$

so that

$$x_2(t) = \ell + A\cos\left(\sqrt{\frac{k}{m_2}}t + \phi\right) \tag{5}$$

where A and ϕ are undetermined constants that need to be solved for in terms of the initial conditions on the position and velocity of mass m_2 at the initial time $t = 0$. Initial conditions are

$$x_2(0) = \ell/2, \qquad \dot{x}_2(0) = 0. \tag{6}$$

Namely, the position of mass m_2 at time 0 is $\ell/2$, and its velocity at time 0 is 0 since it starts at rest. In order to enforce the velocity condition, we need to compute $\dot{x}_2$. We find that,

$$\dot{x}_2(t) = -A\sqrt{\frac{k}{m_2}}\sin\left(\sqrt{\frac{k}{m_2}}t + \phi\right). \tag{7}$$

So enforcing the initial conditions gives

$$\frac{\ell}{2} = \ell + A\cos\phi, \qquad 0 = -A\sqrt{\frac{k}{m_2}}\sin\phi \tag{8}$$

There are many equivalent[1] solutions to these equations, the simplest is

$$\phi = 0, \qquad A = -\frac{\ell}{2}. \tag{9}$$

So we get the following for the position $x_2(t)$ of mass m_2 as a function of time before if passes through the equilibrium position of the spring:

$$x_2(t) = \ell - \frac{\ell}{2}\cos\left(\sqrt{\frac{k}{m_2}}t\right). \tag{10}$$

Therefore, the position $X(t)$ of the center of mass during this period of time is

$$X(t) = \frac{m_1 x_1(t) + m_2 x_2(t)}{m_1 + m_2} = \frac{m_1 \cdot 0 + m_2 x_2(t)}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} x_2(t) \tag{11}$$

so plugging in our solution $x_2(t)$ gives the motion of the center of mass before time t_1 when mass m_2 passes through equilibrium. Before we write than down explicitly, what exactly is that time? Well, it is defined by the condition $x_2(t_1) = \ell$ which gives

$$\ell = \ell - \frac{\ell}{2}\cos\left(\sqrt{\frac{k}{m_2}}t_1\right), \tag{12}$$

which gives

$$\sqrt{\frac{k}{m_2}}t_1 = \frac{\pi}{2} \implies t_1 = \frac{\pi}{2}\sqrt{\frac{m_2}{k}}. \tag{13}$$

so in summary the motion of the center of mass satisfies

$$\boxed{X(t) = \frac{m_2}{m_1 + m_2}\left(\ell - \frac{\ell}{2}\cos\left(\sqrt{\frac{k}{m_2}}t\right)\right), \qquad 0 \le t \le t_1} \tag{14}$$

[1]equivalent here means that they all lead to the same solution to the differential equation

What about for $t > t_1$? Well for that amount of time, the center of mass moves at constant velocity equal to what it's velocity as at time t_1. Let's explicitly solve for that velocity;

$$V_1 = \dot{X}(t_1) = \frac{m_2}{m_1 + m_2} \frac{\ell}{2} \sqrt{\frac{k}{m_2}} \sin\left(\frac{\pi}{2}\right) = \frac{m_2}{m_1 + m_2} \frac{\ell}{2} \sqrt{\frac{k}{m_2}} \tag{15}$$

so the motion for times $t > t_1$ is

$$X(t) = X(t_1) + V_1(t - t_1) = \frac{m_2}{m_1 + m_2}\ell + \frac{m_2}{m_1 + m_2} \frac{\ell}{2} \sqrt{\frac{k}{m_2}}(t - t_1) \tag{16}$$

or after a factorization

$$\boxed{X(t) = \frac{m_2}{m_1 + m_2} \left(\ell - \frac{\ell}{2} \sqrt{\frac{k}{m_2}}(t - t_1)\right), \qquad t > t_1}. \tag{17}$$