Physics 1A: Mechanics

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Discussion: Week 1

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(Alternatively, just e-mail me your questions.)

Let us begin with a few puzzles that are not strictly mechanics-related, but show the kind of thinking you will be taught during this course. Sometimes students are tempted to skip these riddles and move on to "real problems", thinking that after all there will be no riddles of this kind on the exam. While the premise is true, the conclusion is not: students who tend to exercise their brain by solving the riddles generally do better on the exams too. Remember: physics is all about riddles! What is the Earth made of? How did the Universe evolve? How to weigh an atom?... Or, let us start with something a little bit simpler:

Riddle 1

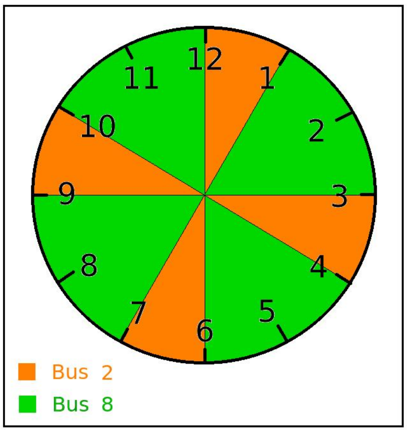
Every day after work, your TA takes a bus home. She always goes to the same bus stop where she can board either Bus 2 or Bus 8 to reach her destination. Each bus arrives every 15 minutes (let us assume that they're never late). Even though the TA leaves at different times and never checks for the arrival time of the next bus, on average she ends up riding Bus 8 twice more often then Bus 2. Can you figure out why?

We know that both buses arrive every 15 minutes - in more physical terms we would say that *the buses arrive with the same frequency.* If the TA arrives at the bus stop randomly and ends up riding Bus 8 more often, it means that somehow it is more probable for her to board Bus 8. How is it possible?

We were not given any information about the relation between the times of arrival of the buses (i.e., we were not given the bus schedule). *They do not have to arrive in equal time intervals*! From the description of the problem we see that the waiting period between arrival of Bus 2 and arrival of Bus 8 must be two times longer than the waiting period between arrival of Bus 8 and Bus 2, such that it is twice more probable to get to the bus stop after Bus 2 has left and before Bus 8 arrives than otherwise. An example schedule would be:

4 : 00	Bus 8	
4 : 05	Bus 2	
4 : 15	Bus 8	
4 : 20	Bus 2	
4 : 30	Bus 8	
4 : 35	Bus 2	
4 : 45	Bus 8	
4 : 50	Bus 2	
5 : 00	Bus 8	...

If, for example, the TA arrives at the bus stop randomly anywhere between 4:00 and 4:15, it is twice more probable for her to arrive in the time interval in which she will wait for bus 8! To make this even more clear, we can annotate the time intervals in which the TA will board Bus 2 and Bus 8 on an hour clock as on Figure 1. Make note that this simple clock picture is also an example of a *diagram*.



Riddle 2

A bucket contains 1 liter of pure water. Let's call it the "water bucket". A second bucket contains one liter of pure alcohol. Let's call it the "alcohol bucket". Note that we have chosen two liquids that may be mixed safely. 100 ml of alcohol is taken from the alcohol bucket and put into the water bucket, then thoroughly mixed. Then the same amount, 100 ml, of this mixture is removed from the water bucket and put into the alcohol bucket. Now both buckets contain again 1 liter of liquid. (We ignore the fact that a mixture of water and alcohol occupies a volume slightly less than the sum of the volumes of the two materials before they were mixed.) Repeat this process 100 times. (This is a thought experiment, so this won't take all that long.)

Now, both buckets contain a mixture of water and alcohol, and both buckets contain 1 liter of liquid. Is there now more alcohol in the water than water in the alcohol?

Hint: Although in principle one can solve this problem by means of detailed mathematics, you are supposed to solve this by means of careful insight. The problem is presented to be much harder than it really is!

Hint 2: Make a careful note of what the questions asks for.

Hint 3: Thinking about two fluids sloshing around may be deceitfully hard. For starters, you may instead consider two cases, each containing 10 dollars. In case 1, there are ten $1 bills. In case 2 there are 40 quarters. The rest of the problem follows similarly.

This is a classic example of a problem in which more information is given than is needed. It's also an example of using insightful approaches versus detailed mathematical approaches. Many people will get out their calculators and start calculating mixture ratios with the numbers given, however, it is not necessary - and much too cumbersome!

It doesn't matter how much is initially in the buckets; they need not have equal volumes. It doesn't matter how much liquid is transferred each time. It isn't necessary to mix the liquid after a transfer. It doesn't even matter how many transfers are made, so long as each bucket ends up with the same volume it had initially.

And the result is... The "water bucket" will end up with as much alcohol as the "alcohol bucket" has water. The reasoning is simple. The water bucket ends up with some alcohol in it. That alcohol must have come from the alcohol bucket. Since the alcohol bucket now has as much liquid as it had initially, the amount of alcohol it lost has been replaced by that same amount of water!

Riddle 3

Imagine the following scenario: You embark on a trip somewhere near the equator. One day, you find yourself on a desert island. While rummaging through a seemingly abandoned hut, you have an unlucky encounter with a poisonous spider - the pesky creature suddenly strikes out of nowhere and bites you! You know that in order to live you have to take an antidote. You find an emergency kit, in which indeed there is an antidote. However, the owner of the hut has a rather sick sense of humor (clearly - he is a physicist). The note on the bottle says:

"There are eight pills. They are all the same size and color. However, all pills but one, which is slightly heavier, are poisonous. The one heavier pill is the antidote."

Slightly angry, you try to find a scale. Indeed, on the other side of the hut you find a balance scale. This, however, also has a note attached:

"I decided to make this joke a bit funnier and twitched the scale so that it will immediately fall apart if you try to use it three times. Have fun!"

Slightly delirious, you begin to believe that the spider and the owner of the hut must be working for the same party. Still, you are a science major for a reason! You are more than capable of dealing with deadly riddles. After a few minutes' thought you figure out how to discover the antidote pill. Phew! That was close.

Now, how did you find which pill was the antidote?

Hint: one could still use the same trick if one was dealing with 9 pills.

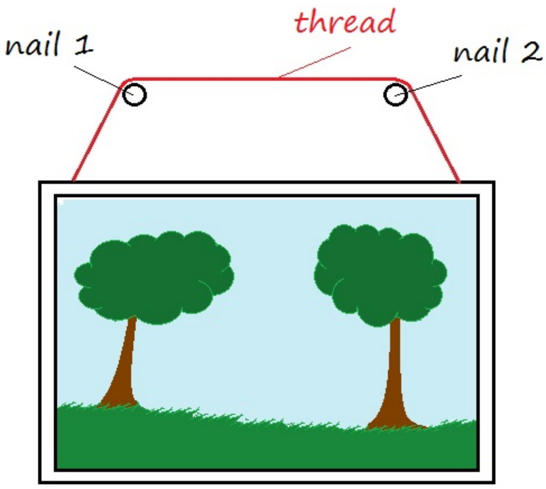
Choose 6 random pills out of 8, and set the remaining 2 aside. Out of the chosen 6, place 3 pills on each side of the balance scale. There are two possible scenarios:

1) If they weigh the same, it means that the antidote pill must be among the 2 pills set aside. You weigh those to find which one is heavier and you're done!

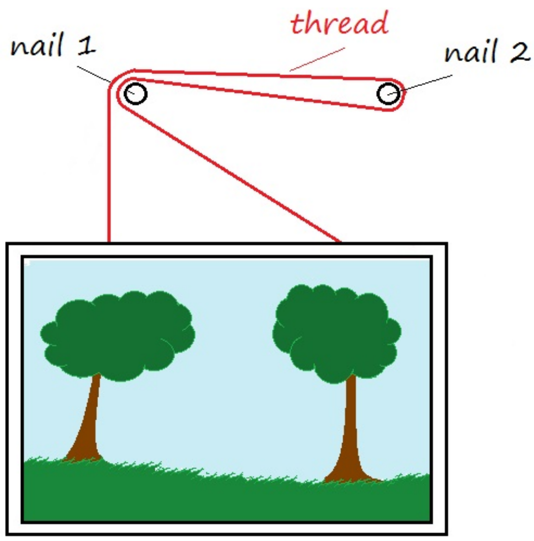
2) If one set of 3 pills weighs more than the other, you know that the antidote pill must be among these 3. You choose (randomly) 2 pills out of those (i.e., you set 1 pill aside) and weigh them. If one of those 2 is heavier, it must be the antidote, and if both weigh the same, the antidote pill must be the one you set aside.

Riddle 4

This is an advanced riddle. I have no doubt that each of you is capable of solving it, but it may take some time.
If one wants to hang a picture using a thread and two nails, the ordinary way of doing this is the following:

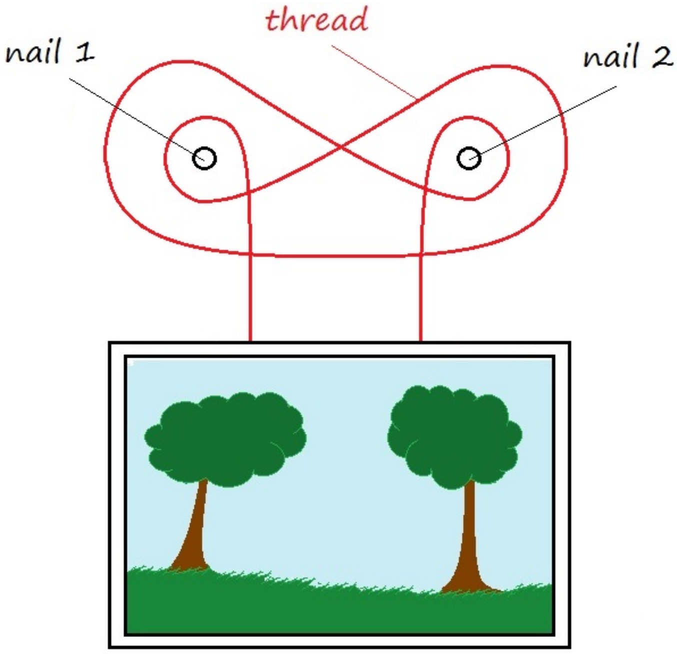


If one were a bit eccentric, one could try the less orthodox way:

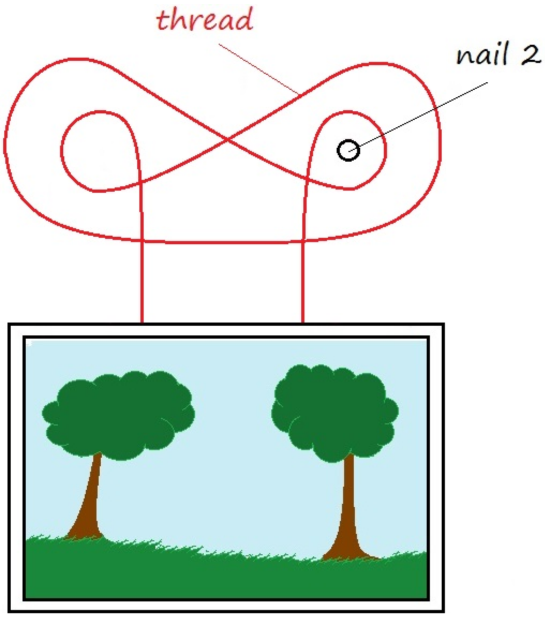


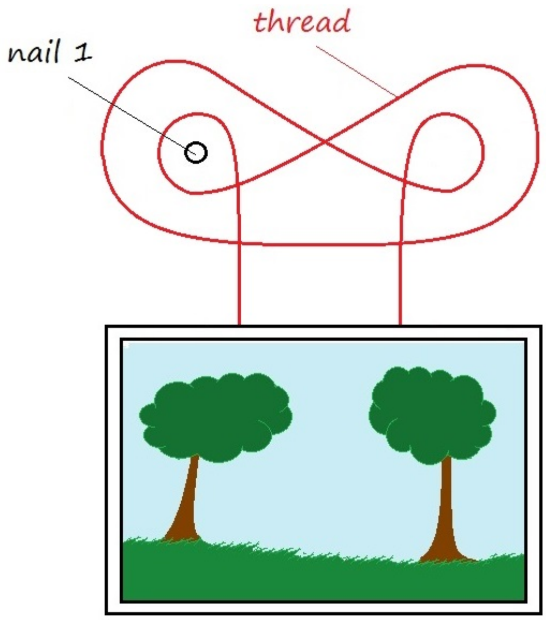
Note that in this situation if one takes out Nail 1, the picture will still be hanging, but if one takes out Nail 2 then the picture will fall down.
What is the way of hanging the picture such that the picture falls down if *either* (not both) of the nails is removed?

First of all: this is a *very* hard riddle. It takes many people, even graduate students or professors, *hours* to figure out. If you have not figured this one out on your own, but still have given it a good thought, you already owe yourself a pat on the back!
The solution is shown in the figure below:



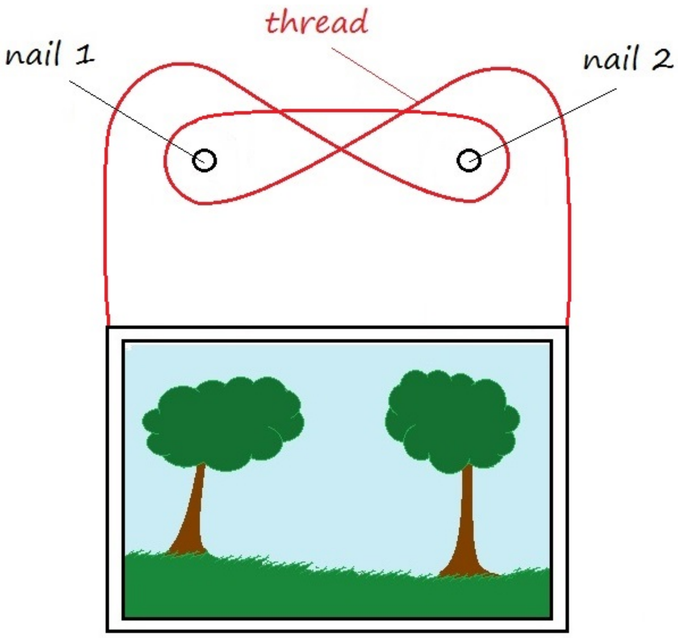
Note how this looks when either Nail 1 or Nail 2 is removed:





Indeed, one can see that without either of these nails the thread is just wound around Nail 2 or Nail 1 (as opposed to *enclosing* them), respectively, and that without the support of Nail 1 or Nail 2 the picture will fall down immediately.

This is only one of quite a few ways of drawing this. The following arrangement is also a valid solution:



You might be interested to know that this riddle has much more to offer than only intellectual exercise. The contour defined by the thread is called the Pochhammer contour, and can be used in complex analysis for contour integration. You might also want to look up the *Borromean link* (or *Borromean rings*) and see how they are connected with this riddle. All these have strong connections to *topology*, a branch of mathematics concerned with the properties of space that are preserved under continuous deformations, such as stretching and bending, but not tearing or gluing. Topology is useful in many branches of physics, such as quantum field theory or cosmology. It is used to form the very basis of our understanding of the Universe, and if someone wants to better this understanding considerably, they should obviously be familiar with its mathematical underlining. (Though clearly you've got a few more years to do that! No pressure ;).)

Problem 1

A train leaves from Chicago heading towards Los Angeles at 100 mph. Two hours later, a train leaves from Los Angeles heading towards Chicago at 200 mph. Assume there's exactly 2000 miles between Los Angeles and Chicago. When they meet, which city are they closer to?

Advice: Do not plug in numbers until the very end of the problem. In fact, this discussion is the last time we will use any numbers at all - it is far more useful to use symbols! You should get used to it as quick as you can.

Let us denote the departure time of the first train as t_1, and the departure time of the second train as t_2. The velocity of the first train is denoted by v_1, and the velocity of the second train is denoted by v_2. Let us also denote the total distance between the cities as s. Until now, we have just assigned symbols to the quantities given in the problem description.

Now, let us denote the time at which the trains meet as t. Note that we do not know the value of t! Still, we can use our basic kinematics knowledge to solve the problem. The distance traveled by the first train (Chicago-LA) is given by

$$s_1 = v_1 \left(t - t_1\right) \ , \tag{1}$$

while the distance traveled by the second train (LA-Chicago) is given by

$$s_2 = v_2 \left(t - t_2\right) \ . \tag{2}$$

Let us solve for t using (1):

$$t = \frac{s_1}{v_1} + t_1 \ . \tag{3}$$

Now, we can plug this into (2):

$$s_2 = v_2 \left(\frac{s_1}{v_1} + t_1 - t_2\right) = s_1 \frac{v_2}{v_1} + v_2 \left(t_1 - t_2\right) \ . \tag{4}$$

Lastly, we use

$$s = s_1 + s_2 \ . \tag{5}$$

We can solve for s_1 and plug this into (4):

$$s_2 = \left(s - s_2\right) \frac{v_2}{v_1} + v_2 \left(t_1 - t_2\right) = s\frac{v_2}{v_1} - s_2 \frac{v_2}{v_1} + v_2 \left(t_1 - t_2\right) \ . \tag{6}$$

This equation can be rewritten as

$$\left(1 + \frac{v_2}{v_1}\right) s_2 = s\frac{v_2}{v_1} + v_2 \left(t_1 - t_2\right) \ . \tag{7}$$

After reducing to a common denominator we get

$$\frac{v_1 + v_2}{v_1} s_2 = \frac{s v_2 + v_1 v_2 (t_1 - t_2)}{v_1} \ , \tag{8}$$

and finally we can divide both sides of the equation by the coefficient in front of s_2 to get

$$s_2 = \frac{s v_2 + v_1 v_2 (t_1 - t_2)}{v_1 + v_2} = s\frac{v_2}{v_1 + v_2} - v_2 (t_2 - t_1) \frac{v_1}{v_1 + v_2} \ . \tag{9}$$

At this point we can plug in the numbers given in the problem description. Please note the efficiency of using symbols: up until now we solved the problem of two trains leaving any city at any velocity and at any time! Only now we plug in specific numbers to check out the result in our specific Chicago-LA case. Noting that $t_2 - t_1 = 2$ [s], we have

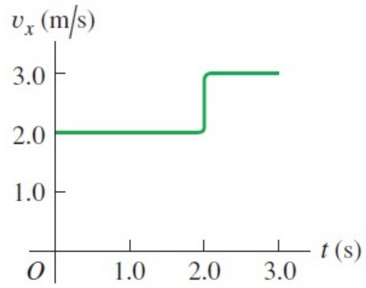
$$s_2 = 2000\ [\mathrm{mi}] \times \frac{200\ \left[\frac{\mathrm{mi}}{\mathrm{h}}\right]}{100\ \left[\frac{\mathrm{mi}}{\mathrm{h}}\right] + 200\ \left[\frac{\mathrm{mi}}{\mathrm{h}}\right]} - 200 \left[\frac{\mathrm{mi}}{\mathrm{h}}\right] \times 2\ [\mathrm{h}] \times \frac{100\ \left[\frac{\mathrm{mi}}{\mathrm{h}}\right]}{100\ \left[\frac{\mathrm{mi}}{\mathrm{h}}\right] + 200\ \left[\frac{\mathrm{mi}}{\mathrm{h}}\right]} = \tag{10}$$

$$= 2000\ [\mathrm{mi}] \times \frac{2}{3} - 400\ [\mathrm{mi}] \times \frac{1}{3} = 1200\ [\mathrm{mi}] \ . \tag{11}$$

Clearly, at this point the trains are closer to Chicago!

Problem 2

A bird moves in a straight line (the x-axis). The graph in the figure shows this bird's velocity as a function of time.
a) What are the bird's average speed and average velocity during the first 3.0 s?
b) Suppose that the bird moved in such a way that the graph segment after 2.0 s was -3.0 m/s instead of $+3.0$ m/s. Find the bird's average speed and average velocity in this case.



What do you think should be the answer? Specifically, why it *cannot* be

$$v_{\text{avg}} = \frac{v_1 + v_2}{2} \text{ ?} \tag{12}$$

Furthermore, is there a special case when the average velocity is indeed described by the above equation, although in general it's wrong?

To start, let us remind ourselves that the average velocity is given by the change in position during some finite time interval Δt divided by this time interval:

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \; . \tag{13}$$

a) From the graph we know that the distance traveled by body is given by

$$\Delta x = v_1 \Delta t_1 + v_2 \Delta t_2 = 2.0 \left[\frac{\text{m}}{\text{s}}\right] \times 2.0 \text{ [s]} + 3.0 \left[\frac{\text{m}}{\text{s}}\right] \times 1.0 \text{ [s]} = 4.0 \text{ [m]} + 3.0 \text{ [m]} = 7.0 \text{ [m]} \; , \tag{14}$$

so that

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{\Delta t_1 + \Delta t_2} = \frac{7.0 \text{ [m]}}{3.0 \text{ [s]}} = 2.33 \left[\frac{\text{m}}{\text{s}}\right] \; . \tag{15}$$

Indeed, this is different from

$$v_{\text{avg}} = \frac{v_1 + v_2}{2} = \frac{2.0 \left[\frac{\text{m}}{\text{s}}\right] + 3.0 \left[\frac{\text{m}}{\text{s}}\right]}{2} = 2.5 \left[\frac{\text{m}}{\text{s}}\right] \; . \tag{16}$$

b) Now we have

$$\Delta x = v_1 \Delta t_1 + v_2 \Delta t_2 = 2.0 \left[\frac{\text{m}}{\text{s}}\right] 2.0 \text{ [s]} - 3.0 \left[\frac{\text{m}}{\text{s}}\right] 1.0 \text{ [s]} = 4.0 \text{ [m]} - 3.0 \text{ [m]} = 1.0 \text{ [m]} \; , \tag{17}$$

so that the average velocity is

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{\Delta t_1 + \Delta t_2} = \frac{1.0 \text{ [m]}}{3.0 \text{ [s]}} = 0.33 \left[\frac{\text{m}}{\text{s}}\right] \; , \tag{18}$$

which is again different from

$$v_{\text{avg}} = \frac{v_1 + v_2}{2} = \frac{2.0\ \left[\frac{\text{m}}{\text{s}}\right] - 3.0\ \left[\frac{\text{m}}{\text{s}}\right]}{2} = -1.0\ \left[\frac{\text{m}}{\text{s}}\right]\ . \tag{19}$$

The average speed will be the same as in part a). That is, the average speed does not take the direction of the motion into the account. Incidentally, this is exactly what a *speedometer* shows!

Can you think of a situation in which the formula

$$v_{\text{avg}} = \frac{v_1 + v_2}{2} \tag{20}$$

would be correct?

This happens when $\Delta t_1 = \Delta t_2$. Let's see:

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{\Delta t_1 + \Delta t_2} = \frac{\Delta x_1 + \Delta x_2}{\Delta t_1 + \Delta t_2} = \frac{v_1 \Delta t_1 + v_2 \Delta t_2}{\Delta t_1 + \Delta t_2}\ . \tag{21}$$

Now since $\Delta t_1 = \Delta t_2$, we have

$$v_{\text{avg}} = \frac{\Big(v_1 + v_2\Big)\Delta t_1}{2\Delta t_1} = \frac{\Big(v_1 + v_2\Big)}{2}\ . \tag{22}$$

However, since it easy to make a grave mistake while in a hurry (for example to mistake equal times with equal distances and similar), it is best just to follow formula (13).

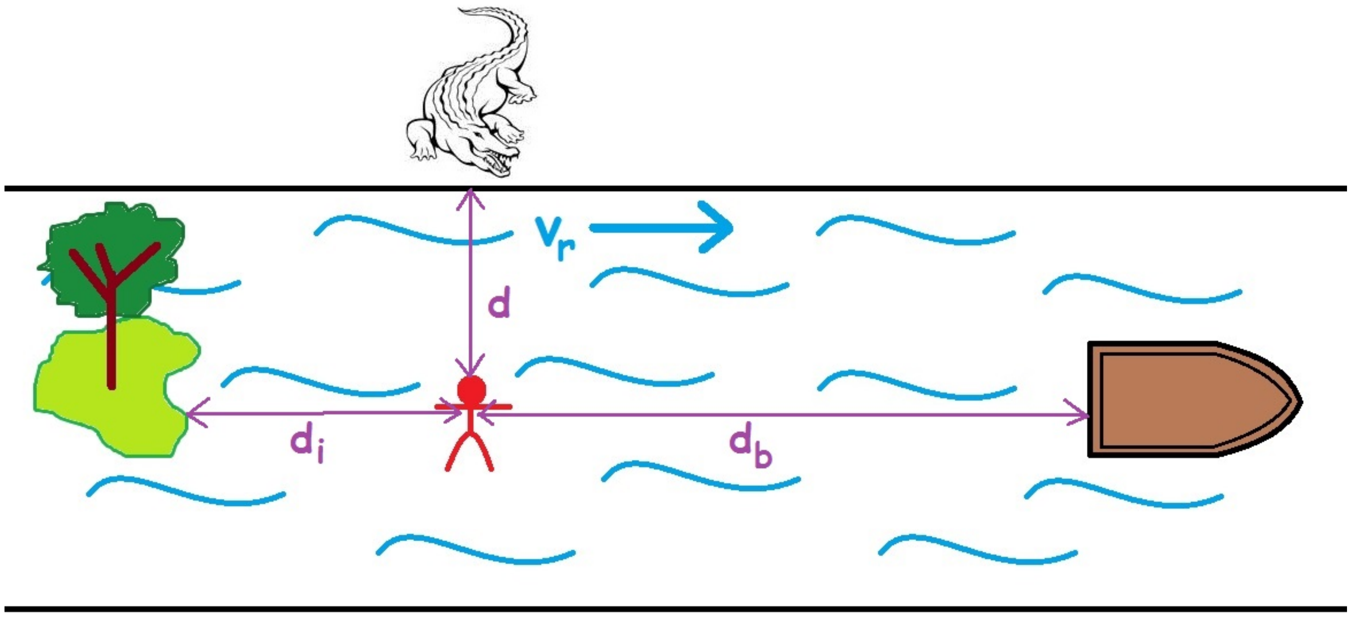
Problem 3

This problem has a very nasty-looking answer even though the setup looks deceptively straightforward. Don't be put off by messy formulas.

You're on vacation in Florida. You're having a great time taking a soak in a river, a distance d from the shore, while suddenly you spot an alligator getting ready to jump into the water and get you. You must think fast! To the left there is an island on the river with a tree, which you could climb to escape the alligator. The island is a distance d_i away. To the right there is an anchored boat, climbing in which would also save you from the alligator. The boat is a distance d_b away from you, and $d_b > d_i$. The river flows to the right with velocity v_r, and your fastest swimming speed is v_s. The alligator can swim with a velocity v_a. It is a clever beast which will immediately swim towards your destination as soon as you decide which way to go. (It will also swim straight towards you if you hesitate too long!)

Which way should you go in order to escape the beast?

As you have surely noticed, there are no numbers present in this problem. Instead, as a solution state a condition under which you could arrive safely at both destinations.



Once again: this problem is *nasty.* I think the main take-away here is that sometimes the solution one gets is just not pretty, but is still valid. During this course I will try to do my best in providing problems yielding both concise and not-so-concise results (though nothing as ugly as this problem should surface again). This should help you during the exams, which tend to concentrate on very specific problems and sometimes also do not have nice, concise formulas for answers.

In order to escape the alligator, it must take you less time to reach a given destination then it will take the alligator to reach it. First, let us calculate how much time it will take you to reach either of the destinations.

If you choose to swim towards the island, you will swim against the river flow. Hence your relative speed will be

$$v' = v_s - v_r \ . \tag{23}$$

When we say "relative speed", we mean here "relative to the shore". That is, your speed with respect to the water is v_s, but since the water itself is moving, your speed as perceived by an observer on the shore will be a *vector sum* of the two.

Now then the time it will take to swim to the island is given by

$$d_i = (v_s - v_r)\, t_i \qquad \Rightarrow \qquad t_i = \frac{d_i}{v_s - v_r} \ . \tag{24}$$

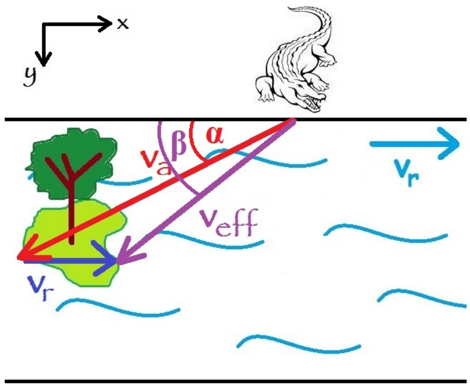
Similarly, if you choose to swim towards the boat, the river flow will work in your favor, meaning that your relative speed will be

$$v'' = v_s + v_r \ , \tag{25}$$

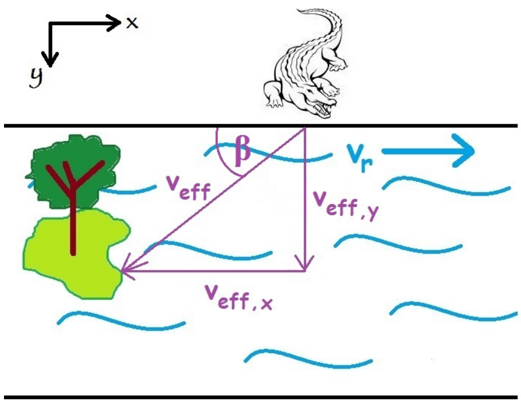
and hence the time it will take to swim to the boat is

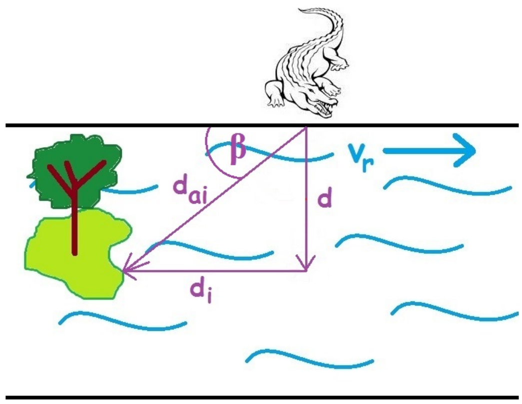
$$t_b = \frac{d_b}{v_s + v_r} \ . \tag{26}$$

Now let us consider the alligator. In order for the beast to swim straight towards the island or towards the boat, it will have to adjust its speed so that it takes into the account the river flow. Let us say that the alligator swims towards the island. Then the vector sum of its speed and the river speed should look like this:



Let us note that the x- and y-component of the effective velocity, v_{eff}, have the same geometrical relations between them as d_i, d, and $d_{ai} = \sqrt{d_i^2 + d^2}$, where the latter is the distance from the alligator to the island:





Since we know d, d_i, as well as d_{ai}, we in fact know $\cos\beta$ and $\sin\beta$.

Now, from the first drawing it should be clear that

$$v_{\text{eff},x} = v_{a,x} - v_r \quad \text{ and } \quad v_{\text{eff},y} = v_{a,y} \ . \tag{27}$$

Of course we cannot calculate $v_{a,x}$ and $v_{a,y}$ until we know something more about v_a, such as what is the angle it makes with the $x-$axis (denoted by α), but meanwhile let us proceed and see whether we can extract any more information from what we already got. Even though we do not know the value of v_{eff} yet, we can use the Pythagoras' law to write

$$v_{\text{eff}}^2 = v_{\text{eff},x}^2 + v_{\text{eff},y}^2 \ . \tag{28}$$

We use (27) to write this as

$$\begin{aligned} v_{\text{eff}}^2 &= \left(v_{a,x} - v_r\right)^2 + v_{a,y}^2 = \left(v_a \cos\alpha - v_r\right)^2 + \left(v_a \sin\alpha\right)^2 = v_a^2\cos^2\alpha - 2v_a v_r \cos\alpha + v_r^2 + v_a^2 \sin^2\alpha = & (29)\\ &= v_a^2 + v_r^2 - 2v_a v_r \cos\alpha \ , & (30) \end{aligned}$$

which makes sense, as this is just the law of cosines, that is the generalized Pythagoras' law.

We do not know what the angle α is, so that the above equation does not help us a lot yet. However, using the first drawing we can write

$$v_{\text{eff}} \cos\beta = v_a \cos\alpha - v_r \qquad \Rightarrow \qquad v_a \cos\alpha = v_{\text{eff}} \cos\beta + v_r \ . \tag{31}$$

(This is just pure geometry, so make sure you understand where the above equation comes from!) Now, we can plug this into (30) to obtain

$$v_{\text{eff}}^2 \ = \ v_a^2 + v_r^2 - 2v_r\big(v_{\text{eff}}\cos\beta + v_r\big) = v_a^2 + v_r^2 - 2v_r v_{\text{eff}}\cos\beta - 2v_r^2 = v_a^2 - v_r^2 - 2v_r v_{\text{eff}}\cos\beta \ , \tag{32}$$

which can be easily put into the form of a quadratic equation for v_{eff}:

$$v_{\text{eff}}^2 + 2v_r\cos\beta \ v_{\text{eff}} + v_r^2 - v_a^2 = 0 \ . \tag{33}$$

We can solve this using the standard procedure for the quadratic equation since we know all the other quantities used: v_r, v_a and $\cos\beta = \frac{d_i}{\sqrt{d_i^2+d^2}}$. Straightforwardly, for the equation of the form $ax^2+bx+c=0$ we have

$$\Delta = b^2 - 4ac = 4v_r^2\cos^2\beta - 4\big(v_r^2 - v_a^2\big) = 4v_r^2\big(\cos^2\beta - 1\big) + 4v_a^2 = 4\big(v_a^2 - v_r^2\sin^2\beta\big) \ , \tag{34}$$

so that

$$v_{\text{eff},1} = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-2v_r\cos\beta - 2\sqrt{v_a^2 - v_r^2\sin^2\beta}}{2} = -v_r\cos\beta - \sqrt{v_a^2 - v_r^2\sin^2\beta} \ , \tag{35}$$

$$v_{\text{eff},2} = \frac{-b+\sqrt{\Delta}}{2a} = -v_r\cos\beta + \sqrt{v_a^2 - v_r^2\sin^2\beta} \ . \tag{36}$$

Since all this time we were considering the magnitudes of the velocities, our solution should be positive, which is why we choose the second solution:

$$v_{\text{eff}} = \sqrt{v_a^2 - v_r^2\sin^2\beta} - v_r\cos\beta \ . \tag{37}$$

It should be clear that there exist values of β and v_r, for which the solution is not a viable one (i.e., $v_{\text{eff}} < 0$). For now we assume that both β and v_r are such that the alligator can reach the island - then the time it will take is given by

$$d_{a_i} = v_{\text{eff}}\ t_{ai} \qquad \Rightarrow \qquad t_{ai} \ = \ \frac{d_{ai}}{v_{\text{eff}}} = \frac{\sqrt{d_i^2+d^2}}{\sqrt{v_a^2 - v_r^2\frac{d^2}{d_i^2+d^2}} - v_r\frac{d_i}{\sqrt{d_i^2+d^2}}} = \frac{\sqrt{d_i^2+d^2}}{\frac{\sqrt{v_a^2\left(d_i^2+d^2\right)-v_r^2d^2}}{\sqrt{d_i^2+d^2}} - v_r\frac{d_i}{\sqrt{d_i^2+d^2}}} = \tag{38}$$

$$= \ \frac{d_i^2+d^2}{\sqrt{v_a^2\big(d_i^2+d^2\big) - v_r^2d^2} - v_rd_i} = \frac{d_i^2+d^2}{\sqrt{\big(v_a^2 - v_r^2\big)d^2 + v_a^2d_i^2} - v_rd_i} \ . \tag{39}$$

Let us now remind ourselves what this whole problem was about. In order for you to escape the alligator - in case you chose to swim towards the island - the time t_{ai} has to be bigger than the time it will take you to reach the island, i.e.

$$t_i < t_{a_i} \qquad \Rightarrow \qquad \frac{d_i}{v_s - v_r} < \frac{d_i^2+d^2}{\sqrt{\big(v_a^2 - v_r^2\big)d^2 + v_a^2d_i^2} - v_rd_i} \ . \tag{40}$$

This, of course, is a monstrous inequality and I do not expect anyone to go any deeper into the subject.

We can similarly calculate the effective velocity of the alligator if the beast swims towards the boat. In this situation, which almost exactly mirrors the previous one, we can easily obtain the solution by substituting $d_i \ \rightarrow \ d_b$ and $-v_r \ \rightarrow \ +v_r$, so that the quadratic equation we deal with becomes

$$v_{\text{eff}}^2 - 2v_r\cos\beta' \ v_{\text{eff}} + v_r^2 - v_a^2 = 0 \ , \tag{41}$$

where

$$\cos\beta' = \frac{d_b}{\sqrt{d_b^2+d^2}} \ . \tag{42}$$

Then the two solutions for the effective velocity are given by

$$v_{\text{eff},1} = \frac{-b-\sqrt{\Delta}}{2a} = \frac{2v_r\cos\beta' - 2\sqrt{v_a^2 - v_r^2\sin^2\beta'}}{2} = v_r\cos\beta' - \sqrt{v_a^2 - v_r^2\sin^2\beta'}\ , \tag{43}$$

$$v_{\text{eff},2} = \frac{-b+\sqrt{\Delta}}{2a} = v_r\cos\beta' + \sqrt{v_a^2 - v_r^2\sin^2\beta'}\ . \tag{44}$$

A quick consideration of these solutions (check, for example, limiting cases $\beta \to 0$ and $\beta \to \frac{\pi}{2}$) shows that again the second solution is the correct one:

$$v_{\text{eff}} = \frac{-b+\sqrt{\Delta}}{2a} = v_r\cos\beta' + \sqrt{v_a^2 - v_r^2\sin^2\beta'}\ . \tag{45}$$

Then

$$t_{ab} = \frac{d_{ab}}{v_{\text{eff}}} = \frac{\sqrt{d_b^2+d^2}}{\sqrt{v_a^2 - v_r^2\frac{d^2}{d_b^2+d^2}} + v_r\frac{d_b}{\sqrt{d_b^2+d^2}}} = \frac{\sqrt{d_b^2+d^2}}{\frac{\sqrt{v_a^2\left(d_b^2+d^2\right)-v_r^2d^2}}{\sqrt{d_b^2+d^2}} + v_r\frac{d_b}{\sqrt{d_b^2+d^2}}} = \tag{46}$$

$$= \frac{d_b^2+d^2}{\sqrt{v_a^2\left(d_b^2+d^2\right) - v_r^2d^2} + v_rd_b} = \frac{d_b^2+d^2}{\sqrt{\left(v_a^2-v_r^2\right)d^2 + v_a^2d_b^2} + v_rd_b}\ , \tag{47}$$

and in order to escape we need

$$t_b < t_{ab} \qquad \Rightarrow \qquad \frac{d_b}{v_s+v_r} < \frac{d_b^2+d^2}{\sqrt{\left(v_a^2-v_r^2\right)d^2 + v_a^2d_b^2} + v_rd_b}\ . \tag{48}$$

Again, this is one of those equations one wants to immediately stop dealing with once it is derived (i.e., it's very nasty and does not provide much insight - it's just a result for a particular situation).