Physics 22: Homework 3 Hints

1. This is a direct application of Gauss's Law. To determine the flux through any of the surfaces, the net charge enclosed, Q_{encl}, must be determined, noting that although charges outside of a given surface will alter the overall spatial electric-field profile, nevertheless this will not have any bearing on the combinatorics of field-line piercings. Once the enclosed net charge is determined, one simply needs to divide by the permittivity of free space, ε_0, in order to determine the net flux. Again, note the power behind Gauss's Law. If such a law was not available at our disposal, the calculation of flux would be a daunting task, as one would have to perform a brute-force calculation of the flux integral of non-trivial electric-field profiles over complex surfaces.

2. This problem is trying to get at the heart of reviewing through the subtle arguments utilized in class to reinforce the use of Gauss's Law for the purpose of determining electric-field profiles. To use Gauss's Law, it is up to the individual to choose an appropriate closed surface (called a "Gaussian surface") to determine the net flux by using the Right-Hand Side (RHS) of Gauss's Law, and then to see if it at all possible to use symmetry to make the implicit expression for $\vec{E}$ on the Left-Hand Side (LHS) of the law into an explicit one by taking the "E" out of the surface integral. Unfortunately, even if charge configurations have specific types of desirable symmetries, it could still be impossible to exploit Gauss's Law for the purposes of harvesting the electric-field profile. As seen in class, there are really only a few common geometries for which Gauss's Law may be used, and those geometries of charge still have to have a desirable charge distribution that mimics the same symmetries as the shape of the configuration itself.

3. Both of these parts echo a point that we discussed extensively in class regarding the condition for electrostatic equilibrium in conductors, as well as the implications of enclosing no charge with a Gaussian surface.

   (a) In this case, if any discharge occurs from, say, the clouds to the metal frame of the car, the charges will only be able to move or remain on the metal shell. So long as the person in the car is well insulated from the chassis, then there is no charge within the metal body of the car. For this reason, the net flux will be zero, which would also imply the absence of an electric field within the metal frame of the car. That being said, no electric field where the person is implies that no electric forces can be generated on the charges present on the person. (Even though our bodies are not the best electrical conductors in the world, our skin is still a fairly good conductor of electricity, primarily because of the fair amount of salt and water concentration. In this way, our bodies can carry charge through the skin. Of course, our neural networks are practically electrical circuits, and it is also well known that our blood carries hemoglobin, an important property of which is the ionized iron that is necessary to bind oxygen.)

   (b) This was discussed a bit in class, noting that since a neutral conductive shell contains conduction electrons capable of being influenced externally, then the shell may be polarized due to the presence of this charge in the cavity. Of course, a critical feature of this polarization are the consequences of electrostatic equilibrium within the conductor, during which the field within the conductor is zero. With the use of Gauss's Law, one can draw a Gaussian surface within the shell, in which case the net flux will be zero since the field is zero within the conductor. However, this would also imply that the net charge enclosed by the surface must be zero. Since this enclosed charge must include the existing charge in the cavity, the only way to make the net charge enclosed zero would be to have an equal amount of the opposite charge on the inner surface of the shell. By virtue of the shell being neutral, this then implies that there must be an equal amount of positive charge on the outer surface. One would expect this based on the effects of polarization, where the charge in the cavity induces a charge separation within the conductor via the movement of conduction electrons in one direction and the movement of the conduction electron vacancies (known as "holes") in the other direction.

4. This was done in class. Note that because the solid sphere is a conductor, it must be in electrostatic equilibrium. Moreover, since the geometry of the sphere is highly symmetric and the surface has a constant curvature, then the charge must also be uniformly distributed.

5. This was done in class. For Part (f), note that the distribution of charge is nonuniform but radial. For the parts involving arguing for the radial nature of the field outside, note that because the distribution is radial, one can imagine the overall sphere is formed by thin spherical shells, each with its own distinct distribution of charge which is uniform for each respective shell. Indeed, since the shells all have spherical symmetry, then

the collection of shells (i.e., the overall sphere) will have it as well, so that the field of this sphere outside will look identical to that of a point charge with the exact same charge placed at the center of the sphere. As for arguing for the field's properties within the sphere, pick a point, without loss of generality, along a diameter that is above the center of the sphere. It helps to again break up the regions into quadrants on the cross-section (technically octants in full three dimensions) by drawing a vertical and horizontal line that cut through the observation point. However, unlike what was done in class for the uniform-charge-distribution case, instead draw a single concentric shell and begin arguing by looking at mirror-image charges lying on the shell. Of course, the charges will only be mirror images across the upper and lower quadrants, but one can immediately come to the conclusion that the field will again be radial and will be dependent on how far away the observation point is from the center of the sphere. For the quantitative portion of this part, note that the charge density, ρ, cannot simply be taken out of the volume integral for determining the enclosed charge,

$$Q_{\text{encl}} = \int_{V_G} \rho d\mathcal{V} \neq \rho \int_{V_G} d\mathcal{V},$$

since the charge density is dependent on the radial integration that will result from the volume integration measure.

6. This is a quantitative manifestation of Problem 3b.

   (a) The field is undefined for $r = 0$. The answer is:

   $$\vec{E} = \begin{cases} -\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r} & \text{for } 0 < r < a \\ \vec{0} & \text{for } a < r < b \\ -\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r} & \text{for } r > b \end{cases}.$$

   (b) Note that because the charge is in the very center of the spherical cavity, the polarized charges must be uniformly distributed on the conducting shell. Thus, with the use of Gauss's Law, the total charge on each of the surfaces may be determined. As such, the surface charge density can simply be determined by taking the total charge on each surface and dividing by the area of each surface.

   (c) This still does not change the fact that the field inside of the conductor is zero, but it does alter the field profile for $r > b$ and the surface-charge densities on the inner and outer surfaces of the conducting shell.

7. In this case, the appropriate Gaussian surface to use is a cylindrical one with end-caps. The simplest choice of end-caps would be disks.

   (a) The arguments provided in class were regarding the insulating sphere, drawn in cross-section. However, the arguments shown for that cross-section are actually really more applicable for the cylinder, because the cross-section is uniform for such a cylinder. For $r > R$, pick a point above the "north pole" of the cylinder and draw a "north-south" line to cross through the center of the cylinder. Note that for every point on the cylinder to the right of this dividing line will have a mirror image on the left side. These mirrored charges will have their electric fields cancel along the "east-west" line at the observation point, but will have their north (or, more appropriately radial) components reinforce. This will be true for all such pairs of points, so that the total electric field will, indeed, be radial. As for $r < R$, pick a point above the center of the cylinder and draw a "north-south" and "east-west" line intersecting commonly at that observation point. Mirror images may be chosen along the "north-south" line and "east-west" line, all of which will cancel in fours. The "cap" above the "east-west" line can be folded over that line to then show all of the charge contributions that would have their net field cancel. All that then remains is the region of the cylinder below the "folded-over" region, in which case the field will again be purely northward (or radial). To show that all points within or outside the cylinder have radially directed fields, note that a rotation of the cylinder with the fixed observation point (called an active rotation)—or, equivalently, a movement of the observation point on a concentric circle with a fixed orientation of the cylinder (called a passive rotation)—maps out the profile for all other equivalent points, so that the field is indeed radially directed no matter what observation point is picked. Of course, because the chosen observation points were arbitrary in terms of distance from the center, then this is going to be true for all observation points.

(b) This follows by using the respective arguments for the relevant regions from Part (a) and seeing that a simple alteration of distance will result in a different electric-field strength which is solely dependent on this radial distance. For example, when outside and picking an observation point farther away, the field contributions from mirror images will not only become more along the "north-south" line, but they will also become weaker. The two effects, although seemingly strengthening the field in the case of the components becoming more concentrated along the "north-south" direction will be seen to be overpowered by the additional distance considerations. Whereas, for example, picking an observation point within that is closer to the center will result in the "folding-over" region becoming a much larger part of the overall cylinder, thus resulting in less charges contributing to the field. Indeed, the field will be zero on the symmetry axis.

(c) As mentioned before, the end-caps are necessary to be able to apply Gauss's Law, since this law looks at closed surfaces.

i. The symmetry arguments from above (particularly from Part (a)) show that the field is radial. Since the normal vectors on the end-caps of the concentric Gaussian cylinder are at right-angles to the radial direction, these end-caps will not contribute. Thus, the only contribution will stem from the curved portion of the cylinder.

ii. The field on the outside is exactly the same as the one determined in class for a thin, very long rod of uniformly distributed charge. For inside the cylinder, the arguments from Parts (a) and (b) for that region can be implemented to show that the appropriate surface again is the Gaussian cylinder with end-caps. Although the Left-Hand-Side of Gauss's Law takes on an identical form as to the case of points outside the cylinder of charge, the Right-Hand-Side of the law must have the enclosed charge integrated over the Gaussian volume. Since ρ and R are the given quantities, then the answers are

$$\vec{E} = \begin{cases} \frac{\rho}{2\varepsilon_0} r\hat{r} & \text{for } 0 \leq r < R \\ \frac{\rho R^2}{2\varepsilon_0}\frac{1}{r}\hat{r} & \text{for } r \geq R \end{cases}.$$

8. This was done in class. There is some lacking information in the problem, as since the sheet of charge is taken to be an insulator, it must have also been specified that the charge distribution was uniform. Another way to have corrected this would have been to call the sheet of charge a conductor, in which case the charges would definitely be uniformly distributed on account of the shuffling of the charges that would ensue until the condition of electrostatic equilibrium was met for this conductor. Thus, whatever the case, the charge distribution will be taken as uniform. Recall that the required symmetries were translational along the plane (so that observation points at a fixed perpendicular distance from the plane would have no distinction when moved around in a plane parallel to the sheet, since all such movements would still make the edges of the plane infinitely far away and would have the charge distribution look indistinguishable), translational perpendicular to the plane (so that observation points taken along a perpendicular line would all be equivalent since the plane would look just as infinite no matter how far away one placed the point), and rotational (so that if the plane was rotated through any angle, the observation point will still see the plane in an identical fashion to before the plane was even rotated). If the sheet is taken to be on the xy-plane with the z-axis, therefore, serving as the perpendicular axis to the plane, then the field will be

$$\vec{E} = \begin{cases} +\hat{z}\frac{Q/A}{2\varepsilon_0} & \text{for } z > 0 \\ -\hat{z}\frac{Q/A}{2\varepsilon_0} & \text{for } z < 0 \end{cases}.$$

Of course, the surface charge density, $\sigma = Q/A$.

9. Note that even though this is a large conducting plate, it has a non-negligible thickness, d.

(a) Since we are presuming that it is in electrostatic equilibrium, then it immediately follows that $\vec{E} = \vec{0}$ within the slab (i.e., for $z < |d/2|$).

(b) Of course, we make use of the results of Problem 8, as mentioned. There are a few ways to approach this problem: (1) draw a Gaussian pillbox that extends from above $z = d/2$ to below $z = -d/2$, or (2) draw a Gaussian pillbox that extends from above $z = d/2$ to within the slab, or (3) draw a Gaussian pillbox that extends from within the slab to below $z = -d/2$. Choosing Method (2), note that the only contribution will come from the top cap of the pillbox, so that we may write

$$\oint_{\partial V_G} \vec{E} \cdot d\vec{A} = \int_{\mathcal{S}_{\text{top}}} \vec{E} \cdot d\vec{A} = E\mathcal{A},$$

where $\mathcal{A}$ is the area of the cross-section of this pillbox. However, note that because the TOTAL charge is Q, then the charge must be divided in half between the top and bottom surfaces of the slab. Thus, the enclosed charge through this Gaussian pillbox will be

$$Q_{\text{encl}} = \sigma' \mathcal{A} = \left(\frac{Q/2}{A}\right) \mathcal{A}.$$

Thus, merging the two sides of Gauss's Law, we find

$$E = \frac{Q}{2\varepsilon_0 A},$$

which will be true for $|z| > d/2$.

(c) We have found the surface-charge density in Part (b), but to reiterate its value, note that we may take the total charge of the slab, Q, and divide by the total area of the plate, $2A$, which presumes that even though the slab technically has faces along the thickness, that those areas are negligible in comparison to the area of the top and bottom of the slab, so that practically all of the charge lives on those top and bottom surfaces. Mathematically,

$$\sigma' = \frac{Q}{2A}.$$

Thus, the result of Part (b) may be rewritten as

$$E = \frac{\sigma'}{\varepsilon_0}.$$

(d) The two expressions appear different because of the fact that surface-charge density in Problem 8, $\sigma = 2\sigma'$, where σ' is the surface-charge density in this problem. That being said, even though the expressions appear different, note that they are not, particularly when written in terms of the total charge.

10. In this case, the slab is an insulating one, so that the field within the slab is no longer zero. Indeed, because the charge is distributed uniformly, this means that the (volume) charge density, ρ, is a constant. In particular,

$$\rho = \frac{Q}{Ad}.$$

For $z \geq |d/2|$, take a Gaussian pillbox that extends across the full thickness of the slab, with the top above $z = d/2$ and the bottom below $z = -d/2$. The field must be symmetric on both sides. Considering the cross-section of the pillbox to be $\mathcal{A}$, one can show upon using Gauss's Law that

$$E = \frac{\rho d}{2\varepsilon_0} = \frac{Q}{2\varepsilon_0 A},$$

which is identical to the result of Problem 8. As for $z < |d/2|$, consider a Gaussian pillbox that is taken symmetrically about $z = 0$, but with the top and bottom end-caps within the slab. Following similar procedures, one can show that the field in this region is given by

$$E = \frac{\rho}{\varepsilon_0} z = \frac{Q}{A\varepsilon_0} \frac{z}{d},$$

so that the field varies linearly in the perpendicular dimension. Note that, as can be argued qualitatively, $E\,(z = 0) = 0$.

11. There are a few techniques that may be used to approach this problem. It helps to divide up the regions into four parts, where Region I is for $z > a$, Region II corresponds to $a < z < 0$, Region III is for $0 < z < -a$, and Region IV corresponds to $z < -a$.

   (a) The great place to start is to take a Gaussian pillbox that has its top end-cap at a point above $z = a$, and its bottom end-cap at a point below $z = -a$, so that the pillbox extends across all three plates. Suppose the cross-sectional area of this pillbox is $\mathcal{A}$. Then, since we know the field profile must be uniform ultimately, note that if we take the top and bottom end-caps to be very far away from the plate configuration, then the system looks as though the plates have been collapsed to a single plate, of total charge $Q+Q-Q = Q$ and cross-sectional area A. Thus, the field for $|z| > a$ is identical to the one for a single such sheet:

$$E_{\mathrm{I}} = E_{\mathrm{IV}} = \frac{Q}{2\varepsilon_0 A}.$$

   Then, take a Gaussian pillbox with its top end-cap at a point above $z = a$, and its bottom end-cap at a point between $z = a$ and $z = 0$. Note that the flux contribution from the top end-cap must be positive, since the field points towards $+z$ for $z > a$. If the charge density $\sigma = Q/A$ is utilized, then Gauss's Law would state

$$E_{\mathrm{II}} = \frac{\sigma}{\varepsilon_0} - E_{\mathrm{I}} = \frac{Q}{2\varepsilon_0 A}.$$

   Note that the positive nature of E_{II} implies that the field in Region II points towards $-z$. Then, using this fact and drawing a Gaussian pillbox with its top end-cap at a point between $z = a$ and $z = 0$ and its bottom end-cap at a point between $z = 0$ and $z = -a$, then Gauss's Law would state

$$E_{\mathrm{III}} = \frac{\sigma}{\varepsilon_0} + E_{\mathrm{II}} = \frac{3Q}{2\varepsilon_0 A},$$

   where the "+" after the first equal sign for the second term results from the fact that the flux contribution of $\vec{E}_{\mathrm{II}}$ from the top end-cap would be negative since this field points along $-z$ while this top end-cap's normal vector points along $+z$.

   (b) These charge densities may be read off from the values of the electric fields directly. The bottom and top of Sheet 1 will have densities $\sigma_1^{(\mathrm{bot})} = \sigma_1^{(\mathrm{top})} = +Q/(2A)$. The bottom and top of Sheet 2 will have a density $\sigma_2^{(\mathrm{bot})} = -Q/(2A)$ and $\sigma_2^{(\mathrm{top})} = +3Q/(2A)$, respectively. Finally, The bottom and top of Sheet 3 will have densities $\sigma_3^{(\mathrm{bot})} = -3Q/(2A)$ and $\sigma_3^{(\mathrm{top})} = +Q/(2A)$, respectively.

   (c) This follows from the fact mentioned in Part (a) that the net effect of these three plates when viewed from afar looks as a single plate with the net charge given by the sum of the charges on all of the plates. For this reason, the field must emanate equally in both directions based on the spatial symmetries of such large plates. Since the fields outside of the external plates must, therefore, be spatially uniform, this also puts a restriction on the charge densities (as well as the charges) on the external surfaces of those outermost plates, as seen in Part (b).

12. This is really a rehashing of all that we have derived and argued when considering the logical deductions used to go from Coulomb's Law down to Gauss's Law for electrostatics. The gravitational field will be defined as the ratio of the gravitational force on a test mass ($\mathcal{M}$) due to a source mass (M) to the test mass

$$\vec{g}_M \equiv \frac{\vec{F}_{\mathcal{M}M}}{\mathcal{M}}.$$

   The flux will then be defined as

$$\Phi_g \equiv \int_{\mathcal{S}} \vec{g} \cdot d\vec{A},$$

   while Gauss's Law will take the form

$$\oint_{\partial V_G} \vec{g} \cdot d\vec{A} = -4\pi G,$$

   where the negative sign on the Right-Hand Side follows because the field lines always point towards masses, since there are no such things as opposite masses. The proof of Part (c) is identical to the result in electrostatics of the field outside of a uniformly charged insulating sphere in Problem 5.

13. We treat the hydrogen atom of this model as consisting of a uniformly charged distribution of positive matter (of total charge $+e = 1.60 \times 10^{-19}$ C) with an electron in the center.

    (a) The electron has to be in the center because, upon recalling the result of Problem 5 for the uniformly charged insulating sphere, the field of such a sphere at its center is zero. (Of course, we can see this qualitatively because for any point chosen within the sphere, there is a mirror reflection about its center which will generate the exact same field in the opposite direction at the center. Thus, the net field will be zero.) As such, the force will also be zero on this electron situated at the center.

    (b) Again recall from Problem 5 that the field inside of a uniformly charged insulating sphere is a linear function of the radial position from the center. So, if one displaced the electron from the center by some amount r, then the force will have the form

    $$\vec{F} = -\kappa r \hat{r},$$

    exactly like a Hooke's-Law force of a spring. Here,

    $$\kappa = \frac{e^2}{4\pi\varepsilon_0 R^3}.$$

    To see the relevance of the minus sign, consider displacing the electron by this amount to the left. Then, the electron would "see" more positive charge on the right than on the left, since the right side has more than a hemisphere of charge while the left side has less than a hemisphere of charge. As such, the sign indicates that this will, indeed, be a restoring force. Using Newton's 2nd Law for displacement along this radial axis

    $$m_e \frac{d^2 r}{dt^2} = -\kappa r,$$

    which is exactly the SHM differential equation.

    (c) The angular frequency of oscillation of the electron is given by

    $$\omega = \sqrt{\frac{\kappa}{m_e}},$$

    where, upon using the fact that the (regular) frequency $f = \omega/(2\pi)$, we have

    $$f = \frac{1}{2\pi}\sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e^2 R^3}}.$$

    Using the above relationship will output the desired result for the radius of the Thomson atom, R.

    (d) In this case, recall that the force law is an inverse-square law when outside of the sphere, since the field of such a uniformly charged ball when outside the ball is exactly that of a point charge. Thus, the motion will not be SHM.

    (e) The point of this problem is to draw a force diagram on one of the charges. The force from the positively charged ball will oppose the force due to the other electron, and both electrons would have to be placed along a diameter of the ball. The field of the ball at the location of one of the electrons must be determined by finding the field within the sphere at the location of that charge (which, of course, has to be some distance away from the center). This, of course, has already been done by virtue of Problem 5. This attractive force due to the ball must then be equated to the repelling force due to the other electron, which is a distance that is twice the distance away from the charge than the center of the ball. The result may then be outputted.

14. Because the charge is radially distributed, Gauss's Law may still be applied by virtue of the arguments utilized in Problem 5f. Of course, the use of Gauss's Law will be forthcoming in Part (c).

(a) To determine the total amount of charge contained within a sphere of some arbitrary radius, r, simply integrate the density over the volume of this sphere, being sure to include the charge of the nucleus, $+q_e$. Mathematically,

$$Q = Q_{\text{nucleus}} + Q_{\text{cloud}} = q_e + \int_V \rho d\mathcal{V}.$$

Here, $d\mathcal{V}$, based on the spherical symmetry, represents the volume bounded by an infinitesimally thin spherical shell. This can be shown to have the form

$$d\mathcal{V} = 4\pi r^2 dr.$$

Thus, for the cloud charge, we have

$$\begin{aligned} Q_{\text{cloud}}(r) &= \int_V \rho d\mathcal{V} \\ &= \int_0^r \rho(r') 4\pi (r')^2 dr' \\ &= -\left(\frac{4q_e}{a_0^3}\right) \int_0^r (r')^2 e^{-2r'/a_0} dr' \\ &= -4q_e \int_0^r \left(\frac{r'}{a_0}\right)^2 e^{-2\left(r'/a_0\right)} d\left(\frac{r'}{a_0}\right) \\ &\equiv -4q_e \int_0^{r/a_0} u^2 e^{-2u} du. \end{aligned}$$

The result of the integral above can be obtain via the Integration-by-Parts technique. The result becomes

$$Q_{\text{cloud}}(r) = -q_e \left[1 - \left(1 + 2\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-2r/a_0}\right].$$

Thus, the total charge then is

$$Q(r) = q_e \left(1 + 2\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-2r/a_0}.$$

(b) Taking $r \to \infty$ of the result to Part (a) results in $Q(r \to \infty) \to 0$, which is expected because the net charge of this infinitely-sized system should be zero. (After all, this is a *neutral* hydrogen atom.)

(c) For this, Gauss's Law must be used with the help of a spherical Gaussian surface, since this system indeed has spherical symmetry. The enclosed charge is given in Part (a), whereas the Left-Hand Side of Gauss's Law has the simplification (based on the spherical-symmetry arguments):

$$\oint_{\partial V_G} \vec{E} \cdot d\vec{A} = \oint_{\partial V_G} E dA = E \oint_{\partial V_G} dA = E\left(4\pi r^2\right).$$

Merging the two sides of Gauss's Law, we find

$$E = \frac{Q(r)}{4\pi\varepsilon_0 r^2} = \frac{q_e}{4\pi\varepsilon_0 r^2}\left(1 + 2\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-2r/a_0}.$$

(d) This is best left to a graphing program, but note that the field does overall drop off to zero as $r \to \infty$, which can be most easily seen by Taylor-expanding the decaying exponential in order to show that the power series in the numerator will tend to zero quicker than the r^2 in the denominator, which comes from the surface area of a sphere.