Physics 1A
Spring 2015

Challenge Problem 14

This problem was designed for you to explore one of the simplest, most important, and most ubiquitous systems in all of physics: the classical **harmonic oscillator**.

Consider a block of mass m sliding on a frictionless surface in the x-direction. Suppose, additionally, that the block is connected to a Hooke's Law spring of spring constant k that is attached to a wall. Let $x = 0$ denote the position of the block when the spring is at its natural length.

(a) Determine the equation of motion for the block. In other words, determine the equation satisfied by the position as a function of time $x(t)$ of the block.

Solution. The equation of motion is the equation that can be solved for the position $x(t)$ of the block as a function of time. Given that the natural length of the spring coincides where the position of the block is $x = 0$, the spring force is

$$F_s = -kx. \tag{1}$$

Newton's Second Law therefore gives the following equation of motion provided we use the fact that the acceleration is the second time derivative of the position:

$$\boxed{-kx = m\ddot{x}}. \tag{2}$$

(b) Define a variable $\omega = \sqrt{k/m}$. What are the units of this variable?

Solution. The units of k are units of force divided by units of displacement which gives kg/s^2. Dividing by m and taking the square root gives units of 1/s.

(c) Write your equation of motion from part (a) in terms of ω.

Solution. Plugging in the definition of ω in (2) gives

$$\ddot{x} = -\omega^2 x \tag{3}$$

(d) Determine the most general solution to your equation from part (c) as follows:

(i) Try solutions to the equation of the form $\cos(\alpha t)$ and $\sin(\alpha t)$, and determine what α needs to be for these trial solutions to work.

Solution. Plugging these trial solutions into the equation of motion gives

$$-\alpha^2 \cos(\alpha t) = -\omega^2 \cos(\omega t) \tag{4}$$
$$-\alpha^2 \sin(\alpha t) = -\omega^2 \sin(\omega t) \tag{5}$$

Se we can identify

$$\boxed{\alpha = \omega} \tag{6}$$

(ii) The general solution is then $x(t) = A\cos(\alpha t) + B\sin(\alpha t)$ (in math, this is called a **linear combination** of the cos and sin solutions), where A and B are constants that are determined by "initial conditions." More concretely, A and B can be solved for once one specifies the initial position $x(0)$ and the initial velocity $\dot{x}(0)$.

(iii) Suppose that the following initial conditions are given:

$$x(0) = x_0, \qquad \dot{x}(0) = v_0. \tag{7}$$

Solve for A and B in your general solution in terms of x_0 and v_0.

Solution. The general solution is of the form

$$x(t) = A\cos(\omega t) + B\sin(\omega t). \tag{8}$$

The corresponding velocity as a function of time is

$$\dot{x}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \tag{9}$$

It follows that

$$x(0) = A, \qquad \dot{x}(0) = B\omega \tag{10}$$

so the initial conditions given imply that

$$A = x_0, \qquad B = \frac{v_0}{\omega} \tag{11}$$

and the general solution becomes

$$\boxed{x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)} \tag{12}$$

(e) In light of your solution, what is a physical interpretation of ω?

Solution. Notice that ωt is the argument of trigonometric functions, so we have think of it as a quantity measured in radians. Therefore, we can think of ω as having units radians per second (even though radians are not a physical unit – this is just for understanding). Thus, ω is like an angular frequency. Physically, the mass is going to oscillate back and forth, and a full cycle of oscillations corresponds to 2π radians. A higher value of ω means that the mass oscillates back and forth faster, so a full such cycle takes less time, and vice versa.

(f) Suppose that at time $t = 0$, the spring is at its natural length, and that the block is given a push so that its initial velocity is v_0.

(i) What is the position $x(t)$ of the block as a function of time? Sketch it.

Solution. Plugging $x_0 = 0$ into the general solution, we obtain

$$\boxed{x(t) = \frac{v_0}{\omega} \sin(\omega t)}. \tag{13}$$

The graph of $x(t)$ versus t looks like a sine curve.

(ii) What is the velocity $v(t)$ of the block as a function of time? Sketch it.

Solution. Computing $\dot{x}$ gives

$$c(t) = \dot{x}(t) = \boxed{v_0 \cos(\omega t)}. \tag{14}$$

The graph looks like an cosine curve.

(iii) What is the acceleration $a(t)$ of the block as a function of time? Sketch it.

Solution. Computing $\dot{v}$ gives

$$a(t) = \dot{v}(t) = \boxed{v_0 \omega \sin(\omega t)} \tag{15}$$

This graph looks like an inverted cosine function.

(iv) What is the magnitude of the largest displacement from its initial position that the block ever reaches? This maximum displacement is called the **amplitude** of its motion.

Solution. The magnitude of the largest displacement is the magnitude of the maximum of $|x(t)|$ which occurs when $\sin(\omega t) = 1$. So the amplitude is $\boxed{|x_{\max}| = |v_0/\omega|}$.

(v) What is the maximum speed of the block during its motion? At what point in its motion does this speed occur?

Solution. Similar reasoning to the last part gives $\boxed{|v_{\max}| = |v_0|}$.

(vi) What is the magnitude of the maximum acceleration of the block during its motion? At what point during its motion does this maximum occur?

Solution. Similar reasoning to the last part gives $\boxed{|a_{\max}| = |v_0 \omega|}$.