Physics 22: Homework 4

The following problems encompass the topics of electric potential energy, electric potential, and capacitance.

1. A point charge, with charge $q = +2\ \mu$C, can be moved from, and to, points A, B, and C, all of which are within a uniform electric field, of magnitude $E = 200$ N/C, that is directed as shown in Figure 1. The length between points A and B is 3 cm, while the length between points B and C is 4 cm.

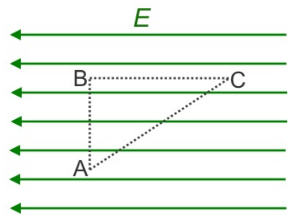

Figure 1: Points within a uniform electric field to which a point charge may be moved to assess energy considerations.

   (a) Fill in the following chart, noting that
$$\Delta U_{\mathrm{i}\to\mathrm{f}} \equiv U_{\mathrm{f}} - U_i,$$
   is the change in electric *potential energy*, and that $W_E^{(\mathrm{i}\to\mathrm{f})}$ represents the work done on the point charge due to the source charges (which are generating the uniform field but not shown in Figure 1) going from initial point, i, to final point, f.

$\Delta U_{\mathrm{A}\to\mathrm{B}}$	$\Delta U_{\mathrm{B}\to\mathrm{C}}$	$\Delta U_{\mathrm{A}\to\mathrm{C}}$	$W_E^{(\mathrm{A}\to\mathrm{B})}$	$W_E^{(\mathrm{B}\to\mathrm{C})}$	$W_E^{(\mathrm{A}\to\mathrm{C})}$

   (b) Repeat Part (a) for the case when the point charge is $q' = -3\ \mu$C.

   (c) Determine the electric *potential* differences V_{AB}, V_{BC}, and V_{AC}, and show that the answers are independent of the amount of charge that is being moved from one point to another.

   (d) Compare the electric potential to the electric potential energy. In particular, discuss what makes the electric potential, in some ways, easier to use than the electric potential energy.

2. As in Figure 2, consider two identical, very large, equally but oppositely charged, flat, parallel conducting plates that are separated by a distance $d = 6.0$ cm and have a potential difference between them that is $V_{\mathrm{BA}} = 36$ V, where B is the positive plate and A is the negative plate that is grounded (i.e., has a potential of zero).

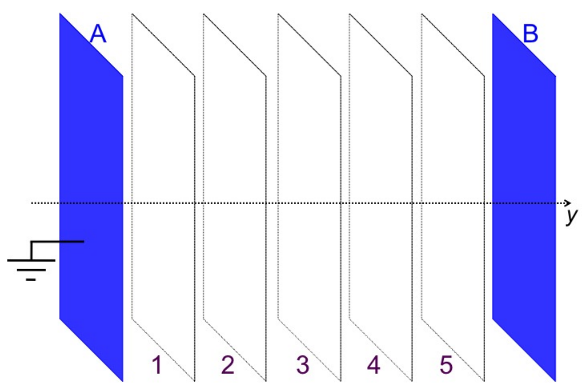

Figure 2: A parallel-plate capacitor with one plate that is grounded. Consideration is to be made regarding the equipotential surfaces of such a capacitor.

   (a) Consider the imaginary parallel planes (outlined with dashed lines) shown in Figure 2 and labeled as 1, 2, 3, 4, and 5, with imaginary plane 1 being closest to the grounded plate (A), and imaginary plane 5 being closest to the positive plate (B). These imaginary planes are equidistant with adjacent imaginary planes/plates.

      i. Why do we call these imaginary planes equipotential surfaces?

      ii. Are the plates themselves equipotential regions? Explain.

iii. What is the purpose of showcasing the equipotential surfaces as equally spaced in Figure 2?

(b) Determine the values of the equipotential surfaces 1 - 5.

(c) Determine all possible potential differences illustrated in Figure 2 (e.g., V_{12}, V_{53}, $V_{2\mathrm{B}}$, etc.).

(d) Determine the desired result in the following cases.

i. Work done by an external agent in moving a 1 μC charge from 1 to 4.

ii. Potential energy, relative to ground, of a -0.5 μC charge at 2.

iii. Kinetic energy of an electron that is released at 4 immediately before it hits the plate to which it is attracted.

iv. Work done by the source charges on the plates,

A. when a $+3$ μC charge is moved from 2 to 4. Is an external force required to allow this to happen in the first place?

B. when a -3 μC charge is moved from 2 to 4. Is an external force required to allow this to happen in the first place?

(e) Determine the electric field (magnitude and direction) between the plates.

(f) Determine the charge density on the plates.

(g) As in Figure 2, the y-axis runs perpendicular to the plates, with $y = 0$ taken at the grounded plate and $+y$ taken towards the positive plate from this origin.

i. Draw a graph of the potential as a function of the position on this y-axis, $V(y)$, including the regions $y < 0$ and $y > d$.

ii. Show that

$$E = E_y = -\frac{dV}{dy},$$

helping to confirm the result in Part (e).

(h) Which answers in all of the previous parts would change if plate B was instead the grounded plate?

3. Consider a configuration of three very large, charged, flat, parallel, conducting plates—labeled as A, B, and C (and ordered in this manner)—set parallel to each other. When moving from one plate to the other, the potential changes according to the graph shown in Figure 3.

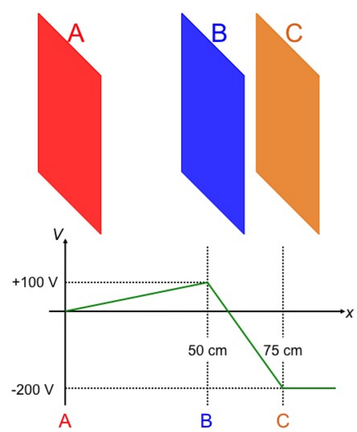

Figure 3: A configuration of three charged parallel plates with a graph of the spatial profile of the potential along a perpendicular line running through the plates. An electron that is "boiled off" of plate A will be made to move through the plates.

(a) Determine the electric fields (magnitudes and directions) everywhere.

(b) Draw a graph of the field as a function of the position along an axis perpendicular to the plates at all points on this axis.

(c) Suppose that in between plates A and B an electron breaks loose, from rest, from plate A and accelerates towards B.

i. Why would an electron accelerate toward B in the first place?

ii. Determine the electron's acceleration in this process.

iii. Determine the speed with which the electron would hit plate B.

(d) Assume that there is a small-enough hole bored into plate B to allow the electron that has flown from plate A (in Part (c)) to slip through plate B, thus entering the region between plates B and C. Describe the subsequent motion of the electron in this region, additionally discussing whether or not it will reach plate C in this journey.

4. The potential in a certain region of space is given by

$$V(x, y) = \alpha xy,$$

for $x \in [0, \infty)$ and $y \in [0, \infty)$, and where $\alpha > 0$ is some constant.

(a) For x and y measured in meters, determine the SI units of α.

(b) Determine the electric field components, E_x and E_y, in this region by using the fact that

$$E_s = -\frac{\partial V}{\partial s},$$

where $s = x$ or y, and the derivative shown here is a partial derivative. In other words, $\partial V/\partial x$ is the derivative of V with respect to x while treating y as a constant, while $\partial V/\partial y$ is the derivative of V with respect to y while treating x as a constant.

(c) Sketch some equipotential lines in the xy-plane. Recall that equipotential surfaces satisfy the condition that $V =$ constant. To do this, try using some specific values of V(e.g., $V_1 = \alpha$, $V_2 = 4\alpha$, $V_3 = 9\alpha$) to obtain the proper functional form of $y = y(x)$.

(d) Use the sketches of equipotential surfaces in Part (c) to draw some representative field lines for this situation.

5. When the electron of a hydrogen atom is "excited" it rises from its spherically symmetric ground-state position (at the radius $a_0 = 5.3 \times 10^{-11}$ m $\equiv 0.53$ Å) to a spherically symmetric position that is four times farther.

(a) Using the standard of $U \to 0$ as $r \to \infty$, determine the electric potential energy of the electron (in both units of Joules (J) and electron-Volts (eV)) at the highlighted initial and final positions.

(b) Using the standard of $V \to 0$ as $r \to \infty$, determine the electric potential at each of these positions.

(c) Determine the work done by the electrostatic force due to the nucleus on the electron in this excitation process.

(d) Determine the change in the electric potential energy of the hydrogen atom in this excitation process.

(e) Draw a graph of the atom's electric potential energy, as well as of the nucleus' electric potential, as functions of the radial position, r, from the nucleus.

(f) Consider, instead, anti-hydrogen, which consists of an antiproton nucleus (i.e., a nucleus made of the antiparticle of a proton which has the same charge as a proton, but negative) and an orbiting positron (i.e., the antiparticle of the electron which has the same charge as an electron, but positive). As long as the anti-particles exist in system where little matter exists, this is a fairly stable "atomic" configuration. How would the results to the parts above differ in the analysis of anti-hydrogen?

6. As shown in Figure 4, consider the arrangement of point charges—$q_1 = +q$, $q_2 = -q$, $q_3 = -2q$, and $q_4 = +3q$—with each charge placed at the corner of a square of side-length, ℓ.

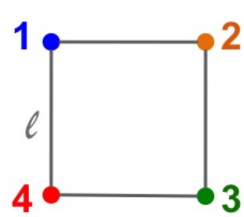

Figure 4: An arrangement of point charges at the corners of a square of side-length ℓ.

(a) Determine the amount of energy required to form such an assembly of charges.

(b) Suppose that q_1 is allowed to move freely.

i. Argue qualitatively that q_1 will always move farther away from the configuration from one point in time to another.

ii. Determine its kinetic energy when it is very far away from the remaining charges (which are presumed to be fixed in place).

7. Consider a solid sphere, of charge Q and radius R.

   (a) If the sphere is conducting, determine the energy necessary in order to form such an assembly of charge in the first place.

   (b) If the sphere is insulating and the charge is distributed uniformly throughout its solid volume, determine the energy necessary in order to form such an assembly of charge in the first place.

8. As shown in Figure 5, consider a charged metal sphere, of radius a and total charge Q, surrounded by an un-charged metal shell, of inner radius b and outer radius c.

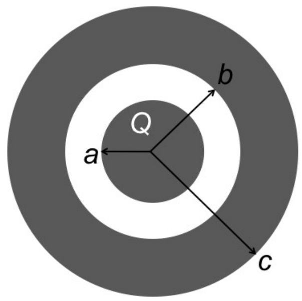

Figure 5: A charged metal sphere is within the cavity of, and arranged concentrically with, an uncharged spherical metal shell.

   (a) In Figure 5, draw electric field lines in the region of space where they exist, and attempt to draw some representative equipotential surfaces.

   (b) Determine the electric field everywhere in space.

   (c) Using the standard reference for the potential of such a finite charge configuration, determine the electric potential everywhere in space, and show that it, unlike the field, is continuous everywhere.

   (d) Graph the field, as well as the electric potential, as functions of the radial position, r, on separate graphs.

9. Consider an insulating, uniformly charged sphere, of radius R and total charge Q.

   (a) Determine the electric field everywhere in space.

   (b) Using the standard reference for the potential of such a finite charge configuration, determine the electric potential everywhere in space, and show that it, unlike the field, is continuous everywhere.

   (c) Graph the field, as well as the electric potential, as functions of the radial position, r, on separate graphs.

   (d) Determine the amount of energy it would take to bring in a point charge, q, from a position $r = 5R$ and place it at a position $r = R/2$.

10. As in Figure 6, consider two conducting spheres that are equally charged, each having a total charge Q, but with different radii a and $b > a$.

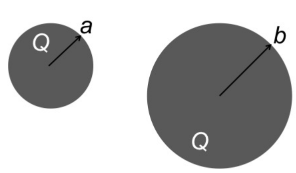

Figure 6: Two conducting spheres of equal charge, but different radii, that will eventually be connected by a very thin conducting wire.

   (a) Which sphere is at the higher potential? Why?

   (b) Describe and determine what will happen to the charges if the two sphere are connected via a thin conducting wire.

   (c) After the connected spheres reach equilibrium, determine the final charge ratio, q_a/q_b, of the charge on the sphere of radius a to the charge on the sphere of radius b.

   (d) Determine the ratio of surface charge densities, σ_a/σ_b.

(e) Use the results of this problem to help explain why, for example, discharge is more prone to occur from sharp protuberances of a surface.

11. Consider two point charges placed on an x-axis, with charge $q_1 = +4q$ placed at the origin, and the charge $q_2 = -q$ placed a distance, d away from q_1 along, say, the $+x$-axis.

 (a) Along the x-axis, find the location(s) (other than $x \to \pm\infty$) at which the potential is zero.

 (b) Are there other finite places in the xy-plane where the potential is zero? Explain.

 (c) Compare the result of Part (a) to the single point at which the field is zero along this x-axis. Moreover, discuss, in general, exactly why the potential and the field are not zero at the same point.

12. Consider a neutral dipole—with poles $+q$ and $-q$ that are separated by a distance $2a$—placed symmetrically about the x-axis, as shown in Figure 7. In this way, the y-axis acts as a perpendicular bisector of this dipole. As usual for such finite charge configurations, take the potential to be zero infinitely far away from the configuration.

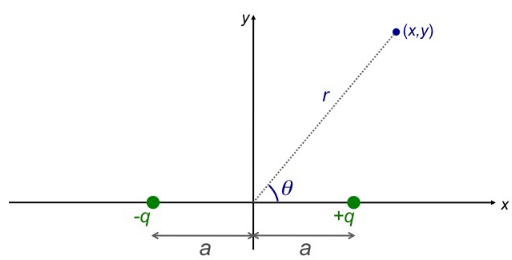

Figure 7: A neutral dipole placed on the x-axis symmetrically about the y-axis. The potential and field are to be determined at various points, including at the arbitrary point (x, y).

 (a) Determine the potential of this configuration along the x-axis.

 (b) Determine the potential of this configuration along the y-axis. Comment on the result.

 (c) Describe the motion of an electron that is released from an arbitrary point on the y-axis.

 (d) Using your knowledge of the electric-field lines of this dipole configuration, attempt to draw some representative equipotential lines in this xy-plane.

 (e) Consider a point at some arbitrary location (x, y) in the xy-plane. Consider also utilizing a polar-coordinate decomposition, where this point is represented by a position vector, of length r, that makes an angle, θ, measured relative to the $+x$-axis. Obtain an exact expression for the electric potential of this dipole, $V(x, y)$, at this point.

 (f) Using the result of Part (e), show that for $r \gg a$, the lowest-order approximation of the potential takes on the form

$$V(x, y) \approx \frac{p}{4\pi\varepsilon_0} \frac{x}{(x^2 + y^2)^{3/2}},$$

 where $p \equiv q(2a) = 2qa$ is the magnitude of this dipole's moment.

 (g) Using the result of Part (f) and the formalism described in Problem 4b, obtain the exact expression for the electric field, $\vec{E}$, of this dipole, which is given, in polar form, by:

$$\vec{E} = E_x\hat{x} + E_y\hat{y} \approx \frac{p}{4\pi\varepsilon_0} \frac{1}{r^3} \left[\left(3\cos^2\theta - 1\right)\hat{x} + 3\sin\theta\cos\theta\hat{y} \right].$$

 (This problem showcases the immense power in calculating the field from the potential, where one is able to completely bypass the vector nature of the field and simply tack it on at the very end by taking a derivative of it. In particular, the vector nature of the field arrives from the fact that the field is related to the gradient of the potential. Mathematically,

$$\vec{E} = -\vec{\nabla}V,$$

 where $\vec{\nabla}$ (known as "nabla" or "del") is the gradient operator defined, in 2D Cartesian components, by

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y},$$

 so that

$$\vec{E} = -\hat{x}\frac{\partial V}{\partial x} - \hat{y}\frac{\partial V}{\partial y}.$$

 Indeed, the computational power associated with the potential is one of the most useful aspects of the potential.)

(h) Show that the result to Part (g) condenses to the results of the field along the x- and y-axes with which you are familiar from past analyses using Coulomb's Law.

13. Consider the identical configuration in Figure 7, except that both poles are positive, with either pole having charge $+q$ and the separation between poles again being $2a$.

    (a) Determine the potential of this configuration along the x-axis, $V(x)$.

    (b) Determine the potential of this configuration along the y-axis, $V(y)$.

    (c) Show that the results of both parts above condense to the result of a point charge, $2q$, when $x \gg a$ and $y \gg a$, respectively.

14. As shown in Figure 8, consider a very long, solid conducting rod, of radius a and linear charge density λ.

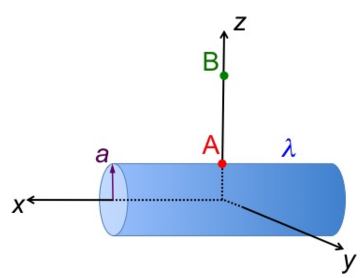

Figure 8: A very long, solid conducting rod centered on the x-axis and arranged symmetrically about the yz-plane. The y- and z-axes serve as radial lines to the cylindrical rod. The rod has a linear charge density λ.

    (a) What is the potential difference between any two points on the surface of this conducting rod, and what does this imply about the electric field at the surface of this rod?

    (b) What is the potential difference between any two points within the conducting rod?

    (c) Does it make sense to define the zero of potential at infinity in this case? Explain.

    (d) Consider the points A and B in Figure 8, where both points lie along the same line that happens to be perpendicular to the axis of symmetry of the rod. Point A is on the surface of the rod, while point B is outside the rod.

        i. Determine the potential difference between points A and B.

        ii. Suppose the zero of the potential is taken at B (i.e., $V_{\mathrm{B}} \equiv 0$).

            A. Determine the potential at any radial position r (i.e., $V(r)$) with $r > a$ (i.e., outside the rod).

            B. Determine the potential at any radial position r (i.e., $V(r)$) with $r \leq a$ (i.e., on and within the rod).

            C. Draw separate graphs of the potential, V, and the field, E, as functions of the radial position, r.

    (e) Suppose that the rod is now surrounded by a very long, thin, cylindrical, conducting shell, of linear charge density $-\lambda$ and cross-sectional radius b (i.e., the shell is placed with point B being a point on the surface of this concentric shell). Upon defining the potential on this shell to be zero (i.e., again defining $V_{\mathrm{B}} \equiv 0$), describe and determine the potential inside and outside this shell.

15. Consider a thin rod, of total charge Q and length ℓ, that is placed on the x-axis symmetrically about the y-axis.

    (a) Assume that the linear density of this rod is uniform.

        i. Determine the potential at any point, x, along the x-axis, such that $|x| > \ell/2$.

        ii. Determine the potential at any point, y, along the y-axis, such that $y \neq 0$.

    (b) Now, assume that the linear density of this rod is non-uniform and given by

$$\lambda(x) = 4\lambda_0 \frac{x^2}{\ell^2},$$

    where $\lambda_0 > 0$ is an unknown constant.

        i. Determine λ_0 in terms of the given quantities.

        ii. Determine the potential at any point, x, along the x-axis, such that $|x| > \ell/2$.

        iii. Determine the potential at any point, y, along the y-axis, such that $y \neq 0$.

16. As in Figure 9, consider a thin conducting ring, of radius a and total charge q. The axis of symmetry of this ring is taken as the z-axis, with the ring located on the xy-plane (i.e., at the plane defined by $z = 0$).

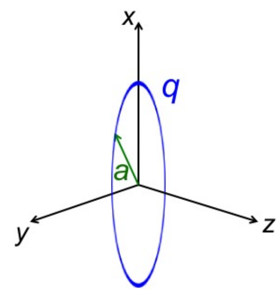

Figure 9: A thin conducting ring, of radius a and total charge q, is placed on the xy-plane with the z-axis serving as its symmetry axis.

(a) Determine the potential (referencing zero at ∞) at any point, z, along the ring's axis of symmetry.

(b) Since the potential along this axis only has a dependence on the spatial coordinate, z, use the formalism of Problem 4b to determine the field along this symmetry axis. Also, verify that the result obtained from this formalism matches the result obtained for the field using Coulomb's Law.

(c) Determine the point along the axis where the potential is a maximum, and compare to the point where the field is a maximum.

(d) Draw separate graphs of the potential, V, and the field, E_z, as a function of positions, z, along the axis of symmetry.

17. As in Figure 10, consider a thin conducting disk, of radius R and total charge Q. The axis of symmetry of this disk is taken as the z-axis, with the disk located on the xy-plane (i.e., at the plane defined by $z = 0$).

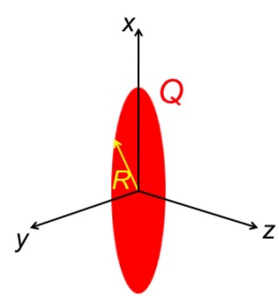

Figure 10: A thin conducting disk, of radius R and total charge Q, is placed on the xy-plane with the z-axis serving as its symmetry axis.

(a) Use the result of Problem 16 to determine the potential of this disk at any point, $z \neq 0$, along its axis of symmetry.

(b) Since the potential along this axis only has a dependence on the spatial coordinate, z, use the formalism of Problem 4b to determine the field along this symmetry axis. Also, verify that the result obtained from this formalism matches the result obtained for the field using Coulomb's Law.

18. As in Figure 11, consider a conducting solid sphere, of radius R and total charge $-Q$, surrounded by a very thin, concentric, conducting spherical shell, of total charge $+2Q$ and radius $2R$.

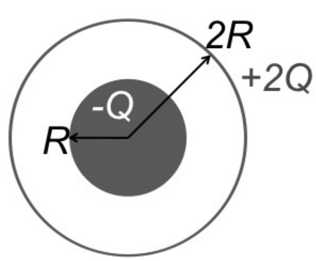

Figure 11: A conducting solid sphere, of radius R and total charge $-Q$, surrounded by a concentric, thin, conducting spherical shell, of radius $2R$ and total charge $+2Q$.

(a) Determine the potential difference between the surface of the solid sphere and the shell.

(b) With the potential taken to be zero at ∞, determine the potential at any radial position, r, for $r \in [0, \infty)$.

(c) Draw separate graphs of the potential and the field as functions of the radial position, r. (The field may either be determined using the formalism of Problem 4b, or by using Gauss's Law.)

19. Consider a very large, insulating slab of charge, of thickness d, with uniform (volume) charge density ρ. As in Figure 12, the z-axis is drawn perpendicular to the largest cross-section of the plate, with the origin of coordinates ($z = 0$) placed at the midpoint of its thickness.

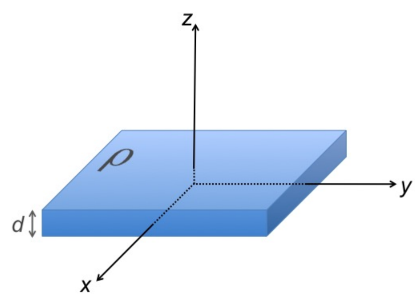

Figure 12: An insulating slab, of thickness d, with a very large cross-sectional area and uniformly charged with a charge density, ρ.

(a) Upon choosing a convenient reference as zero for the potential, determine the potential difference between the center and the surface of the plate (e.g., between $z = 0$ and $z = \pm d/2$).

(b) With the same chosen reference point serving as the potential's zero, graph the potential as a function of any position, z, along the z-axis.

20. Figure 13 shows a sketch, in cross-section, of an old-school oscilloscope tube. An electron "gun" accelerates, from rest, electrons through a potential difference V, which, upon passing through a collimator (i.e., a tiny hole in a conducting plate that only allows passage of electrons moving roughly along the x-axis), generates a fast beam of electrons moving at some speed u_x along this axis. At some point, this beam reaches a very large parallel-plate capacitor, with the parallel plates held at a potential difference V'. The plates are separated by an amount, d, and span a length, L, along the x-axis. This capacitor aims to deflect the beam of electrons from the x-axis to allow vertical movement. After escaping the capacitor region, the beam eventually will collide with a fluorescent screen, which will cause the screen to literally light up where the incidence occurs. The edge of the capacitor assembly is a distance, D, away from the flat screen. Suppose that the beam's point of collision vertically along the screen is labeled via the use of a y-axis, with $y = 0$ coincident with the undeflected beam line.

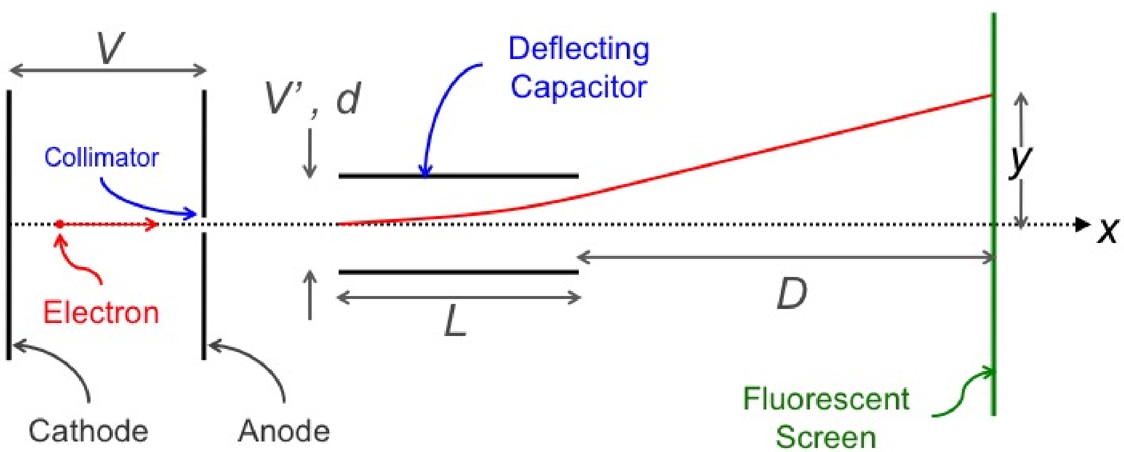

Figure 13: A schematic of an oscilloscope tube with accelerating potential, V, and deflecting potential, V', which are used in conjunction to make electrons eventually hit the fluorescent screen of the oscilloscope.

(a) In terms of the given quantities, determine the final vertical position, y, of the beam when it strikes the screen, and show that it is directly proportional to V'.

(b) Explain why an oscilloscope serves as a device to measure potential differences (or voltages).

(c) How can the electron beam be made to "sweep" back and forth in the horizontal plane?

21. Recall that the capacitance is a physical property of a device. As such, the specific amount of charge does not determine the capacitance, unless it is also known what the simultaneous voltage is between the capacitor plates, in which case

$$C = \frac{Q}{\Delta V_C},$$

but it still will not change the fact that this quantity is only dependent on the geometry of the capacitor. With this in mind, determine the capacitance for each of the following geometries composed of conducting objects, which are commonly used as capacitors.

(a) Two (very large) parallel plates of cross-sectional area, A, and plate separation, d.

(b) A sphere of radius R.

(c) Two concentric spherical shells, with the inner sphere having a radius a, and the outer sphere having a radius b.

(d) Two concentric cylindrical shells, each of (very large) length ℓ, and with the inner cylinder having a radius a, and the outer cylinder having a radius b.

(e) How would the results to the above parts change if each of the capacitor systems was placed, instead, within a liquid, of dielectric constant κ?

22. Explain how dielectrics work and how they increase the capacitance of a capacitor that is initially placed in, for example, air. Be sure to clearly distinguish between "free" and "bound" charges, and from where these aforementioned charges originate.

23. Consider a pair of oppositely, but equally, charged (very large but thin) parallel plates—with charges $+Q$ and $-Q$—oriented with the z-axis running perpendicular to the plates, as shown in Figure 14. The plates each have cross-sectional area, A, and a separation distance, d.

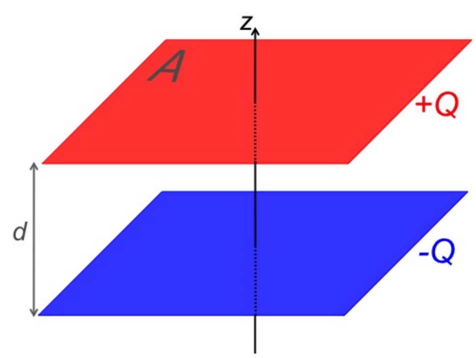

Figure 14: A parallel-plate capacitor of charge Q, plate area A, and separation distance d.

(a) Determine the energy necessary to build up such an assembly of charge in the first place.

(b) Using the fact that the conservative force between two objects that are interacting is given by the negative of the spatial gradient of the potential-energy of interaction, which in this case may be written as

$$F = -\frac{dU}{dz},$$

determine the attractive force between the plates.

(c) Write the expression in Part (b) in terms of the electric field between the plates and the charge on the plates.

(d) Note that the result in Part (c) is not

$$F = QE,$$

as would likely be expected from the definition of the electric field. Resolve this apparent contradiction.