Physics 1A: Mechanics
Winter 2016

A Short Introduction: Vectors

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(Alternatively, just e-mail me your questions.)

1. WHAT'S THE DIFFERENCE?

In physics we encounter two particular types of quantities: *scalars* and *vectors*.

A **scalar** is an ordinary, or undirected, quantity. The physical quantities like

$$\text{mass } m \ , \quad \text{number of particles } n \ , \quad \text{or temperature } T \tag{1.1}$$

are scalars. When we speak about scalars we only speak about *magnitude*.

A **vector** is a quantity that has a magnitude as well as direction assigned to it. Examples of such quantities are

$$\text{displacement } \vec{x} \ , \quad \text{velocity } \vec{v} \ , \quad \text{or force } \vec{F} \ . \tag{1.2}$$

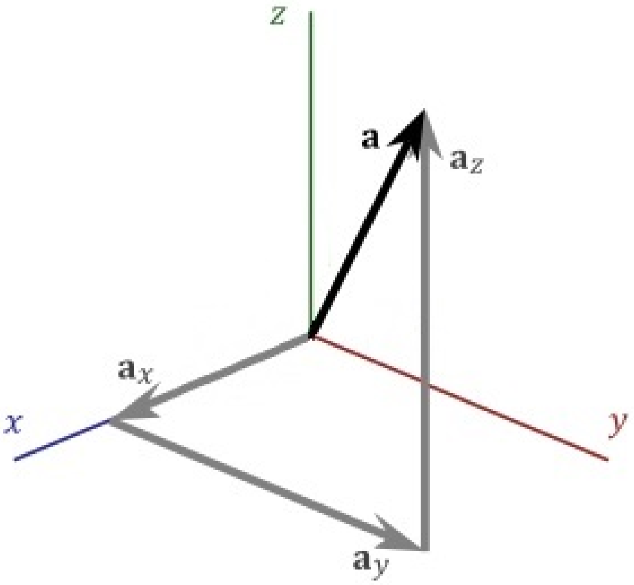
The direction of displacement is important because if we move something, it matters both "how much" (magnitude) we have moved it as well as "where" (direction). When it comes to velocity, if someone steals my money and I want to chase them, the speed with which I am running matters, but the direction is important as well. And if you've ever fallen off of something, you have a very clear idea that gravity - that is the force of gravity - indeed has a direction.

SCALARS	VECTORS
mass m	displacement $\vec{x}$
number of particles n	velocity $\vec{v}$
temperature T	force $\vec{F}$

Professor Corbin provided during the first lecture a very neat way of distinguishing scalars from vectors: if you rotate and the quantity does not change, it's a scalar. If, on the other hand, it changes upon rotation, it's a vector.
As what you're going to concentrate in this quarter is learning about *kinematics* and *dynamics*, which deal with all the quantities in the right column of the table (and many more), it is clear that you need to know how to go about calculations involving vectors.

2. THE VERY BASICS

To describe a quantity with a direction in a three dimensional space we need, in fact, three quantities. Let us see that for a general vector, let's call it $\vec{a}$, starting at the origin as below



the three numbers uniquely specifying this vector are "the lenght that $\vec{a}$ 'takes' in x direction", "the lenght that $\vec{a}$ 'takes' in y direction", and "the lenght that $\vec{a}$ 'takes' in z direction". Mathematically, we can write that as

$$\vec{a} = (a_x, a_y, a_z) \ , \tag{2.1}$$

where a_x, a_y, a_z are the lengths just mentioned. Note that the order of the numbers within the brackets is of crucial importance, for example

$$(a_x, a_y, a_z) \neq (a_y, a_x, a_z) \ . \tag{2.2}$$

It may seem a bit weird to describe a geometrical object just by putting three numbers in a bracket. This is just one of many equivalent ways to do it, but it proves to be a very handy one. In any case, make sure you can convince yourself that a vector is indeed fully described by those three numbers representing lengths in three mutually perpendicular directions!

As our vector is conveniently placed at the origin, we can immediately read that $a_x = x$, $a_y = y$, and $a_z = z$, where x, y, z are the coefficients of the so-called *projections* of $\vec{a}$ onto corresponding axes (more on projections in a moment). However, we can also see that the magnitudes of a_x, a_y, a_z do not depend on whether the vector starts at the origin or not. For a vector placed anywhere in the 3D plane, we have

$$a_x = x_2 - x_1 \ , \quad a_y = y_2 - y_1 \ , \quad a_z = z_2 - z_1 \ , \tag{2.3}$$

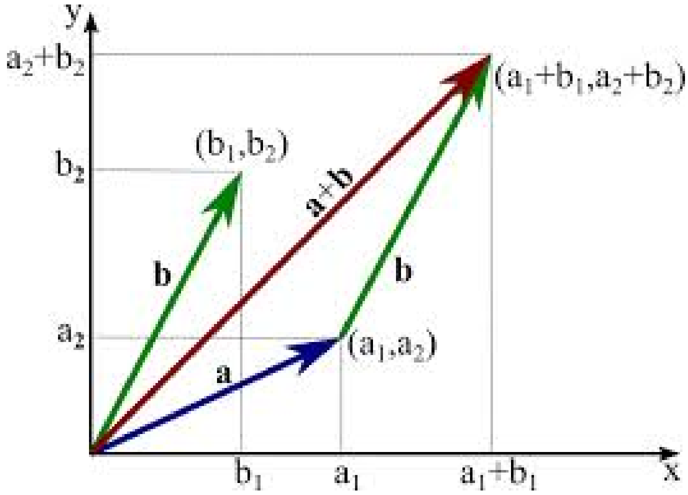
where the subscript "2" marks the coefficients of the "arrow end" (officially known as a *terminal point*) of the vector and subscript "1" marks the coefficients of the "back end" (or *initial point*) of the vector. If this is the same vector $\vec{a}$ just *translated* somewhere away from the origin, a_x, a_y, a_z are going to be the same as before. By choosing the vector to start at the origin, we just conveniently set $x_1 = y_1 = z_1 = 0$. You may want to remember about the possibility to use this convenience while doing your problem sets.

You may have recognized that what we have drawn here is some coordinate system. We may of course choose many - *infinitely* many - different coordinate systems to describe the same physical situation and then the axes coefficients of the vector will change. However, it is important to note already at this point that *the physics is independent of the coordinate system.* This is also a bit of information that you may find useful while doing the assigned problems: if you find that the problem is easier to solve in a specific coordinate system, go for it! For example, when describing a uniform linear motion, I may attempt to describe it by choosing the motion to proceed in a random direction in the 3-dimensional coordinate system, but I also may do something smart instead and choose a coordinate system such that the motion progresses along *one* of the axes, for example x. In this way I deal only with one coefficient instead of three, which is undeniably just easier.

3. BASIC OPERATIONS

A. Addition

Here's how the addition of vectors looks like in 2-dimensional space:



Generalization to 3 dimensions is straightforward. Now, algebraically (in 3D) it looks like this:
If, in some chosen coordinate system, we have a vector

$$\vec{a} = (a_x, a_y, a_x) \ , \tag{3.1}$$

as well as

$$\vec{b} = (b_x, b_y, b_z) \ , \tag{3.2}$$

then the sum of these vectors is

$$\vec{c} = (c_x, c_y, c_z) = \vec{a} + \vec{b} = (a_x, a_y, a_z) + (b_x, b_y, b_z) = (a_x + b_x,\ a_y + b_y,\ a_z + b_z) \ . \tag{3.3}$$

Please note how the above implies

$$c_x = a_x + b_x \ , \qquad c_y = a_y + b_y \ , \qquad c_z = a_z + b_z \ . \tag{3.4}$$

Immediately, from this we can see that

$$\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a} \ , \tag{3.5}$$

that is, that vector addition is *abelian* (don't panic! "Abelian" just means "commutative", or "invariant of order"). This is of course because the ordinary addition is commutative as well, and here we just add the scalar coefficients.

We also have the following property called *associativity*:

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \ . \tag{3.6}$$

That is, it does not matter whether we add $\vec{b}$ to $\vec{c}$ first and only then add the resulting sum to $\vec{a}$, or whether we add $\vec{a}$ to $\vec{b}$ first and then add that to $\vec{c}$.

You may raise your eyebrows at this point as to why we point out something that obvious, however, be aware that not all operations have this property. Actually, you already know one such operation! Clearly, for the operation of division we have $12 : (6 : 2) = 4$, while $(12 : 6) : 2 = 1$. Can you think of any other operation that is not associative?

B. Multiplication by a scalar

If we multiply our vector by an ordinary number n, we get a new vector defined as

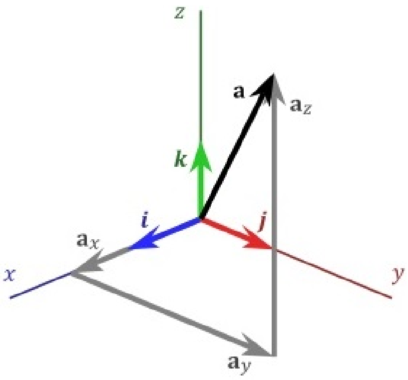
$$n \cdot \vec{a} = n \cdot (a_x, a_y, a_z) = (na_x, na_y, na_z) \ . \tag{3.7}$$

For example, for $n = 2$ that would be $2\vec{a} = (2a_x, 2a_y, 2a_z)$.

Try to convince yourself that in result we just obtain a vector whose magnitude is n times the original magnitude. Note that it is perfectly possible that $n < 1$ or oven $n < 0$ - in the latter case the vector not only changes its magnitude by a factor of $|n|$, but also *reverses direction*.

C. Unit vectors

A unit vector is just a vector of length one and a given orientation. A notion of such unit vectors may become very useful if we want to study the properties of some phenomenon in a given direction. As a very simple, but nonetheless useful example let us consider the unit vector along the x axis, which we will denote $\hat{\imath}$, as well as a unit vector along the y axis, denoted by $\hat{\jmath}$, and a unit vector along the z axis, denoted by $\hat{k}$:



Then, using the addition and multiplication by a scalar, we have

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} \ , \tag{3.8}$$

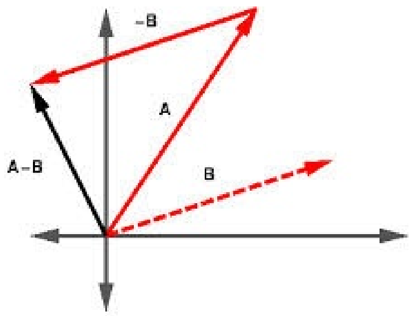
which is an extremely useful way of representing a vector, as you will shortly see yourself.

D. Subtraction

The subtraction of a vector is conveniently defined as addition of a negative vector, that is

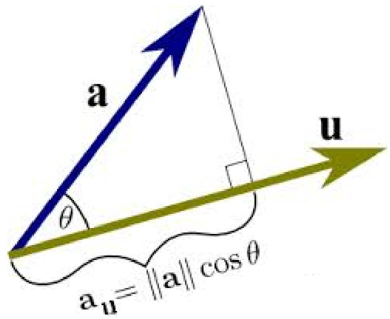
$$\vec{b} - \vec{a} = \vec{b} + (-\vec{a}) = (b_x, b_y, b_z) + (-a_x, -a_y, -a_z) = (b_x - a_x,\ b_y - a_y,\ b_z - a_z) \ . \tag{3.9}$$

Geometrically, it is easy to assume the same strategy in order to subtract a vector:



E. Vector projection

A vector projection is a projection of a vector onto another vector, some direction, or a plane. In case of a projection of the vector $\vec{a}$ onto another vector $\vec{u}$ (we will call this projection a_u) it looks like this:



Intuitively, we may say that a_u is "a component of $\vec{a}$ along the direction of $\vec{u}$. Using basic trigonometry we can see that it is equal

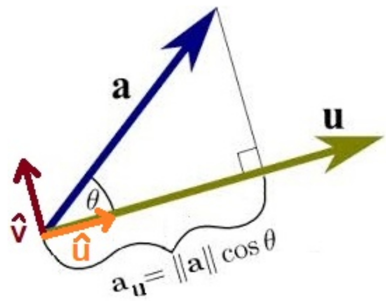
$$a_u = |\vec{a}| \cos\theta \ , \tag{3.10}$$

where θ is the angle between $\vec{a}$ and $\vec{u}$.

Projection is a very useful concept! For example, we can always decompose a vector into its component along some vector and a component perpendicular to this vector. Clearly, in our example this would be

$$\vec{a} = |\vec{a}| \cos\theta \hat{u} + |\vec{a}| \sin\theta \hat{v} \ , \tag{3.11}$$

where by $\hat{v}$ we have denoted the direction perpendicular to $\vec{u}$ (picture below).



You will find yourself using projections later in the quarter, so make sure you understand this concept.

F. Scalar product

Let us first talk about the length of a vector $\vec{r} = (r_x, r_y, r_z)$. We denote this length as $|\vec{r}|$ or sometimes just r. By generalized Pythagoras' theorem we have

$$|\vec{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} \ . \tag{3.12}$$

Naturally, the length of a vector does not depend on the coordinate system and so we have as well

$$|\vec{r}| = \sqrt{r_x'^2 + r_y'^2 + r_z'^2} \ , \tag{3.13}$$

where " $'$ " denotes the coordinates of $\vec{r}$ in some other coordinate system. The above implies that

$$r_x^2 + r_y^2 + r_z^2 = r_x'^2 + r_y'^2 + r_z'^2 \ , \tag{3.14}$$

i.e., the sum of the squares of components of a vector is a quantity that does not depend on the coordinate system.

Now, we *define* a new operation, $\vec{a} \cdot \vec{a}$, to be given by

$$\vec{a} \cdot \vec{a} = a_x^2 + a_y^2 + a_z^2 \ . \tag{3.15}$$

From the above discussion we can see that $\vec{a} \cdot \vec{a}$ is independent of the coordinate system used.

We also define an analogous quantity for two different vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \ , \tag{3.16}$$

and we can again find that this quantity remains the same for all coordinate systems. This quantity (as well as the operation that leads to it) is called a *scalar product*. (This double naming may feel a bit awkward, but it's really similar to "sum" and "summing", "multiple" and "multiplying" etc.) An alternative name for the scalar product is also the *dot product*. In the future we will use both these names.

As any operation, scalar product has a number of properties.

First, we have

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \ . \tag{3.17}$$

Do you know why that is?

Yes, it is because multiplication of ordinary numbers, as in (3.16), is commutative.

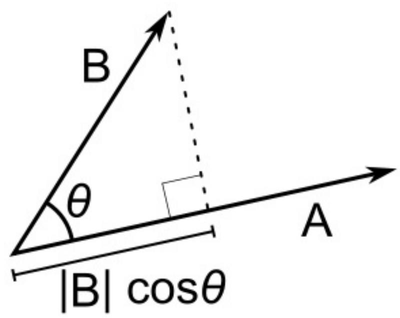
It can be easily shown (try showing it yourself by employing the notation used in (3.16)!) that

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \ , \tag{3.18}$$

i.e., that the scalar product is *distributive* (you will probably never use this word again, but I provide it for completeness).

By choosing a specific coordinate system (and by "specific" we really mean here "easy to visualize") we can easily show the following: Let us take $\vec{a}$ such that it lies along the x axis. Then the only nonzero component of $\vec{a}$ is a_x. Now the x component of $\vec{b}$ is just the projection of $\vec{b}$ onto the x axis, that is

$$b_x = |\vec{b}| \cos\theta \ , \tag{3.19}$$



where θ is the angle between vector vector $\vec{b}$ and x axis (or equivalently, in this very instance, between $\vec{b}$ and $\vec{a}$). Then, realizing that $a_x = |\vec{a}|$, we can write

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta. \tag{3.20}$$

Since we know that $\vec{a} \cdot \vec{b}$ does not depend on the coordinate system, the above equation is true in *every* coordinate system! In this way we have just obtained another, *equivalent* way of carrying out the scalar product operation. That is, the result of a scalar product is given by both

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \ , \tag{3.21}$$

as in defining relation (3.16), and by

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \ , \tag{3.22}$$

where $\theta = \angle(\vec{a}, \vec{b})$. It depends on the problem considered which of these two ways is more convenient to use.

Now, based on these results, can you tell when the scalar product is zero for two non-zero vectors?
Right, when $\cos\theta = 0$, that is when either $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. This means that the scalar product is zero for vectors that are *perpendicular to each other*.

Let us now see that using the unit vectors we can obtain a component of vector $\vec{a}$ along a given axis in a very neat way, that is taking the dot product:

$$\vec{a} \cdot \hat{i} = |\vec{a}|\,|\hat{i}| \cos\theta = |\vec{a}| \cos\theta = a_x \ . \tag{3.23}$$

Let us also note that since the axes of a cartesian coordinate system are perpendicular to each other, we have

$$\hat{i} \cdot \hat{j} = |\hat{i}|\,|\hat{j}| \cos\frac{\pi}{2} = 0 \ , \qquad \hat{j} \cdot \hat{k} = |\hat{j}|\,|\hat{k}| \cos\frac{\pi}{2} = 0 \ , \qquad \hat{k} \cdot \hat{i} = |\hat{k}|\,|\hat{i}| \cos\frac{\pi}{2} = 0 \ , \tag{3.24}$$

while

$$\hat{i} \cdot \hat{i} = |\hat{i}|\,|\hat{i}| \cos 0 = 1 \ , \qquad \hat{j} \cdot \hat{j} = |\hat{j}|\,|\hat{j}| \cos 0 = 1 \ , \qquad \hat{k} \cdot \hat{k} = |\hat{k}|\,|\hat{k}| \cos 0 = 1 \ . \tag{3.25}$$

This is in full agreement with our definition of the scalar product, namely let us see that

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) = \\
&= a_x b_x\, \hat{i}\cdot\hat{i} + a_x b_y\, \hat{i}\cdot\hat{j} + a_x b_z\, \hat{i}\cdot\hat{k} + a_y b_x\, \hat{j}\cdot\hat{i} + a_y b_y\, \hat{j}\cdot\hat{j} + a_y b_z\, \hat{j}\cdot\hat{k} + a_z b_x\, \hat{k}\cdot\hat{i} + a_z b_y\, \hat{k}\cdot\hat{j} + a_z b_z\, \hat{k}\cdot\hat{k} = \\
&= a_x b_x + a_y b_y + a_z b_z \ ,
\end{aligned} \tag{3.26}$$

where we have just used (3.24) and (3.25).

G. Vector product

A *vector product*, known also as a *cross product*, is denoted by

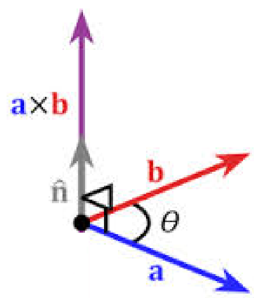
$$\vec{a} \times \vec{b} \ , \tag{3.27}$$

and is itself a vector. That means that the result of the above operation is susceptible to rotations, or simply put that $\vec{a} \times \vec{b}$ has a direction.
We define the vector product operation by defining the magnitude of the resulting vector as well as its direction. The magnitude of the vector product is given by

$$c = |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \ , \tag{3.28}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.
Can you tell when $\vec{a} \times \vec{b}$ will be equal to zero for nonzero vectors $\vec{a}$ and $\vec{b}$?
Yes, when either $\theta = 0$ or $\theta = \pi$. In particular, a vector product of any vector with itself is equal zero!

The direction of $\vec{c} = \vec{a} \times \vec{b}$ is given by the so-called *right-hand rule* as follows: arrange the middle finger, the index finger and the thumb of your right hand such that they are mutually perpendicular to each other. Then, keeping this arrangement fixed, twist your hand is such a way that the index finger points in the direction of the first vector in the vector product (here $\vec{a}$), while the middle finger points in the direction of the second vector in the vector product (here $\vec{b}$). Then the thumb points in the direction of $\vec{c}$. Note that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$!



We should also note that

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \ , \tag{3.29}$$

that is, a vector product is *not* commutative!

Now let us see how the vector product operation looks when we use the unit vectors representation:

$$\begin{aligned} &\vec{a} \times \vec{b} = (a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}) \times (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}) = \qquad (3.30)\\ &= \ a_xb_x\ \hat{\imath} \times \hat{\imath} + a_xb_y\ \hat{\imath} \times \hat{\jmath} + a_xb_z\ \hat{\imath} \times \hat{k} + a_yb_x\ \hat{\jmath} \times \hat{\imath} + a_yb_y\ \hat{\jmath} \times \hat{\jmath} + a_yb_z\ \hat{\jmath} \times \hat{k} + a_zb_x\ \hat{k} \times \hat{\imath} + a_zb_y\ \hat{k} \times \hat{\jmath} + a_zb_z\ \hat{k} \times \hat{k} \ . \end{aligned}$$

As already noted above, the vector product of a vector with itself is equal zero, and it is no different in the case of the unit vectors:

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0 \ . \tag{3.31}$$

Now, one by one, using the right-hand rule, we work out that

$$\begin{aligned} \hat{\imath} \times \hat{\jmath} = \hat{k} \ , &\qquad \hat{\jmath} \times \hat{\imath} = -\hat{k} \ , \\ \hat{\jmath} \times \hat{k} = \hat{\imath} \ , &\qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \ , \\ \hat{k} \times \hat{\imath} = \hat{\jmath} \ , &\qquad \hat{\imath} \times \hat{k} = -\hat{\jmath} \ , \end{aligned} \tag{3.32}$$

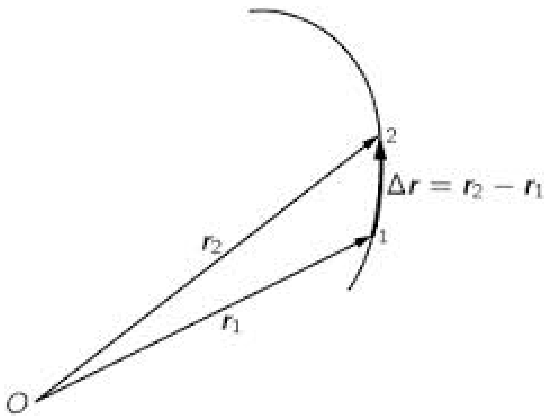
so that (3.30) becomes

$$\begin{aligned} \vec{a} \times \vec{b} \ &= \ a_xb_y\ \hat{k} - a_xb_z\ \hat{\jmath} - a_yb_x\ \hat{k} + a_yb_z\ \hat{\imath} + a_zb_x\ \hat{\jmath} - a_zb_y\ \hat{\imath} = \\ &= \ \big(a_yb_z - a_zb_y\big)\hat{\imath} + \big(a_zb_x - a_xb_z\big)\hat{\jmath} + \big(a_xb_y - a_yb_x\big)\hat{k} \ . \end{aligned} \tag{3.33}$$

The above equation presents an equivalent way of calculating a vector product. Let's note that this prescription gives us a vector, that is a magnitude with a direction. Apart from the initial relations (3.32), which never change, we did not have to use the right-hand rule at all!

H. Differentiation

As an example of vector differentiation let us ask ourselves how to calculate velocity using vectors?



As shown in the figure above, the distance that a particle moves in a short time $\Delta t = t_2 - t_1$, if we find it to be "here" at one point in time (say t_1) and "there" at another (say t_2), is the vector difference of the positions taken by the particle at times t_1 and t_2:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \ . \tag{3.34}$$

By dividing this by the short time interval $\Delta t = t_2 - t_1$ we get the "average velocity" vector. We get the exact quantity (i.e., the velocity at time $t = t_1$) in the limit

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \ . \tag{3.35}$$

If the last equation seems unfamiliar, you are strongly urged to get acquainted with basic calculus. It is possible that you are taking a calculus course this quarter. Try to get ahead and get at least an intuitive notion of differentiation and integration as soon as possible. We will use them very often!

Can you think of any equation you know that involves differentiating a vector?
That would be the Newton second law of motion, of course:

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} \ . \tag{3.36}$$

It took long, but here you see at last the first of many examples illustrating that physics *needs* vector calculus.