Challenge Problem 17

N women, each of mass m, stand on a railway flatcar of mass M. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction.

(a) What is the final velocity of the flatcar if all the women jump at the same time?

(b) What is the final velocity of the flatcar if the women jump off one at a time?

(c) Which answer, either (a) or (b), is greater? Try to make physical and/or intuitive sense of this answer and comment on your thoughts.

Solution. We apply momentum conservation in the x-direction. Before the women jump, the total momentum of the women plus flatcar in the x-direction is zero, and it remains zero after the women jump.

(a) When all women jump at the same time, momentum conservation gives

$$0 = Nm(u + V) + MV \tag{1}$$

where we have used the fact that the velocity of the women relative to the ground after the jump will be $u + V$, their velocity relative to the flatcar plus the velocity of the flatcar relative to the ground. Solving for V gives

$$\boxed{V = -\frac{Nm}{Nm + M}u} \tag{2}$$

(b) Let V_n denote the velocity of the cart just after the n^{th} woman has jumped, then momentum conservation for her jump relates the velocity V_{n-1} of the women plus freight car just after the $(n-1)^{\text{st}}$ jump to the velocity V_n of the women plus freight car just after her the n^{th} jump;

$$[(N - n + 1)m + M]V_{n-1} = [(N - n)m + M]V_n + m(u + V_n) \tag{3}$$

A bit of rearranging of this equation gives

$$V_n = V_{n-1} - \frac{m}{(N-n+1)m + M} u \tag{4}$$

We want to compute V_N. We can do this by using the relation above repeatedly. First, we plug $N = n$ into the relation:

$$V_N = V_{N-1} - \frac{m}{m+M} u \tag{5}$$

Then we replace V_{N-1} by its expression in terms of V_{N-2} using the relation above:

$$V_N = V_{N-2} - \frac{m}{m+M} u - \frac{m}{2m+M} u \tag{6}$$

If we keep doing this, we get

$$\boxed{V_N = -\left[\frac{m}{m+M} + \frac{m}{2m+M} + \cdots + \frac{m}{Nm+M}\right] u} \tag{7}$$

Notice that

$$\begin{aligned}
&\left(\frac{m}{m+M} + \frac{m}{2m+M} + \cdots + \frac{m}{Nm+M}\right) u && (8)\\
&\qquad > \left(\frac{m}{Nm+M} + \frac{m}{Nm+M} + \cdots + \frac{m}{Nm+M}\right) u && (9)\\
&\qquad = \frac{Nm}{Nm+M} u && (10)
\end{aligned}$$

The very left hand side is the factor is the answer for the speed from part (b), and the very right hand side is the speed from part (a), so the cart gets going faster if the women jump one after another.

We can understand the physical intuition behind this in a particular limiting case. Suppose that each woman is very massive compared to the freight car, and that N, the initial number of women, is also large. If all of the women jump at once, then our answer from part (a) in this limiting case gives $V \approx -u$. On the other hand, suppose that $N-1$ women jump first, and then the last woman jumps. Since N is large, $N-1$ women jumping first will again result in the cart having velocity approximately equal to $-u$, but now if the last woman jumps, she has a significant additional effect on the velocity because she is massive compared to the freight car. Therefore, this scheme results in a higher speed than if all women jumped at once.