Physics 1A
Spring 2015

Challenge Problem 16

A freight car of mass M contains a mass of sand m. At $t = 0$, a constant, horizontal force of magnitude F starts being applied, and at the same time a port in the bottom is opened to let the sand flow out at a constant rate γ (γ has dimensions of mass over time). Find the speed of the freight car at the first instant when all of the sand is gone.

Solution. We consider the system consisting of the mass m that is originally in the freight car plus the freight car itself with has mass M. Let's consider a time $t > 0$ and compare the total momentum in the system at this time to the total momentum in the system at some later time $t+\Delta$. The momentum of the car plus the sand that remains inside at time t is

$$P(t) = (M + m - \gamma t)v(t) \tag{1}$$

where $v(t)$ is the velocity of the car at time t. The momentum of the same objects at time $t + \Delta t$ is

$$P(t + \Delta t) = (M + m - \gamma(t + \Delta t))(v(t) + \Delta v) + (\gamma \Delta t)(v(t) + \Delta v) \tag{2}$$

where the second term on the right is the momentum of the mass that fell out. The rate of change of momentum of the system is therefore

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = \lim_{\Delta t \to 0} (M + m - \gamma t)\frac{\Delta v}{\Delta t} \tag{3}$$

$$= (M + m - \gamma t)\frac{dv}{dt} \tag{4}$$

Notice that we did not consider the mass of sand that had already fallen out by time t because despite the fact that it's part of the system, it's momentum doesn't change from t to $t + \Delta t$, so it wouldn't have affected this calculation leading to dP/dt. Now recall that the rate of change of the total momentum of the system equals the net external force on the system, so we get

$$F = (M + m - \gamma t)\frac{dv}{dt}. \tag{5}$$

This equation allows us to solve for the velocity $v(t)$ as a function of time of the cart and then use this to figure out the velocity of the cart at the time when all of the mass has fallen out. We can solve this by separation of variables and integration which gives

$$\int_0^t \frac{F}{M+m-\gamma t}\,dt = \int_0^{v(t)} dv, \tag{6}$$

where we have used the condition $v(0) = 0$. This gives

$$v(t) = -\frac{F}{\gamma}\ln\frac{M+m-\gamma t}{M+m}. \tag{7}$$

The time at which all of the sand has fallen out is when $m = \gamma t$, so the velocity at this time is

$$\boxed{v = \frac{F}{\gamma}\ln\frac{M+m}{M}}. \tag{8}$$