Physics 22: Homework 1 Hints

1. Since 1 mol of particles contains Avogadro's number of such particles, then 1 mol $= 6.02 \times 10^{23}$. Thus, 1.2 mol of H_2 consists of 7.22×10^{23} such pairs of hydrogen. As such, doubling this number would provide one with the number of hydrogen atoms, which would also serve as the number of electrons in such a collection since each hydrogen atom consists of one electron. To get the total charge, this number must be multiplied by the (negative) electron charge.

2. The point is to choose a combination of three quarks from the "up" and "down" variety such that the sum of their charge becomes $+e$ for the proton and 0 for the neutron. As should be the case, there is one unique combination that delivers this result.

3. The moral of the problem is that the gravitational force between objects with small values of mass can be neglected, particularly when such objects have even an insignificant amount of charge.

4. Because the charges are collinear, then all of the electrostatic force vectors will also be collinear due to the so-called "central" nature of the Coulomb Law.

   (a) Note that A will be attracted to both B and C, B will be attracted to A but repelled by C, and C will be attracted to A but repelled by B.

   (b) This is just a direct use of the Coulomb Law, and there is no need to worry about vector components since all the force vectors lie along the line running through the charges. Of course, the direction must be carefully considered, particularly for charge C, which feels an attraction in one direction and a repulsion in the opposite direction.

   (c) Note that B can't be placed to the left of A in such a circumstance, because there would be no way for C's repulsive force to balance A's attractive one. Moreover, the region in between charges A and C will not suffice either, as the forces that B would feel within that region point in the same direction. Thus, the only place the forces on B may balance is to the right of C. Nonetheless, one must also be careful about the condition on this position. In particular, if one is to set the magnitude of the force on B due to C equal to the magnitude of the force on B due to A to get the equilibrium condition, then one will inevitably get two possible solutions for the position. The point is to pick the correct position, as one of the values will be incorrect. The incorrect nature of this solution arises from the condition of setting the force magnitudes equal to one another. Just because the force magnitudes are set equal does not imply that the directions are opposite. Indeed, one of the possible solutions would be in the region between charges A and C, where we know that those forces would not oppose each other, but rather would reinforce each other. As for the notion of stability, the point is to ask whether or not B would want to restore back to its equilibrium position when it is slightly kicked off of that position. For true stability, the restoration should be independent of the kicking direction.

   (d) For this part, the field at the location of a charge must be due to the other charges acting as sources. Indeed, even though it is not explicitly stated, it should be clear that we do not have the proper tools to treat the field of a point charge at its exact position, as the field diverges under such circumstances. Thus, for example, the field at the location of B is the field of A and C (which act as the sources) at the location of B.

   (e) Based on the result of Part (d)—as well as our discussion in class—the field is a source-centric concept. Thus, for example, the field of charges A and C at the location of B is independent of B. Thus, removing the charge, B, has no bearing on the field at that location.

   (f) Because the arrangement is now in two dimensions, the field vectors—unlike in the linear arrangement in the previous parts—will no longer be collinear. As such, proper trigonometry and vector decomposition must be utilized to determine the force vectors (e.g., with the use of an xy-coordinate system).

5. Because the ball-bearings in this problem are presumed to be small (i.e., much smaller in diameter than the 1-m separation at which the forces are typically measured), then one may ignore the effects of polarization in the problem.

(a) Since the polarization effect may be ignored, the only way for the charged objects to attract is if they have opposite signs.

(b) Care must be taken regarding the fact that these ball bearings are identical in terms of geometry, size, and material composition. For this reason, one can use symmetry to get the proper conclusion for the charges when the exchange has occurred while touching one another.

(c) The key in this part is to use charge conservation,

$$q_1 + q_2 = q_1' + q_2',$$

where the "prime" denotes the charges on the ball-bearings after the exchange. Using this condition and the information provided, both in this part and at the beginning of the problem, will uniquely lead to the values of q_1 and q_2.

6. Again, because the metal spheres in this problem are presumed to be small (i.e., much smaller in diameter than the typical separation between them), then one may ignore the effects of polarization in the problem.

(a) This result follows from considering the fact that the weights of the spheres are identical, and that the string-lengths are identical. Even though the charges are presumed to be equal—and, of course, they must share the same sign for such an equilibrium configuration to be viable—they do not have to be in order to preserve the symmetry of the configuration. This is due to Newton's Third Law applied to the electrostatic interaction between the charges.

(b) Using Newton's Second Law and Coulomb's Law, the answer is: $q = \pm\sqrt{16\pi\varepsilon_0 mg\ell^2 \tan(\phi)\sin^2(\phi)}$.

(c) To do this part, it's important to see how changing one of the parameters changes the force diagrams on the spheres. In particular, it's crucial to see how the shifting of each parameter changes the geometry of the vector(s) involved, which then must inevitably result in a potential reconfiguration to output an equilibrium condition.

i. The angle of the string to which the mass that is increased is attached decreases, while the other angle increases.

ii. Both angles increase.

iii. The angle of the shorter string increases, while the other string's angle decreases.

(d) In this case, not only do the spheres feel a mutual repulsion from the Coulomb interaction between one another, each sphere also feels the effect due to the externally applied field. A properly drawn force diagram for one of the spheres will suffice in terms of determining the value of the field necessary to keep this configuration in equilibrium. In other words, it's redundant to do a force analysis on each charge, as the results will be identical.

i. The positive nature of q follows essentially from the direction of the uniform field. In particular, on top of the repulsive force between the charges, the tension, and the weight, the electric field introduces a force on each charge that can only result in the equilibrium configuration shown only if the charges are deemed positive.

ii. After some careful trigonometric considerations and proper vector decompositions, the result is:

$$E = \frac{1}{16\pi\varepsilon_0}\frac{q}{\ell^2}\frac{\cos(\alpha/2)}{\sin^2(\alpha/2)}.$$

7. This problem was (or will be) practically fully done in class. For the field along the y-axis, there are technically three relevant regions: (i) for $y > d/2$, (ii) for $y < |d/2|$, and (iii) for $y < -d/2$. In all cases, the distance to the positive charge will be $r_+ = y - (d/2)$, while the distance to the negative charge will be $r_- = y + (d/2)$. However, the fields oppose for $y > |d/2|$ and will reinforce for $y < |d/2|$. As far as determining the maximum of the field along the y-axis, the result is trivially ambiguous, as the field will technically diverge at the locations of the point charges. Because infinite fields are not realistic, the implication of such a divergence is that there really are no such things as "point charges." In particular, all charges must have some dimension, even if it is too small to measure. For the field along the x-axis, the result may be obtained without any separation

into different regions. In particular, note that the field components along the x-axis cancel, while the field components along the y-axis actually reinforce in a way that the y-component of the total field contribution due to the dipole is twice the y-component contribution of one of the poles. As such, the result for the field maximum along the x-axis is well-defined, as—unlike the result for Part (b)—there are no such divergences at any point along that axis. As for the approximations, proper manipulations must be made to compare 1 to the ratio of length scales. Once morphed into such a form, the binomial approximation must be used, at most, to first order. In particular,

$$(1 \pm z)^n \approx 1 \pm nz,$$

so long as $z \ll 1$.

8. This problem is a slight variation on Problem 7. The procedures are practically identical, but the results are a bit different. For instance, such a system, when viewed from afar, will not seemingly have the charges look to cancel, but would rather look like a monopole of charge $2q$. For the field along the y-axis, there are technically three relevant regions: (i) for $y > d/2$, (ii) for $y < |d/2|$, and (iii) for $y < -d/2$. In all cases, the distance to the upper positive charge will be $r_{\mathrm{u}} = y - (d/2)$, while the distance to the lower negative charge will be $r_{\mathrm{l}} = y + (d/2)$. However, the fields reinforce for $y > |d/2|$ and will oppose for $y < |d/2|$. For the field along the x-axis, the result may be obtained without any separation into different regions. In particular, note that the field components along the y-axis cancel, while the field components along the x-axis actually reinforce in a way that the x-component of the total field contribution due to the dipole is twice the x-component contribution of one of the poles.

9. Since we must determine the field contribution for this dipole, for example, at any observation point $x > d/2$ in order to ultimately take the limit, note that the distance to the negative charge will be x, the distance to the left-hand positive charge will be $x + (d/2)$, and the distance to the right-hand positive charge will be $x - (d/2)$. To get the result desired, the binomial approximation must be used to *second* order, as the use of the first-order (or, for that matter, the zeroth-order) approximation will result in a null result. The binomial approximation under these circumstances takes the form,

$$(1 \pm z)^n \approx 1 \pm nz + \frac{n(n-1)}{2} z^2,$$

so long as $z \ll 1$. The answer is:

$$\vec{E}_{\mathrm{quad}} \approx \hat{x} \frac{3qd^2}{8\pi\varepsilon_0} \frac{1}{x^4}.$$

10. This problem was (or will be) partly worked out in class.

    (a) No matter how the dipole is oriented, the force on the positive pole will be equal and opposite to the force on the negative pole.

    (b) Although it's most transparent to determine the answer from the midpoint of the dipole as the reference, the answer will turn out to be independent of this reference! The result is:

    $$\vec{\tau}_{\mathrm{dip}} = \vec{p} \times \vec{E},$$

    where $\vec{p} = q\vec{d}$ is the dipole, with $\vec{d}$ a vector that has its magnitude as the dipole separation, and has its direction pointing from the negative pole to the positive one.

    (c) The work done on the dipole due to an external agent will exactly be the change in the dipole potential energy, which was (or will be) worked out in class. The result is:

    $$W_{\mathrm{me}}^{(\mathrm{i}\to\mathrm{f})} = \Delta U_{\mathrm{dip}} = \Delta\left(-\vec{p}\cdot\vec{E}\right) = -\Delta\left(\vec{p}\cdot\vec{E}\right) = -\left(\vec{p}\cdot\vec{E}\right)_{\mathrm{f}} + \left(\vec{p}\cdot\vec{E}\right)_{\mathrm{i}},$$

    where $\vec{p}\cdot\vec{E} = pE\cos(\theta) = (qd)\,E\cos(\theta)$, with θ the angle between the tails of $\vec{p}$ and $\vec{E}$.

    (d) The stable equilibrium occurs when the net force and the net torque on the dipole is zero, such that when the dipole is perturbed from such a position, then it wants to restore back to that configuration. This occurs when $\theta = 0$. The unstable equilibrium occurs when the net force and the net torque on the dipole is zero, such that when the dipole is perturbed from such a position, then it does not want to restore back to that configuration. This occurs when $\theta = \pi$.

(e) The SHM differential equation has the form

$$\beta \frac{d^2 z}{dt^2} + \gamma z = 0,$$

for a physical quantity $z = z(t)$ undergoing such motion, with constant coefficients $\beta, \gamma > 0$. Using Newton's 2nd Law for rotational dynamics,

$$\vec{\tau}_{\text{net}/A} = I_{\text{A}} \vec{\alpha},$$

with $\vec{\alpha}$ the angular acceleration, then we find

$$(Eqd) \sin(\theta) = -I \frac{d^2 \theta}{dt^2},$$

where the negative sign arises from the fact that, for example, as θ increases, then the angular speed decreases (since the angular acceleration opposes the angular velocity). Rearranging, we find

$$I \frac{d^2 \theta}{dt^2} + (Eqd) \sin(\theta) = 0,$$

which is clearly not the SHM differential equation due to the $\sin(\theta)$ term. However, recall that if $\theta \ll 1$ (in radians), then by the small-angle approximation (i.e., Taylor's Theorem to leading order),

$$\sin(\theta) \approx \theta.$$

Under such a scheme, the differential equation becomes

$$I \frac{d^2 \theta}{dt^2} + (Eqd)\, \theta = 0,$$

which is SHM. To get the oscillation frequency, first divide by the MI to get

$$\frac{d^2 \theta}{dt^2} + \frac{Eqd}{I} \theta = 0.$$

The coefficient of the term with θ is the square of the angular frequency of oscillation

$$\omega^2 = \frac{Eqd}{I},$$

so that since the relationship between the angular frequency and the frequency is $\omega = 2\pi f$, then we find that

$$f = \frac{1}{2\pi} \sqrt{\frac{Eqd}{I}},$$

as desired.

11. Since the field of a point charge is not spatially uniform, it will turn out that the dipole in this problem will not only feel a net force, but also a net torque.

(a) Note that the forces on the positive and negative poles will have their x-components cancel by symmetry. Thus, the net force will be purely in the $+y$-direction, and will be double the y-component contribution from one of the charges. The result is:

$$\vec{F}_{\text{dip}} = \hat{y} \frac{Qqd}{4\pi\varepsilon_0} \frac{1}{\left[x^2 + (d/2)^2\right]^{3/2}}.$$

(b) Because the field in which the dipole finds itself is non-uniform, then the answer is not simply $\vec{p} \times \vec{E}$. Taking the $+z$-direction to be "out of the page," then referencing the dipole midpoint, the answer is:

$$\vec{\tau}_{\text{dip}} = -\hat{z} \frac{Qqd}{4\pi\varepsilon_0} \frac{x}{\left[x^2 + (d/2)^2\right]^{3/2}}.$$

(c) In this case, the force on Q due to the poles of the dipole must be determined. Of course, the result will be the opposite (in direction) to the result of Part (a).

(d) In this case, a zeroth-order binomial expansion will suffice.

(e) Recall that torque is a quantity quoted with respect to some predetermined reference point. Since the force on Q is directed in the $-\hat{y}$ direction, then any reference point along the y-axis will give zero torque; however, any other point will result in a nonzero torque.

12. In this problem, even though the plate is technically finite in size, it is presumed that one can ignore the field contribution from the edges (i.e., one may ignore so-called "edge effects"). Because the field is uniform and, therefore, the force felt by the proton is constant, then the proton will undergo constant-acceleration motion, with its acceleration directed perpendicular to, and toward, the plate. Thus, one inevitably may use the constant-acceleration kinematic formulas, for which the horizontal movement (i.e., parallel movement to the plate) is uniform, while the vertical movement is at a constant acceleration. As soon as the proton reaches the opposite edge, it will presumably move with a constant velocity. Thus, the angle of deflection is the direction of this final velocity relative to the horizontal.

13. This problem is identical to the one of free-fall movement of, for example, a football being kicked from the ground on a level field. One can, for example, determine that the time it takes for the electron to reach its maximum height from the plate is exactly half the time it takes for the electron to hit the plate after being launched.

14. Since the charge is uniformly distributed on the rod, then $\lambda = Q/\ell$.

(a) Clearly, the field will point along the x-axis pointing away from the rod, whether considering $x > \ell/2$ or $x < -\ell/2$. Each infinitesimal charge $dq = \lambda dx'$ is a distance $x - x'$ away from the observation point $x > 0$, where $x' \in [-\ell/2, +\ell/2]$ denotes the position of this charge in the space of the rod, and dx' is an infinitesimal length of this rod. Using the electric field of an infinitesimal point charge, the expression may be integrated over the aforementioned interval to determine that

$$\vec{E}\,(x) = \begin{cases} \hat{x}\frac{1}{4\pi\varepsilon_0}\frac{Q}{\left[x^2-(\ell/2)^2\right]} & \text{for } x > \ell/2 \\ -\hat{x}\frac{1}{4\pi\varepsilon_0}\frac{Q}{\left[x^2-(\ell/2)^2\right]} & \text{for } x < -\ell/2 \end{cases}.$$

(b) This was done in class. The answer is:

$$\vec{E}\,(y) = \hat{y}\frac{Q}{4\pi\varepsilon_0}\frac{1}{y\sqrt{y^2+(\ell/2)^2}}.$$

(c) This field will have an x-component as well as a y-component. To approach it, both components have to be determined from some arbitrary point on the rod, at, for example, an observation point with $0 < x < \ell/2$ and $y > 0$. Using x' to denote positions of infinitesimal charges on the rod (with, of course, $x' \in [-\ell/2, \ell/2]$) and $dq = \lambda dx'$, then the answers become

$$\vec{E}_x = \hat{x}\frac{Q}{4\pi\varepsilon_0\ell}\left[\frac{1}{\sqrt{\left[(\ell/2)-x\right]^2+y^2}} - \frac{1}{\sqrt{\left[(\ell/2)+x\right]^2+y^2}}\right]$$

and

$$\vec{E}_y = \hat{y}\frac{Q}{4\pi\varepsilon_0\ell}\frac{1}{y}\left[\frac{(\ell/2)-x}{\sqrt{\left[(\ell/2)-x\right]^2+y^2}} + \frac{(\ell/2)+x}{\sqrt{\left[(\ell/2)+x\right]^2+y^2}}\right].$$

(d) This was also done in class. The answer is achieved by using the result to Part (b) (or Part (c)), then taking $\ell \to \infty$ while keeping $\lambda = Q/\ell$ fixed. The result is:

$$\vec{E} = \hat{y}\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{y}.$$

(e) The limit to be taken here is the case with $y \ll \ell$, essentially approaching zero.

15. In this case, since the rod is placed symmetrically about the origin, then from the definition of the absolute-value function, the functional form of this charge density may be explicitly written as

$$\lambda(y) = \begin{cases} \alpha y & \text{for } y \geq 0 \\ -\alpha y & \text{for } y < 0 \end{cases}.$$

For Part (b), one must integrate the charge density over the whole length of the rod to determine the total charge:

$$Q = \int \lambda dy.$$

However, the integral must be split up for $y \in [-L/2, 0]$ and for $y \in (0, L/2]$. In order to determine the field along the perpendicular bisector, note that the charge distribution of the rod has reflective symmetry about $y = 0$ (i.e., the x-axis). For this reason, taking the infinitesimal charges pairwise will eliminate the y-components of the field, only leaving the x-component. Furthermore, it will be true that the x-components of the field for each pair will be twice the x-component of the field of one of the pairs. In this way, the integration can be performed to obtain the final result.

16. Note that since the field is being determined at the center of curvature, all of the distances to the infinitesimal point charges on the arc will be identical, since these distances serve equally as the radius of the arc.

(a) Note that because the charge density is constant, and since all charges are equidistant from the observation point, then it helps to make use of pairs of charges, such that a charge at a point (x, y) and a mirror-image charge at a point $(-x, y)$ end up having their x-components in the electric field cancel. The y-components are identical, so that the total y-component of the field is simply double of one of the pairs. Since the integration will depend on θ, it is worthwhile to convert the arc length, ds, into a form involving the angle, $d\theta$, subtended by dq. Specifically, $ds = Rd\theta$, which follows from the arc-length formula of a circular arc.

(b) This result is an extension of Part (a) for a semicircle, and should, again, output a field that is purely along the y-axis.

(c) This result is an extension of Part (a) for a full circle. The qualitative analysis is to see, without any use of detailed math, that the result should be zero for the field at the center.

17. This problem was (or will be) thoroughly done in class.

(a) This follows by using diametrically opposite dq's to see that the off-axis components cancel in pairs, while the on-axis components reinforce.

(b) The result is:

$$\vec{E} = \hat{z}\frac{Q}{4\pi\varepsilon_0}\frac{z}{(a^2 + z^2)^{3/2}}.$$

(c) To find these extrema, it's a matter of analyzing the functional form of the result to Part (b). Note that the form quoted above is also equally valid for $z < 0$.

(d) Alluding to the hint for Problem 10e, the dynamics law is, instead, the linear version of Newton's 2nd Law (i.e., Newton's 2nd Law). Suppose this charge is located at a position $z > 0$. Thus, the force on this charge will be

$$\vec{F} = -q\vec{E} = -\hat{z}\frac{qQ}{4\pi\varepsilon_0}\frac{z}{(a^2 + z^2)^{3/2}}.$$

Upon performing the binomial approximation to zeroth order, one can show that $F \sim z$, and that the force behaves like a Hooke's Law restoring force: $F = -\kappa z$. In so doing, Newton's 2nd Law becomes

$$m\frac{d^2 z}{dt^2} = -\kappa z,$$

which is exactly the SHM differential equation. Following suit to the procedures of Problem 10e, one may then determine the oscillation period, $\mathcal{T}$, defined in terms of the oscillation frequency as

$$\mathcal{T} = \frac{1}{f}.$$

18. This problem essentially takes the result of Problem 17b, and places one ring at a location $z = -b$ and another at $z = +b$. Specifically, one makes use of the Superposition Principle to determine the electric field at all relevant locations: (i) for $z < -b$, (ii) for $|z| < b$, and (iii) for $z > b$. The key is to properly shift the result of Problem 17b to properly obtain the expression for each electric field. Note that because the rings are charged equally (and uniformly), but oppositely, their fields will reinforce between the rings but will partially cancel outside of that region.

19. This is a fairly challenging problem, because it really takes patience and good visualization skills to resolve the components. Note that one cannot simply cancel out the off-axis components, because the arc lacks the mirror symmetry about the y-axis. Nonetheless, there is still some reflective symmetry present about the x-axis, which, upon taking mirrored pairs about this x-axis, will aim to cancel the field contributions along the y-axis. However, the field will have a component along the x- and z-axes. The component along the z-axis looks eerily similar to the value for the full ring:

$$\vec{E}_z = -\hat{z}\frac{|Q|}{4\pi\varepsilon_0}\frac{z}{\left(z^2+a^2\right)^{3/2}}.$$

Technically, if we started with the half ring and then formed its other half by adding additional charge with the same density, its z-component (which, of course, happens to be its only component) would be double the value of the above result for the half-ring's z-component. In this respect, this is nothing new. However, unlike the full ring, the half ring will have an off-axis component. In the setup provided in this problem, this off-axis component will be purely along the x-axis. The result will require an integration over the angle, ϕ, that may be defined as the angle that a radial line drawn from the origin to a point on the rim makes relative to the y-axis. If an infinitesimal arc of charge, dq, on this rim subtends an angle, $d\phi$, in the xy-plane, then one may represent the arc length encompassed by this charge as $ds = ad\phi$. In particular, there will be a non-trivial integration over this angle. Upon careful inspection and proper setup, the result for the x-component is:

$$\vec{E}_x = \hat{x}\frac{|Q|}{2\pi^2\varepsilon_0}\frac{a}{\left(z^2+a^2\right)^{3/2}}.$$

This result will agree with the answer to Problem 16b when $z = 0$.

20. The various parts of this problem was (or will be) extensively done in class.

    (a) The result is:

$$\vec{E} = \begin{cases} \hat{z}\frac{\mathcal{Q}}{2\pi\varepsilon_0 R^2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right) & \text{for } z>0 \\ -\hat{z}\frac{\mathcal{Q}}{2\pi\varepsilon_0 R^2}\left(1+\frac{z}{\sqrt{R^2+z^2}}\right) & \text{for } z<0 \end{cases}.$$

    (b) One will obtain the exact same result as in Part (c) of this problem, which was the limit taken in class. Ultimately, the answer to leading order is obtained by simply neglecting the second term in the parentheses for the field result in Part (a):

$$\vec{E} = \begin{cases} \hat{z}\frac{\mathcal{Q}}{2\pi\varepsilon_0 R^2} & \text{for } z>0 \\ -\hat{z}\frac{\mathcal{Q}}{2\pi\varepsilon_0 R^2} & \text{for } z<0 \end{cases}.$$

    (c) In this case, $R \to \infty$ while keeping the areal charge density, $\sigma = \mathcal{Q}/\left(\pi R^2\right)$, fixed. The result was found in class, and it is:

$$\vec{E} = \begin{cases} \hat{z}\frac{\sigma}{2\varepsilon_0} & \text{for } z>0 \\ -\hat{z}\frac{\sigma}{2\varepsilon_0} & \text{for } z<0 \end{cases}.$$

21. For the infinite rod, note that the field along the x-axis will have the form $E \sim 1/x$. However, note that each infinitesimal charge, $dq = (Q/\ell)\, dx$, on the finite rod is at a different position, $x \in [a, a+\ell]$. For this reason, the force must be integrated. Note that since the infinite rod is positively charged and the finite rod is negatively charged, the net force on the finite rod will be attractive and, thus, will be directed along the $-\hat{x}$ direction. Performing the integration, the answer is:

$$\vec{F} = -\hat{x}\frac{\lambda\,|Q|}{2\pi\varepsilon_0\ell}\ln\left(1+\frac{\ell}{a}\right).$$