Physics 1A
Spring 2015

Extra Credit 1

River is playing beer pong at his frat house, and he notices that when he bounces a ping-pong ball on the table, it seems to have a smaller speed after each bounce. Josh is a pretty baller physics student, and he figures out that on each successive bounce, the speed of the ball just after each bounce is $1/\sqrt{2}$ times the speed it had just before each bounce. Josh also notices that when River bounces the ball, its velocity just after each bounce makes an angle of $\pi/4$ radians with the table.

At what speed would River initially have had to throw the ping-pong ball so that the total horizontal distance it travels (including all subsequent bounces) after the first bounce is one meter? Neglect air resistance as usual

Solution. Notice, firstly, that if a projectile is thrown with an initial speed v at an angle θ relative to the horizontal, then assuming it is thrown from the origin, its x- and y-positions as a function of time will be given by

$$x(t) = v\cos\theta t \tag{1}$$

$$y(t) = v\sin\theta t - \frac{1}{2}gt^2 \tag{2}$$

If the projectile is thrown on flat ground, then the time at which it hits the ground is the nonzero solution to $y(t) = 0$;

$$0 = v\sin\theta\, t - \frac{1}{2}gt^2, \tag{3}$$

which is

$$t = \frac{2v\sin\theta}{g}. \tag{4}$$

Plugging this time into the equation for $x(t)$ gives the range r of the projectile.

$$r = \frac{v^2}{g}(2\sin\theta\cos\theta) = \frac{v^2}{g}\sin(2\theta). \tag{5}$$

Now, suppose that the projectile is a ball initially with a speed v_0 and that after it is thrown at the ground with this speed, it bounces up at an angle θ and with a speed $v_1 = f v_0$ where f is some fraction. In fact, after this initial bounce, it keeps bouncing at an angle θ, and its successive speeds after bouncing satisfy

$$v_n = f v_{n-1}, \tag{6}$$

Then one finds that

$$\begin{aligned} v_1 &= f v_0 \\ v_2 &= f v_1 = f^2 v_0 \\ &\vdots \\ v_n &= f^n v_0 \\ &\vdots \end{aligned}$$

The distance traveled on the n^{th} bounce is therefore

$$r_n = \frac{v_n^2}{g}\sin(2\theta) = f^{2n}\frac{v_0^2}{g}\sin(2\theta) \tag{7}$$

The ball will bounce an infinite number of times, but the total distance traveled will be finite;

$$\begin{aligned} d &= \sum_{n=1}^{\infty} r_n && (8) \\ &= \sum_{n=1}^{\infty} f^{2n}\frac{v_0^2}{g}\sin(2\theta) && (9) \\ &= \frac{v_0^2}{g}\sin(2\theta)\sum_{n=1}^{\infty}(f^2)^n && (10) \\ &= \frac{v_0^2}{g}\sin(2\theta) f^2 \sum_{n=0}^{\infty}(f^2)^n \\ &= \frac{v_0^2}{g}\sin(2\theta)\frac{f^2}{1-f^2} && (11) \end{aligned}$$

Therefore, if we want the ball to bounce a given total distance d, then the initial speed with which it needs to be thrown is

$$v_0 = \sqrt{\frac{gd}{\sin(2\theta)} \frac{1-f^2}{f^2}} \tag{12}$$

When $d = 1m$, $\theta = \pi/4$, and $f = \frac{1}{\sqrt{2}}$, we get

$$v_0 = \sqrt{g\,\mathrm{m}} = \boxed{\sqrt{9.8}\,\mathrm{m/s}} \tag{13}$$