Physics 22: Homework 7 Hints

1. The point of this problem is to use the magnetic force law on a point charge given the uniform field, $\vec{B} = B_0\hat{x}$, for the different velocity directions given in each of the parts.

   (a) $\vec{F}_B = \vec{0}$

   (b) $\vec{F}_B = -\hat{z}qvB_0$

   (c) $\vec{F}_B = \hat{y}qvB_0$

   (d) $\vec{F}_B = -\hat{z}qvB_0 \sin(\pi/4)$

2. Suppose this particle, of charge q, has a velocity $\vec{v}$ at the instant that it feels a magnetic field, $\vec{B}$ at some spatial location. By the definition of work and the magnetic force law, consider an infinitesimal displacement of this particle, $d\vec{r}$, along its trajectory:
$$dW_B = \vec{F}_B \cdot d\vec{r} = q\left(\vec{v} \times \vec{B}\right) \cdot d\vec{r}.$$
   Since this particle displaces by an amount, $d\vec{r}$, presumably in some time, dt, then we must have that the particle's velocity is given by, $\vec{v} = d\vec{r}/dt$, so that we may rewrite the above work expression as
$$dW_B = q\left(\vec{v} \times \vec{B}\right) \cdot \frac{d\vec{r}}{dt}dt = q\left(\vec{v} \times \vec{B}\right) \cdot \vec{v}dt.$$
   Recall that the cross product $\vec{v} \times \vec{B}$ is orthogonal to both $\vec{v}$ and $\vec{B}$. However, note that the dot product is an operation that is only sensitive to the component of one vector along the other. In particular, the dot product between two orthogonal vectors is zero. Thus,
$$dW_B = q\,(0)\,dt = 0.$$
   Since this infinitesimal work is zero, then the work done by this magnetic force over the entire path will also be zero.

3. Recall that the motion of a charged particle in a plane perpendicular to a uniform magnetic field is that of Uniform Circular Motion (UCM).

   (a) Because the kinetic energies of the particles are presumed the same, then because the kinetic energy, K, is related to the speed and mass, v and m, of the charged particle by
$$K = \frac{1}{2}mv^2,$$
   then
$$v = \sqrt{\frac{2K}{m}},$$
   so that the electron will have the higher speed since it has the smaller mass. So, as far as the magnetic force is concerned, since both particles have the same charge and are feeling the same field, the electron will undergo the larger magnetic force since this force is proportional to the speed of the charged particle. Since the centripetal force is proportional to the product of the mass and the square of the speed, then this product is invariant for either particle since they both have the same kinetic energy. Because the centripetal force is also inversely proportional to the orbital radius, then the particle that feels the larger magnetic force will have the smaller orbital radius. In particular, the electron will have the smaller orbital radius, which will be seen rigorously from the outcome of Part (b) below.

   (b) Since the charges of either the proton or electron have magnitude $e = 1.60 \times 10^{-19}$ C, and the velocity is, by construction, orthogonal to the uniform field, then
$$\left|\vec{F}_B\right| = \left|e\vec{v} \times \vec{B}\right| = evB = eB\sqrt{\frac{2K}{m}}.$$

This magnetic force is the centripetal force of this movement in a circle. In particular,

$$m\frac{v^2}{R} = \frac{m}{R}\frac{2K}{m} = \frac{2K}{R}.$$

Thus,

$$\frac{R}{2K} = \frac{1}{eB}\sqrt{\frac{m}{2K}},$$

and

$$R = \frac{1}{eB}\sqrt{2mK}.$$

4. Suppose this uniform magnetic field has components $\vec{B} = B_x\hat{x}+B_y\hat{y}+B_z\hat{z}$. Take $v_0 = 3.50$ km/s, $F_x = 7.60$ mN, and $F_z = 5.20$ mN.

   (a) Using the magnetic force law, on the one hand we have

$$\begin{aligned}\vec{F}_B = q\vec{v}\times\vec{B} &= q\left(-v_0\hat{y}\right)\times\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)\\ &= -qv_0\left(B_x\hat{y}\times\hat{x}+B_y\hat{y}\times\hat{y}+B_z\hat{y}\times\hat{z}\right)\\ &= -qv_0\left(-B_x\hat{z}+\vec{0}+B_z\hat{x}\right)\\ &= -qv_0B_z\hat{x}+qv_0B_x\hat{z}.\end{aligned}$$

   On the other hand, we have that

$$\vec{F}_B = F_x\hat{x} - F_z\hat{z}.$$

   Thus, we must have

$$\begin{cases} F_x &= -qv_0B_z \\ F_z &= -qv_0B_x \end{cases}.$$

   (b) Note from the analysis above that there could potentially be a magnetic field in the y-direction, but the magnetic force would not be sensitive to this component of the field since components of $\vec{v}$ and $\vec{B}$ that lie along parallel lines give no force since the force is related to the cross product of these vectors.

   (c) Because of the magnetic force law, since $\vec{v}\times\vec{B}$ is a vector that's perpendicular to both $\vec{v}$ and $\vec{B}$, then without performing any calculation, $\vec{F}_B\cdot\vec{B} = 0$.

5. The point here is to break the velocity vector into two components: one that is along the line defined by the magnetic field ($\vec{v}_{\parallel}$), and another that is along the line perpendicular to the magnetic field ($\vec{v}_{\perp}$). Thus, we may write

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}.$$

   (a) The component of the velocity parallel to the magnetic-field line will not administer any force from the magnetic field. Thus, $\vec{v}_{\parallel}$ will be a constant of the motion. However, the component of the velocity that is perpendicular to the magnetic-field line will undergo Uniform Circular Motion (UCM) in this plane that's perpendicular to the field line. Thus, as the particle moves in a circle in the plane perpendicular to the field, it simultaneously drifts with velocity $\vec{v}_{\parallel}$. In this way, the trajectory is helical.

   (b) Note that in the plane perpendicular to the field, the charged particle moves in a circle at a speed of $v_{\perp} = v\sin(\phi)$. Mimicking the process outlined in Problem 3 without regard to kinetic energy, Newton's 2nd Law would read

$$ev_{\perp}B = m_e\frac{v_{\perp}^2}{R},$$

   so that

$$R = \frac{m_e}{e}\frac{v\sin(\phi)}{B}.$$

Moreover, note that the particle undergoes a period

$$\mathfrak{I} = \frac{2\pi R}{v_\perp},$$

so that in this time, the particle undergoes one full revolution in its circular movement. However, since the particle also simultaneously moves along the field line at a speed $v_\parallel = v\cos(\phi)$, then the pitch can be determined from the distance traveled along the field in exactly one period:

$$p = v_\parallel \mathfrak{I}.$$

6. The point of this problem is to get the charged particles moving rightward through the parallel-plate assembly to undergo no net force in their trajectory, implying that the magnetic force must balance the electric force. Without loss of generality, assume that the incoming ion beam consists of positively charged particles. Under these circumstances, the electric force will point downward, since these charges will be repelled by the top plate and attracted to the bottom plate. Since the magnetic field is oriented perpendicular to the plane of the page—and, thus, perpendicular to the electric-field lines in between the plates—then this magnetic field must generate a force upward on these positive charges. The only way this would be feasible is if the magnetic field points into the page, so that $q\vec{v} \times \vec{B}$ is purely upward if $\vec{v}$ is purely rightward. For velocities that are not oriented purely rightward, the electric field—if the only field present—will cause these charges to move parabolically and, thus, for a sufficiently long plate region, will inevitably hit the plate. Moreover, if only the magnetic field is present, this will cause the charges to undergo circular motion, which would also inevitably hit the plates. All in all, any velocity not moving parallel to the length of the plates will naturally be doomed to hit the plates in the presence of both fields. Thus, this system truly does select for proper beam directionality, so that it does act like a collimator. However, the speed must also be well balanced, because if the speed is too low, the electric force will overwhelm the magnetic one, while if the speed is too high, the magnetic force will, instead, overwhelm the electric one. For the proper speed magnitude, the answer is:

$$v = \frac{E}{B}.$$

7. Because the positively charged ions move through a potential difference (from high potential to low potential), they will accelerate. Via the use of mechanical energy conservation, one may write that the exit speed of the ions from the collimator is given by the condition

$$\frac{1}{2}mv^2 = qV,$$

so that the speed may be determined.

   (a) At the instant these ions enter the magnetic-field region, they must feel a rightward magnetic force. For this reason, this field must point outward.

   (b) Because the cations move through an angle of π in exiting the collimator and hitting the collector plate, the distance from the collimator to the collector plate must be the diameter, $D = 2R$, of the circular trajectory undergone by the charges. Using the magnetic force law and Newton's 2$^{\text{nd}}$ Law (noting, of course, that the acceleration is a centripetal one), the answer is:

$$D = \sqrt{\frac{8mV}{q}}\frac{1}{B}.$$

   (c) Note that the distance at which the cations hit is a function of the mass-to-charge ratio of each cation. Note that given a huge collection of such ions with, presumably different such ratios, the target locations will be very different. If there is some knowledge as to what types of ions are present in the system, the collector plate can distinguish one species from another based on a proper calibration, and sensitive measurement, of the touchdown points of each species along the length of the collector plate.

8. Because the negative charges are the ones that are actually moving, then one can see from the figure that they will populate the side of the slab with terminal a, while leaving vacancies on the opposite end of the slab (with terminal b).

   (a) Since terminal a is negatively charged and terminal b is positively charged, then $V_{\text{ab}} < 0$.

   (b) In the steady state, there will be some static concentration of electrons on the side of the slab with terminal a, with an equal amount of vacancies on the side of the slab with terminal b. As such, these static charges will generate an electric field from b to a of magnitude

   $$E = \frac{\Delta V_{\text{Hall}}}{w},$$

   since this configuration would act to generate a uniform field based on the uniform nature of slab, as well as the fact that we are ignoring the edge effects of the slab. For any subsequent electrons moving along the length of the slab agains the conventional current, the steady state situation really occurs when these electrons feel equal magnetic and electric forces. The magnetic force, of course, is generated by the externally applied magnetic field, while the electric force is generated by this charge accumulation on the sides of the slab. The force balance for a given electron with drift speed, v, is written as

   $$\begin{aligned} F_E &= F_B \\ eE &= evB, \end{aligned}$$

   so that

   $$\Delta V_{\text{Hall}} = wvB.$$

   However, the drift speed of such electrons may be written in terms of the current density of such carriers as

   $$J_e = nev,$$

   where n is the electron concentration and e is the magnitude of the electron charge. However, since current may be taken as the flux of current density, then we may write that

   $$I = J_e A = J_e wt,$$

   so that

   $$J_e = \frac{I}{wt}.$$

   The final answer is:

   $$\Delta V_{\text{Hall}} = \frac{IB}{net}.$$

9. In analogy with the electrical analogue of the torque on an electric dipole due to a uniform electric field, recall that the torque aimed to make the dipole moment of the electric dipole moment, $\vec{p}$, align with the direction of $\vec{E}$.

   (a) Using this electrical analog, the magnetic dipole moment, $\vec{\mu}$, will want to align itself with the direction of the uniform magnetic field, $\vec{B}$, since the magnetic dipole torque result

   $$\vec{\tau}_B = \vec{\mu} \times \vec{B},$$

   would aim to do the same thing based on the form of the torque formula. In this way, since the loop is aimed to "rise" from its initial placement in Quadrant I of the xz-plane, then we must have $\vec{B} = -B\hat{x}$ to achieve this.

   (b) Based on the Right-Hand Rule (RHR) for determining the direction of the magnetic moment, note that the tail of $\vec{\mu}$ makes an angle of $\pi/2 - \phi$ relative to the now-determined magnetic-field direction. As such,

   $$\tau_B = \left|\vec{\mu} \times \vec{B}\right| = \mu B \sin\left(\pi/2 - \phi\right) = \mu B \cos\phi.$$

Recalling that the gravitational torque acts at the Center of Mass (CM) of the loop which—based on the fact that the loop has a uniform shape and mass distribution on its perimeter—happens to be at the physical center of the loop, note that if we pick the reference axis about which to analyze the toque to be the z-axis, then the tail of the weight, $\vec{w} = m\vec{g}$, makes an angle ϕ relative to the moment arm, $\vec{r}$, from this axis, with $|\vec{r}| = b/2$. Thus

$$\tau_w = |\vec{r} \times \vec{w}| = \frac{1}{2} bmg \sin\phi,$$

which counters the torque due to the magnetic field. Imposing that the net torque is zero due to the equilibrium situation of the loop means that one will have to set these two torques equal to each other, which will thus output the desired field strength to achieve this torque balance specifically at the angle ϕ outlined.

(c) Again, since the magnetic dipole moment aims to align with the field direction in the torque process, in order to make the loop rise in the same manner from its initial placement in Quadrant I of the xz-plane, then we must have $\vec{B} = -B\hat{y}$. Indeed, if $\vec{B}$ was, instead, oriented along the $+y$-direction, then the loop would not feel any torque. Thus, it would not rise in the first place if a slight kick was given to the loop; however, if $\vec{B}$ points in the $-y$-direction, then a slight kick will make it rise in the same way. The kick is necessary, because then the loop is initially in Quadrant I of the xz-plane, there will be no torque in either circumstance. However, with $\vec{B}$ being antiparallel to $\vec{\mu}$, the loop will want to rise when kicked to allow for the two vectors to potentially align.

(d) When the field is oriented along the $-x$-direction, the equilibrium is stable since any minor kick away from equilibrium will result in a restoring torque to bring the loop back to the equilibrium states. However, when the field is oriented along the $-y$-direction, then the equilibrium is neutral, since the result actually happens to be independent of the angle!

10. The magnetic dipole moment of the bar magnet points in a direction that points form the South to the North pole of the magnet (somewhat analogously to how the electric dipole moment points in a direction from the negative to the positive pole).

(a) Using the fact that

$$\vec{\tau}_B = \vec{\mu} \times \vec{B},$$

then

$$\tau_B = \mu B \sin\phi,$$

which can then be solved for μ.

(b) Recall that the potential energy for such a dipole system (again in analogy to the electrical case) is given by

$$U = -\vec{\mu} \cdot \vec{B}.$$

Now, the work done by the field may be written in terms of the change in potential energy as

$$W_B^{(\mathrm{i}\to\mathrm{f})} = -\Delta U = U_\mathrm{i} - U_\mathrm{f},$$

and since the torque that one exerts to get the dipole to misalign from the field direction must be opposite to the torque provided by the field, then if the dipole starts from rest and ends at rest in the process of misalignment, then

$$W_\mathrm{me}^{(\mathrm{i}\to\mathrm{f})} = -W_B^{(\mathrm{i}\to\mathrm{f})} = U_\mathrm{f} - U_\mathrm{i}.$$

Since the initial state is that of the (stable) equilibrium configuration, then

$$W_\mathrm{me}^{(\mathrm{i}\to\mathrm{f})} = -\mu B\cos\phi - (-\mu B \cos 0) = \mu B\,(1 - \cos\phi)\,.$$

(c) Since the form for the dipole potential energy

$$U = -\vec{\mu} \cdot \vec{B}$$

itself references the zero point (where $\vec{\mu}$ and $\vec{B}$ are orthogonal), then the answer is just the value of U_f above.

(d) No it is not, as the zero point is reserved for when $\vec{\mu}$ and $\vec{B}$ are orthogonal. However, the lowest-energy state is when the two vectors are parallel. Indeed,

$$U_{\rm i} = -\mu B < 0,$$

where this refers to the initial state of this problem.

11. Define an xy-coordinate system with $+y$ upward and $+x$ downward with the center of the quarter-circle serving as the origin. Consider a radial line drawn from the center that hits a point on the quarter-circle, making an angle, θ, relative to the horizontal. Note that $d\vec{\ell} \times \vec{B}$ points towards the center of the circle, so that

$$d\vec{\ell} \times \vec{B} = -\hat{r} B ds,$$

where $\hat{r}$ is a unit vector pointing radially outward from the center, and $ds = R d\theta$ is the arc length of the infinitesimal displacement along the circle. Recall that one may write

$$\hat{r} = \cos\theta \hat{x} + \sin\theta \hat{y},$$

so that

$$I d\vec{\ell} \times \vec{B} = -IRB \left(\cos\theta \hat{x} + \sin\theta \hat{y}\right) d\theta,$$

with $\theta \in [0, \pi/2]$. Integrating this expression over this angular interval will then give the net magnetic force on the wire. Note that by symmetry, one can deduce that the direction of the magnetic force on this segment must point at an angle of 45° in Quadrant III of the xy-coordinate system described.

12. Based on what we have discussed in class, this law centers around the fact that there is no such thing (as far as we know experimentally) as a magnetic monopole. In other words, what we traditionally deem to be North and South "poles" on a magnet are not poles in the same sense as what we call poles for electric fields (i.e., positive and negative charges). Specifically, there is no way to isolate the poles of a magnet. For this reason, we say that there are no sources or sinks of magnetic fields. Unlike electric fields—where we consider positive charges to be sources of electric field (since they have field lines emanate from them), and negative charges to be sinks of electric field (since they have field lines terminate on them)—magnetic field lines are all connected. For this reason, when we consider a closed surface (i.e., a Gaussian surface), then all magnetic field lines that enter this surface also identically exit. Thus, the net influx of such field lines is always the same as the net outflux, leaving no net flux through this closed surface. This is the essence of the magnetic Gauss's Law, which showcases some of the asymmetries between electric and magnetic fields. Nonetheless, for this single asymmetry in the theory of electromagnetism, there are quite a few symmetries as well, the most significant one being that electric and magnetic fields can actually generate one another.