

# Final Project Documentation

Nashir Janmohamed

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## 1 Recurrence Relations/Dynamic Programming

### 1.1 Bell numbers

The Bell numbers represent the number of ways to count partitions of (or equivalently equivalence relations on) an  $n$  element set. The  $n$ -th bell number is given by the recurrence

$$B_n = \sum_{k=1}^n \binom{n-1}{k-1} B_{n-k}$$

for  $n \geq 0$ .

### 1.2 Catalan numbers

The Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects. They can be expressed by the recurrence relation

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$$

for  $n \geq 0$ .

Their closed form is given by

$$\binom{2n}{n} - \binom{2n}{n+1}$$

### 1.3 Fibonacci numbers

The Fibonacci numbers, commonly denoted  $F_n$ , form a sequence such that each number is the sum of the two preceding ones, with  $F_0 = 0$ ,  $F_1 = 1$ , and the recurrence given by

$$F_n = F_{n-1} + F_{n-2}$$

for  $n > 1$ .

## 1.4 Stirling numbers of the first kind

## 1.5 Stirling numbers of the second kind

# 2 Permutations and Combinations

## 2.1 Combinations without repetition

A combination without repetition is a selection of items from a collection, such that the order of selection does not matter. A  $k$ -combination of an  $n$  element set  $S$  is a subset of  $k$  distinct elements. The number of  $k$ -combinations is equal to the binomial coefficient given by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

## 2.2 Permutations without repetition

$k$ -permutations of  $n$  are the different ordered arrangements of a  $k$ -element subset of an  $n$ -set. This number is given by

$$P(n, k) = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

## 2.3 Combinations with repetition

A  $k$ -combination with repetitions allowed is a sequence of  $k$  not necessarily distinct elements of  $S$ , where order is not taken into account, i.e. the number of ways to sample  $k$  elements from a set of  $n$  elements allowing for duplicates but disregarding different orderings. Using the *stars and bars method*, it can be shown that this number is given by

$$\binom{n+k-1}{k}$$

## 2.4 Permutations with repetition

Permutations with repetition are ordered arrangements of  $k$  elements from a set  $S$  with  $n$  elements where repetition is allowed. The number of permutations with repetition of size  $k$  is simply  $k^n$  (except if  $k > n$ , where the result is 1).

## 2.5 Generate permutations of a string

Given a string  $s$  of length  $n$ , generate all  $n!$  permutations of  $s$ . For example, all the permutations of “the” are [“the”, “teh”, “het”, “hte”, “eth”, “eht”].

## 2.6 Generate all bit strings of length $n$

Given an integer  $n$ , all bit strings of length  $n$  can be recursively constructed by appending a 0 and a 1 to all bit strings of length  $n - 1$ .

Thus, there are  $2 * (s_{n-1})$  bit strings of length  $n$ . Since there are 2 bit strings of length 1, there are  $2^n$  bit strings of length  $n$ .

## 3 Relations

### 3.1 # of relations

A relation on an  $n$  element set  $S$  is a subset of  $S \times S$ , or equivalently, an element of the power set of  $S \times S$ . There are

$$2^{|S| \times |S|} = 2^{n^2}$$

such subsets.

### 3.2 # of transitive relations

There is no known closed formula for counting the number of transitive relations. The (perhaps inefficient) approach taken in this algorithm is as follows

- (1) Generate all possible relations for an  $n$  element set (given by the power set of the cartesian product of the set  $\{1, 2, 3, \dots, n\}$ )
- (2) For each relation generated in (1), check that for each  $(a, b)$ , if there is a point of the form  $(b, c)$ , then  $(a, c)$  must be in the relation

### 3.3 # of (ir)reflexive relations

A relation is reflexive if all elements are related to themselves, or equivalently, all entries on the main diagonal of the matrix representation of the relation must be 1. There are  $n^2$  entries in the matrix and  $n$  entries on the main diagonal. For the remaining  $n^2 - n$  off diagonal entries, the ordered pair may or may not be in the relation. Thus, there are

$$2^{n^2 - n}$$

reflexive relations. The argument for irreflexive relations is the same, with the exception that all entries on the main diagonal of the matrix representation of the relation must be 0.

### 3.4 # of symmetric relations

A relation  $R$  is symmetric if for all  $(a, b)$  that are in  $R$ ,  $(b, a)$  is also in  $R$ . Each element on the diagonal may or may not be related to itself, and similarly for

all the  $\binom{n}{2}$  two element subsets (with distinct elements). Thus, there are

$$2^{\binom{n}{2}+n} = 2^{\frac{n(n+1)}{2}}$$

symmetric relations on a set with  $n$  elements.

### 3.5 # of antisymmetric relations

A relation  $R$  is antisymmetric if for all  $(a, b)$  that are in  $R$ , if  $(b, a)$  is in  $R$ , then  $a = b$ . There are two choices for every element on the diagonal. For the remaining  $\binom{n}{2}$  two element subsets (with distinct elements) with elements  $a$  and  $b$ , either  $(a, b) \in R$  and  $(b, a) \notin R$ ,  $(a, b) \notin R$  and  $(b, a) \in R$ , or  $(a, b) \notin R$  and  $(b, a) \notin R$ , so there are 3 choices for each two element subset. Thus, there are

$$2^n 3^{\binom{n}{2}} = 2^n 3^{\frac{n(n-1)}{2}}$$

antisymmetric relations on a set with  $n$  elements.

### 3.6 # of equivalence relations

A relation  $R$  on a set  $A$  is an equivalence relation if it is reflexive, symmetric, and transitive. For each  $a \in A$ , the equivalence class of  $a$  is given by  $[a] = \{x \mid xRa\}$ . The equivalence classes form a partition of  $A$ , and so the number of equivalence relations on a set  $S$  is given by the number of partitions of a set  $S$ . So, this number is equivalent to the Bell number (number of partitions/equivalence relations for an  $n$  element set) which we can compute directly.

$$B_n = \sum_{k=1}^n \binom{n-1}{k-1} B_{n-k}$$

for  $n \geq 0$ .

## 4 Sets

### 4.1 Generate power set

The power set of a set  $S$  is the set of all subsets of  $S$ , denoted by  $\mathcal{P}(S)$ . For each element in  $S$ , it either is or is not in a subset of  $S$ , so there are two choices for each element of  $S$  in each subset. Thus, there are  $2^{|S|}$  subsets of  $S$ .

### 4.2 Generate cartesian product

The Cartesian product of two sets  $A$  and  $B$ , denoted  $A \times B$ , is the set of all ordered pairs  $(a, b)$  where  $a \in A$  and  $b \in B$ . In terms of set-builder notation, that is

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

This function can be called with either one or two sets. If one set  $A$  is passed, then the result will be  $A \times A$ . If two sets are passed, separate them with a semicolon, i.e. “1, 2, 3; 4, 5”.

## 5 Recurrence

### 5.1 Solve linear homogeneous recurrence relations with constant coefficients

## 6 Isomorphisms

maybe total orders?

## 7 Default

No documentation provided.