Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Strategy and Tests for Determining if a Series Converges

This document gives a general set of guidelines for determining if a series converges or diverges. In addition, it summarizes all of the tests available for determining if a series converges or diverges.

These are general guidelines because some series can have more than one test applied to them and, possibly give different results depending on the path taken through this set of guidelines. In fact, because more than one test may apply, you should always go completely through the guidelines and identify all possible tests that can be used on a given series. Once this has been done you can identity the test that you feel will be the easiest for you to use.

Without further delay, here is the set of guidelines for determining the convergence of a series.

n^{th} term test

1. With a quick glance does it look like the series terms don't converge to zero in the limit, *i.e.* does $\lim_{n\to\infty} a_n \neq 0$? If so, use the Divergence Test. Note that you should only do the divergence test if a quick glance suggests that the series terms may not converge to zero in the limit.

2. Is the series a *p*-series ($\sum \frac{1}{n^p}$) or a geometric series ($\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$)? If so use the fact that *p*-series will only converge if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.

3. Is the series similar to a *p*-series or a geometric series? If so, try the Direct Comparison Test.

4. Is the series a rational expression involving only polynomials or polynomials under radicals (*i.e.* a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test and/or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.

5. Does the series contain factorials or constants raised to powers involving *n*? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we've got that will work is the Ratio Test.

6. Can the series terms be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$? If so, then the Alternating Series Test may work.

7. Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.

8. If $a_n = f(n)$ for some positive, decreasing function and $\int_a^{\infty} f(x)\,dx$ is easy to evaluate then the Integral Test may work.

Note that these are only guidelines and are not hard and fast rules to use when trying to determine the best test to use on a series. If more than one test can be used try to use the test that will be the easiest for you to use and remember that what is easy for someone else may not be easy for you!

The Tests

Divergence Test (nth Term Test)

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge.

Integral Test

Suppose that $f(x)$ is a positive, decreasing function on the interval $[k,\infty)$ and that $f(n)=a_n$ then,

1. If $\int_k^\infty f(x)\,dx$ is convergent so is $\sum_{n=k}^{\infty} a_n$.
2. If $\int_k^\infty f(x)\,dx$ is divergent so is $\sum_{n=k}^{\infty} a_n$.

Direct Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n. Define,

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If c is positive (*i.e.* $c > 0$) and is finite (*i.e.* $c < \infty$) then either both series converge or both series diverge.

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim\limits_{n\to\infty} b_n = 0$ and,
2. $\{b_n\}$ is eventually a decreasing sequence, $\frac{a_{n+1}}{a_n} \leq 1$

the series $\sum a_n$ is convergent

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, convergent, or absolutely convergent.

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

4. if $L < 1$ the series is absolutely convergent (and hence convergent).
5. if $L > 1$ the series is divergent.
6. if $L = 1$ the series may be divergent, convergent, or absolutely convergent.