23 April 2021 Friday BC Calc

9.4 Integral Test/Telescoping Series Test

Homework
Day 12

* MC Quiz on 9.1, 9.2, 9.3, RT, nth term test (located in modules) - may use calculators. No notes.

8.3 Integral Test, P Series, & Harmonic Series

Integral Test:

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$ either both converge or diverge.

(good test to use if function is easy to integrate)

$$\sum_{n=5}^{\infty} a_n \qquad \int_5^{\infty} f(x)\,dx$$

$f(x) \leftarrow a_n$

1. Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} + \cdots$$

$f(n) = \frac{1}{n} =$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{a \to \infty} \int_1^{a} \frac{1}{x}\,dx = \lim_{a \to \infty} (\ln a - \ln 1)$$

$$= \infty$$

diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} & \text{cov} \quad p > 1 \\ \infty & \text{div} \quad p \le 1 \end{cases}$$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{\cdot} \cdots \frac{1}{n^p} + \cdots$$

p-series cov for $p > 1$

div for $p \le 1$

Ex

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{\cdots} \cdots \frac{1}{n^p} \cdots$$

p-series cvr for $p > 1$

div for $p \leq 1$

Ex

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

a) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ b) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$p = 1/2 < 1$ $p = \frac{3}{2} > 1$

p series div cvr

Integral Test:

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or diverge.

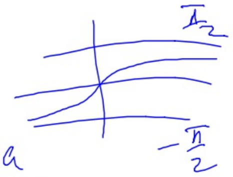
ex. does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

ex. Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{a \to \infty} \int_1^{a} \frac{dx}{x^2+1}$$



$$= \lim_{a \to \infty} \arctan x \Big|_1^a =$$

$$= \lim_{a \to \infty} \left(\arctan a - \arctan 1 \right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

cor

ex. Does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge or diverge

$$\int_1^{\infty} \frac{\cancel{n}}{x^2+1}\,dx$$

$$= \lim_{a \to \infty} \int_1^{a} \frac{x}{x^2+1}\,dx = \lim_{a \to \infty} \frac{1}{2} \ln(x^2+1) \Big|_1^a$$

$u = x^2+1$ $\quad x\,dx = \frac{1}{2}du$

$du = 2x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln\left(x^2+1\right)$$

Telescopic Series

When a Series is convergence?

A series is convergence if

$$\lim_{n\to\infty} S_n = L$$

L any real number

S_n is the partial sum of order n

Telescoping Series - special case

(used when function can be split up using partial fractions)

when telescoping series is expanded, it is of the form

$(b_1-b_2) + (b_2-b_3) + (b_3-b_4) + \ldots$

* the nth partial sum is $b_1 - b_{n+1}$
* the series will converge if b_n approaches a finite number
* if the series converges, its sum is

$$S = b_1 - \lim_{x \to \infty} b_{n+1}$$

Steps

-

ex. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

1. break up function using partial fractions.

2. factor out constant (if applicable)

3. expand series to make sure it fits pattern

4. find sum using formula S_n

5. find $\lim_{n\to\infty} S_n = L \quad , \quad L \in R$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}$$

S_n

Partial fractions

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} = \frac{1}{k} - \frac{1}{k+1}, \quad \frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2}$$

$1 = A(k+1) + Bk.$

$k=0 \quad A = 1$

$k=-1 \quad B = -1$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} =$$

$$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

The series is cv

ex. $\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)}$

ex. $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

$$\sum_{n=0}^{\infty} \frac{8}{(n+1)(n+2)}$$

$$S_n = 8\left[\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \frac{1}{(n-1)n} + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)}\right]$$

$$\frac{A}{n+1} + \frac{B}{n+2} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$A(n+2) + B(n+1) = 1$$

$$n = -2 \quad B = -1$$

$$n = -1 \quad A = 1$$

$$S_n = 8\left[\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\cdots} - \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right]$$

$$= 8\left[\frac{1}{2} - \frac{1}{n+2}\right] \quad \lim_{n\to\infty} S_n = 8\left[\frac{1}{2} - \frac{1}{n+2}\right] = 4$$

Conv

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)}$

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

$$S_n = \sum_{k=2}^{n} \frac{1}{k^2-1}$$

$$\frac{1}{k^2-1} = \frac{A}{k+1} + \frac{B}{k-1} = \frac{1}{2}\left[\frac{1}{k-1} - \frac{1}{k+1}\right]$$

$$1 = A(k-1) + B(k+1)$$

$$K=1 \quad B = \frac{1}{2}$$

$$K=-1 \quad A = -\frac{1}{2}$$

$$\frac{1}{4^2-1} \qquad \frac{1}{(n-2)^2-1}$$

$$S_n = \sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2^2-1} + \frac{1}{3^2-1} + \cdots + \frac{1}{(n-1)^2-1} + \frac{1}{n^2-1}$$

ex. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

$$S_n = \frac{1}{2}\left[\frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n-2}} - \frac{1}{n} + \cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right]$$

$$S_n = \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{n+1} - \frac{1}{n}\right]$$

$$\lim_{n\to\infty} \frac{1}{2}\left[1 + \frac{1}{2} + \underbrace{\frac{1}{n+1} - \frac{1}{n}}\right]$$

$$\lim_{n\to\infty} \frac{1}{2}\left[\frac{3}{2} + \frac{\cancel{n} - \cancel{n} - 1}{n(n+1)}\right] = \frac{3}{4} \text{ sov}$$

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Question 3

The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n (n-1)^2} \text{ for } n \geq 2.$$

he graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

ı) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

) Write the third-degree Taylor polynomial for f about $x = 0$.

Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

{ 1 : answer

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

(a) f has a relative maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(b) $f(0) = 6,\ f'(0) = 0$

$$f''(0) = -\frac{3!}{5^2\,1^2} = -\frac{6}{25},\ f'''(0) = \frac{4!}{5^3\,2^2}$$

$$P(x) = 6 - \frac{3!x^2}{5^2\,2!} + \frac{4!x^3}{5^3\,2^2\,3!} = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$$

$3 : P(x)$
$\langle -1 \rangle$ each incorrect term
Note: $\langle -1 \rangle$ max for use of ex

(c) $$u_n = \frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\dfrac{(-1)^{n+2}(n+2)}{5^{n+1}n^2}x^{n+1}}{\dfrac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n}\right|$$

$$= \left(\frac{n+2}{n+1}\right)\left(\frac{n-1}{n}\right)^2\frac{1}{5}|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{5}|x| < 1 \text{ if } |x| < 5.$$

The radius of convergence is 5.

$4 : \begin{cases} 1 : \text{general term} \\ 1 : \text{sets up ratio} \\ 1 : \text{computes limit} \\ 1 : \text{applies ratio te} \\ \quad \text{radius of conv} \end{cases}$