NONCOMMUTATIVE SELF MAPS OF THE OPERATOR UNIT DISK

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We show that the operator unit disk is a right-admissible noncommutative set and endow it with a ”distance” that makes every noncommutative self map on the unit disk a contraction.

Let $\mathscr{A}$ be a C*-algebra over $\mathbb{C}$ and $\mathbb{D}(\mathscr{A}) = (\mathbb{D}(M_n(\mathscr{A})))_{n\in\mathbb{N}}$ be the noncommutative unit disk in $\mathscr{A}$, where $\mathbb{D}(M_n(\mathscr{A})) = \{a \in M_n(\mathscr{A}) : \|a\| < 1\}$.

Proposition 1. *$\mathbb{D}(\mathscr{A})$ is a right-admissible noncommutative set over $\mathscr{A}$.*

Proof. Suppose that $a \in \mathbb{D}(M_m(\mathscr{A}))$ and $b \in \mathbb{D}(M_n(\mathscr{A}))$. Clearly $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathbb{D}(M_{m+n}(\mathscr{A}))$.

For $w \in M_{m\times n}(\mathscr{A})$, $\begin{pmatrix} a & w \\ 0 & b \end{pmatrix} \in \mathbb{D}(M_{m+n}(\mathscr{A}))$ i.e. $\left\|\begin{pmatrix} a & w \\ 0 & b \end{pmatrix}\right\| < 1$ if and only if $\begin{pmatrix} a & w \\ 0 & b \end{pmatrix}^* \begin{pmatrix} a & w \\ 0 & b \end{pmatrix} < 1$.

Equivalently,

$$\begin{pmatrix} a^* & 0 \\ w^* & b^* \end{pmatrix}\begin{pmatrix} a & w \\ 0 & b \end{pmatrix} < 1 \Leftrightarrow \begin{pmatrix} a^*a & a^*w \\ w^*a & w^*w + b^*b \end{pmatrix} < 1 \Leftrightarrow \begin{pmatrix} 1 - a^*a & -a^*w \\ -w^*a & 1 - w^*w - b^*b \end{pmatrix} > 0$$

Given $p > 0$, we have $\begin{pmatrix} p & q \\ r & s \end{pmatrix} > 0$ if and only if $s - rp^{-1}q > 0$.

As $1 - a^*a > 0$, the above condition implies the following equivalent statements:

$$1 - w^*w - b^*b - w^*a(1 - a^*a)^{-1}a^*w > 0$$
$$w^*(1 + a(1 - a^*a)^{-1}a^*)w < 1 - b^*b$$
$$(1 - b^*b)^{-1/2}w^*(1 + a(1 - a^*a)^{-1}a^*)w(1 - b^*b)^{-1/2} < 1$$

We claim that $1+a(1-a^*a)^{-1}a^* = (1-aa^*)^{-1}$. This can be seen by writing $(1-a^*a)^{-1} = \sum_{k=0}^{\infty}(a^*a)^k$.

$$1 + a(1 - a^*a)^{-1}a^* = 1 + \sum_{k=0}^{\infty} a(a^*a)^k a^* = 1 + \sum_{k=1}^{\infty}(aa^*)^k = \sum_{k=0}^{\infty}(aa^*)^k = (1 - aa^*)^{-1}$$

This further implies the equivalent statements:

$$(1 - b^*b)^{-1/2}w^*(1 - aa^*)^{-1}w(1 - b^*b)^{-1/2} < 1$$
$$\|(1 - aa^*)^{-1/2}w(1 - b^*b)^{-1/2}\| < 1$$

Set $\varepsilon = \varepsilon(a, b, w) = \dfrac{1}{\|(1 - aa^*)^{-1/2}w(1 - b^*b)^{-1/2}\|}$. Then $\begin{pmatrix} a & \lambda w \\ 0 & b \end{pmatrix} \in \mathbb{D}(M_{m+n}(\mathscr{A}))$ for $|\lambda| \leq \varepsilon$. □

Proposition 2. *Let $f : \mathbb{D}(\mathcal{A}) \to \mathbb{D}(\mathcal{A})$ be a noncommutative function on $\mathbb{D}(\mathcal{A})$ into itself. Fix $n \in \mathbb{N}$ and $a, b \in \mathbb{D}(M_n(\mathcal{A}))$. Then the linear map $\varphi : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ defined by*

$$w \mapsto (1 - f(a)f(a)^*)^{-1/2}\Delta f(a,b)((1-aa^*)^{1/2}w(1-b^*b)^{1/2})(1-f(b)^*f(b))^{-1/2} =: \varphi(w)$$

is a complete contraction.

Proof. As f is a noncommutative function that maps $\mathbb{D}(\mathcal{A})$ into itself, for $|\lambda| \le \varepsilon(a,b,w)$

$$f\begin{pmatrix} a & \lambda w \\ 0 & b \end{pmatrix} = \begin{pmatrix} f(a) & \Delta f(a,b)(\lambda w) \\ 0 & f(b) \end{pmatrix} \in \mathbb{D}(M_{2n}(\mathcal{A}))$$

By similar arguments as in **Proposition 1** and linearity of $\Delta f(a,b)$, we have for $|\lambda| \le \varepsilon(a,b,w)$

$$|\lambda| \,.\, \|(1-f(a)f(a)^*)^{-1/2}\Delta f(a,b)(w)(1-f(b)^*f(b))^{-1/2}\| < 1$$

Taking limit $\lambda \uparrow \varepsilon(a,b,w)$, we obtain

$$\|(1-f(a)f(a)^*)^{-1/2}\Delta f(a,b)(w)(1-f(b)^*f(b))^{-1/2}\| \le \frac{1}{\varepsilon(a,b,w)}$$

$$\|(1-f(a)f(a)^*)^{-1/2}\Delta f(a,b)(w)(1-f(b)^*f(b))^{-1/2}\| \le \|(1-aa^*)^{-1/2}w(1-b^*b)^{-1/2}\|$$

We conclude that the map is a contraction by replacing w by $(1-aa^*)^{1/2}w(1-b^*b)^{1/2}$.

Next we show that the map is a complete contraction.

Consider for $k \in \mathbb{N}$ the map $\varphi \otimes i_k$ on $M_k(M_n(\mathcal{A}))$, where i_k is the identity map on $M_k(\mathbb{C})$. Let $\mathbf{w} = (w_{ij})_{1\le i,j\le k} \in M_k(M_n(\mathcal{A}))$ and denote by $\mathbf{a}$ and $\mathbf{b}$ the ampliations $a \otimes 1_k$ and $b \otimes 1_k$ respectively.

As f is noncommutative, $f(\mathbf{a}) = f(a)\otimes 1_k$, $f(\mathbf{b}) = f(b)\otimes 1_k$ and $\Delta f(\mathbf{a},\mathbf{b})(\mathbf{w}) = (\Delta f(a,b)(w_{ij}))_{1\le i,j\le k}$.

Moreover, as composition of noncommutative functions is noncommutative, we have

$$\begin{aligned}
&(1_k - f(\mathbf{a})f(\mathbf{a})^*)^{-1/2}\Delta f(\mathbf{a},\mathbf{b})(\mathbf{w})(1_k - f(\mathbf{b})^*f(\mathbf{b}))^{-1/2} \\
&= \Big((1-f(a)f(a)^*)^{-1/2}\otimes 1_k\Big)\Big((\Delta f(a,b)(w_{ij}))_{1\le i,j\le k}\Big)\Big((1-f(b)^*f(b))^{-1/2}\otimes 1_k\Big) \\
&= \Big((1-f(a)f(a)^*)^{-1/2}\Delta f(a,b)(w_{ij})(1-f(b)^*f(b))^{-1/2}\Big)_{1\le i,j\le k}
\end{aligned}$$

It is now clear that

$$\|(1_k - f(\mathbf{a})f(\mathbf{a})^*)^{-1/2}\Delta f(\mathbf{a},\mathbf{b})(\mathbf{w})(1_k - f(\mathbf{b})^*f(\mathbf{b}))^{-1/2}\| \le \|(1-\mathbf{a}\mathbf{a}^*)^{-1/2}\mathbf{w}(1-\mathbf{b}^*\mathbf{b})^{-1/2}\|$$

as for each $i, j = 1, 2, \cdots, k$ the norm of the $(i,j)^{th}$ entry of the matrix on L.H.S. is dominated by the norm of the $(i,j)^{th}$ entry of the matrix on R.H.S. Finally, $\varphi \otimes i_k$ is a contraction.

This completes the proof that φ is a complete contraction. □

Corollary 3. *Let $f : \mathbb{D}(\mathcal{A}) \to \mathbb{D}(\mathcal{A})$ be a noncommutative function, $n \in \mathbb{N}$, and $a, b \in \mathbb{D}(M_n(\mathcal{A}))$. Then*

$$\|(1-f(a)f(a)^*)^{-1/2}(f(a)-f(b))(1-f(b)^*f(b))^{-1/2}\| \le \|(1-aa^*)^{-1/2}(a-b)(1-b^*b)^{-1/2}\|$$

Proof. This follows directly from **Proposition 2** by setting $w = a - b$ and $\Delta f(a,b)(w) = f(a) - f(b)$. □

The above corollary suggests that a noncommutative self map of the operator ball is a contraction with respect to the distance $\varrho(a,b) = \|(1-aa^*)^{-1/2}(a-b)(1-b^*b)^{-1/2}\|$ for a and b in $\mathbb{D}(\mathcal{A})$.

Note that this distance is not symmetric, however we do have $\varrho(b,a) = \varrho(a^*,b^*)$.

In the following proposition, we show that the balls with respect to the topology induced by this distance are norm closed, convex and bounded away from the topological boundary of $\mathbb{D}(\mathcal{A})$.

Proposition 4. *Denote the (closed) ϱ- ball in $\mathbb{D}(\mathcal{A})$ centered at a of radius r by*

$$B_\varrho(a,r) = \{b \in \mathbb{D}(\mathcal{A}) : \varrho(a,b) \le r\}.$$

Then for a in $\mathbb{D}(\mathcal{A})$ and $r > 0$, $B_\varrho(a,r)$ is a convex, norm-closed noncommutative set bounded away from the topological boundary $\mathbb{T}(\mathcal{A}) = \{u : \|u\| = 1\}$ of $\mathbb{D}(\mathcal{A})$.

Proof. First show bounded away from the boundary.

Let $b_n \in B_\varrho(a,r)$ such that $\|b_n - b\| \to 0$ as $n \to \infty$. Clearly, $\|b_n^* - b^*\| \to 0$ as $n \to \infty$. Moreover, as $\|b_n\| < 1$, we have as $n \to \infty$

$$\|b_n^*b_n - b^*b\| \le \|(b_n^* - b^*)b_n\| + \|b^*(b_n - b)\| \le \|b_n^* - b^*\| + \|b^*\|.\|b_n - b\| \to 0.$$

bounded away condition: Therefore $1 - b^*b \ge \mathit{something} > 0$ so that $b \in \mathbb{D}(\mathcal{A})$.

By analytic functional calculus, $(1 - b_n^*b_n)^{-1/2} \to (1-b^*b)^{-1/2}$ in norm as $n \to \infty$. This proves that $\varrho(a,b) \le r$, so that $b \in B_\varrho(a,r)$. Therefore $B_\varrho(a,r)$ is closed in the norm topology of $\mathcal{A}$.

To show that $B_\rho(a,r)$ is convex, it suffices to show that it is midpoint convex, since it is closed.

Suppose that $b_1, b_2 \in \mathbb{D}(\mathcal{A})$ such that $\|(1-aa^*)^{-1/2}(a-b_i)(1-b_i^*b_i)^{-1/2}\| \le r$ for $i = 1,2$.

This is equivalent to saying that for $i = 1,2$,

$$(1-b_i^*b_i)^{-1/2}(a-b_i)^*(1-aa^*)^{-1}(a-b_i)(1-b_i^*b_i)^{-1/2} \le r^2.1$$

or that

$$(a-b_i)^*(1-aa^*)^{-1}(a-b_i) \le r^2(1-b_i^*b_i). \tag{0.1}$$

We claim that $\dfrac{b_1+b_2}{2} \in B_\varrho(a,r)$. From above, we know that this again amounts to showing that

$$\left(a - \frac{b_1+b_2}{2}\right)^* (1-aa^*)^{-1}\left(a - \frac{b_1+b_2}{2}\right) \le r^2\left(1 - \left(\frac{b_1+b_2}{2}\right)^*\left(\frac{b_1+b_2}{2}\right)\right).$$

i.e.

$$\big((a-b_1)+(a-b_2)\big)^*(1-aa^*)^{-1}\big((a-b_1)+(a-b_2)\big) \le r^2\big(4 - (b_1+b_2)^*(b_1+b_2)\big) \tag{0.2}$$

Adding equation (0.1) for $i = 1,2$ and multiplying by 2, we get

$$2(a-b_1)^*(1-aa^*)^{-1}(a-b_1) + 2(a-b_2)^*(1-aa^*)^{-1}(a-b_2) \le r^2(4 - 2b_1^*b_1 - 2b_2^*b_2) \tag{0.3}$$

We show that L.H.S. of (0.2) is dominated by L.H.S. of (0.3) and that R.H.S. of (0.3) is dominated by R.H.S. of (0.2), thereby proving our claim.

Direct computations show that the difference between L.H.S. of (0.3) and L.H.S. of (0.2) is $(b_1 - b_2)^*(1-aa^*)^{-1}(b_1-b_2) \ge 0$.

Similarly, the difference between R.H.S. of (0.2) and R.H.S. of (0.3) is $r^2(b_1-b_2)^*(b_1-b_2) \ge 0$. □