Early Fourier Analysis

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Complex Numbers, a starting point

- $$e^{iy} = \cos y + i \sin y \,.$$

- **Theorem 2.1.** *Let q be a positive integer. Then for any integer k,*
$$\sum_{n=1}^{q} e(kn/q) = \begin{cases} q & (q|k), \\ 0 & (otherwise). \end{cases}$$

Discrete Fourier Transform

- $$\widehat{f}(k) = \sum_{n=1}^{q} f(n) e(-nk/q)$$

- $$f(n) = \frac{1}{q} \sum_{k=1}^{q} \widehat{f}(k) e(kn/q) .$$

Convolution of Arithmetic Periodic Function

- Suppose that $f(n)$ and $g(n)$ are arithmetic functions with period q. The *convolution* of f and g is

$$(f * g)(n) = \sum_{a=1}^{q} f(n-a)g(a), \tag{2.16}$$

- **Theorem 2.5.** *Suppose that $f(n)$ and $g(n)$ are arithmetic functions with period q, and that $h = f * g$. Then*

$$\widehat{h}(k) = \widehat{f}(k)\widehat{g}(k).$$

Fourier Coefficients

- $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$

- **Definition 3.1.** If f has period 1 and $\int_0^1 |f(x)|\, dx < \infty$, then for each integer n we put

$$\widehat{f}(n) = \int_0^1 f(x)e(-nx)\, dx. \tag{3.6}$$

Fourier Series

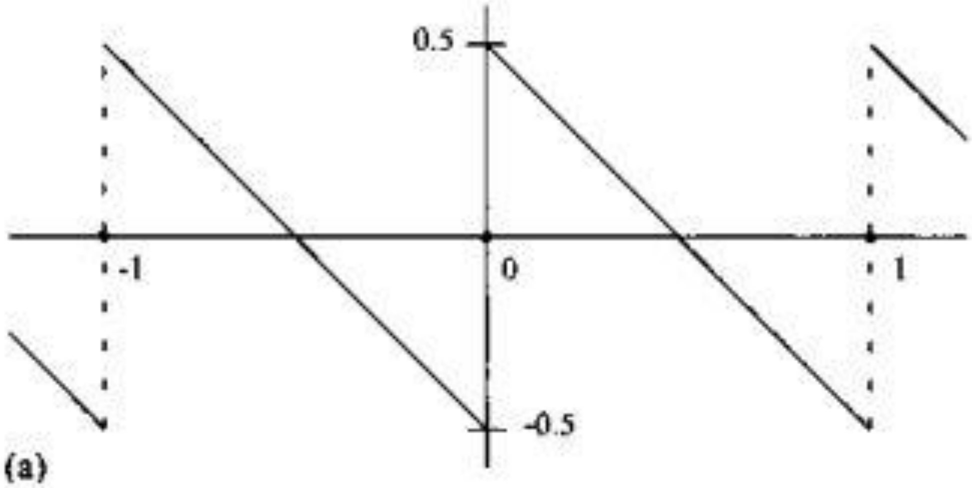
The fourier series of a periodic function f is defined as:

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e(nx).$$

L^p and L^1 Functions

- $$\int_S |f(x)|^p \, dx < \infty \, .$$
- $\int_0^1 |f(x)| \, dx < \infty$", we shall simply say "$f \in L^1(\mathbb{T})$"
- $$\|f\|_1 = \int_0^1 |f(x)| \, dx \, .$$ L^1 norm

Sawtooth Function



s(x) = 0 if x is integer

s(x) = ½ - {x} otherwise, where {x} is the fractional part of x

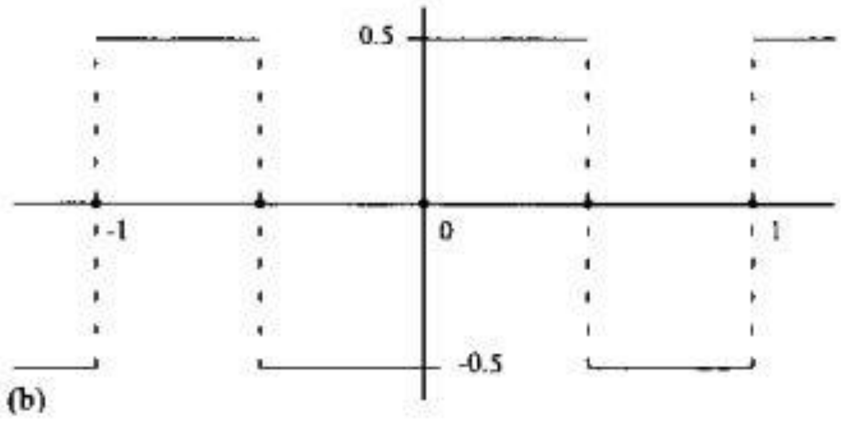
Sawtooth Fourier Series

$$\widehat{s}(n) = \int_0^1 (1/2 - x)e(-nx)\,dx = \frac{1}{2\pi i n}.$$

Fourier Series:

$$\sum_{n\neq 0} \frac{e(nx)}{2\pi i n} = \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n}.$$

Square Wave Function



$$w(x) = \begin{cases} 1/2 & (0 < x < 1/2), \\ -1/2 & (1/2 < x < 1), \\ 0 & (x = 0 \text{ or } 1/2), \end{cases}$$

Square Wave Fourier Series

$$\widehat{w}(n) = \frac{1}{2}\int_0^{1/2} e(-nx)\,dx - \frac{1}{2}\int_{1/2}^{1} e(-nx)\,dx = \begin{cases} \dfrac{1}{\pi i n} & (n \text{ odd}), \\ 0 & (n \text{ even}). \end{cases}$$

$$\sum_{n \text{ odd}} \frac{e(nx)}{\pi i n} = \sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)} \sin 2\pi(2n-1)x\,.$$

Riemann-Lebesgue Lemma

Theorem 3.6. (The Riemann–Lebesgue Lemma) *Suppose that* $f \in L^1(\mathbb{T})$. *Then*

$$\lim_{n \to \pm\infty} \widehat{f}(n) = 0\,.$$

Convolution

Suppose that $f \in L^1(\mathbb{T})$, and that $g \in L^1(\mathbb{T})$. The *convolution* of f and g is the function

$$h(x) = (f * g)(x) = \int_0^1 f(u)g(x-u)\,du\,. \tag{3.29}$$

Trigonometric Polynomials

$$T(x) = \sum_{n=-N}^{N} t_n e(nx).$$

Lemma 3.14. *Suppose that $f \in L^1(\mathbb{T})$, and that $T(x)$ is a trigonometric polynomial, as defined in (3.42). Then*

$$(T * f)(x) = \sum_{n=-N}^{N} \widehat{f}(n) t_n e(nx).$$

The 20th Fourier series

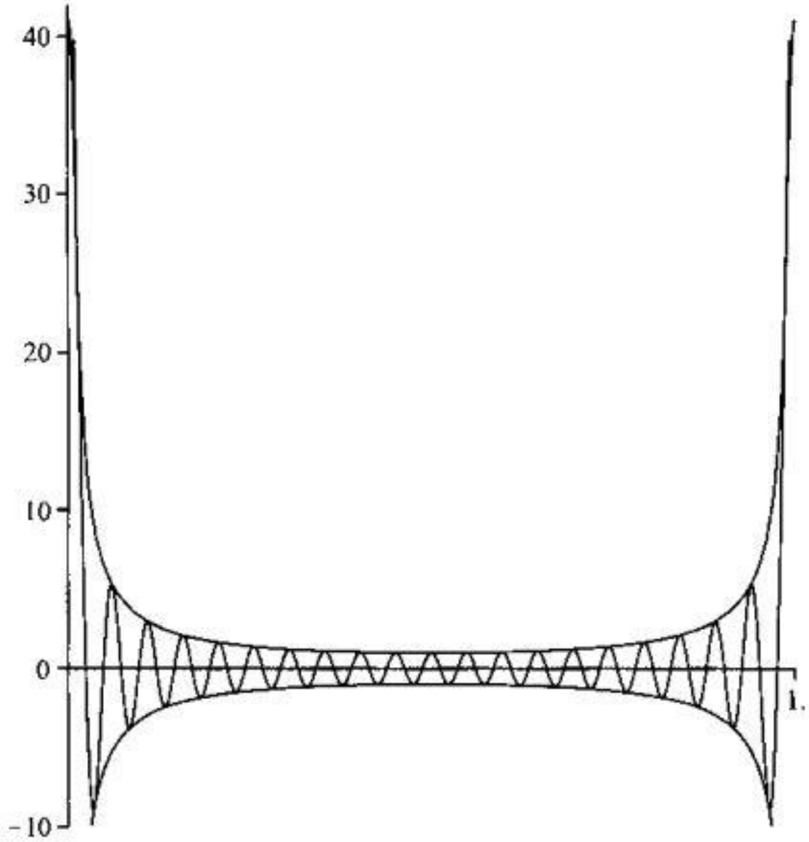


Figure 3.2. Graph of $D_{20}(x)$ for $0 \le x \le 1$, with its envelopes $\pm 1/\sin \pi x$.

Gibbs Phenomenon

- The tendency for partial sums to overshoot and undershoot the function as x increases from 0

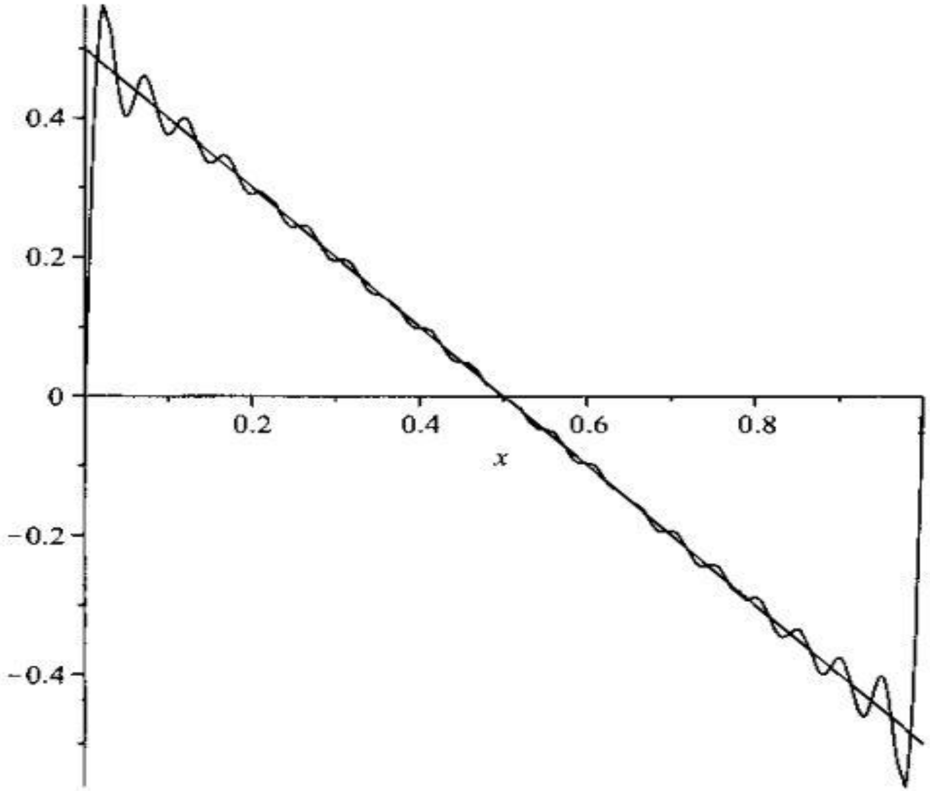


Figure 3.3. Graph of the sawtooth function $s(x) = 1/2 - x$ and its Fourier approximation $s_{20}(x)$.

Cesaro Summability

$$s_N = \sum_{n=0}^{N} a_n \qquad \sigma_N = \frac{1}{N} \sum_{n=0}^{N-1} s_n$$

We say that the series $\sum_{n=0}^{\infty} a_n$ is *Cesàro summable to a*, and write

$$\sum_{n=0}^{\infty} a_n = a \qquad (C)$$

if

$$\lim_{N \to \infty} \sigma_N = a\,.$$

Example

$$\sum_{n=0}^{\infty}(-1)^n.$$

- This series diverges yet has a Cesaro sum
- Averages of partial sums: 1, ½, ⅔, ½, ⅗, ½, etc

$$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2} \qquad (C).$$

Fejer Kernel

$$\Delta_N(x) = \sum_{n=-N}^{N} (1 - |n|/N) e(nx) = 1 + 2\sum_{n=1}^{N} (1 - n/N) \cos 2\pi n x \,.$$

$$\sigma_N(x) = \frac{s_0(x) + s_1(x) + \cdots + s_{N-1}(x)}{N} = \sum_{n=-N}^{N} (1 - |n|/N) \widehat{f}(n) e(nx)$$

Weierstrass Theorem

Corollary 4.6. (Weierstrass) *Suppose that f is continuous on the interval $[a,b]$. For any $\varepsilon > 0$ there is a polynomial $P(x) = \sum_{n=0}^{N} a_n x^n$ such that $|f(x) - P(x)| < \varepsilon$ uniformly for $x \in [a,b]$.*

Theorem 4.7

Theorem 4.7. *Suppose that* $f \in L^1(\mathbb{T})$. *Then*

$$\lim_{N\to\infty} \int_0^1 |f(x) - \sigma_N(x)|\, dx = 0\,.$$

Hardy’s Theorem

Theorem 4.32. (Hardy) *Suppose that the series* $\sum_{n=1}^{\infty} a_n$ *is Cesàro-summable to* a *and that there is a constant* $C > 0$ *such that*

$$|a_n| \leq \frac{C}{n} \tag{4.58}$$

for all n. *Then* $\sum_{n=1}^{\infty} a_n$ *converges to* a.

Summability Kernels

- These theorems help us determine the summability of kernels
- Fejer’s Kernel is summable
- Dirichlet is not

Fourier Transform and Inversion

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)e(-tx)\,dx\,.$$

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(t)e(xt)\,dt\,.$$

Plancherel's Theorem

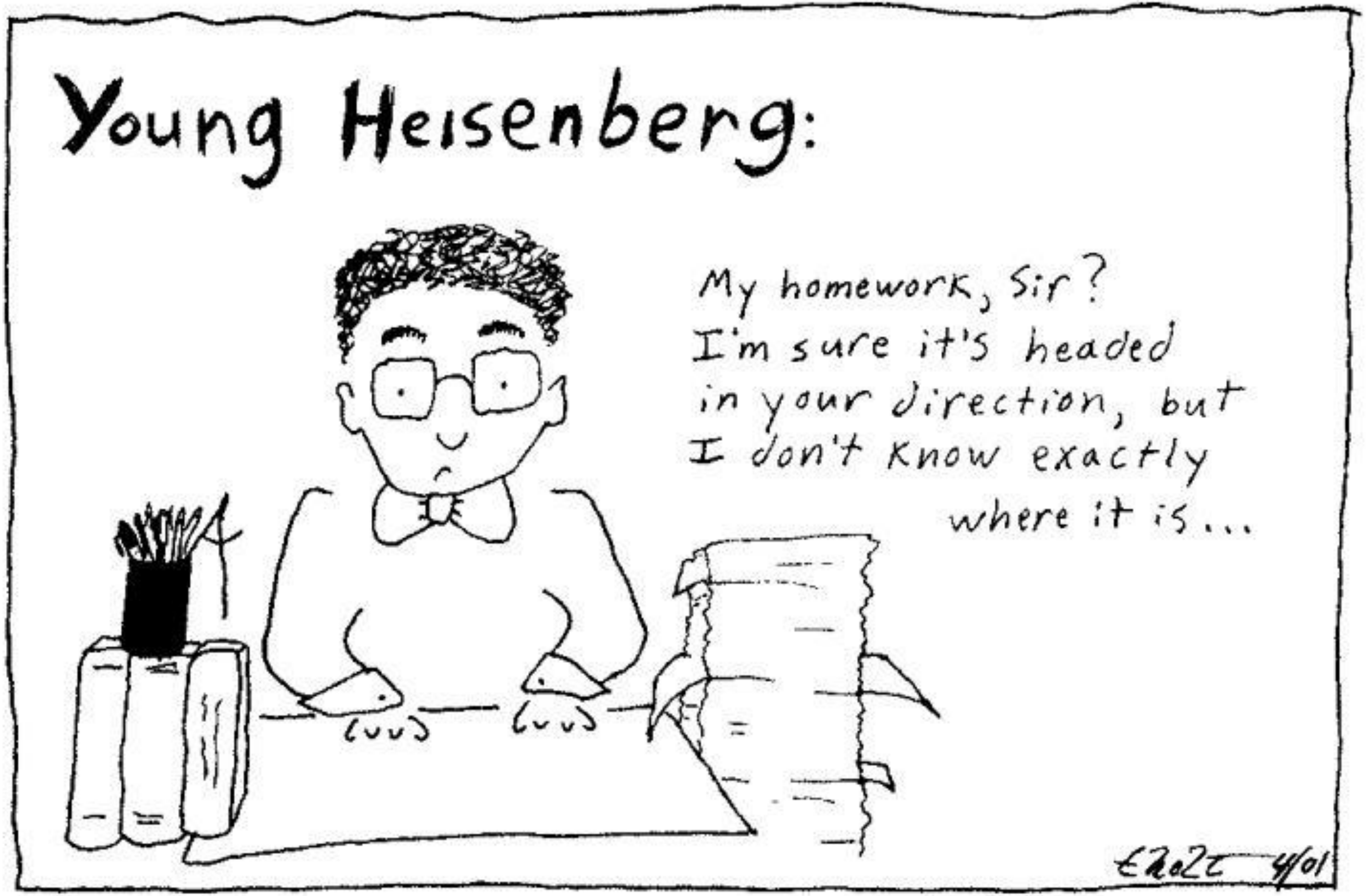
Theorem 10.26. (Plancherel) *If* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then*

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\widehat{f}(t)|^2 \, dt \, .$$

Heisenberg Uncertainty Principle

Theorem 10.29. (The Heisenberg Uncertainty Principle) *If* $f \in L^1(\mathbb{R})$, $xf(x) \in L^2(\mathbb{R})$, *and* $f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then*

$$\left(\int_{-\infty}^{\infty} |xf(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |t\widehat{f}(t)|^2\, dt\right) \geq \frac{1}{16\pi^2}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)^2.$$


Young Heisenberg:
My homework, Sir? I'm sure it's headed in your direction, but I don't know exactly where it is...
4/01

Further Application

- Signal processing
- Acoustics
- Quantum mechanics
- Heat equation

Questions?

Also huge thank you to my mentor Nasheed!