Timing Rayleigh Quotient minimization in R

true

2023-9-6

Abstract

This vignette is simply to record the methods and results for timing various Rayleigh Quotient minimizations with R using different functions and different ways of running the computations, in particular trying Fortran subroutines and the R byte compiler. It has been updated from a 2012 document to reflect changes in R and its packages that make it awkward to reprocess the original document on newer computers and which show that timing profiles of R commands have changed in the interim.

The computational task

The maximal and minimal eigensolutions of a symmetric matrix A are extrema of the Rayleigh Quotient

$$R(x) = (x'Ax)/(x'x)$$

We could also deal with generalized eigenproblems of the form

$$Ax = eBx$$

where B is symmetric and positive definite by using the Rayleigh Quotient (RQ)

$$R_g(x) = (x'Ax)/(x'Bx)$$

In this document, B will always be an identity matrix, but some programs we test assume that it is present.

Note that the objective is scaled by the parameters, in fact by by their sum of squares. Alternatively, we may think of requiring the **normalized** eigensolution, which is given as

$$x_{normalized} = x/sqrt(x'x)$$

Timings and speedups

In R, execution times can be measured by the function `system.time`, and in particular the third element of the object this function returns. However, various factors influence computing times in a modern computational system, so we generally want to run replications of the times. The R packages `rbenchmark` and `microbenchmark` can be used for this. I have a preference for the latter. However, to keep the time to prepare this vignette with `Sweave` or `knitR` reasonable, many of the timings will be done with only `system.time`.

There are some ways to speed up R computations.

- The code can be modified to use more efficient language structures. We show some of these below, in particular, to use vector operations.
- We can use the R byte code compiler by Luke Tierney, which has been part of the R distribution since version 2.14.
- We can use compiled code in other languages. Here we show how Fortran subroutines can be used.

Our example matrix

We will use a matrix called the Moler matrix Nash (1979, Appendix 1). This is a positive definite symmetric matrix with one small eigenvalue. We will show a couple of examples of computing the small eigenvalue solution, but will mainly perform timings using the maximal eigenvalue solution, which we will find by minimizing the RQ of (-1) times the matrix. (The eigenvalue of this matrix is the negative of the maximal eigenvalue of the original, but the eigenvectors are equivalent to within a scaling factor for non-degenerate eigenvalues.)

Here is the code for generating the Moler matrix.

```
molermat<-function(n){
   A<-matrix(NA, nrow=n, ncol=n)
   for (i in 1:n){
      for (j in 1:n) {
          if (i == j) A[i,i]<-i
          else A[i,j]<-min(i,j) - 2
      }
   }
   A
}
```

However, since R is more efficient with vectorized code, the following routine by Ravi Varadhan should do much better.

```
molerfast <- function(n) {
# A fast version of `molermat'
  A <- matrix(0, nrow = n, ncol = n)
  j <- 1:n
  for (i in 1:n) {
    A[i, 1:i] <- pmin(i, 1:i) - 2
  }
  A <- A + t(A)
  diag(A) <- 1:n
  A
}
```

Time to build the matrix

Let us see how long it takes to build the Moler matrix of different sizes. In 2012 we used the byte-code compiler, but that now seems to be active by default and NOT to give worthwhile improvements. We also include times for the `eigen()` function that computes the full set of eigensolutions very quickly.

```
## Loading required package: microbenchmark
```

```
##       n   osize buildi buildir eigentime eigentimr bfast bfastr
## 1    50   20216   1173     855       510       247   512   1045
## 2   100   80216   3345     582      1559        63   677     46
## 3   150  180216   7324     710      4354       190  1023     45
## 4   200  320216  12994     872      9007       718  1446     53
## 5   250  500216  20180     553     15885       510  2151    262
## 6   300  720216  29251     682     26235       912  2657    685
## 7   350  980216  39864    1390     40224       711  4601   7208
## 8   400 1280216  51991     958     58801      1053  5140   7171
## 9   450 1620216  67892    6795     82032      1092  7362   9582
## 10  500 2000216  81840    2055    110934      1028  6888   7099
```

```
## osize - matrix size in bytes
```

```
## eigentime - all eigensolutions time

## buildi - interpreted build time, range

## bfast - interpreted vectorized build time

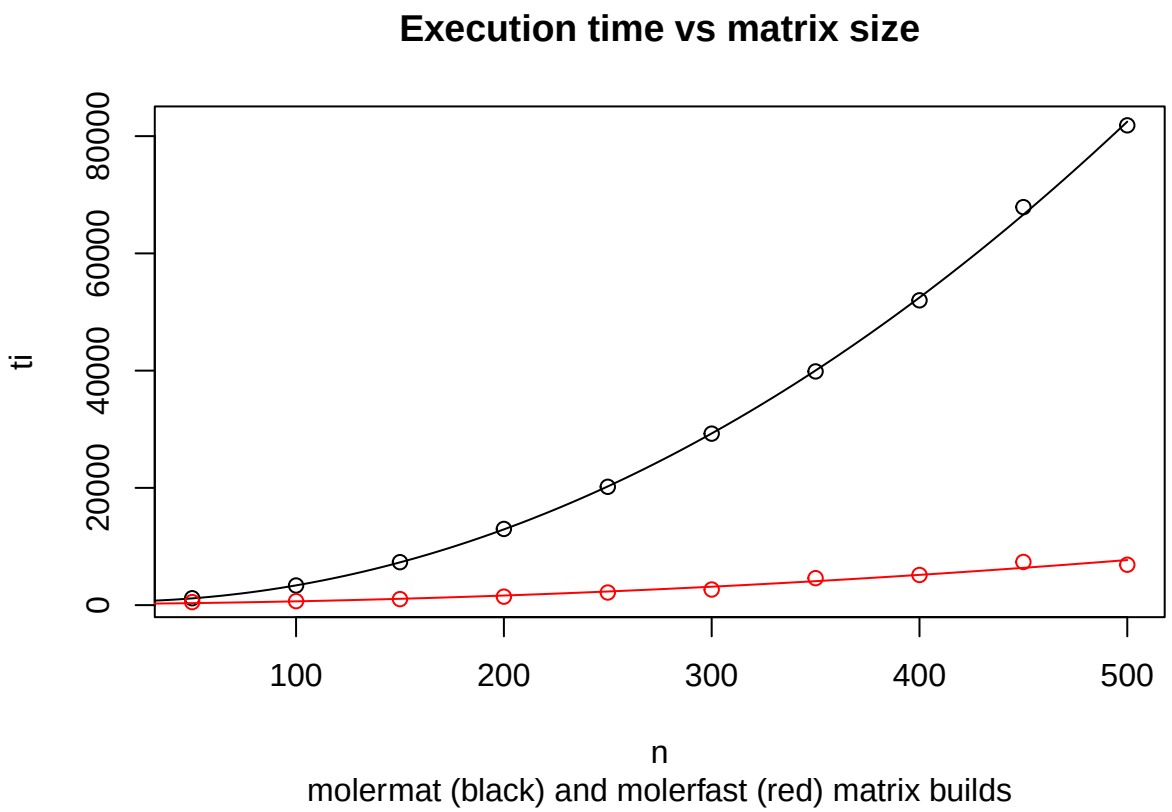
## Times converted to milliseconds
```

It does not appear that the compiler has much effect, or else it is being automatically invoked.

We can graph the times. The code, which is not echoed here, also models the times and the object size created as almost perfect quadratic models in `n`.

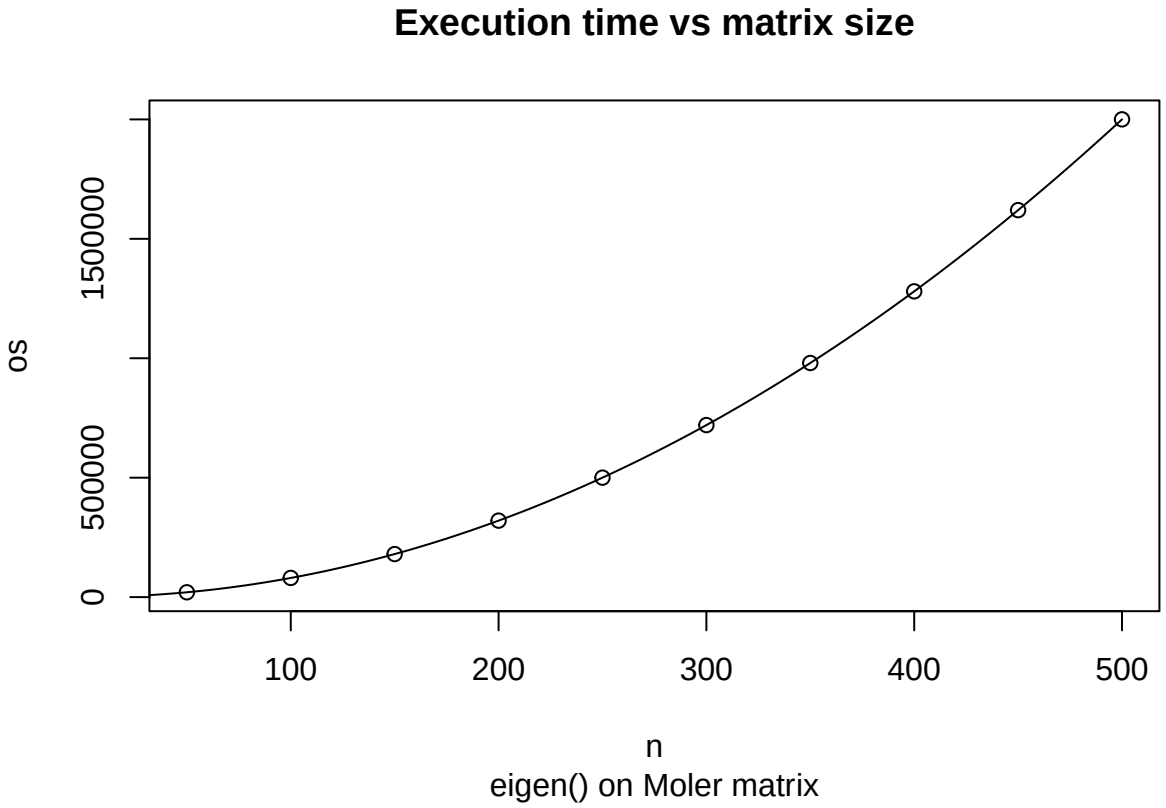
```
##
## Call:
## lm(formula = ti ~ n + n2)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -629.39 -139.57  -30.15   29.11 1274.38
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 635.383333 672.693364   0.945    0.376
## n            -6.740318   5.618902  -1.200    0.269
## n2            0.340817   0.009956  34.231 4.71e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 571.9 on 7 degrees of freedom
## Multiple R-squared:  0.9997, Adjusted R-squared:  0.9996
## F-statistic: 1.088e+04 on 2 and 7 DF,  p-value: 5.963e-13
```

```
##
## Call:
## lm(formula = tf ~ n + n2)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -802.38 -185.60  -35.85  136.25 1001.59
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 166.36667  666.21131   0.250   0.8100
## n             2.21318    5.56476   0.398   0.7027
## n2            0.02567    0.00986   2.603   0.0353 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 566.4 on 7 degrees of freedom
## Multiple R-squared:  0.9622, Adjusted R-squared:  0.9514
## F-statistic: 89.12 on 2 and 7 DF,  p-value: 1.049e-05
```




```
## Warning in summary.lm(osize): essentially perfect fit: summary may be
## unreliable
```

```
##
## Call:
## lm(formula = os ~ n + n2)
##
## Residuals:
##        Min         1Q     Median         3Q        Max
## -2.654e-12 -1.314e-13  3.293e-13  7.262e-13  1.211e-12
##
## Coefficients:
##              Estimate Std. Error   t value Pr(>|t|)
## (Intercept) 2.160e+02  1.617e-12 1.336e+14  < 2e-16 ***
## n           5.127e-13  1.351e-14 3.795e+01 2.29e-09 ***
## n2          8.000e+00  2.394e-17 3.342e+17  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.375e-12 on 7 degrees of freedom
## Multiple R-squared:      1,  Adjusted R-squared:      1
## F-statistic: 1.112e+36 on 2 and 7 DF,  p-value: < 2.2e-16
```



Computing the Rayleigh Quotient

The Rayleigh Quotient requires the quadratic form $x'Ax$ divided by the inner product $x'x$. R lets us form this in several ways.

```
rqdir<-function(x, AA){
  rq<-0.0
  n<-length(x) # assume x, AA conformable
  for (i in 1:n) {
     for (j in 1:n) {
        rq<-rq+x[i]*AA[[i,j]]*x[j]
     }
  }
  rq
}
```

Somewhat better (as we shall show below) is

```
ray1<-function(x, AA){
    rq<-  t(x)%*%AA%*%x
}
```

and (believed) better still is

```
ray2<-function(x, AA){
    rq<-  as.numeric(crossprod(x, crossprod(AA,x)))
}
```

Note that we could implicitly include the minus sign in these routines to allow for finding the maximal eigenvalue by minimizing the Rayleigh Quotient of $-A$. However, such shortcuts often rebound when the

implicit negation is overlooked.

If we already have the inner product $ A x$ as vector `ax` from some other computation, then we can simply use

```
ray3<-function(x, AA, ax=axftn){
    # ax is a function to form AA%*%x
    rq<- - as.numeric(crossprod(x, ax(x, AA)))
}
```

Matrix-vector products

In generating the RQ, we do not actually need the matrix itself, but simply the inner product with a vector `x`, from which a second inner produce with `x` gives us the quadratic form $ x' A x$. If `n} is the order of the problem, then for largen`, we avoid storing and manipulating a very large matrix if we use **implicit inner product** formation. We do this with the following code. For future reference, we include the multiplication by an identity.

```
ax<-function(x, AA){
   u<- as.numeric(AA%*%x)
}

axx<-function(x, AA){
   u<- as.numeric(crossprod(AA, x))
}
```

Note that second argument, supposedly communicating the matrix which is to be used in the matrix-vector product, is ignored in the following implicit product routine. It is present only to provide a common syntax when we wish to try different routines within other computations.

```
aximp<-function(x, AA=1){ # implicit moler A*x
   n<-length(x)
   y<-rep(0,n)
   for (i in 1:n){
      tt<-0.
      for (j in 1:n) {
          if (i == j) tt<-tt+i*x[i]
          else tt<-tt+(min(i,j) - 2)*x[j]
      }
      y[i]<-tt
   }
   y
}
ident<-function(x, B=1) x # identity
```

However, Ravi Varadhan has suggested the following vectorized code for the implicit matrix-vector product.

```
axmolerfast <- function(x, AA=1) {
# A fast and memory-saving version of A%*%x
# For Moler matrix. Note we need a matrix argument to match other functions
n <- length(x)
j <- 1:n
ax <- rep(0, n)
for (i in 1:n) {
term <- x * (pmin(i, j) - 2)
ax[i] <- sum(term[-i])
}
```

```
ax <- ax + j*x
ax
}
```

We can also use external language routines, for example in Fortran. However, this needs a Fortran **subroutine** which outputs the result as one of the returned components. The subroutine is in file `moler.f`.

```
      subroutine moler(n, x, ax)
      integer n, i, j
      double precision x(n), ax(n), sum
c     return ax = A * x for A = moler matrix
c     A[i,j]=min(i,j)-2 for i<>j, or i for i==j
      do 20 i=1,n
         sum=0.0
         do 10 j=1,n
            if (i.eq.j) then
               sum = sum+i*x(i)
            else
               sum = sum+(min(i,j)-2)*x(j)
            endif
 10      continue
         ax(i)=sum
 20   continue
      return
      end
```

This is then compiled in a form suitable for R use by the command (this is a command-line tool, and was run in Ubuntu Linux in a directory containing the file `moler.f` but outside this vignette):

```
R CMD SHLIB moler.f
```

This creates files `moler.o` and `moler.so`, the latter being the dynamically loadable library we need to bring into our R session.

```
dyn.load("moler.so")
cat("Is the mat multiply loaded? ",is.loaded("moler"),"\n")
```

```
## Is the mat multiply loaded?  TRUE
```

```
axftn<-function(x, AA=1) { # ignore second argument
   n<-length(x) # could speed up by having this passed
   vout<-rep(0,n) # purely for storage
   res<-(.Fortran("moler", n=as.integer(n), x=as.double(x), vout=as.double(vout)))$vout
}
```

We can also byte compile each of the routines above

Now it is possible to time the different approaches to the matrix-vector product.

```
dyn.load("moler.so")
cat("Is the mat multiply loaded? ",is.loaded("moler"),"\n")
```

```
## Is the mat multiply loaded?  TRUE
```

```
require(microbenchmark)
nmax<-10
ptable<-matrix(NA, nrow=nmax, ncol=11) # to hold results
# loop over sizes
for (ni in 1:nmax){
```

```
  n<-50*ni
  x<-runif(n) # generate a vector
  ptable[[ni, 1]]<-n
  AA<-molermat(n)
  tax<- microbenchmark(oax<-ax(x, AA), times=mbt)$time
  taxx<-microbenchmark(oaxx<-axx(x, AA), times=mbt)$time
  if (! identical(oax, oaxx)) stop("oaxx NOT correct")
  taxftn<-microbenchmark(oaxftn<-axftn(x, AA=1), times=mbt)$time
  if (! identical(oax, oaxftn)) stop("oaxftn NOT correct")
  taximp<-microbenchmark(oaximp<-aximp(x, AA=1), times=mbt)$time
  if (! identical(oax, oaximp)) stop("oaximp NOT correct")
  taxmfi<-microbenchmark(oaxmfi<-axmolerfast(x, AA=1), times=mbt)$time
  if (! identical(oax, oaxmfi)) stop("oaxmfi NOT correct")
  ptable[[ni, 2]]<-msect(tax); ptable[[ni,3]]<-msecr(tax)
  ptable[[ni, 4]]<-msect(taxx); ptable[[ni, 5]]<-msecr(taxx)
  ptable[[ni, 6]]<-msect(taxftn); ptable[[ni, 7]]<-msecr(taxftn)
  ptable[[ni, 8]]<-msect(taximp); ptable[[ni,9]]<-msecr(taximp)
  ptable[[ni, 10]]<-msect(taxmfi); ptable[[ni,11]]<-msecr(taxmfi)
}

axtym<-data.frame(n=ptable[,1], ax=ptable[,2], sd_ax=ptable[,3],  axx=ptable[,4],
                  sd_axx=ptable[,5],  axftn=ptable[,6], sd_axftn=ptable[,7],
                  aximp=ptable[,8], sd_aximp=ptable[,9],
                  axmfast=ptable[,10], sd_axmfast=ptable[,11])
print(axtym)
```

```
##       n  ax sd_ax axx sd_axx axftn sd_axftn aximp sd_aximp axmfast sd_axmfast
## 1   50  71   335  62    281   135      615  1071     1242     511        981
## 2  100  12    23  10      3    31        5  3262      529     642         25
## 3  150  24    59  20      3    62        6  7249      716    1085         79
## 4  200  39    76  36      6   107        4 12928      879    1579        148
## 5  250  35    10  59     19   166       11 19862      803    2250       1058
## 6  300  48    15  80     15   234        4 29597     7263    2649        901
## 7  350  65    24 106     10   314        4 38524      934    3296        921
## 8  400  82    25 139     12   411        6 50403     1188    4057       1054
## 9  450 104    32 180     30   519        9 64972     7008    4793       1358
## 10 500 124    26 222     43   638        4 80379     7400    5408       1104
```

```
## ax = R matrix * vector  A %*% x
## axx = R crossprod A, x
## axftn = Fortran version of implicit Moler A * x
## aximp = implicit moler A*x in R
## axmfast = A fast and memory-saving version of A %*% x
## Times in milliseconds from microbenchmark
```

From the above output, we see that the `crossprod` variant of the matrix-vector product appears to be the fastest. However, we have omitted the time to build the matrix. If we must build the matrix, then we need somehow to include that time. Apportioning "fixed costs" to timings is never a trivial decision. Similarly if, where and how to store large matrices if we do build them, and whether it is worth building them more than once if storage is an issue, are all questions that may need to be addressed if performance becomes important.

```
## Times (in millisecs) adjusted for matrix build
```

```
##       n axbld axxbld axftn aximp
## 1   50  1244   1235   135  1071
## 2  100  3357   3355    31  3262
```

```
## 3  150  7348   7344    62  7249
## 4  200 13033  13030   107 12928
## 5  250 20215  20239   166 19862
## 6  300 29299  29331   234 29597
## 7  350 39929  39970   314 38524
## 8  400 52073  52130   411 50403
## 9  450 67996  68072   519 64972
## 10 500 81964  82062   638 80379
```

Out of all this, we see that the Fortran implicit matrix-vector product is the overall winner at all values of `n`. Moreover, it does NOT require the creation and storage of the matrix. However, using Fortran does involve rather more work for the user, and for most applications it is likely we could live with the use of either

- the interpreted matrix-product based on `crossprod` and an actual matrix is good enough, especially if a fast matrix build is used and we have plenty of memory, or
- the interpreted or byte-code compiled implicit matrix-vector multiply `axmolerfast`.

RQ computation times

We have set up three versions of a Rayleigh Quotient calculation in addition to the direct form. The third form is set up to use the `axftn` routine that we have already shown is efficient. We could also use this with the implicit matrix-vector product `axmolerfast`.

It seems overkill to show the RQ computation time for all versions and matrices, so we will do the timing simply for a matrix of order 500.

```
## Direct algorithm:  17514 sd= 469
```

```
## ray1: mat-mult algorithm:  240 sd= 160
```

```
## ray2: crossprod algorithm:  238 sd= 169
```

```
## ray3: ax Fortran + crossprod:  682.6669
```

```
## ray3: ax fast R implicit + crossprod:  5694 sd= 1538
```

Here we see that the use of either the matrix multiplication in `ray1` or of crossprodinray2is very fast, and this is interpreted code. Once again, we note that all timings except those forray3should have some adjustment for the building of the matrix. If storage is an issue, thenray3, which uses the implicit matrix-vector product in Fortran, is the approach of choice. My own preference would be to use this option if the Fortran matrix-vector product subroutine is already available for the matrix required. I would not, however, generally choose to write the Fortran subroutine for a "new" problem matrix. The fast implicit matrix-vector tool withray3‘ is also useful and quite fast if we need to minimize memory use.

Solution by spg

To actually solve the eigensolution problem we will first use the projected gradient method `spg` from `BB`. We repeat the RQ function so that it is clear which routine we are using.

```
# spgRQ.R
molerfast <- function(n) {
  # A fast version of `molermat'
  A <- matrix(0, nrow = n, ncol = n)
  j <- 1:n
  for (i in 1:n) {
    A[i, 1:i] <- pmin(i, 1:i) - 2
  }
```

```
  A <- A + t(A)
  diag(A) <- 1:n
  A
}

rqfast<-function(x){
  rq<-as.numeric(t(x) %*% axmolerfast(x))
  rq
}
rqneg<-function(x) { -rqfast(x)}
proj <- function(x) {sign(x[1]) * x/sqrt(c(crossprod(x))) } # from ravi
# Note that the c() is needed in denominator to avoid error msgs
require(BB)
n<-100
x<-rep(1,n)
x<-x/as.numeric(sqrt(crossprod(x)))
AA<-molerfast(n)
teig<-microbenchmark(evs<-eigen(AA), times=mbt)$time
cat("eigen time =", msect(teig),"sd=",msecr(teig),"\n")
```

```
## eigen time = 2494 sd= 647
```

```
tmin<-microbenchmark(amin<-spg(x, fn=rqfast, project=proj,
                               control=list(trace=FALSE)), times=mbt)$time
tmax<-microbenchmark(amax<-spg(x, fn=rqneg, project=proj,
                               control=list(trace=FALSE)), times=mbt)$time
evalmax<-evs$values[1]
evecmax<-evs$vectors[,1]
evecmax<-sign(evecmax[1])*evecmax/sqrt(as.numeric(crossprod(evecmax))) # normalize
emax<-list(evalmax=evalmax, evecmax=evecmax)
evalmin<-evs$values[n]
evecmin<-evs$vectors[,n]
evecmin<-sign(evecmin[1])*evecmin/sqrt(as.numeric(crossprod(evecmin)))
avecmax<-amax$par
avecmin<-amin$par
avecmax<-sign(avecmax[1])*avecmax/sqrt(as.numeric(crossprod(avecmax)))
avecmin<-sign(avecmin[1])*avecmin/sqrt(as.numeric(crossprod(avecmin)))
cat("minimal eigensolution: Value=",amin$value,"in time ",
      msect(tmin),"sd=",msecr(tmin),"\n")
```

```
## minimal eigensolution: Value= 5.939165e-08 in time  26682887 sd= 237580
```

```
cat("Eigenvalue - result from eigen=",amin$value-evalmin,"  vector max(abs(diff))=",
    max(abs(avecmin-evecmin)),"\n")
```

```
## Eigenvalue - result from eigen= 5.93916e-08   vector max(abs(diff))= 0.000135496
```

```
#print(amin$par)
cat("maximal eigensolution: Value=",-amax$value,"in time ",
     msect(tmax),"sd=",msecr(tmax),"\n")
```

```
## maximal eigensolution: Value= 3934.277 in time  500534 sd= 17217
```

```
cat("Eigenvalue - result from eigen=",-amax$value-evalmax,"  vector max(abs(diff))=",
    max(abs(avecmax-evecmax)),"\n")
```

```
## Eigenvalue - result from eigen= -3.761099e-06   vector max(abs(diff))= 4.747616e-06
```

```
nmax<-10
stable<-matrix(NA, nrow=nmax, ncol=4) # to hold results
# =========== works to here, but spg is slower than eigen
# loop over sizes
for (ni in 1:nmax){
  n<-50*ni
  x<-runif(n) # generate a vector
  AA<-molerfast(n) # make sure defined
  stable[[ni, 1]]<-n
  tbld<-microbenchmark(AA<-molerfast(n), times=mbt)
  tspg<-microbenchmark(aspg<-spg(x, fn=rqneg, project=proj,
                                 control=list(trace=FALSE)), times=mbt)
  teig<-microbenchmark(aseig<-eigen(AA), times=mbt)
  stable[[ni, 2]]<-msect(tspg$time)
  stable[[ni, 3]]<-msect(tbld$time)
  stable[[ni, 4]]<-msect(teig$time)
}
spgtym<-data.frame(n=stable[,1], spgrqt=stable[,2], tbld=stable[,3], teig=stable[,4])
print(round(spgtym,0))
```

```
##       n   spgrqt tbld  teig
## 1   50   187973  288   786
## 2  100   774689 1131  2144
## 3  150  1887099 1773  4014
## 4  200  3391862 2152  6242
## 5  250  5641603 2657  8685
## 6  300  8485127 3448 14392
## 7  350 11944818 4634 15770
## 8  400 16344867 5414 22431
## 9  450 21153870 5371 26005
## 10 500 27165006 5731 40519
```

Solution by other optimizers

We can try other optimizers, but we must note that unlike `spg` they do not take account of the scaling. However, we can build in a transformation, since our function is always the same for all sets of parameters scaled by the square root of the parameter inner product. The function `nobj` forms the quadratic form that is the numerator of the Rayleigh Quotient using the more efficient `code{crossprod()` function

```
rq<- as.numeric(crossprod(y, crossprod(AA,y)))
```

but we first form

```
y<-x/sqrt(as.numeric(crossprod(x)))
```

to scale the parameters.

Since we are running a number of gradient-based optimizers in the wrapper `optimx::opm()`, we have reduced the matrix sizes and numbers.

```
require(optimx)
nobj<-function(x, AA=-AA){
   y<-x/sqrt(as.numeric(crossprod(x)))
   rq<- as.numeric(crossprod(y, crossprod(AA,y)))
}

ngrobj<-function(x, AA=-AA){
```

```
   y<-x/sqrt(as.numeric(crossprod(x)))
   n<-length(x)
   dd<-sqrt(as.numeric(crossprod(x)))
   T1<-diag(rep(1,n))/dd
   T2<- x%o%x/(dd*dd*dd)
   gt<-T1-T2
   gy<- as.vector(2.*crossprod(AA,y))
   gg<-as.numeric(crossprod(gy, gt))
}
mset<-c("L-BFGS-B", "BFGS", "ncg", "spg", "ucminf", "nlm", "nlminb", "nvm")
nmax<-5
for (ni in 1:nmax){
  n<-20*ni
  x<-runif(n) # generate a vector
  AA<-molerfast(n) # make sure defined
  aall<-opm(x, fn=nobj, gr=ngrobj, method=mset, AA=-AA,
     control=list(trace=1,starttests=FALSE, dowarn=FALSE, kkt=FALSE))
  # optansout(aall, NULL)
  summary(aall, order=value, )
  cat("Above for n=",n," \n")
}
```

```
## opm: wrapper to call optimr to run multiple optimizers
## Method:  L-BFGS-B
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## final  value -140.899147
## converged
## Post processing for method  L-BFGS-B
## Successful convergence!
## Method:  BFGS
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## initial  value -76.869360
## final  value -140.899147
## converged
## Post processing for method  BFGS
## Successful convergence!
## Method:  ncg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## ncg -- J C Nash 2023 - bounds constraint version of new CG
## an R implementation of Alg 22 with Yuan/Dai modification
## stepredn = 0.2
## Initial function value= -76.86936
## Initial fn= -76.86936
## 1   0   1   -76.86936   last decrease= NA
## *4   1   2   -107.4959   last decrease= 30.62652
## 6   2   3   -140.0924   last decrease= 32.5965
## Yuan/Dai cycle reset
## 6   3   1   -140.0924   last decrease= NA
## 8   4   2   -140.8975   last decrease= 0.8051046
## 10   5   3   -140.8991   last decrease= 0.001661851
## 12   6   4   -140.8991   last decrease= 2.326985e-06
```

```
## 14   7   5   -140.8991   last decrease= 1.0767e-09
## Very small gradient -- gradsqr = 3.1649910998823e-13
## ncg seems to have converged
## Post processing for method  ncg
## Successful convergence!
## Method:  spg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## iter:  0  f-value:  -76.86936  pgrad:  25.37794
## Post processing for method  spg
## Successful convergence!
## Method:  ucminf
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## ucminf message: Stopped by small gradient (grtol).
## Post processing for method  ucminf
## Successful convergence!
## Method:  nlm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## iteration = 0
## Step:
##  [1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## Parameter:
##  [1] 0.28810290 0.86452112 0.02563072 0.53299656 0.79551715 0.68073549
##  [7] 0.68852975 0.16399912 0.27836728 0.34715762 0.55899409 0.50125182
## [13] 0.41166590 0.38020118 0.02963047 0.08426403 0.63174654 0.56266313
## [19] 0.58696982 0.51706989
## Function Value
## [1] -76.86936
## Gradient:
##  [1]  11.905536  25.377937  -2.174347   9.627475  14.481791   8.620040
##  [7]   6.606270 -10.695838  -9.262225  -9.051956  -4.525679  -7.659593
## [13] -11.528471 -13.538076 -24.725087 -24.058510  -8.955616 -11.625155
## [19] -11.345751 -13.590400
##
## iteration = 1
## Step:
##  [1]  -5.947178 -12.677052   1.086153  -4.809217  -7.234096  -4.305972
##  [7]  -3.300033   5.342897   4.626763   4.521728   2.260715   3.826200
## [13]   5.758822   6.762681  12.350934  12.017959   4.473603   5.807119
## [19]   5.667548   6.788819
## Parameter:
##  [1]  -5.659075 -11.812531   1.111783  -4.276221  -6.438579  -3.625237
##  [7]  -2.611503   5.506896   4.905131   4.868886   2.819709   4.327452
## [13]   6.170488   7.142882  12.380565  12.102223   5.105350   6.369782
## [19]   6.254518   7.305889
## Function Value
## [1] -78.59738
## Gradient:
##  [1] -0.84291684 -2.01154585  0.02227162 -1.04046110 -1.56676313 -1.26524918
##  [7] -1.27629354 -0.09078013 -0.36881958 -0.54031849 -1.04171017 -0.93493204
## [13] -0.76185111 -0.72247682  0.05449635 -0.07465814 -1.31405619 -1.14410825
## [19] -1.19357415 -1.03179131
```

```
##
## iteration = 2
## Step:
##  [1]  0.383397782  0.923339737 -0.004983391  0.487275964  0.733800566
##  [6]  0.601094454  0.612122577  0.074439234  0.207804989  0.291932247
## [11]  0.527060591  0.482992077  0.408208304  0.394321412  0.041507000
## [16]  0.103456109  0.673841053  0.597285696  0.620945365  0.547247182
## Parameter:
##  [1]  -5.275677 -10.889192   1.106800  -3.788945  -5.704778  -3.024143
##  [7]  -1.999381   5.581335   5.112936   5.160818   3.346769   4.810444
## [13]   6.578697   7.537204  12.422072  12.205679   5.779191   6.967068
## [19]   6.875464   7.853136
## Function Value
## [1] -88.51651
## Gradient:
##  [1] -0.86019700 -2.05904806  0.02969005 -1.05545443 -1.58849738 -1.26962855
##  [7] -1.26869962 -0.03934781 -0.31140482 -0.47526864 -0.97802947 -0.85717981
## [13] -0.66860273 -0.61799020  0.18730895  0.06037577 -1.20776293 -1.03155103
## [19] -1.08078773 -0.91401651
##
## iteration = 3
## Step:
##  [1] 10.9647962 26.4688643 -0.2377277 13.8321657 20.8197100 16.8785114
##  [7] 17.0324832  1.4113007  5.0687850  7.3260591 13.9277798 12.5126641
## [13] 10.2193170  9.6861440 -0.5581324  1.1476751 17.4893160 15.2642459
## [19] 15.9234531 13.7953176
## Parameter:
##  [1]  5.6891190 15.5796727  0.8690722 10.0432209 15.1149321 13.8543688
##  [7] 15.0331026  6.9926359 10.1817208 12.4868769 17.2745491 17.3231080
## [13] 16.7980136 17.2233479 11.8639394 13.3533538 23.2685067 22.2313137
## [19] 22.7989167 21.6484534
## Function Value
## [1] -122.4459
## Gradient:
##  [1]  0.401831207  0.775830275 -0.065124274  0.280087456  0.425764894
##  [6]  0.263293690  0.227637148 -0.259529219 -0.186298820 -0.152756054
## [11]  0.009183401 -0.057039810 -0.144602745 -0.178055635 -0.491604742
## [16] -0.460239550 -0.005056866 -0.084035038 -0.074176150 -0.140216659
##
## iteration = 4
## Step:
##  [1] -1.37895016 -3.24350614  0.05287279 -1.64275224 -2.47460060 -1.97343366
##  [7] -1.97743231 -0.05377702 -0.48984266 -0.75473568 -1.55220861 -1.37046377
## [13] -1.08297473 -1.01204184  0.24699640  0.04494312 -1.94738534 -1.66923652
## [19] -1.74621095 -1.48408370
## Parameter:
##  [1]  4.310169 12.336167  0.921945  8.400469 12.640331 11.880935 13.055670
##  [8]  6.938859  9.691878 11.732141 15.722340 15.952644 15.715039 16.211306
## [15] 12.110936 13.398297 21.321121 20.562077 21.052706 20.164370
## Function Value
## [1] -127.1688
## Gradient:
##  [1]  0.39605344  0.76311677 -0.06296702  0.27685787  0.42103655  0.26305093
##  [7]  0.23023324 -0.24595052 -0.17155306 -0.13634753  0.02492756 -0.03798895
```

```
## [13] -0.12190772 -0.15281046 -0.45928069 -0.42742421  0.02036346 -0.05690752
## [19] -0.04695499 -0.11169237
##
## iteration = 5
## Step:
##  [1] -1.46040004 -3.44166792  0.05346639 -1.74843779 -2.63371224 -2.10440856
##  [7] -2.11103444 -0.07209390 -0.53699075 -0.82014893 -1.66830293 -1.47723704
## [13] -1.17379246 -1.09996060  0.23476317  0.01927021 -2.09572559 -1.80126889
## [19] -1.88340700 -1.60548872
## Parameter:
##  [1]  2.8497687  8.8944986  0.9754114  6.6520309 10.0066192  9.7765266
##  [7] 10.9446359  6.8667650  9.1548874 10.9119923 14.0540375 14.4754072
## [13] 14.5412465 15.1113454 12.3456990 13.4175671 19.2253958 18.7608082
## [19] 19.1692988 18.5588810
## Function Value
## [1] -132.2671
## Gradient:
##  [1]  0.36724269  0.70166040 -0.05740987  0.25400231  0.38670427  0.24336021
##  [7]  0.21595143 -0.21767011 -0.14618143 -0.11094996  0.03985913 -0.01494122
## [13] -0.08897499 -0.11454255 -0.39348279 -0.36297779  0.04832181 -0.02236281
## [19] -0.01300861 -0.07225579
##
## iteration = 6
## Step:
##  [1]  -4.43192139 -10.45449148   0.15673850  -5.32177730  -8.01633843
##  [6]  -6.41458017  -6.44099212  -0.25417061  -1.67239727  -2.53823705
## [11]  -5.11990120  -4.54470491  -3.62782453  -3.40792343   0.64250195
## [16]  -0.01494487  -6.44124241  -5.54838688  -5.79894202  -4.95532928
## Parameter:
##  [1] -1.582153 -1.559993  1.132150  1.330254  1.990281  3.361946  4.503644
##  [8]  6.612594  7.482490  8.373755  8.934136  9.930702 10.913422 11.703422
## [15] 12.988201 13.402622 12.784153 13.212421 13.370357 13.603552
## Function Value
## [1] -140.1117
## Gradient:
##  [1] -0.0777677185 -0.2556323206 -0.0010000575 -0.1507472453 -0.2237851197
##  [6] -0.1787143609 -0.1670934341  0.0071780981 -0.0123226586 -0.0195340354
## [11] -0.0707084179 -0.0404043283 -0.0006559071  0.0205964553  0.1361207256
## [16]  0.1253592349 -0.0376395483 -0.0153048150 -0.0213838394  0.0005279385
##
## iteration = 7
## Step:
##  [1]  0.80474696  1.92515656 -0.02391766  0.99258084  1.49422997  1.20141001
##  [7]  1.20673790  0.06513583  0.32578296  0.48484757  0.96018292  0.85290864
## [13]  0.68220272  0.63994352 -0.10812456  0.01323200  1.19937767  1.03582297
## [19]  1.08261077  0.92727405
## Parameter:
##  [1] -0.7774057  0.3651637  1.1082322  2.3228345  3.4845108  4.5633565
##  [7]  5.7103817  6.6777302  7.8082731  8.8586028  9.8943192 10.7836109
## [13] 11.5956246 12.3433655 12.8800764 13.4158542 13.9835311 14.2482443
## [19] 14.4529675 14.5308258
## Function Value
## [1] -140.8421
## Gradient:
```

```
##  [1]  0.0627467911  0.0519368224 -0.0161986278 -0.0150710611 -0.0191717664
##  [6] -0.0316117345 -0.0294561402 -0.0473129379 -0.0346502053 -0.0254504512
## [11] -0.0091454967 -0.0036971746  0.0018075188  0.0100918499  0.0007355851
## [16]  0.0046442733  0.0271176853  0.0203776169  0.0197954458  0.0159245078
##
## iteration = 8
## Step:
##  [1] -0.0698166013 -0.1187823048  0.0089615755 -0.0409243921 -0.0633504322
##  [6] -0.0432677947 -0.0448347022  0.0189340690 -0.0001694979 -0.0126347521
## [11] -0.0438909031 -0.0416038205 -0.0362662389 -0.0384951029  0.0021464623
## [16] -0.0057656744 -0.0742991824 -0.0631161746 -0.0651298256 -0.0557269198
## Parameter:
##  [1] -0.8472223  0.2463814  1.1171938  2.2819101  3.4211603  4.5200887
##  [7]  5.6655470  6.6966643  7.8081036  8.8459681  9.8504283 10.7420071
## [13] 11.5593584 12.3048704 12.8822228 13.4100886 13.9092319 14.1851282
## [19] 14.3878377 14.4750988
## Function Value
## [1] -140.852
## Gradient:
##  [1]  0.052171681  0.034935666 -0.014435267 -0.019903968 -0.026685871
##  [6] -0.035748017 -0.033311792 -0.041567710 -0.031132887 -0.023219444
## [11] -0.010899971 -0.004678342  0.001999195  0.010335726  0.007107337
## [16]  0.010173124  0.023104878  0.018142198  0.017392118  0.014922240
##
## iteration = 9
## Step:
##  [1] -0.153932496 -0.202348392  0.028328834 -0.034372336 -0.057778557
##  [6] -0.020080998 -0.024945908  0.072058760  0.034167876  0.008212369
## [11] -0.048516867 -0.052271730 -0.052326472 -0.064643182 -0.005079466
## [16] -0.019392447 -0.128164798 -0.107272131 -0.109223964 -0.093535684
## Parameter:
##  [1] -1.00115478  0.04403299  1.14552263  2.24753774  3.36338177  4.50000767
##  [7]  5.64060105  6.76872304  7.84227143  8.85418042  9.80191147 10.68973533
## [13] 11.50703192 12.24022722 12.87714337 13.39069610 13.78106708 14.07785603
## [19] 14.27861372 14.38156315
## Function Value
## [1] -140.8692
## Gradient:
##  [1]  0.028950411  0.005402756 -0.009837783 -0.023790943 -0.033340534
##  [6] -0.036287496 -0.033897950 -0.027378730 -0.021715162 -0.016809703
## [11] -0.011965692 -0.005555217  0.001801560  0.008976563  0.014957702
## [16]  0.016444105  0.014005009  0.012395717  0.011570446  0.011480582
##
## iteration = 10
## Step:
##  [1] -0.137770505 -0.115872881  0.034906012  0.030642822  0.038282008
##  [6]  0.064895399  0.059050398  0.098186939  0.068889286  0.047286071
## [11]  0.009780701 -0.003284439 -0.016352586 -0.035556063 -0.015198863
## [16] -0.024655434 -0.075749964 -0.061219804 -0.060275077 -0.051767585
## Parameter:
##  [1] -1.13892529 -0.07183989  1.18042864  2.27818056  3.40166378  4.56490307
##  [7]  5.69965145  6.86690998  7.91116072  8.90146650  9.81169217 10.68645089
## [13] 11.49067933 12.20467116 12.86194451 13.36604067 13.70531711 14.01663622
## [19] 14.21833864 14.32979557
```

```
## Function Value
## [1] -140.8844
## Gradient:
##  [1]  0.008582034 -0.011768928 -0.004967117 -0.019635289 -0.028108086
##  [6] -0.027133905 -0.025516077 -0.013083806 -0.011468254 -0.009409059
## [11] -0.009700620 -0.004819483  0.001004847  0.005726240  0.015029019
## [16]  0.015426009  0.005739014  0.006451103  0.005884899  0.007102233
##
## iteration = 11
## Step:
##  [1] -0.107046850 -0.015579600  0.038320620  0.094286812  0.132964021
##  [6]  0.145656690  0.139506558  0.115616880  0.098080208  0.083058324
## [11]  0.068904926  0.047545836  0.022401059 -0.001993448 -0.023869131
## [16] -0.027535896 -0.014654593 -0.008209935 -0.004467742 -0.004190466
## Parameter:
##  [1] -1.24597214 -0.08741949  1.21874926  2.37246737  3.53462780  4.71055976
##  [7]  5.83915801  6.98252686  8.00924093  8.98452482  9.88059709 10.73399672
## [13] 11.51308039 12.20267771 12.83807538 13.33850477 13.69066252 14.00842629
## [19] 14.21387090 14.32560510
## Function Value
## [1] -140.8966
## Gradient:
##  [1] -0.0063161521 -0.0141000196 -0.0003825307 -0.0076821216 -0.0112481651
##  [6] -0.0090771506 -0.0088691625 -0.0003509222 -0.0015433044 -0.0018238937
## [11] -0.0042294507 -0.0024713086 -0.0002795113  0.0009473056  0.0070971733
## [16]  0.0070243594 -0.0006901319  0.0009827125  0.0009785817  0.0022359480
##
## iteration = 12
## Step:
##  [1]  0.002976101  0.060615138  0.008363165  0.054840527  0.080372719
##  [6]  0.073752429  0.072741320  0.029675206  0.034337331  0.036061552
## [11]  0.048372351  0.039739903  0.028154339  0.021000016 -0.008964346
## [16] -0.006074572  0.035794692  0.031587788  0.033833315  0.028684555
## Parameter:
##  [1] -1.24299604 -0.02680435  1.22711242  2.42730790  3.61500052  4.78431219
##  [7]  5.91189933  7.01220206  8.04357826  9.02058637  9.92896944 10.77373663
## [13] 11.54123473 12.22367773 12.82911103 13.33243020 13.72645721 14.04001408
## [19] 14.24770421 14.35428966
## Function Value
## [1] -140.8989
## Gradient:
##  [1] -0.0051128749 -0.0052020317  0.0001696606 -0.0009895777 -0.0014253278
##  [6] -0.0007487194 -0.0011313327  0.0007090025 -0.0001339854 -0.0004616921
## [11] -0.0013250507 -0.0010436994 -0.0007192478 -0.0006805860  0.0009577448
## [16]  0.0011772943 -0.0005390935  0.0004148744  0.0006725003  0.0011442075
##
## iteration = 13
## Step:
##  [1]  0.0163216294  0.0287145950 -0.0012897631  0.0113316680  0.0172397320
##  [6]  0.0127735800  0.0133269036 -0.0019890643  0.0024521774  0.0051770345
## [11]  0.0123235992  0.0113500008  0.0096147655  0.0097048843 -0.0004613268
## [16]  0.0011024304  0.0173048699  0.0144313815  0.0147948140  0.0124769483
## Parameter:
##  [1] -1.226674407  0.001910242  1.225822660  2.438639569  3.632240250
```

```
##  [6]  4.797085769  5.925226229  7.010213000  8.046030437  9.025763407
## [11]  9.941293044 10.785086628 11.550849497 12.233382611 12.828649707
## [16] 13.333532634 13.743762085 14.054445458 14.262499028 14.366766605
## Function Value
## [1] -140.8991
## Gradient:
##  [1] -2.517663e-03 -1.005359e-03 -1.697402e-04  3.361075e-04  6.009256e-04
##  [6]  4.873518e-04  5.877314e-05 -4.417369e-04 -7.533391e-04 -7.972736e-04
## [11] -7.282853e-04 -6.831198e-04 -6.937038e-04 -7.144191e-04 -6.177849e-04
## [16] -2.342178e-04  3.441278e-04  8.355711e-04  1.117082e-03  1.238294e-03
##
## iteration = 14
## Step:
##  [1]  4.506108e-03  4.195151e-03 -5.229342e-04  1.125515e-04  2.506179e-04
##  [6] -3.856975e-04 -2.910949e-05 -1.368589e-03 -4.599926e-04  7.603181e-05
## [11]  1.086513e-03  1.200370e-03  1.293165e-03  1.605110e-03  5.740882e-04
## [16]  6.201942e-04  2.164577e-03  1.494180e-03  1.355463e-03  1.008850e-03
## Parameter:
##  [1] -1.222168299  0.006105392  1.225299726  2.438752120  3.632490868
##  [6]  4.796700071  5.925197120  7.008844411  8.045570444  9.025839438
## [11]  9.942379557 10.786286998 11.552142662 12.234987721 12.829223795
## [16] 13.334152828 13.745926662 14.055939638 14.263854491 14.367775455
## Function Value
## [1] -140.8991
## Gradient:
##  [1] -1.843221e-03 -3.912079e-04 -2.514752e-04  3.355268e-04  6.094078e-04
##  [6]  3.927441e-04  4.924594e-06 -7.011795e-04 -8.933297e-04 -8.724441e-04
## [11] -6.696645e-04 -6.197922e-04 -6.274661e-04 -6.120899e-04 -6.727530e-04
## [16] -2.894862e-04  5.067845e-04  8.967583e-04  1.155612e-03  1.225411e-03
##
## iteration = 15
## Step:
##  [1]  0.0057887095  0.0038650711 -0.0002523381 -0.0004474734 -0.0006958815
##  [6] -0.0010259228 -0.0003309311 -0.0006643045  0.0003056452  0.0007482803
## [11]  0.0013921825  0.0014727238  0.0016026379  0.0018873554  0.0011456665
## [16]  0.0007969606  0.0012209158  0.0002596896 -0.0001265344 -0.0004825968
## Parameter:
##  [1] -1.216379589  0.009970464  1.225047388  2.438304647  3.631794986
##  [6]  4.795674148  5.924866189  7.008180106  8.045876089  9.026587719
## [11]  9.943771739 10.787759722 11.553745300 12.236875077 12.830369462
## [16] 13.334949788 13.747147578 14.056199328 14.263727956 14.367292858
## Function Value
## [1] -140.8991
## Gradient:
##  [1] -9.851914e-04  1.761812e-04 -2.891412e-04  2.621023e-04  4.920513e-04
##  [6]  2.189214e-04 -7.756187e-05 -8.419872e-04 -9.039323e-04 -8.291224e-04
## [11] -5.424264e-04 -4.889811e-04 -4.843676e-04 -4.325509e-04 -6.038518e-04
## [16] -2.730002e-04  5.838543e-04  8.345669e-04  1.037946e-03  1.056536e-03
##
## iteration = 16
## Step:
##  [1]  0.0065275236  0.0026033959  0.0004098418 -0.0009082694 -0.0015976592
##  [6] -0.0013008485 -0.0001748805  0.0011393301  0.0019784755  0.0021265322
## [11]  0.0019855917  0.0019031681  0.0019634873  0.0020473488  0.0018169959
```

```
## [16]  0.0008412644 -0.0006392754 -0.0019075621 -0.0026333061 -0.0029468036
## Parameter:
##  [1] -1.20985207  0.01257386  1.22545723  2.43739638  3.63019733  4.79437330
##  [7]  5.92469131  7.00931944  8.04785456  9.02871425  9.94575733 10.78966289
## [13] 11.55570879 12.23892243 12.83218646 13.33579105 13.74650830 14.05429177
## [19] 14.26109465 14.36434605
## Function Value
## [1] -140.8991
## Gradient:
##  [1] -2.342839e-05  5.610634e-04 -2.270592e-04  1.289142e-04  2.539549e-04
##  [6]  1.747515e-05 -1.233085e-04 -7.048682e-04 -6.517658e-04 -5.602042e-04
## [11] -2.965084e-04 -2.554842e-04 -2.405731e-04 -1.729900e-04 -3.720092e-04
## [16] -1.752355e-04  4.750962e-04  5.494563e-04  6.532753e-04  6.294218e-04
##
## iteration = 17
## Step:
##  [1]  0.0025793911 -0.0009765894  0.0009819441 -0.0007841235 -0.0015120921
##  [6] -0.0006135127  0.0002358228  0.0027022521  0.0027424437  0.0023920804
## [11]  0.0013305610  0.0010856987  0.0009956960  0.0007408556  0.0013287715
## [16]  0.0002567018 -0.0025939590 -0.0033210458 -0.0039503471 -0.0039778369
## Parameter:
##  [1] -1.20727267  0.01159727  1.22643917  2.43661225  3.62868523  4.79375979
##  [7]  5.92492713  7.01202169  8.05059701  9.03110633  9.94708789 10.79074859
## [13] 11.55670448 12.23966328 12.83351523 13.33604775 13.74391434 14.05097072
## [19] 14.25714430 14.36036822
## Function Value
## [1] -140.8991
## Gradient:
##  [1]  3.507902e-04  4.219753e-04 -7.970947e-05  2.289476e-05  4.449126e-05
##  [6] -6.081843e-05 -7.821787e-05 -3.013312e-04 -2.392704e-04 -1.926010e-04
## [11] -7.456481e-05 -5.922412e-05 -4.637390e-05 -3.746717e-06 -1.045758e-04
## [16] -4.927230e-05  2.015872e-04  1.821232e-04  2.028486e-04  1.791610e-04
##
## iteration = 18
## Step:
##  [1] -1.112120e-03 -2.126481e-03  6.150094e-04 -2.048667e-04 -4.682713e-04
##  [6]  1.398364e-04  2.926948e-04  1.707318e-03  1.324384e-03  9.582276e-04
## [11]  1.387560e-04 -2.674307e-05 -1.352119e-04 -3.950706e-04  2.074997e-04
## [16] -2.100301e-04 -1.866702e-03 -1.848287e-03 -2.017631e-03 -1.886355e-03
## Parameter:
##  [1] -1.208384794  0.009470789  1.227054184  2.436407387  3.628216963
##  [6]  4.793899623  5.925219826  7.013729006  8.051921392  9.032064559
## [11]  9.947226648 10.790721846 11.556569271 12.239268211 12.833722729
## [16] 13.335837724 13.742047642 14.049122433 14.255126672 14.358481863
## Function Value
## [1] -140.8991
## Gradient:
##  [1]  1.820592e-04  1.119912e-04  1.229298e-05  3.138046e-07 -1.207055e-05
##  [6] -2.528036e-05 -1.654820e-05 -2.997865e-05 -1.750869e-05 -1.660969e-05
## [11] -9.176963e-06 -9.318506e-06 -3.695639e-06  9.576773e-06  4.229399e-06
## [16]  7.081991e-06  2.531038e-05  1.453517e-05  1.474905e-05  1.209918e-05
##
## iteration = 19
## Step:
```

```
##  [1] -8.785082e-04 -7.450680e-04  9.785277e-05  3.001370e-05  3.413145e-05
##  [6]  1.509253e-04  8.610547e-05  2.979285e-04  1.351411e-04  3.918890e-05
## [11] -1.370828e-04 -1.684048e-04 -2.015196e-04 -2.739103e-04 -1.182631e-04
## [16] -1.321264e-04 -3.879568e-04 -2.731342e-04 -2.506537e-04 -1.939435e-04
## Parameter:
##  [1] -1.209263303  0.008725721  1.227152036  2.436437401  3.628251095
##  [6]  4.794050549  5.925305931  7.014026935  8.052056533  9.032103748
## [11]  9.947089565 10.790553441 11.556367752 12.238994300 12.833604466
## [16] 13.335705598 13.741659685 14.048849299 14.254876019 14.358287919
## Function Value
## [1] -140.8991
## Gradient:
##  [1]  5.144178e-05  2.856767e-06  2.691109e-05  6.504343e-06 -3.824753e-06
##  [6]  1.422131e-06  2.390600e-06  2.146732e-05  1.234584e-05  1.501989e-06
## [11] -1.443313e-05 -1.703264e-05 -1.430411e-05 -9.841296e-06  8.836505e-06
## [16]  1.097004e-05 -6.803280e-06 -1.506124e-07  3.793431e-06  9.583257e-06
##
## iteration = 20
## Step:
##  [1] -1.601260e-04 -7.247435e-05 -1.558645e-05  1.088915e-05  2.634277e-05
##  [6]  2.462217e-05  5.069438e-06 -1.416355e-05 -2.877455e-05 -2.959101e-05
## [11] -2.725373e-05 -2.620278e-05 -3.007923e-05 -3.698701e-05 -3.654502e-05
## [16] -2.443671e-05 -5.609977e-06  1.415448e-05  2.342509e-05  2.740799e-05
## Parameter:
##  [1] -1.209423429  0.008653247  1.227136450  2.436448290  3.628277438
##  [6]  4.794075171  5.925311001  7.014012771  8.052027759  9.032074157
## [11]  9.947062311 10.790527238 11.556337673 12.238957313 12.833567921
## [16] 13.335681161 13.741654075 14.048863453 14.254899444 14.358315327
## Function Value
## [1] -140.8991
## Gradient:
##  [1]  2.779103e-05 -7.837311e-06  2.462969e-05  8.201970e-06  2.624034e-07
##  [6]  5.431893e-06  3.758657e-06  2.024544e-05  9.190495e-06 -1.613302e-06
## [11] -1.708125e-05 -1.942890e-05 -1.719160e-05 -1.369175e-05  5.051957e-06
## [16]  8.907646e-06 -6.193271e-06  3.262168e-06  8.499585e-06  1.483926e-05
##
## iteration = 21
## Step:
##  [1] -7.086804e-05 -1.458804e-05 -2.311871e-05 -1.101834e-06  1.013338e-05
##  [6]  4.634256e-06 -1.387830e-06 -2.239454e-05 -1.919424e-05 -1.143146e-05
## [11]  1.622354e-06  3.966887e-06  1.448484e-06 -2.572046e-06 -1.623636e-05
## [16] -1.352927e-05  7.062185e-06  7.636979e-06  7.824979e-06  4.706307e-06
## Parameter:
##  [1] -1.209494297  0.008638659  1.227113331  2.436447188  3.628287571
##  [6]  4.794079805  5.925309613  7.013990377  8.052008564  9.032062725
## [11]  9.947063934 10.790531205 11.556339121 12.238954741 12.833551684
## [16] 13.335667632 13.741661137 14.048871090 14.254907269 14.358320034
## Function Value
## [1] -140.8991
## Gradient:
##  [1]  1.734358e-05 -1.003028e-05  2.126244e-05  8.073181e-06  1.804924e-06
##  [6]  6.225719e-06  3.732675e-06  1.722174e-05  6.661633e-06 -3.004712e-06
## [11] -1.657866e-05 -1.858610e-05 -1.671076e-05 -1.378953e-05  2.972141e-06
## [16]  7.206700e-06 -4.931859e-06  4.585981e-06  9.837602e-06  1.571938e-05
```

```
##
## iteration = 22
## Step:
##  [1] -1.304358e-04  3.777499e-08 -7.308369e-05 -1.706341e-05  1.065195e-05
##  [6] -5.452422e-06 -8.668783e-06 -6.374872e-05 -3.856712e-05 -8.943239e-06
## [11]  3.558698e-05  4.280737e-05  3.580248e-05  2.472659e-05 -2.669660e-05
## [16] -3.118824e-05  2.087044e-05  3.048187e-06 -6.929816e-06 -2.269413e-05
## Parameter:
##  [1] -1.209624732  0.008638697  1.227040247  2.436430125  3.628298223
##  [6]  4.794074353  5.925300944  7.013926628  8.051969997  9.032053782
## [11]  9.947099521 10.790574012 11.556374924 12.238979468 12.833524988
## [16] 13.335636444 13.741682008 14.048874138 14.254900339 14.358297339
## Function Value
## [1] -140.8991
## Gradient:
##  [1] -1.873577e-06 -1.016119e-05  1.064637e-05  5.652901e-06  3.451404e-06
##  [6]  5.585438e-06  2.671868e-06  8.198342e-06  1.266684e-06 -4.166316e-06
## [11] -1.135118e-05 -1.235144e-05 -1.149033e-05 -1.013719e-05 -7.780536e-07
## [16]  2.842694e-06 -1.728558e-06  5.228455e-06  9.061786e-06  1.267759e-05
##
## iteration = 23
## Parameter:
##  [1] -1.209663624  0.008670052  1.226982168  2.436406769  3.628291371
##  [6]  4.794055880  5.925291144  7.013881612  8.051955544  9.032066432
## [11]  9.947149639 10.790629692 11.556425604 12.239022257 12.833521163
## [16] 13.335619538 13.741696531 14.048860540 14.254871176 14.358251464
## Function Value
## [1] -140.8991
## Gradient:
##  [1] -7.591951e-06 -5.647669e-06  2.231583e-06  2.282484e-06  2.454550e-06
##  [6]  2.870104e-06  1.166187e-06  1.517363e-06 -1.082395e-06 -2.701216e-06
## [11] -4.553024e-06 -4.802334e-06 -4.665916e-06 -4.404511e-06 -1.716745e-06
## [16]  9.927266e-08  1.705166e-07  3.135551e-06  4.786630e-06  6.023042e-06
##
## Relative gradient close to zero.
## Current iterate is probably solution.
##
## Post processing for method  nlm
## Successful convergence!
## Method:  nlminb
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##   0:    -76.869360: 0.288103 0.864521 0.0256307 0.532997 0.795517 0.680735 0.688530 0.163999 0.27836
##   1:    -138.90566: -0.243439 -0.268518 0.122708 0.103162 0.148954 0.295880 0.393582 0.641532 0.6918
##   2:    -140.85447: -0.0705158 0.0428205 0.0952033 0.208926 0.311860 0.398185 0.489494 0.554678 0.64
##   3:    -140.89181: -0.0872204 0.0177139 0.0976791 0.202181 0.301402 0.392537 0.483998 0.562743 0.64
##   4:    -140.89526: -0.0854350 0.00236016 0.100313 0.193109 0.286807 0.381988 0.475379 0.568162 0.653
##   5:    -140.89685: -0.107788 -0.000776900 0.0981015 0.199795 0.298269 0.391849 0.481912 0.566308 0.6
##   6:    -140.89828: -0.0943912 0.000464697 0.103064 0.199545 0.295423 0.390354 0.481451 0.570981 0.65
##   7:    -140.89841: -0.101452 0.000888017 0.0957094 0.195005 0.292054 0.385832 0.477793 0.564445 0.64
##   8:    -140.89908: -0.0970403 0.000613628 0.0996417 0.196965 0.293104 0.387309 0.478503 0.566739 0.6
##   9:    -140.89915: -0.0979409 0.000707192 0.0992121 0.197099 0.293544 0.387862 0.479413 0.567471 0.6
##  10:    -140.89915: -0.0978393 0.000704303 0.0992975 0.197138 0.293569 0.387892 0.479412 0.567498 0.6
##  11:    -140.89915: -0.0978800 0.000698206 0.0992791 0.197134 0.293567 0.387886 0.479410 0.567487 0.6
```

```
##  12:    -140.89915: -0.0978695 0.000704740 0.0992739 0.197128 0.293564 0.387885 0.479414 0.567492 0.6
##  13:    -140.89915: -0.0978695 0.000704740 0.0992739 0.197128 0.293564 0.387885 0.479414 0.567492 0.6
## Post processing for method  nlminb
## Successful convergence!
## Method:  nvm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## nvm -- J C Nash 2009-2015 - an R implementation of Alg 21
## Problem of size n= 20
## Initial fn= -76.86936
## ig= 1   gnorm= 60.06035     1   1   -76.86936
## *ig= 2   gnorm= 10.67287     3   2   -93.38227
## ig= 3   gnorm= 1.04231     4   3   -140.217
## ig= 4   gnorm= 0.5795961     5   4   -140.6824
## ig= 5   gnorm= 0.1325275     6   5   -140.8875
## ig= 6   gnorm= 0.03272263     7   6   -140.8984
## ig= 7   gnorm= 0.006846649     8   7   -140.8991
## ig= 8   gnorm= 0.001269862     9   8   -140.8991
## ig= 9   gnorm= 0.0001299495     10   9   -140.8991
## ig= 10   gnorm= 1.853187e-05     11   10   -140.8991
## ig= 11   gnorm= 3.727895e-06     12   11   -140.8991
## ig= 12   gnorm= 3.6803e-07     13   12   -140.8991
## ig= 13   gnorm= 3.12921e-08     14   13   -140.8991
## ******ig= 14   gnorm= 3.129031e-08     21   14   -140.8991
## **********No acceptable point
## Converged
## Seem to be done nvm
## Post processing for method  nvm
## Successful convergence!
## Above for n= 20
## opm: wrapper to call optimr to run multiple optimizers
## Method:  L-BFGS-B
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
## final  value -602.868475
## converged
## Post processing for method  L-BFGS-B
## Successful convergence!
## Method:  BFGS
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
## initial  value -356.378778
## iter  10 value -596.954168
## final  value -602.868475
## converged
## Post processing for method  BFGS
## Successful convergence!
## Method:  ncg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
## ncg -- J C Nash 2023 - bounds constraint version of new CG
```

```
## an R implementation of Alg 22 with Yuan/Dai modification
## stepredn = 0.2
## Initial function value= -356.3788
## Initial fn= -356.3788
## 1   0   1   -356.3788   last decrease= NA
## **5   1   2   -574.5443   last decrease= 218.1655
## 7   2   3   -602.7303   last decrease= 28.18595
## Yuan/Dai cycle reset
## 7   3   1   -602.7303   last decrease= NA
## 9   4   2   -602.8684   last decrease= 0.1381359
## 11   5   3   -602.8685   last decrease= 7.827836e-05
## 13   6   4   -602.8685   last decrease= 2.129048e-08
## Very small gradient -- gradsqr = 3.04881992176794e-11
## ncg seems to have converged
## Post processing for method  ncg
## Successful convergence!
## Method:  spg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
## iter:  0  f-value:  -356.3788  pgrad:  50.45269
## Post processing for method  spg
## Successful convergence!
## Method:  ucminf
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
## ucminf message: Stopped by zero step from line search
## Post processing for method  ucminf
## Successful convergence!
## Method:  nlm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
## iteration = 0
## Step:
##  [1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [39] 0 0
## Parameter:
##  [1] 0.66053683 0.61757532 0.34124602 0.24052150 0.08777792 0.44890366
##  [7] 0.58335670 0.04729403 0.23407865 0.76820977 0.33491161 0.79778887
## [13] 0.28436449 0.28091129 0.48984313 0.98112915 0.72160778 0.99114883
## [19] 0.02624768 0.69438112 0.63034815 0.40784006 0.71864476 0.19951371
## [25] 0.82882704 0.15341923 0.48397990 0.08291179 0.72825316 0.88413282
## [31] 0.20141489 0.12477368 0.76753281 0.35876707 0.21929292 0.31907798
## [37] 0.71113275 0.38461151 0.13083802 0.53056717
## Function Value
## [1] -356.3788
## Gradient:
##  [1]  43.7542040  38.0268877  18.0802893   8.9730443  -3.2859442  16.0163062
##  [7]  21.4810622 -14.1155588  -5.3289609  24.8215148  -4.2846537  21.6826977
## [13] -12.1461379 -14.6189843  -4.0050179  24.0273054   6.1381156  20.8526558
## [19] -40.0425421  -0.6824849  -6.1492589 -21.2355012  -3.5114401 -36.6119901
## [25]   0.8241181 -41.6919995 -22.4259269 -47.9901292  -9.3017142  -0.5348132
```

```
## [31] -43.0955984 -48.4149207  -9.5492069 -35.1028175 -44.0626798 -38.2966194
## [37] -14.5330778 -34.7591615 -50.4526918 -26.0055444
##
## iteration = 1
## Step:
##  [1] -3.96637317 -3.44718481 -1.63900078 -0.81341766  0.29787494 -1.45189814
##  [7] -1.94728509  1.27959302  0.48307696 -2.25010128  0.38840920 -1.96556359
## [13]  1.10106255  1.32522916  0.36305987 -2.17810520 -0.55642783 -1.89031926
## [19]  3.62990639  0.06186810  0.55743799  1.92502467  0.31831642  3.31892258
## [25] -0.07470734  3.77943176  2.03293825  4.35036508  0.84321200  0.04848148
## [31]  3.90666975  4.38887295  0.86564752  3.18211419  3.99433688  3.47163632
## [37]  1.31744163  3.15096135  4.57359945  2.35743504
## Parameter:
##  [1] -3.3058363 -2.8296095 -1.2977548 -0.5728962  0.3856529 -1.0029945
##  [7] -1.3639284  1.3268870  0.7171556 -1.4818915  0.7233208 -1.1677747
## [13]  1.3854270  1.6061405  0.8529030 -1.1969761  0.1651799 -0.8991704
## [19]  3.6561541  0.7562492  1.1877861  2.3328647  1.0369612  3.5184363
## [25]  0.7541197  3.9328510  2.5169181  4.4332769  1.5714652  0.9326143
## [31]  4.1080846  4.5136466  1.6331803  3.5408813  4.2136298  3.7907143
## [37]  2.0285744  3.5355729  4.7044375  2.8880022
## Function Value
## [1] -387.839
## Gradient:
##  [1]  -9.8107094  -8.7925297  -4.5267877  -2.7714985  -0.2968482  -5.0905967
##  [7]  -6.7085942   1.1166218  -1.2724354  -8.5793537  -2.2539980  -8.6129886
## [13]  -1.2131477  -1.0281138  -3.8474344 -10.6768368  -6.9455383 -10.7301072
## [19]   2.8875721  -6.5613154  -5.6834931  -2.5856182  -7.0310512   0.2339692
## [25]  -8.7239570   0.7354325  -4.0076370   1.5927724  -7.5737526  -9.8410402
## [31]  -0.2842659   0.7244570  -8.4105285  -2.6989016  -0.7846645  -2.2309072
## [37]  -7.7955632  -3.2163066   0.3438161  -5.3065964
##
## iteration = 2
## Step:
##  [1]  0.67283987  0.60683950  0.31726119  0.20100687  0.03523488  0.37070363
##  [7]  0.48732090 -0.04709891  0.12277675  0.63176630  0.20085135  0.64468639
## [13]  0.13985706  0.13248407  0.33216354  0.80836691  0.55649434  0.82303189
## [19] -0.11037384  0.54662435  0.49142928  0.28306934  0.59472975  0.09873684
## [25]  0.72120653  0.07352183  0.40490227  0.02270914  0.65844590  0.81828135
## [31]  0.16266035  0.09612802  0.72867564  0.33722011  0.20730154  0.30869520
## [37]  0.69373262  0.37904252  0.13428759  0.52423455
## Parameter:
##  [1] -2.63299648 -2.22277000 -0.98049357 -0.37188929  0.42088774 -0.63229085
##  [7] -0.87660748  1.27978814  0.83993236 -0.85012522  0.92417216 -0.52308833
## [13]  1.52528411  1.73862452  1.18506654 -0.38860915  0.72167429 -0.07613854
## [19]  3.54578023  1.30287358  1.67921542  2.61593407  1.63169093  3.61717312
## [25]  1.47532624  4.00637282  2.92182042  4.45598600  2.22991106  1.75089565
## [31]  4.27074500  4.60977465  2.36185597  3.87810137  4.42093135  4.09940951
## [37]  2.72230700  3.91461538  4.83872506  3.41223676
## Function Value
## [1] -479.906
## Gradient:
##  [1] -8.6655843 -7.7580371 -3.9697960 -2.3949077 -0.1815080 -4.3919703
##  [7] -5.7908836  1.1622006 -0.9145698 -7.3379570 -1.7059470 -7.2834660
## [13] -0.6943069 -0.4829324 -2.9289456 -8.9179943 -5.5653305 -8.8506880
```

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## [19]  3.2512552 -5.0401026 -4.2027233 -1.4020878 -5.2704665  1.2125167
## [25] -6.6479672  1.7715673 -2.3678768  2.6313633 -5.4286876 -7.3898842
## [31]  1.1026023  2.0334019 -6.0090038 -0.9287889  0.7906095 -0.4671352
## [37] -5.3701528 -1.3076084  1.8496868 -3.1416716
##
## iteration = 3
## Step:
##  [1]  2.83475456  2.55040662  1.32033266  0.81801297  0.10607700  1.50499845
##  [7]  1.98038096 -0.28829771  0.40996807  2.53642254  0.70190700  2.55081810
## [13]  0.40210689  0.34886534  1.16846930  3.15219756  2.06664945  3.16249791
## [19] -0.79765930  1.94259763  1.68295300  0.77792045  2.06378772 -0.05210028
## [25]  2.54443752 -0.20890661  1.16364146 -0.46753991  2.19141853  2.84586724
## [31]  0.06496827 -0.23225756  2.41789888  0.75544189  0.19638761  0.61444204
## [37]  2.22929367  0.89745521 -0.13790520  1.50309598
## Parameter:
##  [1] 0.2017581 0.3276366 0.3398391 0.4461237 0.5269647 0.8727076 1.1037735
##  [8] 0.9914904 1.2499004 1.6862973 1.6260792 2.0277298 1.9273910 2.0874899
## [15] 2.3535358 2.7635884 2.7883237 3.0863594 2.7481209 3.2454712 3.3621684
## [22] 3.3938545 3.6954787 3.5650728 4.0197638 3.7974662 4.0854619 3.9884461
## [29] 4.4213296 4.5967629 4.3357133 4.3775171 4.7797549 4.6335433 4.6173190
## [36] 4.7138515 4.9516007 4.8120706 4.7008199 4.9153327
## Function Value
## [1] -601.541
## Gradient:
##  [1]  1.076431964  0.885149640  0.388547747  0.147388990 -0.160429757
##  [6]  0.249053345  0.352969191 -0.464792824 -0.277647464  0.395483928
## [11] -0.264082312  0.330184864 -0.421213064 -0.461053256 -0.205561407
## [16]  0.449176862  0.076442810  0.453856446 -0.872043572  0.069554003
## [21] -0.001563568 -0.286978342  0.171406113 -0.519557282  0.384614679
## [26] -0.520207194 -0.030730936 -0.561951209  0.354900163  0.596835340
## [31] -0.318538089 -0.397038824  0.516922351 -0.027781721 -0.200811853
## [36] -0.048743306  0.505666707  0.064147247 -0.279388892  0.277422255
##
## iteration = 4
## Step:
##  [1] -0.204298448 -0.177160207 -0.086230115 -0.046089843  0.008106429
##  [6] -0.083053412 -0.111301081  0.048518202  0.004305453 -0.138307085
## [11] -0.009260716 -0.134036071  0.015414861  0.020615638 -0.034178177
## [16] -0.169433657 -0.094614983 -0.170741328  0.100076886 -0.089719089
## [21] -0.073484489 -0.013215304 -0.103748348  0.039429129 -0.141373659
## [26]  0.045490623 -0.051108372  0.059144014 -0.125207252 -0.172031784
## [31]  0.016882956  0.035370055 -0.148337219 -0.035624803  0.001389848
## [36] -0.028292571 -0.140015016 -0.049258763  0.021323734 -0.091622809
## Parameter:
##  [1] -0.002540362  0.150476418  0.253608976  0.400033836  0.535071164
##  [6]  0.789654192  0.992472396  1.040008631  1.254205881  1.547990242
## [11]  1.616818443  1.893693696  1.942805862  2.108105498  2.319357658
## [16]  2.594154759  2.693708755  2.915618048  2.848197819  3.155752117
## [21]  3.288683933  3.380639215  3.591730306  3.604501968  3.878390098
## [26]  3.842956837  4.034353503  4.047590107  4.296122336  4.424731108
## [31]  4.352596225  4.412887141  4.631417635  4.597918455  4.618708804
## [36]  4.685558977  4.811585661  4.762811829  4.722143594  4.823709928
## Function Value
## [1] -602.5201
```

```
## Gradient:
##  [1]  0.529454302  0.418448353  0.169416116  0.041949934 -0.115331370
##  [6]  0.062309975  0.099655187 -0.290227523 -0.211826248  0.093706960
## [11] -0.219172321  0.053993985 -0.297457095 -0.316805461 -0.198602118
## [16]  0.107053987 -0.062610677  0.120456787 -0.487570979 -0.039641978
## [21] -0.062707194 -0.184494816  0.040677670 -0.269024009  0.164105330
## [26] -0.244997559 -0.005512686 -0.242791049  0.193008243  0.315206774
## [31] -0.100352915 -0.127493384  0.305850140  0.059771201 -0.014041761
## [36]  0.061621537  0.323546877  0.121650771 -0.035648735  0.224459511
##
## iteration = 5
## Step:
##  [1] -0.1700385858 -0.1441413289 -0.0674681504 -0.0322623633  0.0140770129
##  [6] -0.0571022162 -0.0778363117  0.0529303748  0.0189032105 -0.0947165218
## [11]  0.0106194028 -0.0893632668  0.0317116732  0.0361862952 -0.0077833268
## [16] -0.1169882689 -0.0571307799 -0.1197474718  0.0973930451 -0.0573797286
## [21] -0.0459091351  0.0009247385 -0.0741076741  0.0394954323 -0.1084934378
## [26]  0.0403412965 -0.0395768317  0.0477684180 -0.1025380633 -0.1419317737
## [31]  0.0088002479  0.0221471313 -0.1274718532 -0.0377383866 -0.0089891736
## [36] -0.0337637163 -0.1245936569 -0.0519414429  0.0045948650 -0.0867337456
## Parameter:
##  [1] -0.172578948  0.006335089  0.186140825  0.367771473  0.549148177
##  [6]  0.732551976  0.914636085  1.092939006  1.273109091  1.453273720
## [11]  1.627437846  1.804330430  1.974517536  2.144291794  2.311574332
## [16]  2.477166490  2.636577975  2.795870576  2.945590864  3.098372388
## [21]  3.242774798  3.381563953  3.517622632  3.643997400  3.769896660
## [26]  3.883298133  3.994776672  4.095358525  4.193584273  4.282799335
## [31]  4.361396473  4.435034273  4.503945782  4.560180069  4.609719630
## [36]  4.651795261  4.686992004  4.710870386  4.726738458  4.736976183
## Function Value
## [1] -602.791
## Gradient:
##  [1]  0.049774957  0.017184353 -0.012842611 -0.036803920 -0.059747433
##  [6] -0.074333985 -0.089205750 -0.110495686 -0.121311261 -0.126121246
## [11] -0.141107923 -0.140645075 -0.150701733 -0.152593097 -0.151423149
## [16] -0.144123800 -0.142670639 -0.129085739 -0.129476755 -0.107238566
## [21] -0.094155267 -0.081684992 -0.060980200 -0.051154938 -0.025446721
## [26] -0.017420279  0.003508152  0.012367743  0.033901435  0.049665422
## [31]  0.055514830  0.067899709  0.087830513  0.093041331  0.100796820
## [36]  0.109145651  0.119949162  0.120759469  0.121089023  0.127775301
##
## iteration = 6
## Step:
##  [1] -0.0074151551 -0.0035760965  0.0006156489  0.0036611263  0.0067367449
##  [6]  0.0074933037  0.0088998157  0.0128408244  0.0136610955  0.0128756165
## [11]  0.0158475988  0.0146789134  0.0173295209  0.0176934021  0.0171430397
## [16]  0.0151496621  0.0158457928  0.0137651155  0.0165856813  0.0124655142
## [21]  0.0113464423  0.0107245046  0.0077404453  0.0082117002  0.0038017074
## [26]  0.0048879456  0.0018079531  0.0020438752 -0.0019726698 -0.0040278480
## [31] -0.0027148042 -0.0037804492 -0.0076452034 -0.0070339713 -0.0074459835
## [36] -0.0085848198 -0.0108001752 -0.0099681964 -0.0092919459 -0.0111063205
## Parameter:
##  [1] -0.179994103  0.002758993  0.186756474  0.371432599  0.555884922
##  [6]  0.740045280  0.923535900  1.105779830  1.286770186  1.466149337
```

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## [11]  1.643285445  1.819009343  1.991847056  2.161985196  2.328717371
## [16]  2.492316152  2.652423768  2.809635692  2.962176546  3.110837902
## [21]  3.254121240  3.392288458  3.525363077  3.652209101  3.773698367
## [26]  3.888186079  3.996584625  4.097402401  4.191611603  4.278771487
## [31]  4.358681669  4.431253823  4.496300579  4.553146097  4.602273647
## [36]  4.643210441  4.676191829  4.700902189  4.717446513  4.725869862
## Function Value
## [1] -602.8263
## Gradient:
##  [1]  0.029487309  0.006980951 -0.011980234 -0.028132829 -0.043165934
##  [6] -0.056401625 -0.068042171 -0.078858375 -0.088008805 -0.095653541
## [11] -0.102738591 -0.106068834 -0.108997922 -0.110179639 -0.110815266
## [16] -0.109343441 -0.106006797 -0.098358719 -0.090695834 -0.080050652
## [21] -0.069974353 -0.059046641 -0.046548008 -0.035074264 -0.021551471
## [26] -0.010074377  0.002487667  0.012412907  0.022937172  0.033254493
## [31]  0.043207858  0.052924073  0.062214142  0.069467024  0.076323007
## [36]  0.081672201  0.086376027  0.089694422  0.092042134  0.093625235
##
## iteration = 7
## Step:
##  [1] -0.0122392229 -0.0028862084  0.0050486274  0.0118319725  0.0181699080
##  [6]  0.0237418534  0.0286943600  0.0333788815  0.0373375930  0.0406682365
## [11]  0.0438664251  0.0454800695  0.0470268872  0.0478352264  0.0484167129
## [16]  0.0481322978  0.0471671211  0.0444200793  0.0417798032  0.0377986674
## [21]  0.0341137718  0.0300986073  0.0254004779  0.0211841187  0.0160336302
## [26]  0.0118134432  0.0070565504  0.0034157945 -0.0005728997 -0.0044548325
## [31] -0.0081388806 -0.0118010089 -0.0153762207 -0.0180633648 -0.0206405976
## [36] -0.0226570147 -0.0244672339 -0.0256865666 -0.0265508602 -0.0271933492
## Parameter:
##  [1] -0.1922333262 -0.0001272158  0.1918051014  0.3832645717  0.5740548301
##  [6]  0.7637871329  0.9522302605  1.1391587115  1.3241077795  1.5068175733
## [11]  1.6871518698  1.8644894126  2.0388739437  2.2098204220  2.3771340842
## [16]  2.5404484502  2.6995908893  2.8540557710  3.0039563489  3.1486365698
## [21]  3.2882350123  3.4223870650  3.5507635554  3.6733932193  3.7897319974
## [26]  3.8999995222  4.0036411751  4.1008181951  4.1910387035  4.2743166542
## [31]  4.3505427882  4.4194528144  4.4809243580  4.5350827325  4.5816330489
## [36]  4.6205534263  4.6517245949  4.6752156226  4.6908956523  4.6986765131
## Function Value
## [1] -602.8685
## Gradient:
##  [1] -2.544764e-03 -1.264037e-03 -4.752086e-04 -1.206919e-04  1.451697e-04
##  [6]  1.210443e-04  4.593048e-05  1.689279e-04  5.579412e-05 -1.580280e-04
## [11]  1.969991e-06 -3.720332e-04 -3.220557e-04 -3.988409e-04 -3.487122e-04
## [16] -4.205155e-04 -3.339146e-04 -7.695668e-04 -6.805962e-04 -1.216375e-03
## [21] -1.304469e-03 -1.318834e-03 -1.558867e-03 -1.339358e-03 -1.621381e-03
## [26] -1.236709e-03 -1.234428e-03 -6.711208e-04 -4.801007e-04 -1.974574e-04
## [31]  2.597429e-04  5.029920e-04  5.105488e-04  9.258731e-04  1.170135e-03
## [36]  1.400487e-03  1.462871e-03  1.699190e-03  1.850040e-03  1.739936e-03
##
## iteration = 8
## Step:
##  [1]  3.007848e-04  2.233463e-04  1.342184e-04  1.135636e-04  8.785821e-05
##  [6]  1.892739e-04  2.456869e-04  1.459486e-04  2.117074e-04  3.545538e-04
## [11]  2.656332e-04  3.997004e-04  2.961170e-04  3.047500e-04  3.448252e-04
```

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## [16]  4.493864e-04  3.828399e-04  4.619979e-04  2.425875e-04  4.064654e-04
## [21]  3.824638e-04  3.179715e-04  3.794036e-04  2.327486e-04  3.611344e-04
## [26]  1.680615e-04  2.122874e-04  6.276478e-05  1.588585e-04  1.470758e-04
## [31] -5.103053e-05 -1.059352e-04  7.824135e-06 -1.255048e-04 -1.880622e-04
## [36] -1.975840e-04 -1.314298e-04 -2.250409e-04 -2.945145e-04 -2.059949e-04
## Parameter:
##  [1] -1.919325e-01  9.613046e-05  1.919393e-01  3.833781e-01  5.741427e-01
##  [6]  7.639764e-01  9.524759e-01  1.139305e+00  1.324319e+00  1.507172e+00
## [11]  1.687418e+00  1.864889e+00  2.039170e+00  2.210125e+00  2.377479e+00
## [16]  2.540898e+00  2.699974e+00  2.854518e+00  3.004199e+00  3.149043e+00
## [21]  3.288617e+00  3.422705e+00  3.551143e+00  3.673626e+00  3.790093e+00
## [26]  3.900168e+00  4.003853e+00  4.100881e+00  4.191198e+00  4.274464e+00
## [31]  4.350492e+00  4.419347e+00  4.480932e+00  4.534957e+00  4.581445e+00
## [36]  4.620356e+00  4.651593e+00  4.674991e+00  4.690601e+00  4.698471e+00
## Function Value
## [1] -602.8685
## Gradient:
##  [1] -0.0016639522 -0.0006312018 -0.0001223147  0.0001464745  0.0003128711
##  [6]  0.0005490520  0.0006079259  0.0004246522  0.0004737690  0.0006411077
## [11]  0.0005287622  0.0005140298  0.0002533515  0.0001845204  0.0003329093
## [16]  0.0005432817  0.0004304821  0.0002091330 -0.0003294330 -0.0004081051
## [21] -0.0005682974 -0.0007674239 -0.0008347511 -0.0010289908 -0.0009461633
## [26] -0.0011036681 -0.0009719638 -0.0008259488 -0.0003576044 -0.0001018361
## [31] -0.0001975225 -0.0001025903  0.0002329379  0.0002777863  0.0003510465
## [36]  0.0005594313  0.0008128466  0.0007879496  0.0007447594  0.0008858559
##
## iteration = 9
## Step:
##  [1]  2.454654e-04  1.144291e-04  4.056667e-05  7.047775e-06 -1.585541e-05
##  [6] -2.234384e-05 -1.740841e-05 -1.501629e-05 -7.327247e-06  1.647358e-06
## [11] -2.188326e-06  2.555535e-05  3.552456e-05  4.514418e-05  3.605010e-05
## [16]  3.218078e-05  3.204848e-05  7.205523e-05  9.069840e-05  1.305811e-04
## [21]  1.436104e-04  1.532234e-04  1.719734e-04  1.651764e-04  1.796539e-04
## [26]  1.597358e-04  1.525039e-04  1.065065e-04  7.105548e-05  3.905098e-05
## [31]  1.121255e-05 -1.074796e-05 -2.760728e-05 -5.881416e-05 -7.960913e-05
## [36] -1.054874e-04 -1.218865e-04 -1.372714e-04 -1.458691e-04 -1.449884e-04
## Parameter:
##  [1] -0.1916870760  0.0002105595  0.1919798865  0.3833851831  0.5741268329
##  [6]  0.7639540630  0.9524585390  1.1392896438  1.3243121596  1.5071737744
## [11]  1.6874153146  1.8649146683  2.0392055853  2.2101703162  2.3775149596
## [16]  2.5409300174  2.7000057776  2.8545898241  3.0042896348  3.1491736162
## [21]  3.2887610865  3.4228582599  3.5513149325  3.6737911443  3.7902727857
## [26]  3.9003273195  4.0040059664  4.1009874663  4.1912686175  4.2745027810
## [31]  4.3505029702  4.4193361312  4.4809045748  4.5348984136  4.5813653776
## [36]  4.6202503549  4.6514712786  4.6748533104  4.6904552687  4.6983255298
## Function Value
## [1] -602.8685
## Gradient:
##  [1] -9.637170e-04 -3.065794e-04 -1.114505e-05  1.584252e-04  2.556451e-04
##  [6]  4.689739e-04  5.371849e-04  3.559799e-04  4.220650e-04  6.099702e-04
## [11]  4.819908e-04  5.408795e-04  3.037016e-04  2.575477e-04  3.759250e-04
## [16]  5.712235e-04  4.540688e-04  3.419789e-04 -1.473743e-04 -1.163741e-04
## [21] -2.421860e-04 -4.159132e-04 -4.312938e-04 -6.449911e-04 -5.207103e-04
## [26] -7.331407e-04 -6.197537e-04 -6.009184e-04 -2.294445e-04 -6.046775e-05
```

```
## [31] -2.309815e-04 -1.942110e-04  9.752178e-05  5.789155e-05  7.582778e-05
## [36]  2.141433e-04  4.237876e-04  3.574161e-04  2.912913e-04  4.355580e-04
##
## iteration = 10
## Parameter:
##  [1] -0.1913661801  0.0003265353  0.1919975143  0.3833494788  0.5740587895
##  [6]  0.7638329635  0.9523226482  1.1391961346  1.3242054120  1.5070258762
## [11]  1.6872949663  1.8647905402  2.0391398413  2.2101182904  2.3774310462
## [16]  2.5407983233  2.6999006363  2.8545256409  3.0043461398  3.1492377414
## [21]  3.2888590474  3.4229999729  3.5514670859  3.6739901591  3.7904481731
## [26]  3.9005443482  4.0041936524  4.1011529014  4.1913337132  4.2745161434
## [31]  4.3505454199  4.4193616311  4.4808556594  4.5348468033  4.5813016827
## [36]  4.6201445199  4.6513103248  4.6747019727  4.6903160892  4.6981530837
## Function Value
## [1] -602.8685
## Gradient:
##  [1] -5.800573e-05  2.276268e-05  4.362444e-05  6.595267e-05  7.510399e-05
##  [6]  1.414832e-04  1.703390e-04  1.107517e-04  1.407918e-04  2.132497e-04
## [11]  1.633248e-04  2.111416e-04  1.380284e-04  1.293710e-04  1.560851e-04
## [16]  2.143999e-04  1.697627e-04  1.704155e-04  1.887856e-05  6.834806e-05
## [21]  3.541156e-05 -1.694865e-05 -4.458818e-06 -8.661784e-05 -2.911501e-05
## [26] -1.230016e-04 -9.070218e-05 -1.320255e-04 -4.066622e-05 -1.410225e-05
## [31] -9.850790e-05 -1.055642e-04 -1.997708e-05 -6.324936e-05 -7.574003e-05
## [36] -5.311003e-05  3.663710e-06 -3.334750e-05 -6.363862e-05 -1.259620e-05
##
## Relative gradient close to zero.
## Current iterate is probably solution.
##
## Post processing for method  nlm
## Successful convergence!
## Method:  nlminb
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
##   0:    -356.37878: 0.660537 0.617575 0.341246 0.240521 0.0877779 0.448904 0.583357 0.0472940 0.2340
##   1:    -586.61183: -0.368849 -0.277066 -0.0841208 0.0294167 0.165085 0.0720953 0.0779815 0.379384 0
##   2:    -593.35487: 0.241629 0.249160 0.168613 0.159603 0.131106 0.305669 0.392526 0.216741 0.325610
##   3:    -602.07222: 0.0295863 0.0720714 0.0886781 0.125858 0.157187 0.247332 0.311638 0.299196 0.369
##   4:    -602.86256: -0.0533531 -0.000351213 0.0544935 0.109259 0.164231 0.217269 0.271020 0.326856 0
##   5:    -602.86832: -0.0558205 0.000203684 0.0557474 0.111197 0.166364 0.221692 0.276377 0.329948 0.3
##   6:    -602.86846: -0.0554874 0.000119745 0.0556102 0.111008 0.166196 0.221214 0.275797 0.329761 0.3
##   7:    -602.86847: -0.0554331 0.000106510 0.0555857 0.110972 0.166161 0.221129 0.275693 0.329716 0.3
##   8:    -602.86847: -0.0553921 9.66418e-05 0.0555666 0.110944 0.166133 0.221062 0.275612 0.329679 0.3
##   9:    -602.86847: -0.0553835 9.46513e-05 0.0555624 0.110938 0.166126 0.221047 0.275594 0.329669 0.3
##  10:    -602.86847: -0.0554178 7.48704e-05 0.0555499 0.110926 0.166114 0.221031 0.275574 0.329648 0.3
##  11:    -602.86847: -0.0553449 0.000102505 0.0555601 0.110929 0.166115 0.221022 0.275565 0.329655 0.3
##  12:    -602.86847: -0.0553449 0.000102505 0.0555601 0.110929 0.166115 0.221022 0.275565 0.329655 0.3
## Post processing for method  nlminb
## Successful convergence!
## Method:  nvm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1
## nvm -- J C Nash 2009-2015 - an R implementation of Alg 21
```

```
## Problem of size n= 40
## Initial fn= -356.3788
## ig= 1   gnorm= 170.0207     1   1   -356.3788
## **ig= 2   gnorm= 56.19917     4   2   -511.8358
## ig= 3   gnorm= 7.926748     5   3   -526.4665
## ig= 4   gnorm= 7.764388     6   4   -531.4956
## ig= 5   gnorm= 7.503298     7   5   -542.6033
## ig= 6   gnorm= 7.418499     8   6   -547.0357
## ****ig= 7   gnorm= 6.719158     13   7   -567.9764
## **ig= 8   gnorm= 5.835863     16   8   -581.3832
## *ig= 9   gnorm= 4.21468     18   9   -594.1439
## ig= 10   gnorm= 3.788676     19   10   -596.9542
## ig= 11   gnorm= 1.545449     20   11   -601.8268
## ig= 12   gnorm= 0.6403444     21   12   -602.6901
## ig= 13   gnorm= 0.02394048     22   13   -602.8682
## ig= 14   gnorm= 0.00441148     23   14   -602.8685
## ig= 15   gnorm= 5.526373e-05     24   15   -602.8685
## ig= 16   gnorm= 3.61282e-06     25   16   -602.8685
## ig= 17   gnorm= 5.410141e-07     26   17   -602.8685
## *ig= 18   gnorm= 4.327274e-07     28   18   -602.8685
## ************No acceptable point
## Converged
## Seem to be done nvm
## Post processing for method  nvm
## Successful convergence!
## Above for n= 40
## opm: wrapper to call optimr to run multiple optimizers
## Method:  L-BFGS-B
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## final  value -1389.103176
## converged
## Post processing for method  L-BFGS-B
## Successful convergence!
## Method:  BFGS
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## initial  value -838.493687
## iter  10 value -1389.037740
## final  value -1389.103176
## converged
## Post processing for method  BFGS
## Successful convergence!
## Method:  ncg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## ncg -- J C Nash 2023 - bounds constraint version of new CG
## an R implementation of Alg 22 with Yuan/Dai modification
## stepredn = 0.2
## Initial function value= -838.4937
## Initial fn= -838.4937
```

```
## 1   0   1   -838.4937   last decrease= NA
## **5   1   2   -1277.872   last decrease= 439.3786
## 7   2   3   -1388.724   last decrease= 110.8516
## 9   3   4   -1389.103   last decrease= 0.3789732
## 11   4   5   -1389.103   last decrease= 0.0003261981
## 13   5   6   -1389.103   last decrease= 1.943577e-07
## 15   6   7   -1389.103   last decrease= 2.416982e-10
## Very small gradient -- gradsqr = 7.51866821083332e-12
## ncg seems to have converged
## Post processing for method  ncg
## Successful convergence!
## Method:  spg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## iter:  0  f-value:  -838.4937  pgrad:  79.80798
## Post processing for method  spg
## Successful convergence!
## Method:  ucminf
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## ucminf message: Stopped by zero step from line search
## Post processing for method  ucminf
## Successful convergence!
## Method:  nlm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## iteration = 0
## Step:
##  [1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [39] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## Parameter:
##  [1] 0.192963022 0.914003425 0.556179334 0.992219951 0.368120915 0.969791849
##  [7] 0.552209644 0.329179156 0.573042927 0.301415088 0.501146834 0.296306973
## [13] 0.837116586 0.739563565 0.908670064 0.192367316 0.959015846 0.541757693
## [19] 0.330947852 0.042234776 0.463909006 0.824626383 0.404271266 0.045855058
## [25] 0.953939790 0.603875664 0.785667696 0.592434716 0.332064490 0.070443688
## [31] 0.282933972 0.760642139 0.348571392 0.013634119 0.941242807 0.748566780
## [37] 0.454219206 0.698581499 0.325399732 0.914505825 0.953172947 0.349156214
## [43] 0.823520204 0.109782191 0.253210641 0.612182602 0.087779896 0.278918379
## [49] 0.755836887 0.880393141 0.246726318 0.577551396 0.601066523 0.841216632
## [55] 0.825234668 0.672790011 0.746321424 0.008867881 0.944684900 0.274806756
## Function Value
## [1] -838.4937
## Gradient:
##  [1]  16.977374  66.952959  37.999074  67.231717  18.913030  60.394529
##  [7]  27.317015   8.535633  23.975574   1.714206  13.998743  -3.301216
## [13]  34.031611  24.687919  34.937501 -19.575366  34.530963   2.019110
## [19] -15.326291 -38.347878  -9.341311  15.241760 -17.303984 -45.278305
## [25]  19.502233  -7.772901   3.954751 -11.713369 -32.246347 -52.841837
## [31] -38.710141  -5.130726 -36.646726 -62.483423   4.142281 -11.192466
## [37] -33.907485 -17.130895 -45.518325  -3.405143  -1.522707 -46.623089
```

```
## [43] -12.718840 -65.751877 -56.001630 -30.444117 -69.526008 -56.198320
## [49] -21.917417 -13.375215 -60.283594 -36.536837 -35.245033 -18.035574
## [55] -19.510027 -30.905931 -25.693817 -79.807982 -11.381186 -60.463056
##
## iteration = 1
## Step:
##  [1] -0.66515214 -2.62313265 -1.48875589 -2.63405402 -0.74098872 -2.36618160
##  [7] -1.07024624 -0.33441537 -0.93933281 -0.06716043 -0.54845312  0.12933748
## [13] -1.33331566 -0.96724159 -1.36880730  0.76693820 -1.35287966 -0.07910620
## [19]  0.60046478  1.50242158  0.36598080 -0.59715298  0.67794831  1.77394699
## [25] -0.76407296  0.30453247 -0.15494218  0.45891504  1.26337126  2.07027666
## [31]  1.51661459  0.20101537  1.43577261  2.44802184 -0.16228936  0.43850674
## [37]  1.32845257  0.67116689  1.78335067  0.13340920  0.05965774  1.82663391
## [43]  0.49830816  2.57607575  2.19407337  1.19276216  2.72394148  2.20177941
## [49]  0.85869679  0.52402408  2.36183531  1.43146727  1.38085600  0.70661108
## [55]  0.76437829  1.21085547  1.00665142  3.12677626  0.44590054  2.36886642
## Parameter:
##  [1] -0.472189121 -1.709129229 -0.932576552 -1.641834069 -0.372867808
##  [6] -1.396389747 -0.518036599 -0.005236213 -0.366289883  0.234254654
## [11] -0.047306282  0.425644452 -0.496199073 -0.227678023 -0.460137232
## [16]  0.959305515 -0.393863816  0.462651497  0.931412627  1.544656352
## [21]  0.829889811  0.227473401  1.082219575  1.819802046  0.189866834
## [26]  0.908408138  0.630725518  1.051349758  1.595435753  2.140720344
## [31]  1.799548563  0.961657512  1.784343999  2.461655954  0.778953445
## [36]  1.187073515  1.782671772  1.369748390  2.108750406  1.047915025
## [41]  1.012830687  2.175790128  1.321828360  2.685857941  2.447284014
## [46]  1.804944765  2.811721374  2.480697787  1.614533676  1.404417218
## [51]  2.608561629  2.009018670  1.981922527  1.547827709  1.589612956
## [56]  1.883645486  1.752972842  3.135644144  1.390585442  2.643673177
## Function Value
## [1] -1106.8
## Gradient:
##  [1]  -6.4392308 -26.4334490 -15.3186268 -27.1907009  -8.4981279 -25.2576074
##  [7] -12.6295209  -5.6609222 -12.2040736  -3.8839599  -9.1998293  -2.8499759
## [13] -18.0607062 -14.8727659 -19.4342522   1.5398046 -20.3433021  -8.0654093
## [19]  -1.7765613   6.7593362  -5.2170639 -15.4446728  -3.1382079   7.3717986
## [25] -18.6996784  -8.4559635 -13.6033151  -7.9445025  -0.3619896   7.2613850
## [31]   1.2081839 -12.5001876  -0.5189741   9.2394068 -17.4586966 -11.8172516
## [37]  -3.2603741 -10.2724767   0.5458531 -16.4356895 -17.5422327  -0.1102487
## [43] -13.8337849   6.7561297   2.6039121  -7.7571414   7.4059700   1.9231434
## [49] -11.7993966 -15.3554225   2.9757439  -6.5454959  -7.1873508 -14.0945490
## [55] -13.6219479  -9.2268100 -11.3721717   9.9065761 -17.1322731   2.2061786
##
## iteration = 2
## Step:
##  [1]  0.163500873  0.676575781  0.393664407  0.699185024  0.222654817
##  [6]  0.653527421  0.332455648  0.156317172  0.326022669  0.115403290
## [11]  0.253788165  0.093655629  0.485331947  0.406233333  0.525553912
## [16] -0.008554055  0.554154444  0.242678618  0.084501186 -0.131220662
## [21]  0.177935463  0.442318705  0.130035812 -0.136307658  0.533475012
## [26]  0.273898766  0.408226362  0.266014582  0.074549766 -0.118041607
## [31]  0.039255792  0.392376414  0.088013110 -0.159552511  0.525761063
## [36]  0.383424140  0.166457184  0.347872673  0.072913408  0.509333544
## [41]  0.539467720  0.095152892  0.448097816 -0.077175624  0.030611834
```

```
## [46]  0.297157758 -0.089619118  0.051818052  0.403991034  0.495924105
## [51]  0.027669998  0.272097671  0.289183619  0.466538823  0.454988489
## [56]  0.343011744  0.398359947 -0.145850847  0.546337696  0.051557851
## Parameter:
##  [1] -0.30868825 -1.03255345 -0.53891215 -0.94264905 -0.15021299 -0.74286233
##  [7] -0.18558095  0.15108096 -0.04026721  0.34965794  0.20648188  0.51930008
## [13] -0.01086713  0.17855531  0.06541668  0.95075146  0.16029063  0.70533012
## [19]  1.01591381  1.41343569  1.00782527  0.66979211  1.21225539  1.68349439
## [25]  0.72334185  1.18230690  1.03895188  1.31736434  1.66998552  2.02267874
## [31]  1.83880435  1.35403393  1.87235711  2.30210344  1.30471451  1.57049766
## [37]  1.94912896  1.71762106  2.18166381  1.55724857  1.55229841  2.27094302
## [43]  1.76992618  2.60868232  2.47789585  2.10210252  2.72210226  2.53251584
## [49]  2.01852471  1.90034132  2.63623163  2.28111634  2.27110615  2.01436653
## [55]  2.04460144  2.22665723  2.15133279  2.98979330  1.93692314  2.69523103
## Function Value
## [1] -1295.639
## Gradient:
##  [1]  -3.9758875 -16.4873092  -9.5754153 -16.9957580  -5.3322405 -15.7818339
##  [7]  -7.8833047  -3.5089956  -7.5618814  -2.3359198  -5.6159766  -1.6140739
## [13] -11.0630323  -9.0247235 -11.8142424   1.3364982 -12.2531324  -4.5210949
## [19]  -0.5231640   4.8751691  -2.5301170  -8.8441751  -1.0938788   5.5343653
## [25] -10.6670964  -4.2050905  -7.3438254  -3.7367182   1.0713004   5.9010771
## [31]   2.1901442  -6.3015485   1.2366763   7.3839813  -9.2228392  -5.6485044
## [37]  -0.2533014  -4.5741994   2.2297454  -8.3153377  -8.9521734   1.9814852
## [43]  -6.5272912   6.3736718   3.8292205  -2.5958990   6.9015857   3.5085721
## [49]  -5.0319289  -7.2289162   4.2318279  -1.6915158  -2.0775135  -6.3742299
## [55]  -6.0648782  -3.3066540  -4.6297247   8.6612533  -8.2037409   3.8684918
##
## iteration = 3
## Step:
##  [1]  0.24599568  1.02762579  0.59859817  1.06286526  0.33763228  0.99067322
##  [7]  0.50007356  0.22912128  0.48341928  0.15936532  0.36547543  0.11769901
## [13]  0.70829226  0.58287709  0.75835510 -0.05972781  0.78887827  0.30850402
## [19]  0.06085545 -0.27408968  0.18899477  0.58404721  0.10245753 -0.30921350
## [25]  0.70205412  0.30070085  0.49772305  0.27425242 -0.02409812 -0.32382701
## [31] -0.09124004  0.43930250 -0.02931760 -0.41127061  0.62499387  0.40334601
## [37]  0.06816393  0.33853998 -0.08452360  0.57372345  0.61435144 -0.06624749
## [43]  0.46493992 -0.33836549 -0.17900872  0.22218917 -0.36909599 -0.15696530
## [49]  0.37596301  0.51345497 -0.20037943  0.16927527  0.19374512  0.46191501
## [55]  0.44293710  0.27126686  0.35392550 -0.47434645  0.57693105 -0.17547380
## Parameter:
##  [1] -0.06269257 -0.00492766  0.05968603  0.12021622  0.18741929  0.24781089
##  [7]  0.31449261  0.38020224  0.44315207  0.50902326  0.57195732  0.63699909
## [13]  0.69742513  0.76143240  0.82377178  0.89102365  0.94916889  1.01383414
## [19]  1.07676926  1.13934601  1.19682005  1.25383932  1.31471292  1.37428089
## [25]  1.42539597  1.48300776  1.53667493  1.59161676  1.64588740  1.69885173
## [31]  1.74756431  1.79333643  1.84303951  1.89083284  1.92970838  1.97384366
## [37]  2.01729289  2.05616104  2.09714022  2.13097202  2.16664985  2.20469553
## [43]  2.23486609  2.27031683  2.29888713  2.32429169  2.35300627  2.37555054
## [49]  2.39448772  2.41379629  2.43585220  2.45039161  2.46485127  2.47628155
## [55]  2.48753855  2.49792409  2.50525829  2.51544685  2.51385419  2.51975723
## Function Value
## [1] -1388.976
## Gradient:
```

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##  [1]  0.062416083 -0.075424082 -0.109441792 -0.204741746 -0.197518644
##  [6] -0.291556432 -0.287481285 -0.294710922 -0.339641017 -0.335416434
## [11] -0.370173465 -0.366707815 -0.426283741 -0.423916132 -0.438519102
## [16] -0.369605232 -0.429069227 -0.379263858 -0.344612830 -0.303622371
## [21] -0.327566348 -0.345320037 -0.290903429 -0.241912436 -0.306099387
## [26] -0.256162581 -0.249836956 -0.207374381 -0.157686694 -0.109813752
## [31] -0.107849878 -0.131374189 -0.075666256 -0.028769287 -0.096452716
## [36] -0.063258661 -0.018885446 -0.021934044  0.029520430 -0.004497244
## [41]  0.012787069  0.089678131  0.071235504  0.157317100  0.163868926
## [46]  0.147544810  0.206841346  0.198282885  0.160992957  0.155520524
## [51]  0.218113170  0.193339167  0.194151466  0.175970934  0.182241430
## [56]  0.202496612  0.203787469  0.275766761  0.196566120  0.258543796
##
## iteration = 4
## Step:
##  [1] -0.0002828363  0.0106695954  0.0085257433  0.0155061414  0.0096010119
##  [6]  0.0180097026  0.0140457346  0.0121929008  0.0157636705  0.0130936873
## [11]  0.0159306940  0.0138824261  0.0205917686  0.0195366427  0.0214257227
## [16]  0.0126262910  0.0213443908  0.0158515076  0.0127071334  0.0086587039
## [21]  0.0131197120  0.0168321895  0.0111673106  0.0062386328  0.0163912698
## [26]  0.0115096400  0.0128298899  0.0095986599  0.0055290445  0.0015123884
## [31]  0.0032630547  0.0082344930  0.0026243235 -0.0019999700  0.0084694949
## [36]  0.0055774614  0.0014063821  0.0036271870 -0.0014788444  0.0048567731
## [41]  0.0045696340 -0.0034406319  0.0013554646 -0.0079356595 -0.0069197705
## [46] -0.0032128100 -0.0099083803 -0.0079498943 -0.0024784338 -0.0012111485
## [51] -0.0089794545 -0.0052225218 -0.0050585842 -0.0023256615 -0.0026930403
## [56] -0.0047436777 -0.0041434168 -0.0131360157 -0.0021486742 -0.0101980489
## Parameter:
##  [1] -0.062975408  0.005741936  0.068211770  0.135722360  0.197020298
##  [6]  0.265820595  0.328538346  0.392395137  0.458915741  0.522116949
## [11]  0.587888009  0.650881515  0.718016903  0.780969044  0.845197500
## [16]  0.903649942  0.970513285  1.029685648  1.089476397  1.148004719
## [21]  1.209939760  1.270671510  1.325880226  1.380519523  1.441787240
## [26]  1.494517396  1.549504822  1.601215417  1.651416448  1.700364115
## [31]  1.750827368  1.801570924  1.845663829  1.888832867  1.938177872
## [36]  1.979421126  2.018699271  2.059788231  2.095661373  2.135828791
## [41]  2.171219480  2.201254894  2.236221559  2.262381172  2.291967358
## [46]  2.321078878  2.343097890  2.367600641  2.392009288  2.412585142
## [51]  2.426872741  2.445169089  2.459792681  2.473955884  2.484845508
## [56]  2.493180413  2.501114875  2.502310834  2.511705517  2.509559182
## Function Value
## [1] -1389.076
## Gradient:
##  [1]  0.061490183  0.086263777  0.016492758  0.023799625 -0.061491238
##  [6] -0.031021408 -0.089758763 -0.127660628 -0.120939901 -0.159475319
## [11] -0.153394810 -0.182957185 -0.142692302 -0.157936316 -0.145326724
## [16] -0.210957154 -0.139354138 -0.173621749 -0.187283576 -0.208168390
## [21] -0.164931374 -0.126706701 -0.158246077 -0.183933180 -0.094173149
## [26] -0.117958620 -0.091245623 -0.097237841 -0.108608651 -0.120930301
## [31] -0.091737651 -0.039185901 -0.067677915 -0.090010465  0.001801964
## [36] -0.007912283 -0.025774529  0.005823631 -0.019074955  0.043934637
## [41]  0.057926840  0.014527015  0.069834023  0.016187671  0.039129846
## [46]  0.079906274  0.038619454  0.060524845  0.106853142  0.121255093
## [51]  0.066771775  0.099488785  0.103283313  0.126960485  0.128074259
```

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## [56]  0.117629358  0.128339404  0.064312480  0.151759146  0.091859002
##
## iteration = 5
## Step:
##  [1] -0.0030277895 -0.0020153497  0.0009262708  0.0019851314  0.0049213512
##  [6]  0.0051515276  0.0071982552  0.0086645144  0.0090683455  0.0104004200
## [11]  0.0106868998  0.0117101149  0.0111249815  0.0116564456  0.0114361186
## [16]  0.0128410193  0.0111469202  0.0117027185  0.0117361360  0.0119352405
## [21]  0.0107506591  0.0096566467  0.0100424443  0.0102931532  0.0080057279
## [26]  0.0081729389  0.0071507575  0.0067910536  0.0065218890  0.0063084627
## [31]  0.0052508352  0.0037131839  0.0039574809  0.0041022536  0.0017744415
## [36]  0.0016581017  0.0016778028  0.0005964411  0.0007673658 -0.0010022840
## [41] -0.0017384209 -0.0012582007 -0.0029649119 -0.0022477558 -0.0031534471
## [46] -0.0043787595 -0.0037367560 -0.0044036826 -0.0055426932 -0.0059863533
## [51] -0.0049230116 -0.0057491077 -0.0059021764 -0.0064975083 -0.0066270194
## [56] -0.0065366664 -0.0069333262 -0.0056529436 -0.0076628807 -0.0063916191
## Parameter:
##  [1] -0.066003197  0.003726586  0.069138041  0.137707492  0.201941650
##  [6]  0.270972123  0.335736601  0.401059651  0.467984087  0.532517369
## [11]  0.598574909  0.662591630  0.729141884  0.792625490  0.856633618
## [16]  0.916490961  0.981660205  1.041388366  1.101212533  1.159939959
## [21]  1.220690419  1.280328157  1.335922671  1.390812676  1.449792967
## [26]  1.502690335  1.556655579  1.608006470  1.657938337  1.706672578
## [31]  1.756078203  1.805284108  1.849621310  1.892935121  1.939952313
## [36]  1.981079228  2.020377074  2.060384672  2.096428739  2.134826507
## [41]  2.169481059  2.199996693  2.233256647  2.260133416  2.288813911
## [46]  2.316700119  2.339361134  2.363196959  2.386466595  2.406598789
## [51]  2.421949730  2.439419981  2.453890505  2.467458375  2.478218488
## [56]  2.486643747  2.494181549  2.496657890  2.504042637  2.503167563
## Function Value
## [1] -1389.102
## Gradient:
##  [1]  0.0174373545  0.0556859035  0.0286122886  0.0500929895  0.0074567519
##  [6]  0.0396218262  0.0101730539 -0.0071575166  0.0041271005 -0.0156744388
## [11] -0.0065937447 -0.0218714862  0.0084289218  0.0002685497  0.0086966114
## [16] -0.0363545670  0.0089967384 -0.0172970079 -0.0307739178 -0.0488347447
## [21] -0.0236173073 -0.0019219463 -0.0274543720 -0.0490726653  0.0063968152
## [26] -0.0143854677 -0.0026067690 -0.0134129677 -0.0281510571 -0.0429250531
## [31] -0.0289135664  0.0012373551 -0.0226057215 -0.0417558154  0.0158048223
## [36]  0.0053782536 -0.0111156922  0.0051797547 -0.0160343283  0.0212539050
## [41]  0.0251796311 -0.0098683679  0.0206285496 -0.0211218958 -0.0109081703
## [46]  0.0122587802 -0.0183835471 -0.0057165403  0.0241679144  0.0326048617
## [51] -0.0050669609  0.0157678627  0.0178174953  0.0329814324  0.0325718131
## [56]  0.0238639812  0.0288578938 -0.0155425263  0.0415950606  0.0010351053
##
## iteration = 6
## Step:
##  [1] -1.035505e-03 -2.918996e-03 -1.406792e-03 -2.447088e-03 -1.547020e-04
##  [6] -1.781742e-03 -2.048748e-04  7.291170e-04  1.631492e-04  1.215479e-03
## [11]  7.558172e-04  1.565410e-03 -9.176038e-06  4.164167e-04 -3.500841e-05
## [16]  2.298705e-03 -9.248656e-05  1.251407e-03  1.921832e-03  2.834450e-03
## [21]  1.492404e-03  3.344173e-04  1.638748e-03  2.740040e-03 -1.811799e-04
## [26]  8.760087e-04  2.273680e-04  7.538155e-04  1.484212e-03  2.221039e-03
## [31]  1.462787e-03 -1.338496e-04  1.089092e-03  2.071209e-03 -9.471565e-04
```

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## [36] -4.177388e-04  4.247027e-04 -4.456080e-04  6.434393e-04 -1.322264e-03
## [41] -1.550557e-03  2.512436e-04 -1.370600e-03  7.815237e-04  2.261936e-04
## [46] -9.990499e-04  5.901074e-04 -7.478897e-05 -1.633170e-03 -2.073458e-03
## [51] -1.083454e-04 -1.195891e-03 -1.300226e-03 -2.090741e-03 -2.072237e-03
## [56] -1.624938e-03 -1.896194e-03  4.132287e-04 -2.575495e-03 -4.612007e-04
## Parameter:
##  [1] -0.0670387022  0.0008075905  0.0677312489  0.1352604042  0.2017869476
##  [6]  0.2691903811  0.3355317263  0.4017887680  0.4681472357  0.5337328480
## [11]  0.5993307264  0.6641570404  0.7291327080  0.7930419062  0.8565986098
## [16]  0.9187896668  0.9815677183  1.0426397734  1.1031343649  1.1627744090
## [21]  1.2221828228  1.2806625741  1.3375614183  1.3935527160  1.4496117875
## [26]  1.5035663437  1.5568829472  1.6087602858  1.6594225487  1.7088936165
## [31]  1.7575409904  1.8051502583  1.8507104017  1.8950063296  1.9390051569
## [36]  1.9806614888  2.0208017763  2.0599390640  2.0970721784  2.1335042431
## [41]  2.1679305013  2.2002479364  2.2318860476  2.2609149398  2.2890401049
## [46]  2.3157010690  2.3399512414  2.3631221695  2.3848334255  2.4045253312
## [51]  2.4218413842  2.4382240893  2.4525902792  2.4653676345  2.4761462510
## [56]  2.4850188088  2.4922853548  2.4970711188  2.5014671421  2.5027063624
## Function Value
## [1] -1389.103
## Gradient:
##  [1]  0.0017448022  0.0114940131  0.0072848010  0.0129810627  0.0049755431
##  [6]  0.0124470069  0.0067779830  0.0034946317  0.0061317137  0.0021781969
## [11]  0.0042353708  0.0011531881  0.0075803968  0.0058204709  0.0073793525
## [16] -0.0023868520  0.0067605199  0.0007943186 -0.0025347153 -0.0067609735
## [21] -0.0018035789  0.0024338640 -0.0032843752 -0.0081436995  0.0032317889
## [26] -0.0014311525  0.0006577794 -0.0020501440 -0.0055948757 -0.0090619873
## [31] -0.0063481752 -0.0001683799 -0.0053082410 -0.0093882377  0.0027156636
## [36]  0.0005195860 -0.0030106754  0.0003296513 -0.0041885000  0.0035673797
## [41]  0.0042429256 -0.0033456519  0.0027947236 -0.0062062412 -0.0042132633
## [46]  0.0005959113 -0.0058176447 -0.0030317323  0.0034474463  0.0053841920
## [51] -0.0024034693  0.0021107031  0.0027080527  0.0060197797  0.0059800277
## [56]  0.0041094599  0.0050471496 -0.0043702457  0.0075645883 -0.0009893534
##
## iteration = 7
## Step:
##  [1] -1.651760e-04 -7.891666e-04 -4.692408e-04 -8.326869e-04 -2.693809e-04
##  [6] -7.671041e-04 -3.751055e-04 -1.499188e-04 -3.302499e-04 -6.286279e-05
## [11] -2.060509e-04  8.079855e-07 -4.414421e-04 -3.284828e-04 -4.458409e-04
## [16]  2.062984e-04 -4.316328e-04 -4.118229e-05  1.686235e-04  4.421060e-04
## [21]  9.328076e-05 -2.053363e-04  1.737961e-04  4.960267e-04 -2.860010e-04
## [26]  2.444415e-05 -1.272250e-04  4.674981e-05  2.777844e-04  5.063749e-04
## [31]  3.181440e-04 -1.037132e-04  2.464658e-04  5.268248e-04 -2.931035e-04
## [36] -1.392501e-04  1.038550e-04 -1.215113e-04  1.891540e-04 -3.367561e-04
## [41] -3.824486e-04  1.325374e-04 -2.889916e-04  3.219013e-04  1.859991e-04
## [46] -1.391245e-04  3.024672e-04  1.201287e-04 -3.117927e-04 -4.356060e-04
## [51]  1.016523e-04 -1.990642e-04 -2.323610e-04 -4.520460e-04 -4.460600e-04
## [56] -3.182300e-04 -3.840789e-04  2.558039e-04 -5.585733e-04  2.349932e-05
## Parameter:
##  [1] -6.720388e-02  1.842384e-05  6.726201e-02  1.344277e-01  2.015176e-01
##  [6]  2.684233e-01  3.351566e-01  4.016388e-01  4.678170e-01  5.336700e-01
## [11]  5.991247e-01  6.641578e-01  7.286913e-01  7.927134e-01  8.561528e-01
## [16]  9.189960e-01  9.811361e-01  1.042599e+00  1.103303e+00  1.163217e+00
## [21]  1.222276e+00  1.280457e+00  1.337735e+00  1.394049e+00  1.449326e+00
```

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## [26]  1.503591e+00  1.556756e+00  1.608807e+00  1.659700e+00  1.709400e+00
## [31]  1.757859e+00  1.805047e+00  1.850957e+00  1.895533e+00  1.938712e+00
## [36]  1.980522e+00  2.020906e+00  2.059818e+00  2.097261e+00  2.133167e+00
## [41]  2.167548e+00  2.200380e+00  2.231597e+00  2.261237e+00  2.289226e+00
## [46]  2.315562e+00  2.340254e+00  2.363242e+00  2.384522e+00  2.404090e+00
## [51]  2.421943e+00  2.438025e+00  2.452358e+00  2.464916e+00  2.475700e+00
## [56]  2.484701e+00  2.491901e+00  2.497327e+00  2.500909e+00  2.502730e+00
## Function Value
## [1] -1389.103
## Gradient:
##  [1] -8.278508e-04 -4.545642e-04  2.411222e-04  4.919541e-04  1.061509e-03
##  [6]  1.042753e-03  1.344150e-03  1.501847e-03  1.439422e-03  1.560557e-03
## [11]  1.476219e-03  1.549448e-03  1.305565e-03  1.274742e-03  1.072486e-03
## [16]  1.189282e-03  6.922208e-04  6.449918e-04  4.997956e-04  4.231850e-04
## [21]  1.142741e-04 -1.534955e-04 -1.191056e-04 -8.530807e-05 -5.280566e-04
## [26] -4.739970e-04 -6.635005e-04 -7.215006e-04 -7.520019e-04 -7.388495e-04
## [31] -8.492786e-04 -1.026993e-03 -8.380557e-04 -6.435300e-04 -9.160108e-04
## [36] -7.501146e-04 -5.687907e-04 -6.076835e-04 -3.916868e-04 -5.640604e-04
## [41] -5.506910e-04 -3.178111e-04 -5.319319e-04 -2.612676e-04 -2.984745e-04
## [46] -3.819730e-04 -8.283059e-05 -2.626017e-05 -5.341280e-05  3.787808e-05
## [51]  4.076129e-04  3.946417e-04  5.109950e-04  5.175586e-04  5.840171e-04
## [56]  6.593181e-04  6.073994e-04  8.798949e-04  4.921514e-04  7.496380e-04
##
## iteration = 8
## Step:
##  [1]  2.745337e-05 -4.258271e-06 -2.266427e-05 -4.282634e-05 -4.998122e-05
##  [6] -6.365622e-05 -6.535083e-05 -6.612084e-05 -6.951517e-05 -6.767567e-05
## [11] -6.913048e-05 -6.699326e-05 -7.038289e-05 -6.673902e-05 -6.281120e-05
## [16] -5.004627e-05 -4.894347e-05 -3.693064e-05 -2.603731e-05 -1.591289e-05
## [21] -1.375346e-05 -1.170765e-05 -2.667643e-06  4.934888e-06  7.296387e-07
## [26]  7.384370e-06  1.076910e-05  1.804161e-05  2.586707e-05  3.204453e-05
## [31]  3.168358e-05  2.759587e-05  3.059000e-05  3.152890e-05  2.036934e-05
## [36]  1.901953e-05  1.950111e-05  1.562719e-05  1.664373e-05  9.685401e-06
## [41]  8.685469e-06  1.453987e-05  1.190814e-05  1.889695e-05  1.729940e-05
## [46]  1.232539e-05  1.363735e-05  7.216317e-06 -2.815376e-06 -9.032040e-06
## [51] -7.926716e-06 -1.508570e-05 -1.992963e-05 -2.572805e-05 -2.774972e-05
## [56] -2.687961e-05 -2.651474e-05 -1.935023e-05 -2.665326e-05 -2.057956e-05
## Parameter:
##  [1] -6.717642e-02  1.416556e-05  6.723934e-02  1.343849e-01  2.014676e-01
##  [6]  2.683596e-01  3.350913e-01  4.015727e-01  4.677475e-01  5.336023e-01
## [11]  5.990555e-01  6.640909e-01  7.286209e-01  7.926467e-01  8.560900e-01
## [16]  9.189459e-01  9.810871e-01  1.042562e+00  1.103277e+00  1.163201e+00
## [21]  1.222262e+00  1.280446e+00  1.337733e+00  1.394054e+00  1.449327e+00
## [26]  1.503598e+00  1.556766e+00  1.608825e+00  1.659726e+00  1.709432e+00
## [31]  1.757891e+00  1.805074e+00  1.850987e+00  1.895565e+00  1.938732e+00
## [36]  1.980541e+00  2.020925e+00  2.059833e+00  2.097278e+00  2.133177e+00
## [41]  2.167557e+00  2.200395e+00  2.231609e+00  2.261256e+00  2.289243e+00
## [46]  2.315574e+00  2.340267e+00  2.363250e+00  2.384519e+00  2.404081e+00
## [51]  2.421935e+00  2.438010e+00  2.452338e+00  2.464890e+00  2.475672e+00
## [56]  2.484674e+00  2.491875e+00  2.497308e+00  2.500882e+00  2.502709e+00
## Function Value
## [1] -1389.103
## Gradient:
##  [1] -4.206120e-04 -5.187342e-04 -9.306169e-05 -1.390787e-04  3.300832e-04
```

```
##  [6]  1.116748e-04  3.941030e-04  5.461143e-04  4.375509e-04  5.910276e-04
## [11]  4.884165e-04  5.970006e-04  3.040664e-04  3.299086e-04  1.878873e-04
## [16]  4.980226e-04  1.719444e-05  1.508374e-04  1.691611e-04  2.441457e-04
## [21] -3.393026e-05 -2.727351e-04 -1.036185e-04  4.311381e-05 -4.654112e-04
## [26] -3.127041e-04 -4.529865e-04 -4.027885e-04 -3.165234e-04 -2.112720e-04
## [31] -3.282407e-04 -5.685701e-04 -3.347305e-04 -1.260834e-04 -5.672645e-04
## [36] -4.213305e-04 -2.320349e-04 -3.286839e-04 -9.623301e-05 -3.727073e-04
## [41] -3.731254e-04 -5.016646e-05 -3.025268e-04  7.567962e-05  1.622720e-05
## [46] -1.404462e-04  1.808236e-04  1.425959e-04 -3.393975e-05 -3.430478e-05
## [51]  3.545275e-04  2.354486e-04  2.805789e-04  2.012508e-04  2.387151e-04
## [56]  3.284990e-04  2.831201e-04  6.648102e-04  1.670170e-04  5.166817e-04
##
## iteration = 9
## Parameter:
##  [1] -6.715166e-02  4.012339e-05  6.724137e-02  1.343868e-01  2.014447e-01
##  [6]  2.683463e-01  3.350633e-01  4.015369e-01  4.677168e-01  5.335641e-01
## [11]  5.990224e-01  6.640524e-01  7.285970e-01  7.926219e-01  8.560729e-01
## [16]  9.189146e-01  9.810805e-01  1.042550e+00  1.103265e+00  1.163186e+00
## [21]  1.222263e+00  1.280458e+00  1.337738e+00  1.394052e+00  1.449351e+00
## [26]  1.503615e+00  1.556791e+00  1.608848e+00  1.659746e+00  1.709447e+00
## [31]  1.757912e+00  1.805107e+00  1.851009e+00  1.895575e+00  1.938764e+00
## [36]  1.980566e+00  2.020940e+00  2.059852e+00  2.097286e+00  2.133198e+00
## [41]  2.167577e+00  2.200400e+00  2.231627e+00  2.261255e+00  2.289245e+00
## [46]  2.315584e+00  2.340261e+00  2.363244e+00  2.384521e+00  2.404082e+00
## [51]  2.421917e+00  2.437997e+00  2.452322e+00  2.464877e+00  2.475658e+00
## [56]  2.484655e+00  2.491858e+00  2.497272e+00  2.500871e+00  2.502681e+00
## Function Value
## [1] -1389.103
## Gradient:
##  [1] -4.697772e-05 -1.254764e-04 -6.027703e-05 -1.055379e-04 -9.868216e-06
##  [6] -8.272438e-05 -1.995218e-05  1.454289e-05 -1.540343e-05  2.346861e-05
## [11] -2.327631e-06  2.610871e-05 -4.727375e-05 -3.557120e-05 -6.178909e-05
## [16]  3.078214e-05 -7.815371e-05 -2.748541e-05 -5.204897e-06  2.844662e-05
## [21] -3.287688e-05 -8.545752e-05 -3.234275e-05  1.313000e-05 -1.119750e-04
## [26] -6.659407e-05 -9.361487e-05 -6.993394e-05 -3.677538e-05 -2.079501e-06
## [31] -2.968176e-05 -9.190255e-05 -3.298875e-05  1.668367e-05 -1.033953e-04
## [36] -7.168774e-05 -2.650776e-05 -5.414926e-05  2.191829e-06 -7.169031e-05
## [41] -7.169070e-05  1.458575e-05 -4.542692e-05  5.571350e-05  4.162351e-05
## [46] -7.313080e-07  7.717558e-05  5.911647e-05  2.932501e-06 -6.216386e-06
## [51]  8.651046e-05  4.841637e-05  5.172864e-05  2.479464e-05  3.115384e-05
## [56]  5.450344e-05  4.595420e-05  1.467342e-04  2.067045e-05  1.116149e-04
##
## Relative gradient close to zero.
## Current iterate is probably solution.
##
## Post processing for method  nlm
## Successful convergence!
## Method:  nlminb
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##   0:    -838.49369: 0.192963 0.914003 0.556179 0.992220 0.368121 0.969792 0.552210 0.329179 0.573043
##   1:    -1387.7647: -0.0498562 -0.0435928 0.0126967 0.0306368 0.0976169 0.105998 0.161507 0.207098 0
##   2:    -1387.7711: -0.00975845 0.0497605 0.0523312 0.0985898 0.0889093 0.145865 0.148943 0.163077 0
```

```
##   3:    -1389.0986: -0.0321752 -0.00313100 0.0294669 0.0592691 0.0925239 0.121508 0.153778 0.185236 0
##   4:    -1389.1008: -0.0300053 0.00257881 0.0322877 0.0642232 0.0932840 0.125685 0.155264 0.185300 0
##   5:    -1389.1021: -0.0319086 -0.000691658 0.0309364 0.0618900 0.0932552 0.123745 0.154636 0.185263
##   6:    -1389.1024: -0.0301832 0.000672717 0.0311559 0.0622096 0.0928040 0.124065 0.154792 0.185547 0
##   7:    -1389.1026: -0.0333450 -0.00113945 0.0312797 0.0626313 0.0943097 0.124818 0.155722 0.186278 0
##   8:    -1389.1030: -0.0297285 0.000665815 0.0309436 0.0617752 0.0923565 0.123478 0.154241 0.185022 0
##   9:    -1389.1031: -0.0318503 -0.000379432 0.0309115 0.0619397 0.0929001 0.123667 0.154387 0.184993
##  10:    -1389.1031: -0.0304683 0.000283248 0.0311517 0.0621534 0.0931254 0.124112 0.154982 0.185743 0
##  11:    -1389.1032: -0.0313090 -4.15462e-05 0.0309932 0.0620639 0.0929523 0.123942 0.154725 0.185430
##  12:    -1389.1032: -0.0307365 8.29250e-05 0.0311215 0.0620829 0.0931290 0.123950 0.154792 0.185490 0
##  13:    -1389.1032: -0.0310881 -4.50704e-06 0.0310274 0.0620530 0.0930076 0.123933 0.154745 0.185452
##  14:    -1389.1032: -0.0310132 2.27021e-05 0.0310591 0.0620730 0.0930425 0.123944 0.154757 0.185459 0
##  15:    -1389.1032: -0.0310132 2.27021e-05 0.0310591 0.0620730 0.0930425 0.123944 0.154757 0.185459 0
## Post processing for method  nlminb
## Successful convergence!
## Method:  nvm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## nvm -- J C Nash 2009-2015 - an R implementation of Alg 21
## Problem of size n= 60
## Initial fn= -838.4937
## ig= 1   gnorm= 279.6709     1   1   -838.4937
## **ig= 2   gnorm= 92.10029     4   2   -1098.431
## ig= 3   gnorm= 9.982973     5   3   -1241.819
## ig= 4   gnorm= 9.878119     6   4   -1248.008
## ig= 5   gnorm= 9.707768     7   5   -1273.44
## ig= 6   gnorm= 9.435443     8   6   -1283.982
## ig= 7   gnorm= 7.705637     9   7   -1347.195
## ig= 8   gnorm= 3.505001     10   8   -1382.333
## ig= 9   gnorm= 3.56731     11   9   -1382.455
## ig= 10   gnorm= 0.3481236     12   10   -1389.038
## *ig= 11   gnorm= 0.2373563     14   11   -1389.073
## ig= 12   gnorm= 0.03771066     15   12   -1389.102
## ig= 13   gnorm= 0.002851096     16   13   -1389.103
## ig= 14   gnorm= 5.784306e-05     17   14   -1389.103
## ig= 15   gnorm= 9.398079e-07     18   15   -1389.103
## ig= 16   gnorm= 3.823014e-09     19   16   -1389.103
## *****ig= 17   gnorm= 3.821786e-09     25   17   -1389.103
## ********No acceptable point
## Converged
## Seem to be done nvm
## Post processing for method  nvm
## Successful convergence!
## Above for n= 60
## opm: wrapper to call optimr to run multiple optimizers
## Method:  L-BFGS-B
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
## final  value -2499.574655
## converged
## Post processing for method  L-BFGS-B
```

```
## Successful convergence!
## Method:  BFGS
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
## initial  value -1399.762688
## iter  10 value -2489.874111
## final  value -2499.574655
## converged
## Post processing for method  BFGS
## Successful convergence!
## Method:  ncg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
## ncg -- J C Nash 2023 - bounds constraint version of new CG
## an R implementation of Alg 22 with Yuan/Dai modification
## stepredn = 0.2
## Initial function value= -1399.763
## Initial fn= -1399.763
## 1   0   1   -1399.763   last decrease= NA
## **5   1   2   -2076.588   last decrease= 676.8253
## 7   2   3   -2490.958   last decrease= 414.3695
## Yuan/Dai cycle reset
## 7   3   1   -2490.958   last decrease= NA
## 9   4   2   -2499.552   last decrease= 8.594024
## 11   5   3   -2499.575   last decrease= 0.02310082
## 13   6   4   -2499.575   last decrease= 1.064152e-05
## 15   7   5   -2499.575   last decrease= 4.411049e-10
## Very small gradient -- gradsqr = 2.28138170576345e-12
## ncg seems to have converged
## Post processing for method  ncg
## Successful convergence!
## Method:  spg
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
## iter:  0  f-value:  -1399.763  pgrad:  111.779
## Post processing for method  spg
## Successful convergence!
## Method:  ucminf
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
## ucminf message: Stopped by zero step from line search
## Post processing for method  ucminf
## Successful convergence!
## Method:  nlm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

```
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
## iteration = 0
## Step:
##  [1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [39] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## [77] 0 0 0 0
## Parameter:
##  [1] 0.037882636 0.265316635 0.442313184 0.884269409 0.843263548 0.660107203
##  [7] 0.223222751 0.421134858 0.055171347 0.777846952 0.592580450 0.485393078
## [13] 0.408743502 0.343692706 0.681456559 0.185403687 0.092300558 0.847092341
## [19] 0.360159821 0.634536916 0.649034155 0.899875906 0.935264228 0.557453242
## [25] 0.638407025 0.280128007 0.311175453 0.984888318 0.680165719 0.878082018
## [31] 0.482901257 0.346433849 0.571409037 0.858365631 0.407714578 0.572897997
## [37] 0.480396118 0.016096071 0.759840551 0.018338764 0.012832313 0.572931843
## [43] 0.657047224 0.028375826 0.565959939 0.488266491 0.336708136 0.112928051
## [49] 0.501914454 0.629630584 0.510539281 0.935274005 0.959071004 0.303493582
## [55] 0.088824211 0.134125171 0.870145474 0.274016408 0.148480287 0.112721429
## [61] 0.021104704 0.622961885 0.663162624 0.081939084 0.243645584 0.954852959
## [67] 0.346430985 0.761354128 0.957190044 0.035051432 0.064981400 0.854627677
## [73] 0.906042915 0.969149427 0.150087489 0.356846136 0.829337755 0.410218885
## [79] 0.359941408 0.002741989
## Function Value
## [1] -1399.763
## Gradient:
##  [1]    7.089823   28.783097   45.025772   90.044208   82.742075   60.085646
##  [7]    9.957349   28.705779  -13.679380   62.029606   39.310159   25.106314
## [13]   14.252719    4.689019   38.847213  -17.389774  -29.902497   49.567136
## [19]   -5.593232   21.857225   21.166523   46.163602   47.859259    4.805804
## [25]   11.560007  -29.281916  -27.870007   43.279026    8.368943   28.035278
## [31]  -16.566302  -33.066538  -10.332491   19.169318  -31.274068  -14.883318
## [37]  -26.400458  -78.211033    1.021978  -80.797770  -82.779180  -23.405624
## [43]  -15.619869  -85.102495  -28.073637  -37.742941  -55.386829  -80.838822
## [49]  -39.812356  -27.088167  -41.087491    3.944201    5.556855  -66.451037
## [55]  -90.607982  -86.557816   -7.571629  -73.022491  -87.404633  -92.036685
## [61] -102.719158  -38.175580  -34.509202  -98.200206  -81.295696   -4.765304
## [67]  -71.307282  -26.815263   -6.029591 -106.441650 -103.575582  -18.294430
## [73]  -13.026385   -6.419818  -95.431046  -73.154989  -22.025919  -67.549719
## [79]  -73.031702 -111.779035
##
## iteration = 1
## Step:
##  [1] -0.64959279 -2.63720166 -4.12540870 -8.25014536 -7.58110000 -5.50524368
##  [7] -0.91232497 -2.63011748  1.25334958 -5.68335573 -3.60172555 -2.30032277
## [13] -1.30588080 -0.42962332 -3.55930893  1.59330804  2.73976475 -4.54150341
## [19]  0.51247022 -2.00263060 -1.93934615 -4.22966046 -4.38502211 -0.44032352
## [25] -1.05916574  2.68290511  2.55354136 -3.96536611 -0.76679002 -2.56868401
## [31]  1.51785885  3.02966453  0.94669669 -1.75635573  2.86543253  1.36365837
## [37]  2.41889649  7.16595102 -0.09363696  7.40295639  7.58449968  2.14449993
## [43]  1.43114355  7.79736939  2.57219859  3.45813186  5.07472275  7.40671771
## [49]  3.64773845  2.48190659  3.76457053 -0.36138065 -0.50913723  6.08846170
## [55]  8.30180017  7.93071069  0.69373745  6.69055985  8.00829880  8.43270260
## [61]  9.41146575  3.49777171  3.16184615  8.99742458  7.44857800  0.43661279
## [67]  6.53340678  2.45690221  0.55245089  9.75253269  9.48993414  1.67619558
```

```
## [73]  1.19352006  0.58820472  8.74370508  6.70269977  2.01808690  6.18912650
## [79]  6.69140379 10.24156134
## Parameter:
##  [1] -0.61171016 -2.37188502 -3.68309551 -7.36587595 -6.73783645 -4.84513648
##  [7] -0.68910222 -2.20898262  1.30852093 -4.90550878 -3.00914510 -1.81492969
## [13] -0.89713730 -0.08593061 -2.87785237  1.77871173  2.83206531 -3.69441107
## [19]  0.87263004 -1.36809369 -1.29031200 -3.32978455 -3.44975789  0.11712972
## [25] -0.42075871  2.96303311  2.86471681 -2.98047779 -0.08662430 -1.69060200
## [31]  2.00076011  3.37609838  1.51810573 -0.89799010  3.27314711  1.93655637
## [37]  2.89929260  7.18204709  0.66620359  7.42129516  7.59733199  2.71743177
## [43]  2.08819078  7.82574522  3.13815853  3.94639835  5.41143088  7.51964576
## [49]  4.14965291  3.11153718  4.27510981  0.57389335  0.44993378  6.39195528
## [55]  8.39062439  8.06483586  1.56388292  6.96457626  8.15677909  8.54542403
## [61]  9.43257045  4.12073359  3.82500878  9.07936366  7.69222358  1.39146575
## [67]  6.87983776  3.21825634  1.50964094  9.78758412  9.55491554  2.53082326
## [73]  2.09956297  1.55735415  8.89379257  7.05954590  2.84742466  6.59934539
## [79]  7.05134520 10.24430333
## Function Value
## [1] -1437.188
## Gradient:
##  [1]  -0.70136898  -3.51189092  -5.66022743 -11.32315649 -10.61208840
##  [6]  -8.03580880  -2.11396183  -4.59526152   0.37833645  -9.05295506
## [11]  -6.48294903  -4.95548388  -3.83909827  -2.88142397  -7.25779021
## [16]  -0.61063337   0.70428478  -9.19880482  -2.68322964  -6.24474372
## [21]  -6.37543041  -9.64176101 -10.06996321  -5.04381706  -6.09412445
## [26]  -1.33934041  -1.73629749 -10.63778778  -6.60535530  -9.23145235
## [31]  -4.02024774  -2.23554428  -5.23387436  -9.05683597  -3.12920244
## [36]  -5.35165059  -4.16825379   1.93297851  -7.94426750   1.82502595
## [41]   1.86255981  -5.57671143  -6.72064470   1.56276882  -5.57877885
## [46]  -4.58125023  -2.60735681   0.32426419  -4.84611554  -6.56018141
## [51]  -5.01173134 -10.66054406 -11.01311234  -2.38533925   0.41141887
## [56]  -0.22421828  -9.99304538  -2.14145682  -0.51279424  -0.06527859
## [61]   1.12901656  -6.84372812  -7.38653639   0.29075475  -1.85443790
## [66] -11.26565300  -3.22319585  -8.71893002 -11.32023846   0.86378049
## [71]   0.45906763  -9.98958801 -10.67582325 -11.52413737  -0.70755696
## [76]  -3.45778093  -9.72084982  -4.19127794  -3.53999251   1.17619006
##
## iteration = 2
## Step:
##  [1]  0.05447018  0.28043832  0.45355515  0.90735982  0.85226729  0.64830730
##  [7]  0.17725007  0.37721102 -0.01809200  0.73685414  0.53352622  0.41348246
## [13]  0.32628670  0.25177713  0.60320638  0.07441835 -0.02859438  0.76414833
## [19]  0.24586968  0.53226334  0.54471764  0.80757878  0.84385050  0.44462353
## [25]  0.53061881  0.15307552  0.18688389  0.89977280  0.57989893  0.79173052
## [31]  0.37776188  0.23739213  0.47892859  0.78630463  0.31509199  0.49465318
## [37]  0.40223729 -0.08290161  0.70780582 -0.07030394 -0.07138354  0.52450230
## [43]  0.61767128 -0.04196728  0.53003485  0.45208952  0.29614599  0.06367679
## [49]  0.47809799  0.61648992  0.49433907  0.94686875  0.97648570  0.28900833
## [55]  0.06709110  0.11917759  0.90049207  0.27477211  0.14590138  0.11127023
## [61]  0.01692547  0.65449732  0.69873911  0.08655788  0.25863589  1.01084927
## [67]  0.36934554  0.80883220  1.01716916  0.04486093  0.07769545  0.91246841
## [73]  0.96767517  1.03581295  0.17247844  0.39239769  0.89276029  0.45143591
## [79]  0.39959331  0.02309524
## Parameter:
```

```
##  [1] -0.5572400 -2.0914467 -3.2295404 -6.4585161 -5.8855692 -4.1968292
##  [7] -0.5118522 -1.8317716  1.2904289 -4.1686546 -2.4756189 -1.4014472
## [13] -0.5708506  0.1658465 -2.2746460  1.8531301  2.8034709 -2.9302627
## [19]  1.1184997 -0.8358304 -0.7455944 -2.5222058 -2.6059074  0.5617532
## [25]  0.1098601  3.1161086  3.0516007 -2.0807050  0.4932746 -0.8988715
## [31]  2.3785220  3.6134905  1.9970343 -0.1116855  3.5882391  2.4312096
## [37]  3.3015299  7.0991455  1.3740094  7.3509912  7.5259485  3.2419341
## [43]  2.7058621  7.7837779  3.6681934  4.3984879  5.7075769  7.5833225
## [49]  4.6277509  3.7280271  4.7694489  1.5207621  1.4264195  6.6809636
## [55]  8.4577155  8.1840134  2.4643750  7.2393484  8.3026805  8.6566943
## [61]  9.4494959  4.7752309  4.5237479  9.1659215  7.9508595  2.4023150
## [67]  7.2491833  4.0270885  2.5268101  9.8324450  9.6326110  3.4432917
## [73]  3.0672381  2.5931671  9.0662710  7.4519436  3.7401849  7.0507813
## [79]  7.4509385 10.2673986
## Function Value
## [1] -1695.171
## Gradient:
##  [1]  -0.7060571  -3.5228131  -5.6745452 -11.3507977 -10.6305589  -8.0370388
##  [7]  -2.0857792  -4.5639028   0.4372393  -9.0122947  -6.4209083  -4.8742291
## [13]  -3.7391548  -2.7627869  -7.1370402  -0.4517671   0.8841493  -9.0339663
## [19]  -2.4800449  -6.0349725  -6.1474256  -9.4046256  -9.8132617  -4.7487876
## [25]  -5.7795378  -0.9862339  -1.3613700 -10.2682784  -6.1989044  -8.8084935
## [31]  -3.5548858  -1.7386123  -4.7209018  -8.5302386  -2.5572002  -4.7603132
## [37]  -3.5466962   2.6001373  -7.2834772   2.5421105   2.6036722  -4.8367483
## [43]  -5.9619388   2.3702225  -4.7728829  -3.7512710  -1.7505215   1.2102261
## [49]  -3.9580285  -5.6593741  -4.0878788  -9.7367209 -10.0717388  -1.3971411
## [55]   1.4270631   0.8064580  -8.9776474  -1.0850578   0.5638627   1.0264603
## [61]   2.2366423  -5.7510033  -6.2853791   1.4262769  -0.7172705 -10.1510378
## [67]  -2.0752356  -7.5813696 -10.1837433   2.0466601   1.6466913  -8.8307415
## [73]  -9.5144800 -10.3602414   0.4964913  -2.2580079  -8.5374324  -2.9866093
## [79]  -2.3303180   2.4027417
##
## iteration = 3
## Step:
##  [1]  0.7127920  3.6457882  5.8908537 11.7838619 11.0581411  8.3946304
##  [7]  2.2568510  4.8426378 -0.3114814  9.4890857  6.8292872  5.2510453
## [13]  4.0990248  3.1113933  7.6611015  0.7658397 -0.5936170  9.6932793
## [19]  2.9339747  6.6361357  6.7759456 10.1704315 10.6174455  5.4008944
## [25]  6.4922643  1.5565105  1.9692117 11.2110532  7.0243424  9.7496930
## [31]  4.3378043  2.4830003  5.5938311  9.5604566  3.4036180  5.7078568
## [37]  4.4760108 -1.8614309  8.3900394 -1.7552910 -1.7967786  5.9245950
## [43]  7.1101744 -1.4917038  5.9207088  4.8832680  2.8321423 -0.2130412
## [49]  5.1535509  6.9318733  5.3228942 11.1857314 11.5493961  2.5891758
## [55] -0.3172257  0.3404350 10.4804022  2.3268965  0.6341384  0.1681954
## [61] -1.0723565  7.2047576  7.7681822 -0.2024646  2.0249106 11.7960110
## [67]  3.4466232  9.1522821 11.8525987 -0.7974063 -0.3774857 10.4704165
## [73] 11.1827561 12.0626356  0.8316805  3.6858430 10.1871569  4.4452661
## [79]  3.7679726 -1.1290672
## Parameter:
##  [1]  0.1555520  1.5543415  2.6613133  5.3253458  5.1725719  4.1978012
##  [7]  1.7449988  3.0108662  0.9789475  5.3204311  4.3536684  3.8495980
## [13]  3.5281742  3.2772398  5.3864555  2.6189697  2.2098539  6.7630166
## [19]  4.0524745  5.8003054  6.0303512  7.6482257  8.0115381  5.9626476
## [25]  6.6021244  4.6726192  5.0208124  9.1303482  7.5176171  8.8508215
```

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## [31]  6.7163263  6.0964908  7.5908655  9.4487711  6.9918571  8.1390663
## [37]  7.7775407  5.2377146  9.7640488  5.5957003  5.7291699  9.1665291
## [43]  9.8160364  6.2920741  9.5889022  9.2817559  8.5397192  7.3702813
## [49]  9.7813018 10.6599004 10.0923431 12.7064935 12.9758156  9.2701395
## [55]  8.1404898  8.5244485 12.9447772  9.5662449  8.9368189  8.8248897
## [61]  8.3771395 11.9799885 12.2919301  8.9634570  9.9757701 14.1983260
## [67] 10.6958065 13.1793706 14.3794088  9.0350388  9.2551253 13.9137082
## [73] 14.2499942 14.6558027  9.8979515 11.1377866 13.9273419 11.4960474
## [79] 11.2189112  9.1383314
## Function Value
## [1] -2369.132
## Gradient:
##  [1]  0.33451511  1.22697309  1.88960279  3.78218757  3.45309881  2.47689078
##  [7]  0.33539659  1.12982056 -0.67786277  2.54563276  1.58083219  0.98271019
## [13]  0.53004408  0.13419541  1.60242677 -0.77697561 -1.29392262  2.10684747
## [19] -0.22374090  0.96624392  0.96009758  2.05144623  2.15505528  0.35726078
## [25]  0.68352038 -1.01584735 -0.91575308  2.15503558  0.71199087  1.59683062
## [31] -0.25261087 -0.90421432  0.11506933  1.42383792 -0.67015060  0.08311129
## [37] -0.35183431 -2.50363043  0.92400178 -2.50826782 -2.54302639  0.03205453
## [43]  0.40959813 -2.50391807 -0.03133737 -0.40036048 -1.10956962 -2.15330076
## [49] -0.36833521  0.21027297 -0.34908886  1.60639928  1.71523164 -1.30927840
## [55] -2.29730777 -2.08785408  1.31055543 -1.44275822 -2.02287174 -2.19151848
## [61] -2.62230310  0.14705083  0.32271817 -2.37108115 -1.63548690  1.63744275
## [67] -1.18269017  0.72586559  1.62566351 -2.63557969 -2.50149992  1.13919831
## [73]  1.37272628  1.66588372 -2.11201234 -1.15232728  1.03434519 -0.89542180
## [79] -1.12096798 -2.76632332
##
## iteration = 4
## Step:
##  [1] -0.11175405 -0.53495421 -0.85735933 -1.71522948 -1.60242399 -1.20571746
##  [7] -0.29956607 -0.67344310  0.08813846 -1.34445726 -0.94916401 -0.71279895
## [13] -0.53914487 -0.38973714 -1.05287458 -0.03702009  0.16661108 -1.33848858
## [19] -0.34283153 -0.88218424 -0.89905836 -1.39367665 -1.45616417 -0.68801942
## [25] -0.84554029 -0.11892059 -0.17703519 -1.53059649 -0.91422411 -1.31220629
## [31] -0.51655397 -0.24289118 -0.69781110 -1.27841928 -0.37394686 -0.71094840
## [37] -0.52924016  0.40151308 -1.10154258  0.38788571  0.39502095 -0.73665472
## [43] -0.90948319  0.35358893 -0.73283530 -0.57956637 -0.27762502  0.17019062
## [49] -0.61605960 -0.87588788 -0.63884758 -1.49816592 -1.55091390 -0.23592700
## [55]  0.19086711  0.09488928 -1.39223153 -0.19551607  0.05331795  0.12228026
## [61]  0.30504705 -0.90845443 -0.99029182  0.17988404 -0.14606253 -1.57876778
## [67] -0.35305601 -1.18952190 -1.58521902  0.27105205  0.20985281 -1.38119604
## [73] -1.48535057 -1.61434852  0.03330700 -0.38552052 -1.33944238 -0.49721519
## [79] -0.39808416  0.32021169
## Parameter:
##  [1]  0.04379795  1.01938727  1.80395396  3.61011631  3.57014790  2.99208372
##  [7]  1.44543274  2.33742314  1.06708593  3.97597381  3.40450435  3.13679909
## [13]  2.98902934  2.88750267  4.33358095  2.58194964  2.37646498  5.42452800
## [19]  3.70964292  4.91812111  5.13129285  6.25454904  6.55537393  5.27462818
## [25]  5.75658411  4.55369857  4.84377716  7.59975173  6.60339295  7.53861521
## [31]  6.19977232  5.85359965  6.89305436  8.17035186  6.61791020  7.42811792
## [37]  7.24830057  5.63922766  8.66250622  5.98358596  6.12419080  8.42987433
## [43]  8.90655325  6.64566303  8.85606691  8.70218949  8.26209419  7.54047191
## [49]  9.16524224  9.78401249  9.45349554 11.20832761 11.42490167  9.03421245
## [55]  8.33135688  8.61933776 11.55254563  9.37072883  8.99013683  8.94716992
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## [61]  8.68218651 11.07153403 11.30163828  9.14334100  9.82970755 12.61955825
## [67] 10.34275045 11.98984871 12.79418980  9.30609083  9.46497808 12.53251215
## [73] 12.76464367 13.04145421  9.93125854 10.75226611 12.58789950 10.99883219
## [79] 10.82082700  9.45854304
## Function Value
## [1] -2435.68
## Gradient:
##  [1]  0.26899088  0.96463269  1.47991188  2.96259787  2.69974928  1.92910492
##  [7]  0.24307926  0.86522896 -0.55785838  1.97414478  1.21433591  0.74329377
## [13]  0.38696267  0.07555222  1.22971674 -0.64033063 -1.04630177  1.62738589
## [19] -0.20385250  0.73264268  0.72922143  1.58898101  1.67293039  0.26283568
## [25]  0.52278371 -0.80934502 -0.72695589  1.69072015  0.56063043  1.26090556
## [31] -0.18776789 -0.69469662  0.11182838  1.14609707 -0.49415489  0.10387073
## [37] -0.23204651 -1.91760720  0.78248030 -1.90988408 -1.93188970  0.09723507
## [43]  0.39890524 -1.88644124  0.06184803 -0.22366789 -0.77663961 -1.59279940
## [49] -0.18570618  0.27300292 -0.16276579  1.37844460  1.46840553 -0.90436649
## [55] -1.67647933 -1.50775388  1.16742209 -0.99310478 -1.44554299 -1.57500112
## [61] -1.91106794  0.26795497  0.40805547 -1.70752258 -1.12772524  1.44635819
## [67] -0.76907889  0.73258039  1.44142570 -1.90656186 -1.79989709  1.06281329
## [73]  1.24734003  1.47910838 -1.48907505 -0.73341702  0.98653177 -0.52931674
## [79] -0.70564281 -1.99840903
##
## iteration = 5
## Step:
##  [1] -0.139995191 -0.669128024 -1.072215866 -2.145111231 -2.003978223
##  [6] -1.507828785 -0.374529865 -0.842306494  0.110115966 -1.681933083
## [11] -1.187725940 -0.892353822 -0.675454239 -0.488905217 -1.318702994
## [16] -0.048465204  0.205849540 -1.677099561 -0.432161817 -1.107223847
## [21] -1.128805188 -1.747985971 -1.826719724 -0.866562683 -1.064251250
## [26] -0.156084888 -0.229453438 -1.923158878 -1.152932529 -1.651491279
## [31] -0.657100097 -0.315618864 -0.885438330 -1.612481298 -0.482026700
## [36] -0.904401766 -0.677983598  0.485342650 -1.395488689  0.466655320
## [41]  0.474800594 -0.941423459 -1.158324635  0.420787543 -0.938796054
## [46] -0.747771127 -0.370778614  0.188699202 -0.795343173 -1.120927718
## [51] -0.825033404 -1.900457213 -1.967059559 -0.322937131  0.210266043
## [56]  0.089643323 -1.770969629 -0.274664429  0.036068048  0.121869910
## [61]  0.350076170 -1.168103693 -1.270794880  0.192531443 -0.215431637
## [66] -2.007685033 -0.474820388 -1.521296269 -2.016469584  0.305088984
## [71]  0.228342731 -1.761874940 -1.892304137 -2.053834192  0.006842933
## [76] -0.517182126 -1.710469108 -0.657148025 -0.533263747  0.365107365
## Parameter:
##  [1] -0.09619724  0.35025924  0.73173810  1.46500508  1.56616968  1.48425493
##  [7]  1.07090287  1.49511665  1.17720190  2.29404072  2.21677841  2.24444527
## [13]  2.31357510  2.39859746  3.01487796  2.53348443  2.58231452  3.74742844
## [19]  3.27748110  3.81089726  4.00248766  4.50656307  4.72865420  4.40806549
## [25]  4.69233286  4.39761368  4.61432372  5.67659285  5.45046042  5.88712393
## [31]  5.54267223  5.53798078  6.00761603  6.55787057  6.13588350  6.52371615
## [37]  6.57031697  6.12457031  7.26701753  6.45024128  6.59899140  7.48845087
## [43]  7.74822861  7.06645057  7.91727086  7.95441837  7.89131558  7.72917112
## [49]  8.36989907  8.66308477  8.62846213  9.30787039  9.45784212  8.71127532
## [55]  8.54162292  8.70898108  9.78157600  9.09606440  9.02620488  9.06903983
## [61]  9.03226269  9.90343034 10.03084340  9.33587244  9.61427592 10.61187322
## [67]  9.86793006 10.46855244 10.77772022  9.61117981  9.69332082 10.77063721
## [73] 10.87233953 10.98762001  9.93810148 10.23508399 10.87743040 10.34168417
```

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## [79] 10.28756325  9.82365041
## Function Value
## [1] -2491.069
## Gradient:
##  [1]  0.133684178  0.413017683  0.615888555  1.233972403  1.107307450
##  [6]  0.765477744  0.033484840  0.289746781 -0.328480438  0.745865364
## [11]  0.413085951  0.205100574  0.047076186 -0.091109871  0.398357089
## [16] -0.405242446 -0.582345083  0.558217368 -0.227684250  0.171240824
## [21]  0.168853889  0.536879134  0.574493475 -0.025638042  0.089630103
## [26] -0.475681568 -0.435777496  0.603486765  0.126019292  0.432774253
## [31] -0.178856319 -0.387179068 -0.033501057  0.418098246 -0.273766930
## [36] -0.007698705 -0.140956747 -0.851730752  0.313365565 -0.828621468
## [41] -0.828900176  0.047471591  0.184803113 -0.784577842  0.056793548
## [46] -0.057654103 -0.286558893 -0.628525382 -0.019996584  0.182638733
## [51]  0.002791830  0.669094844  0.715439535 -0.291029064 -0.613010983
## [56] -0.533506935  0.617675633 -0.299855072 -0.487033167 -0.537033294
## [61] -0.676623676  0.258811253  0.321823196 -0.580255733 -0.330003859
## [66]  0.772953685 -0.172872185  0.471633187  0.777383969 -0.652072162
## [71] -0.604395541  0.621491262  0.701932390  0.803537602 -0.463307231
## [76] -0.137505014  0.600436676 -0.045801146 -0.119154916 -0.670905077
##
## iteration = 6
## Step:
##  [1] -0.109564764 -0.510098485 -0.814660658 -1.629993800 -1.520234502
##  [6] -1.140179858 -0.274675937 -0.629722865  0.097710975 -1.266928152
## [11] -0.889030790 -0.662889219 -0.496699719 -0.353740351 -0.985599276
## [16] -0.016804651  0.177573563 -1.257214029 -0.307872737 -0.822179353
## [21] -0.838488287 -1.310494858 -1.370770752 -0.639364015 -0.790675748
## [26] -0.099124003 -0.155763932 -1.447561298 -0.861371586 -1.242500647
## [31] -0.485766619 -0.226781449 -0.662454741 -1.218047781 -0.357863319
## [36] -0.681389921 -0.510404514  0.374763553 -1.060440953  0.357480272
## [41]  0.362287103 -0.718510625 -0.885114860  0.317296433 -0.720262819
## [46] -0.575834999 -0.289646674  0.135713696 -0.615385898 -0.864546879
## [51] -0.640010234 -1.460826709 -1.512801149 -0.260883440  0.144274321
## [56]  0.051204916 -1.368089191 -0.228560453  0.007317648  0.071893765
## [61]  0.245202519 -0.912535219 -0.991285229  0.123684383 -0.187627925
## [66] -1.554051448 -0.385918355 -1.183920797 -1.561753787  0.207430351
## [71]  0.148615759 -1.368622590 -1.468273520 -1.591777242 -0.021470458
## [76] -0.421298862 -1.331215595 -0.528661683 -0.434549102  0.250038945
## Parameter:
##  [1] -0.20576200 -0.15983924 -0.08292256 -0.16498872  0.04593518  0.34407508
##  [7]  0.79622693  0.86539378  1.27491288  1.02711257  1.32774762  1.58155605
## [13]  1.81687538  2.04485711  2.02927868  2.51667978  2.75988809  2.49021441
## [19]  2.96960836  2.98871791  3.16399938  3.19606821  3.35788345  3.76870148
## [25]  3.90165711  4.29848968  4.45855979  4.22903155  4.58908884  4.64462328
## [31]  5.05690561  5.31119933  5.34516129  5.33982279  5.77802018  5.84232623
## [37]  6.05991246  6.49933387  6.20657658  6.80772155  6.96127850  6.76994025
## [43]  6.86311375  7.38374701  7.19700804  7.37858337  7.60166890  7.86488481
## [49]  7.75451317  7.79853789  7.98845190  7.84704369  7.94504097  8.45039188
## [55]  8.68589724  8.76018600  8.41348681  8.86750395  9.03352253  9.14093359
## [61]  9.27746520  8.99089512  9.03955817  9.45955682  9.42664799  9.05782177
## [67]  9.48201171  9.28463165  9.21596643  9.81861016  9.84193657  9.40201462
## [73]  9.40406601  9.39584277  9.91663102  9.81378512  9.54621480  9.81302248
## [79]  9.85301415 10.07368935
```

```
## Function Value
## [1] -2495.236
## Gradient:
##  [1] -0.022026961 -0.219163767 -0.373913006 -0.746584848 -0.717821025
##  [6] -0.569507342 -0.209939829 -0.374794016 -0.072738969 -0.670802308
## [11] -0.516701369 -0.426043846 -0.359855263 -0.302723784 -0.578253765
## [16] -0.163363259 -0.081741877 -0.701479302 -0.293293016 -0.514268258
## [21] -0.519575896 -0.719409237 -0.739675675 -0.416852127 -0.472751802
## [26] -0.164878412 -0.179111890 -0.724817224 -0.460204293 -0.610400285
## [31] -0.269140443 -0.141615462 -0.313123372 -0.535552290 -0.147205167
## [36] -0.268079075 -0.175679823  0.224062957 -0.376031418  0.252328372
## [41]  0.271035981 -0.179089458 -0.235586389  0.297498570 -0.134771617
## [46] -0.058331899  0.079050965  0.275608613 -0.035491360 -0.130737446
## [51] -0.021763905 -0.362457820 -0.370746762  0.183586581  0.372756700
## [56]  0.345762883 -0.253356012  0.249486271  0.362516915  0.400194673
## [61]  0.482804943 -0.009322297 -0.037089457  0.449006602  0.319618085
## [66] -0.264944781  0.242255046 -0.097159827 -0.255074835  0.512066900
## [71]  0.490683867 -0.160145161 -0.200106698 -0.248979947  0.432291767
## [76]  0.264358889 -0.123925240  0.225372765  0.269226081  0.566111857
##
## iteration = 7
## Step:
##  [1]  0.0442606665  0.2207188849  0.3555401631  0.7112251042  0.6662445767
##  [6]  0.5039810241  0.1314079384  0.2869662181 -0.0260378972  0.5663145785
## [11]  0.4043404127  0.3078617738  0.2371999358  0.1765001052  0.4510385792
## [16]  0.0327168320 -0.0504664103  0.5714183336  0.1613491552  0.3846247209
## [21]  0.3922517147  0.5969343236  0.6232033480  0.3066273896  0.3719470026
## [26]  0.0723999420  0.0966512565  0.6554233873  0.4012589762  0.5655697778
## [31]  0.2372768491  0.1243408063  0.3120302458  0.5515438889  0.1782073126
## [36]  0.3171138029  0.2419601528 -0.1422754819  0.4777774836 -0.1369718167
## [41] -0.1400390168  0.3268525092  0.3980823662 -0.1232021187  0.3250160532
## [46]  0.2616954009  0.1370197732 -0.0478284226  0.2765726919  0.3837541004
## [51]  0.2858876515  0.6403813627  0.6619866154  0.1191558999 -0.0571680090
## [56] -0.0177235819  0.5957825235  0.1018298069 -0.0009809749 -0.0295176442
## [61] -0.1049479763  0.3958203256  0.4296223262 -0.0532063768  0.0813550122
## [66]  0.6726214023  0.1669056603  0.5120954015  0.6753686008 -0.0906338105
## [71] -0.0653822421  0.5911808671  0.6341667311  0.6873287555  0.0073349043
## [76]  0.1800597106  0.5735954087  0.2259562308  0.1849425774 -0.1115230070
## Parameter:
##  [1] -0.16150133  0.06087964  0.27261760  0.54623638  0.71217975  0.84805610
##  [7]  0.92763487  1.15236000  1.24887498  1.59342715  1.73208803  1.88941782
## [13]  2.05407532  2.22135721  2.48031726  2.54939661  2.70942168  3.06163275
## [19]  3.13095752  3.37334263  3.55625109  3.79300253  3.98108680  4.07532887
## [25]  4.27360411  4.37088962  4.55521105  4.88445494  4.99034781  5.21019306
## [31]  5.29418246  5.43554014  5.65719153  5.89136667  5.95622749  6.15944003
## [37]  6.30187261  6.35705838  6.68435406  6.67074974  6.82123948  7.09679276
## [43]  7.26119612  7.26054489  7.52202409  7.64027877  7.73868867  7.81705639
## [49]  8.03108586  8.18229199  8.27433955  8.48742505  8.60702758  8.56954778
## [55]  8.62872923  8.74246242  9.00926933  8.96933376  9.03254156  9.11141595
## [61]  9.17251723  9.38671545  9.46918050  9.40635045  9.50800300  9.73044317
## [67]  9.64891737  9.79672705  9.89133503  9.72797635  9.77655433  9.99319549
## [73] 10.03823274 10.08317153  9.92396592  9.99384483 10.11981021 10.03897871
## [79] 10.03795672  9.96216635
## Function Value
```

```
## [1] -2498.898
## Gradient:
##  [1]  0.047224552  0.078151084  0.095431996  0.192373991  0.151227386
##  [6]  0.071809280 -0.079518314 -0.043846217 -0.172339215  0.018554065
## [11] -0.054315949 -0.102303781 -0.139902635 -0.173071809 -0.087285484
## [16] -0.244231374 -0.282994176 -0.073391322 -0.225739595 -0.154082965
## [21] -0.157121464 -0.089138046 -0.081742913 -0.192892033 -0.168254501
## [26] -0.271195714 -0.260040713 -0.060306753 -0.145079156 -0.080847187
## [31] -0.188517104 -0.219740407 -0.144983727 -0.051221680 -0.172007205
## [36] -0.111936539 -0.126801939 -0.250541665 -0.021331156 -0.226758585
## [41] -0.217881523 -0.044779251 -0.010859207 -0.185307707 -0.019165227
## [46] -0.033204042 -0.068812465 -0.126193554 -0.005217522  0.039159838
## [51]  0.011725819  0.144091725  0.160694267 -0.020134231 -0.072336445
## [56] -0.049995331  0.173554663  0.007557854 -0.021276990 -0.025330541
## [61] -0.047534358  0.131721870  0.146523626 -0.020514598  0.028658872
## [66]  0.237945505  0.061624499  0.184980086  0.245010135 -0.021372626
## [71] -0.010396694  0.221582716  0.238183361  0.259823788  0.024315165
## [76]  0.088302188  0.229507342  0.110173701  0.098567040 -0.003955569
##
## iteration = 8
## Step:
##  [1] -0.0080148678 -0.0257069764 -0.0386512038 -0.0774542714 -0.0699417300
##  [6] -0.0490706248 -0.0039414524 -0.0203234063  0.0177066704 -0.0496003030
## [11] -0.0293590678 -0.0169025029 -0.0075681429  0.0005226103 -0.0304550721
## [16]  0.0190244238  0.0295153767 -0.0419935254  0.0063705173 -0.0190074738
## [21] -0.0194224208 -0.0429035814 -0.0458438089 -0.0091164177 -0.0169201662
## [26]  0.0176231191  0.0145054916 -0.0508037837 -0.0217390603 -0.0414903602
## [31] -0.0041008707  0.0081887975 -0.0144915810 -0.0432679304 -0.0008919478
## [36] -0.0181280343 -0.0105113277  0.0330524775 -0.0401069957  0.0303124833
## [41]  0.0297013951 -0.0254432128 -0.0345889287  0.0251439438 -0.0277909070
## [46] -0.0212334680 -0.0075434246  0.0131990782 -0.0251883108 -0.0383042856
## [51] -0.0276106781 -0.0695678643 -0.0729474895 -0.0108107648  0.0087462417
## [56]  0.0033456841 -0.0687243228 -0.0120468094 -0.0008008755  0.0019360352
## [61]  0.0102789457 -0.0482496241 -0.0524718432  0.0033782150 -0.0124561521
## [66] -0.0813337510 -0.0227051950 -0.0630308707 -0.0822681785  0.0064852448
## [71]  0.0033428038 -0.0730927070 -0.0782374840 -0.0846991597 -0.0059955776
## [76] -0.0263811833 -0.0723937867 -0.0322562042 -0.0277518590  0.0065493038
## Parameter:
##  [1] -0.16951620  0.03517267  0.23396640  0.46878211  0.64223802  0.79898547
##  [7]  0.92369342  1.13203659  1.26658165  1.54382685  1.70272896  1.87251532
## [13]  2.04650717  2.22187982  2.44986219  2.56842104  2.73893705  3.01963922
## [19]  3.13732804  3.35433516  3.53682867  3.75009895  3.93524299  4.06621245
## [25]  4.25668395  4.38851274  4.56971654  4.83365115  4.96860875  5.16870270
## [31]  5.29008158  5.44372894  5.64269995  5.84809874  5.95533555  6.14131200
## [37]  6.29136128  6.39011086  6.64424707  6.70106222  6.85094088  7.07134955
## [43]  7.22660719  7.28568883  7.49423319  7.61904530  7.73114525  7.83025547
## [49]  8.00589755  8.14398770  8.24672887  8.41785718  8.53408009  8.55873701
## [55]  8.63747547  8.74580810  8.94054501  8.95728695  9.03174068  9.11335199
## [61]  9.18279617  9.33846582  9.41670866  9.40972866  9.49554685  9.64910942
## [67]  9.62621217  9.73369618  9.80906685  9.73446159  9.77989714  9.92010278
## [73]  9.95999526  9.99847237  9.91797034  9.96746365 10.04741642 10.00672251
## [79] 10.01020487  9.96871565
## Function Value
## [1] -2499.076
```

```
## Gradient:
##  [1]  0.0362142860  0.0453951308  0.0469709993  0.0952492175  0.0644238515
##  [6]  0.0123248219 -0.0807917717 -0.0652785954 -0.1446099256 -0.0387551594
## [11] -0.0851341938 -0.1165566867 -0.1415712149 -0.1637403923 -0.1167276132
## [16] -0.2099452588 -0.2346453285 -0.1154417205 -0.2054722270 -0.1654463183
## [21] -0.1683140004 -0.1295516722 -0.1252182784 -0.1889018890 -0.1735357651
## [26] -0.2317980608 -0.2239393067 -0.1067321836 -0.1538062728 -0.1140774051
## [31] -0.1734608790 -0.1883774370 -0.1418654164 -0.0841193039 -0.1502811602
## [36] -0.1115381976 -0.1160734920 -0.1836880601 -0.0470579666 -0.1621493525
## [41] -0.1534270926 -0.0499523724 -0.0270645957 -0.1248027687 -0.0254928987
## [46] -0.0305789559 -0.0481579346 -0.0785307191 -0.0058826988  0.0223516253
## [51]  0.0090924661  0.0885375755  0.1013419888  0.0001588998 -0.0266573794
## [56] -0.0107314917  0.1214556636  0.0281016949  0.0140153220  0.0138618045
## [61]  0.0026940201  0.1077825686  0.1175839936  0.0220575376  0.0514043341
## [66]  0.1732709307  0.0719484934  0.1442113014  0.1799879961  0.0269056571
## [71]  0.0340930442  0.1688953321  0.1791166149  0.1926655139  0.0575364272
## [76]  0.0956446282  0.1783026870  0.1101517435  0.1043086358  0.0454952015
##
## iteration = 9
## Step:
##  [1] -0.0153703206 -0.0350267963 -0.0483708058 -0.0971575527 -0.0833771944
##  [6] -0.0518610782  0.0123417336 -0.0072688104  0.0470097959 -0.0420717511
## [11] -0.0123410088  0.0065393543  0.0209855445  0.0336065510 -0.0070435859
## [16]  0.0615609853  0.0771085110 -0.0188918100  0.0479476045  0.0143429520
## [21]  0.0145123848 -0.0169276624 -0.0207508462  0.0290660844  0.0181241319
## [26]  0.0646500863  0.0599383792 -0.0294168775  0.0093392620 -0.0185393704
## [31]  0.0310782876  0.0464672621  0.0142730686 -0.0263217454  0.0296817128
## [36]  0.0045339451  0.0131265463  0.0706181906 -0.0305538323  0.0635401046
## [41]  0.0611770237 -0.0151793340 -0.0289894314  0.0508481408 -0.0224450393
## [46] -0.0148103847  0.0025219761  0.0295318278 -0.0237448621 -0.0426310903
## [51] -0.0291868098 -0.0874272509 -0.0934024814 -0.0104522563  0.0146467878
## [56]  0.0060073856 -0.0931767431 -0.0173013195 -0.0031356508 -0.0003442087
## [61]  0.0103097015 -0.0697836357 -0.0760053470 -0.0005275031 -0.0223784551
## [66] -0.1162723413 -0.0368302674 -0.0919852780 -0.1185581792  0.0016407951
## [71] -0.0029687565 -0.1070790359 -0.1143162258 -0.1235494821 -0.0170862183
## [76] -0.0452859998 -0.1082621759 -0.0541210265 -0.0484079385 -0.0020246102
## Parameter:
##  [1] -0.1848865229  0.0001458703  0.1855955941  0.3716245584  0.5588608288
##  [6]  0.7471243967  0.9360351538  1.1247677833  1.3135914447  1.5017550968
## [11]  1.6903879519  1.8790546716  2.0674927166  2.2554863721  2.4428186050
## [16]  2.6299820237  2.8160455628  3.0007474129  3.1852756418  3.3686781089
## [21]  3.5513410565  3.7331712890  3.9144921436  4.0952785349  4.2748080778
## [26]  4.4531628259  4.6296549175  4.8042342751  4.9779480161  5.1501633294
## [31]  5.3211598725  5.4901961992  5.6569730190  5.8217769986  5.9850172581
## [36]  6.1458459425  6.3044878267  6.4607290517  6.6136932358  6.7646023234
## [41]  6.9121179023  7.0561702125  7.1976177590  7.3365369719  7.4717881477
## [46]  7.6042349151  7.7336672263  7.8597872949  7.9821526885  8.1013566137
## [51]  8.2175420642  8.3304299327  8.4406776105  8.5482847567  8.6521222620
## [56]  8.7518154868  8.8473682634  8.9399856289  9.0286050294  9.1130077769
## [61]  9.1931058750  9.2686821880  9.3407033104  9.4092011597  9.4731683977
## [66]  9.5328370770  9.5893819041  9.6417108999  9.6905086731  9.7361023899
## [71]  9.7769283796  9.8130237460  9.8456790311  9.8749228861  9.9008841254
## [76]  9.9221776515  9.9391542469  9.9526014823  9.9617969267  9.9666910407
## Function Value
```

```
## [1] -2499.332
## Gradient:
##  [1]  1.577157e-02  9.347003e-05 -1.504155e-02 -2.932619e-02 -4.184632e-02
##  [6] -5.273862e-02 -6.239956e-02 -7.180459e-02 -8.050515e-02 -8.938007e-02
## [11] -9.686206e-02 -1.034193e-01 -1.092950e-01 -1.146718e-01 -1.197340e-01
## [16] -1.237431e-01 -1.278131e-01 -1.321876e-01 -1.352246e-01 -1.380668e-01
## [21] -1.401161e-01 -1.413986e-01 -1.413988e-01 -1.400549e-01 -1.382127e-01
## [26] -1.356718e-01 -1.332343e-01 -1.308735e-01 -1.271359e-01 -1.227544e-01
## [31] -1.172753e-01 -1.115744e-01 -1.059530e-01 -9.995113e-02 -9.295067e-02
## [36] -8.596975e-02 -7.863210e-02 -7.113333e-02 -6.453233e-02 -5.715949e-02
## [41] -5.067763e-02 -4.510069e-02 -3.923656e-02 -3.291209e-02 -2.753783e-02
## [46] -2.191961e-02 -1.626172e-02 -1.088419e-02 -6.296980e-03 -1.664482e-03
## [51]  3.261464e-03  8.178232e-03  1.400000e-02  2.078377e-02  2.711647e-02
## [56]  3.256473e-02  3.718549e-02  4.259645e-02  4.746334e-02  5.154778e-02
## [61]  5.477926e-02  5.691697e-02  5.925797e-02  6.188133e-02  6.351203e-02
## [66]  6.448533e-02  6.636079e-02  6.774705e-02  6.956214e-02  7.225718e-02
## [71]  7.382136e-02  7.432423e-02  7.546378e-02  7.729430e-02  7.999864e-02
## [76]  8.178811e-02  8.313022e-02  8.505856e-02  8.664014e-02  8.781531e-02
##
## iteration = 10
## Step:
##  [1] -0.0118473311 -0.0141547172 -0.0141073115 -0.0286452219 -0.0185749654
##  [6] -0.0019216199  0.0274816645  0.0230100265  0.0480845117  0.0154295047
## [11]  0.0301913566  0.0402610372  0.0483107106  0.0554563403  0.0410223812
## [16]  0.0702725993  0.0781612977  0.0411934687  0.0694134916  0.0570789850
## [21]  0.0580905194  0.0460979862  0.0447979648  0.0646599236  0.0598598743
## [26]  0.0779903856  0.0755074061  0.0389337480  0.0535322663  0.0410474980
## [31]  0.0594264562  0.0639333625  0.0492893308  0.0311292884  0.0515664788
## [36]  0.0392972242  0.0405075939  0.0613766189  0.0185957180  0.0542661185
## [41]  0.0513615550  0.0189425730  0.0116371826  0.0419227802  0.0108042104
## [46]  0.0122234069  0.0175351077  0.0268426997  0.0040553747 -0.0048878474
## [51] -0.0009046589 -0.0258231019 -0.0299933101  0.0013438193  0.0095095414
## [56]  0.0043760246 -0.0369784325 -0.0080424512 -0.0038026234 -0.0038832312
## [61] -0.0005046778 -0.0333372793 -0.0364693301 -0.0067728678 -0.0159768505
## [66] -0.0540047903 -0.0224784981 -0.0450538468 -0.0562668311 -0.0086274776
## [71] -0.0109194327 -0.0529643655 -0.0561888036 -0.0604707590 -0.0184268074
## [76] -0.0303631860 -0.0561745216 -0.0349869574 -0.0332134538 -0.0149134301
## Parameter:
##  [1] -0.19673385 -0.01400885  0.17148828  0.34297934  0.54028586  0.74520278
##  [7]  0.96351682  1.14777781  1.36167596  1.51718460  1.72057931  1.91931571
## [13]  2.11580343  2.31094271  2.48384099  2.70025462  2.89420686  3.04194088
## [19]  3.25468913  3.42575709  3.60943158  3.77926928  3.95929011  4.15993846
## [25]  4.33466795  4.53115321  4.70516232  4.84316802  5.03148028  5.19121083
## [31]  5.38058633  5.55412956  5.70626235  5.85290629  6.03658374  6.18514317
## [37]  6.34499542  6.52210567  6.63228895  6.81886844  6.96347946  7.07511279
## [43]  7.20925494  7.37845975  7.48259236  7.61645832  7.75120233  7.88662999
## [49]  7.98620806  8.09646877  8.21663741  8.30460683  8.41068430  8.54962858
## [55]  8.66163180  8.75619151  8.81038983  8.93194318  9.02480241  9.10912455
## [61]  9.19260120  9.23534491  9.30423398  9.40242829  9.45719155  9.47883229
## [67]  9.56690341  9.59665705  9.63424184  9.72747491  9.76600895  9.76005938
## [73]  9.78949023  9.81445213  9.88245732  9.89181447  9.88297973  9.91761452
## [79]  9.92858347  9.95177761
## Function Value
## [1] -2499.501
```

```
## Gradient:
##  [1]  0.0008773713 -0.0183208080 -0.0339211644 -0.0676469852 -0.0676214465
##  [6] -0.0574173403 -0.0294023637 -0.0451713192 -0.0217997840 -0.0736541649
## [11] -0.0624431777 -0.0563943383 -0.0522704675 -0.0487984872 -0.0730473742
## [16] -0.0394115894 -0.0335805633 -0.0863488045 -0.0529826997 -0.0721266270
## [21] -0.0730920512 -0.0901725826 -0.0920421227 -0.0650241154 -0.0695471739
## [26] -0.0435199432 -0.0443632718 -0.0895786232 -0.0668488791 -0.0786693945
## [31] -0.0492337914 -0.0375861354 -0.0508904225 -0.0683615607 -0.0346232812
## [36] -0.0434122874 -0.0342943831  0.0005713163 -0.0482046686  0.0058253898
## [41]  0.0088265656 -0.0274334130 -0.0307367583  0.0153125056 -0.0194218970
## [46] -0.0115920557  0.0013456932  0.0192086989 -0.0054436338 -0.0120447573
## [51] -0.0015447445 -0.0286409492 -0.0278634780  0.0200365275  0.0373519012
## [56]  0.0364924062 -0.0122971783  0.0310954920  0.0418227576  0.0461469875
## [61]  0.0541149396  0.0138968487  0.0124895332  0.0540382573  0.0440013257
## [66] -0.0041865920  0.0389533196  0.0112335850 -0.0013046910  0.0635437754
## [71]  0.0623219484  0.0083349640  0.0054420773  0.0018349176  0.0593156746
## [76]  0.0456711953  0.0135191780  0.0430450932  0.0469580696  0.0719397598
##
## iteration = 11
## Step:
##  [1] -0.004355851  0.005410401  0.013970743  0.027695455  0.030959835
##  [6]  0.030795576  0.025040338  0.032221652  0.027619361  0.045395869
## [11]  0.044116365  0.044147096  0.044590235  0.045115202  0.053802147
## [16]  0.045060777  0.044602892  0.061629059  0.052772999  0.059467729
## [21]  0.060584893  0.066307678  0.067209065  0.059214698  0.060471271
## [26]  0.052502109  0.052550073  0.065801215  0.058555607  0.061413872
## [31]  0.051764616  0.047350646  0.050373469  0.054547102  0.043280503
## [36]  0.044647338  0.040617099  0.028911530  0.042259466  0.024907530
## [41]  0.022927239  0.032828982  0.032878119  0.018164132  0.027662085
## [46]  0.024464334  0.019736820  0.013614229  0.020303131  0.021617270
## [51]  0.017774587  0.025082351  0.023879216  0.008437703  0.002175484
## [56]  0.001519292  0.015283365  0.001477658 -0.002511345 -0.004416968
## [61] -0.007203261  0.004570668  0.004758517 -0.007890319 -0.004997060
## [66]  0.009366016 -0.003620275  0.004526030  0.008038853 -0.011655965
## [71] -0.011477028  0.004609484  0.005351657  0.006109270 -0.011503926
## [76] -0.007799632  0.001488873 -0.007696281 -0.009214907 -0.016904554
## Parameter:
##  [1] -0.201089705 -0.008598446  0.185459025  0.370674791  0.571245699
##  [6]  0.775998353  0.988557157  1.179999462  1.389295318  1.562580470
## [11]  1.764695674  1.963462805  2.160393663  2.356057914  2.537643133
## [16]  2.745315400  2.938809753  3.103569941  3.307462132  3.485224823
## [21]  3.670016469  3.845576953  4.026499173  4.219153156  4.395139223
## [26]  4.583655321  4.757712397  4.908969238  5.090035890  5.252624700
## [31]  5.432350945  5.601480208  5.756635818  5.907453389  6.079864239
## [36]  6.229790505  6.385612520  6.551017201  6.674548420  6.843775972
## [41]  6.986406696  7.107941767  7.242133061  7.396623884  7.510254444
## [46]  7.640922656  7.770939154  7.900244223  8.006511194  8.118086036
## [51]  8.234411992  8.329689181  8.434563517  8.558066279  8.663807287
## [56]  8.757710803  8.825673196  8.933420836  9.022291061  9.104707577
## [61]  9.185397936  9.239915577  9.308992497  9.394537973  9.452194488
## [66]  9.488198303  9.563283131  9.601183084  9.642280695  9.715818947
## [71]  9.754531919  9.764668864  9.794841885  9.820561397  9.870953392
## [76]  9.884014833  9.884468598  9.909918244  9.919368565  9.934873057
## Function Value
```

```
## [1] -2499.568
## Gradient:
##  [1] -3.717618e-03 -1.123566e-02 -1.673056e-02 -3.357568e-02 -3.035011e-02
##  [6] -2.141482e-02 -1.971565e-03 -9.353023e-03  6.990111e-03 -2.256557e-02
## [11] -1.398834e-02 -8.830762e-03 -5.032659e-03 -1.754472e-03 -1.548156e-02
## [16]  5.878623e-03  1.030729e-02 -2.093959e-02  8.196303e-05 -1.100898e-02
## [21] -1.119380e-02 -2.142685e-02 -2.272536e-02 -6.749008e-03 -1.016375e-02
## [26]  4.920873e-03  3.662939e-03 -2.464575e-02 -1.181857e-02 -2.027419e-02
## [31] -3.821281e-03  1.728718e-03 -7.907912e-03 -2.017662e-02 -1.428916e-03
## [36] -8.638008e-03 -4.985406e-03  1.438269e-02 -1.706236e-02  1.411009e-02
## [41]  1.439805e-02 -9.002789e-03 -1.233475e-02  1.436666e-02 -7.995104e-03
## [46] -4.407306e-03  2.297376e-03  1.210903e-02 -3.809112e-03 -8.705676e-03
## [51] -3.243597e-03 -2.076274e-02 -2.153996e-02  6.170884e-03  1.532359e-02
## [56]  1.364242e-02 -1.708229e-02  8.273097e-03  1.382059e-02  1.568980e-02
## [61]  2.004367e-02 -4.724136e-03 -5.854306e-03  1.916069e-02  1.294059e-02
## [66] -1.641136e-02  9.748523e-03 -7.273593e-03 -1.517940e-02  2.390995e-02
## [71]  2.295107e-02 -9.945783e-03 -1.184281e-02 -1.442824e-02  2.001735e-02
## [76]  1.125183e-02 -8.712146e-03  8.806694e-03  1.074973e-02  2.567199e-02
##
## iteration = 12
## Step:
##  [1]  0.0010600666  0.0066202558  0.0109285317  0.0218548550  0.0207522830
##  [6]  0.0161198279  0.0051690400  0.0100519325  0.0008728491  0.0188422849
## [11]  0.0141740146  0.0114579987  0.0095051814  0.0078407166  0.0162208566
## [16]  0.0038066780  0.0014356823  0.0202037733  0.0080050913  0.0147909872
## [21]  0.0150951491  0.0212952297  0.0221272319  0.0126592431  0.0146323837
## [26]  0.0056490523  0.0063682360  0.0231098402  0.0154532126  0.0203348360
## [31]  0.0104276913  0.0069700985  0.0125215422  0.0196177895  0.0083210274
## [36]  0.0123809667  0.0100129278 -0.0016313019  0.0168449875 -0.0017213072
## [41] -0.0019410559  0.0119357539  0.0139412444 -0.0018291204  0.0114806227
## [46]  0.0094451310  0.0055676815 -0.0001140246  0.0094811872  0.0125608096
## [51]  0.0094887510  0.0199837873  0.0204900855  0.0040632299 -0.0013676679
## [56] -0.0003205410  0.0179504162  0.0030046633 -0.0002042956 -0.0011781095
## [61] -0.0035501112  0.0113688177  0.0122851093 -0.0022910644  0.0016589404
## [66]  0.0193140710  0.0040677953  0.0143485189  0.0191750345 -0.0038696346
## [71] -0.0031722336  0.0164700077  0.0177110677  0.0192580374 -0.0011837131
## [76]  0.0039592689  0.0157324826  0.0052785453  0.0040281920 -0.0048756981
## Parameter:
##  [1] -0.20002964 -0.00197819  0.19638756  0.39252965  0.59199798  0.79211818
##  [7]  0.99372620  1.19005139  1.39016817  1.58142275  1.77886969  1.97492080
## [13]  2.16989884  2.36389863  2.55386399  2.74912208  2.94024543  3.12377371
## [19]  3.31546722  3.50001581  3.68511162  3.86687218  4.04862641  4.23181240
## [25]  4.40977161  4.58930437  4.76408063  4.93207908  5.10548910  5.27295954
## [31]  5.44277864  5.60845031  5.76915736  5.92707118  6.08818527  6.24217147
## [37]  6.39562545  6.54938590  6.69139341  6.84205466  6.98446564  7.11987752
## [43]  7.25607431  7.39479476  7.52173507  7.65036779  7.77650684  7.90013020
## [49]  8.01599238  8.13064685  8.24390074  8.34967297  8.45505360  8.56212951
## [55]  8.66243962  8.75739026  8.84362361  8.93642550  9.02208677  9.10352947
## [61]  9.18184782  9.25128439  9.32127761  9.39224691  9.45385343  9.50751237
## [67]  9.56735093  9.61553160  9.66145573  9.71194931  9.75135968  9.78113887
## [73]  9.81255295  9.83981943  9.86976968  9.88797410  9.90020108  9.91519679
## [79]  9.92339676  9.92999736
## Function Value
## [1] -2499.574
```

```
## Gradient:
##  [1] -1.948595e-03 -2.658851e-03 -2.964124e-03 -6.040307e-03 -4.616679e-03
##  [6] -2.044794e-03  2.858723e-03  1.451955e-03  5.558591e-03 -1.063776e-03
## [11]  1.130960e-03  2.442658e-03  3.389637e-03  4.195228e-03  1.011857e-03
## [16]  5.986442e-03  7.041738e-03 -1.934204e-04  4.734791e-03  2.148122e-03
## [21]  2.078605e-03 -3.933280e-04 -8.734488e-04  2.589979e-03  1.490290e-03
## [26]  4.704098e-03  4.147188e-03 -2.701133e-03 -7.922918e-06 -2.346739e-03
## [31]  1.075666e-03  1.955580e-03 -6.769393e-04 -3.933819e-03  8.927995e-06
## [36] -2.107104e-03 -1.682114e-03  2.446688e-03 -5.219320e-03  1.754327e-03
## [41]  1.609214e-03 -3.964741e-03 -4.842262e-03  1.293207e-03 -3.964031e-03
## [46] -3.144677e-03 -1.589721e-03  7.376421e-04 -2.874691e-03 -3.901141e-03
## [51] -2.531526e-03 -6.562740e-03 -6.781228e-03 -4.367107e-04  1.596329e-03
## [56]  1.186443e-03 -5.954840e-03 -3.694236e-05  1.278876e-03  1.811584e-03
## [61]  3.016797e-03 -2.502340e-03 -2.517514e-03  3.549315e-03  2.379271e-03
## [66] -4.167639e-03  2.185591e-03 -1.572247e-03 -3.271760e-03  5.928524e-03
## [71]  5.834870e-03 -1.656637e-03 -1.977262e-03 -2.580485e-03  5.372435e-03
## [76]  3.257094e-03 -1.465600e-03  2.508839e-03  2.832944e-03  6.223545e-03
##
## iteration = 13
## Step:
##  [1]  6.324849e-04  1.556137e-03  2.202567e-03  4.425991e-03  3.880224e-03
##  [6]  2.553751e-03 -2.198292e-04  7.247209e-04 -1.610631e-03  2.411900e-03
## [11]  1.163300e-03  3.959487e-04 -1.766423e-04 -6.715988e-04  1.193313e-03
## [16] -1.792208e-03 -2.429912e-03  1.874882e-03 -1.048376e-03  4.803142e-04
## [21]  5.065049e-04  1.932128e-03  2.131039e-03 -5.151653e-05  4.577626e-04
## [26] -1.587669e-03 -1.366918e-03  2.603090e-03  8.868447e-04  2.124774e-03
## [31] -7.795971e-05 -7.666792e-04  6.500581e-04  2.435877e-03 -6.610824e-05
## [36]  1.028722e-03  6.231077e-04 -1.956458e-03  2.497771e-03 -1.714636e-03
## [41] -1.653606e-03  1.684044e-03  2.244531e-03 -1.351378e-03  1.842746e-03
## [46]  1.444532e-03  6.148722e-04 -6.467740e-04  1.650347e-03  2.420863e-03
## [51]  1.757816e-03  4.276755e-03  4.481159e-03  7.444157e-04 -4.258222e-04
## [56] -1.018654e-04  4.238361e-03  8.145342e-04  1.291713e-04 -5.301043e-05
## [61] -5.860356e-04  2.905954e-03  3.123193e-03 -2.812235e-04  6.324349e-04
## [66]  4.743715e-03  1.169515e-03  3.570597e-03  4.708988e-03 -6.588536e-04
## [71] -4.890887e-04  4.095405e-03  4.387453e-03  4.775926e-03  3.789055e-05
## [76]  1.276469e-03  4.060635e-03  1.653981e-03  1.399604e-03 -6.575481e-04
## Parameter:
##  [1] -0.1993971538 -0.0004220534  0.1985901237  0.3969556377  0.5958782060
##  [6]  0.7946719319  0.9935063674  1.1907761157  1.3885575352  1.5838346550
## [11]  1.7800329888  1.9753167521  2.1697222017  2.3632270320  2.5550573023
## [16]  2.7473298700  2.9378155227  3.1256485957  3.3144188475  3.5004961244
## [21]  3.6856181234  3.8688043103  4.0507574446  4.2317608830  4.4102293695
## [26]  4.5877167047  4.7627137154  4.9346821685  5.1063759469  5.2750843096
## [31]  5.4427006762  5.6076836271  5.7698074187  5.9295070554  6.0881191586
## [36]  6.2432001934  6.3962485553  6.5474294403  6.6938911782  6.8403400288
## [41]  6.9828120345  7.1215615652  7.2583188367  7.3934433862  7.5235778126
## [46]  7.6518123195  7.7771217078  7.8994834247  8.0176427278  8.1330677085
## [51]  8.2456585591  8.3539497233  8.4595347608  8.5628739242  8.6620137973
## [56]  8.7572883967  8.8478619734  8.9372400336  9.0222159372  9.1034764574
## [61]  9.1812617889  9.2541903485  9.3244007989  9.3919656853  9.4544858628
## [66]  9.5122560882  9.5685204414  9.6191021990  9.6661647175  9.7112904593
## [71]  9.7508705962  9.7852342768  9.8169404058  9.8445953607  9.8698075699
## [76]  9.8892505717  9.9042617159  9.9168507707  9.9247963614  9.9293398105
## Function Value
```

```
## [1] -2499.575
## Gradient:
##  [1] -0.0010847706 -0.0006481180 -0.0001638045 -0.0004118831  0.0002646368
##  [6]  0.0010828707  0.0023645552  0.0021393587  0.0031905950  0.0017258354
## [11]  0.0022696023  0.0025526887  0.0027228851  0.0028519035  0.0020403615
## [16]  0.0031210592  0.0033147045  0.0016019057  0.0027154352  0.0020650766
## [21]  0.0019910247  0.0013227771  0.0010625617  0.0016703705  0.0011926610
## [26]  0.0017282183  0.0014199320 -0.0003367671  0.0001032780 -0.0006716897
## [31] -0.0001293627 -0.0001735967 -0.0010106036 -0.0019948130 -0.0013175997
## [36] -0.0020524265 -0.0021841486 -0.0014203063 -0.0033652526 -0.0018658058
## [41] -0.0019651240 -0.0032610610 -0.0034475532 -0.0019896010 -0.0031529375
## [46] -0.0028797259 -0.0024275601 -0.0017607731 -0.0024364171 -0.0024975707
## [51] -0.0020134413 -0.0028183686 -0.0027985094 -0.0013049220 -0.0008067298
## [56] -0.0008213395 -0.0023787386 -0.0009052421 -0.0004955500 -0.0002188576
## [61]  0.0002770271 -0.0007517924 -0.0005058353  0.0011448821  0.0011371448
## [66] -0.0001160856  0.0016043210  0.0009338209  0.0006922295  0.0029474016
## [71]  0.0030623892  0.0014827048  0.0015305714  0.0014227206  0.0032505039
## [76]  0.0027313128  0.0016021939  0.0024662339  0.0024613575  0.0031946617
##
## iteration = 14
## Step:
##  [1]  2.092051e-04  2.606934e-04  2.732035e-04  5.583943e-04  4.049103e-04
##  [6]  1.410418e-04 -3.497041e-04 -2.216628e-04 -6.330130e-04  6.913971e-06
## [11] -2.154218e-04 -3.499820e-04 -4.478829e-04 -5.312522e-04 -2.232550e-04
## [16] -7.133701e-04 -8.194414e-04 -1.133981e-04 -5.976436e-04 -3.451307e-04
## [21] -3.382554e-04 -9.491594e-05 -4.509021e-05 -3.799774e-04 -2.670026e-04
## [26] -5.758499e-04 -5.152891e-04  1.617497e-04 -9.452173e-05  1.431986e-04
## [31] -1.822794e-04 -2.583109e-04  9.762713e-06  3.394981e-04 -3.507979e-05
## [36]  1.840661e-04  1.546104e-04 -2.376654e-04  5.247728e-04 -1.465714e-04
## [41] -1.214704e-04  4.344362e-04  5.303931e-04 -6.040618e-05  4.640187e-04
## [46]  3.931722e-04  2.502236e-04  3.114580e-05  3.930823e-04  5.015885e-04
## [51]  3.755929e-04  7.788252e-04  8.095192e-04  1.982105e-04  8.787272e-06
## [56]  5.755478e-05  7.648740e-04  1.933110e-04  7.198912e-05  2.632167e-05
## [61] -8.643969e-05  4.583117e-04  4.638547e-04 -1.263556e-04 -8.686702e-06
## [66]  6.350145e-04  1.557593e-05  3.863645e-04  5.559364e-04 -3.416150e-04
## [71] -3.298598e-04  4.054699e-04  4.388083e-04  5.006976e-04 -2.748541e-04
## [76] -6.492751e-05  3.998912e-04  1.307243e-05 -1.650753e-05 -3.472037e-04
## Parameter:
##  [1] -0.19918795 -0.00016136  0.19886333  0.39751403  0.59628312  0.79481297
##  [7]  0.99315666  1.19055445  1.38792452  1.58384157  1.77981757  1.97496677
## [13]  2.16927432  2.36269578  2.55483405  2.74661650  2.93699608  3.12553520
## [19]  3.31382120  3.50015099  3.68527987  3.86870939  4.05071235  4.23138091
## [25]  4.40996237  4.58714085  4.76219843  4.93484392  5.10628143  5.27522751
## [31]  5.44251840  5.60742532  5.76981718  5.92984655  6.08808408  6.24338426
## [37]  6.39640317  6.54719177  6.69441595  6.84019346  6.98269056  7.12199600
## [43]  7.25884923  7.39338298  7.52404183  7.65220549  7.77737193  7.89951457
## [49]  8.01803581  8.13356930  8.24603415  8.35472855  8.46034428  8.56307213
## [55]  8.66202258  8.75734595  8.84862685  8.93743334  9.02228793  9.10350278
## [61]  9.18117535  9.25464866  9.32486465  9.39183933  9.45447718  9.51289110
## [67]  9.56853602  9.61948856  9.66672065  9.71094884  9.75054074  9.78563975
## [73]  9.81737921  9.84509606  9.86953272  9.88918564  9.90466161  9.91686384
## [79]  9.92477985  9.92899261
## Function Value
## [1] -2499.575
```

```
## Gradient:
##  [1] -8.128249e-04 -3.113529e-04  1.875802e-04  3.064013e-04  7.836141e-04
##  [6]  1.260204e-03  1.907334e-03  1.846497e-03  2.365415e-03  1.725616e-03
## [11]  1.980833e-03  2.088612e-03  2.130662e-03  2.150069e-03  1.734046e-03
## [16]  2.179477e-03  2.233434e-03  1.429277e-03  1.914266e-03  1.586499e-03
## [21]  1.517647e-03  1.159792e-03  9.600207e-04  1.131438e-03  7.955046e-04
## [26]  9.279640e-04  6.933298e-04 -1.938724e-04 -8.951099e-05 -5.622755e-04
## [31] -4.449776e-04 -5.922034e-04 -1.087963e-03 -1.651292e-03 -1.462563e-03
## [36] -1.919164e-03 -2.093608e-03 -1.840949e-03 -2.806020e-03 -2.177995e-03
## [41] -2.249433e-03 -2.832079e-03 -2.899029e-03 -2.208129e-03 -2.698364e-03
## [46] -2.520537e-03 -2.256665e-03 -1.876355e-03 -2.088147e-03 -2.012523e-03
## [51] -1.694188e-03 -1.981277e-03 -1.924266e-03 -1.222181e-03 -9.705721e-04
## [56] -9.241652e-04 -1.570073e-03 -8.362226e-04 -5.846409e-04 -3.683238e-04
## [61] -1.944686e-05 -3.462165e-04 -9.429914e-05  7.932704e-04  9.364652e-04
## [66]  5.134735e-04  1.433246e-03  1.240887e-03  1.217789e-03  2.313674e-03
## [71]  2.443464e-03  1.812825e-03  1.903406e-03  1.875369e-03  2.701766e-03
## [76]  2.453680e-03  1.924798e-03  2.289488e-03  2.246603e-03  2.553031e-03
##
## iteration = 15
## Step:
##  [1]  5.077537e-04  4.156724e-04  2.763568e-04  5.883881e-04  2.512588e-04
##  [6] -2.216071e-04 -1.025686e-03 -8.524966e-04 -1.519748e-03 -5.400936e-04
## [11] -8.943144e-04 -1.095565e-03 -1.231625e-03 -1.342833e-03 -8.396411e-04
## [16] -1.581093e-03 -1.730512e-03 -6.188836e-04 -1.364856e-03 -9.563670e-04
## [21] -9.289826e-04 -5.224478e-04 -4.021783e-04 -8.684106e-04 -6.287814e-04
## [26] -1.050894e-03 -9.063884e-04  1.957733e-04 -1.527336e-04  2.826350e-04
## [31] -1.548328e-04 -2.046014e-04  2.754371e-04  8.520453e-04  3.327846e-04
## [36]  7.409410e-04  7.572655e-04  1.970390e-04  1.429070e-03  4.112342e-04
## [41]  4.637578e-04  1.325431e-03  1.463988e-03  5.265074e-04  1.321506e-03
## [46]  1.180416e-03  9.244264e-04  5.393388e-04  1.049792e-03  1.160820e-03
## [51]  9.093156e-04  1.493609e-03  1.514547e-03  5.487961e-04  2.395435e-04
## [56]  2.860152e-04  1.353944e-03  4.275368e-04  2.021968e-04  8.074738e-05
## [61] -1.636205e-04  6.071064e-04  5.378875e-04 -4.582307e-04 -3.563460e-04
## [66]  5.654305e-04 -4.765231e-04  3.901011e-05  2.552677e-04 -1.182796e-03
## [71] -1.207417e-03 -1.098386e-04 -9.602139e-05 -1.102801e-05 -1.217098e-03
## [76] -8.829824e-04 -1.496507e-04 -7.367633e-04 -7.600870e-04 -1.260449e-03
## Parameter:
##  [1] -0.1986801950  0.0002543125  0.1991396840  0.3981024201  0.5965343751
##  [6]  0.7945913666  0.9921309775  1.1897019564  1.3864047744  1.5833014755
## [11]  1.7789232526  1.9738712048  2.1680426939  2.3613529469  2.5539944062
## [16]  2.7450354073  2.9352655696  3.1249163140  3.3124563477  3.4991946267
## [21]  3.6843508855  3.8681869465  4.0503101760  4.2305124949  4.4093335854
## [26]  4.5860899611  4.7612920379  4.9350396915  5.1061286916  5.2755101432
## [31]  5.4423635641  5.6072207148  5.7700926185  5.9306985988  6.0884168634
## [36]  6.2441252005  6.3971604312  6.5473888139  6.6958450214  6.8406046917
## [41]  6.9831543219  7.1233214322  7.2603132179  7.3939094875  7.5253633374
## [46]  7.6533859076  7.7782963579  7.9000539094  8.0190856022  8.1347301169
## [51]  8.2469434677  8.3562221573  8.4618588270  8.5636209307  8.6622621281
## [56]  8.7576319667  8.8499807911  8.9378608815  9.0224901231  9.1035835264
## [61]  9.1810117287  9.2552557666  9.3254025412  9.3913810990  9.4541208301
## [66]  9.5134565332  9.5680594942  9.6195275736  9.6669759216  9.7097660481
## [71]  9.7493333198  9.7855299081  9.8172831927  9.8450850303  9.8683156177
## [76]  9.8883026618  9.9045119564  9.9161270798  9.9240197668  9.9277321574
## Function Value
```

```
## [1] -2499.575
## Gradient:
##  [1] -1.593152e-04  2.256703e-04  5.473384e-04  1.071866e-03  1.116826e-03
##  [6]  9.860181e-04  5.979393e-04  7.633154e-04  4.227403e-04  1.049321e-03
## [11]  8.481995e-04  6.967090e-04  5.631054e-04  4.383127e-04  6.707746e-04
## [16]  1.570400e-04  1.550947e-05  6.433673e-04  1.613207e-04  3.565753e-04
## [21]  3.181279e-04  4.797969e-04  4.296239e-04 -6.920629e-06 -3.978163e-05
## [26] -4.590859e-04 -5.143509e-04  1.397434e-05 -3.392921e-04 -2.575398e-04
## [31] -7.126775e-04 -9.317809e-04 -8.153818e-04 -6.417076e-04 -1.130637e-03
## [36] -1.067147e-03 -1.227089e-03 -1.704043e-03 -1.084272e-03 -1.775935e-03
## [41] -1.784687e-03 -1.259548e-03 -1.151793e-03 -1.674924e-03 -1.141758e-03
## [46] -1.148625e-03 -1.217207e-03 -1.335579e-03 -8.893571e-04 -6.709569e-04
## [51] -6.773862e-04 -2.094985e-04 -1.242016e-04 -6.671640e-04 -8.125784e-04
## [56] -7.037234e-04  3.197665e-05 -4.271608e-04 -4.630549e-04 -3.999554e-04
## [61] -3.629808e-04  3.090597e-04  4.754742e-04  8.090859e-05  3.595883e-04
## [66]  1.130628e-03  7.089483e-04  1.186083e-03  1.446044e-03  6.888848e-04
## [71]  7.902065e-04  1.579576e-03  1.690642e-03  1.774952e-03  1.046534e-03
## [76]  1.231854e-03  1.651431e-03  1.259406e-03  1.187449e-03  8.484096e-04
##
## iteration = 16
## Step:
##  [1]  1.819711e-04 -3.043106e-06 -1.715141e-04 -3.253256e-04 -4.197706e-04
##  [6] -4.695212e-04 -4.837192e-04 -5.153321e-04 -5.195431e-04 -5.677847e-04
## [11] -5.618087e-04 -5.429241e-04 -5.173101e-04 -4.901726e-04 -4.781312e-04
## [16] -4.267544e-04 -4.019366e-04 -4.232989e-04 -3.874236e-04 -3.822241e-04
## [21] -3.629309e-04 -3.443058e-04 -2.995564e-04 -2.214561e-04 -1.576835e-04
## [26] -7.746324e-05 -2.591757e-05 -9.079768e-06  5.891755e-05  1.168539e-04
## [31]  2.083456e-04  2.858992e-04  3.388919e-04  3.890923e-04  4.766397e-04
## [36]  5.362797e-04  6.035773e-04  6.770720e-04  6.841286e-04  7.469467e-04
## [41]  7.582031e-04  7.229723e-04  7.040727e-04  7.135708e-04  6.605811e-04
## [46]  6.285509e-04  5.973476e-04  5.586408e-04  4.797609e-04  4.088240e-04
## [51]  3.528596e-04  2.819843e-04  2.485077e-04  2.624152e-04  2.539177e-04
## [56]  2.166823e-04  1.409145e-04  1.284937e-04  9.268158e-05  3.797836e-05
## [61] -3.307077e-05 -1.476648e-04 -2.344971e-04 -2.894373e-04 -3.856796e-04
## [66] -5.083150e-04 -5.617656e-04 -6.497852e-04 -7.120638e-04 -7.108134e-04
## [71] -7.593603e-04 -8.502375e-04 -8.943944e-04 -9.110601e-04 -8.700607e-04
## [76] -8.740256e-04 -8.885788e-04 -8.525682e-04 -8.268506e-04 -7.941072e-04
## Parameter:
##  [1] -0.1984982239  0.0002512694  0.1989681699  0.3977770946  0.5961146045
##  [6]  0.7941218454  0.9916472584  1.1891866243  1.3858852313  1.5827336908
## [11]  1.7783614439  1.9733282807  2.1675253838  2.3608627744  2.5535162750
## [16]  2.7446086529  2.9348636330  3.1244930151  3.3120689241  3.4988124026
## [21]  3.6839879546  3.8678426407  4.0500106196  4.2302910388  4.4091759020
## [26]  4.5860124978  4.7612661204  4.9350306118  5.1061876091  5.2756269971
## [31]  5.4425719096  5.6075066140  5.7704315104  5.9310876911  6.0888935031
## [36]  6.2446614801  6.3977640086  6.5480658859  6.6965291499  6.8413516384
## [41]  6.9839125250  7.1240444045  7.2610172906  7.3946230582  7.5260239186
## [46]  7.6540144585  7.7788937055  7.9006125502  8.0195653631  8.1351389409
## [51]  8.2472963273  8.3565041416  8.4621073347  8.5638833459  8.6625160458
## [56]  8.7578486490  8.8501217056  8.9379893751  9.0225828047  9.1036215048
## [61]  9.1809786579  9.2551081018  9.3251680441  9.3910916617  9.4537351505
## [66]  9.5129482182  9.5674977286  9.6188777885  9.6662638579  9.7090552346
## [71]  9.7485739594  9.7846796707  9.8163887982  9.8441739702  9.8674455570
## [76]  9.8874286362  9.9036233776  9.9152745116  9.9231929162  9.9269380501
```

```
## Function Value
## [1] -2499.575
## Gradient:
##  [1]  7.111921e-05  2.218405e-04  3.306041e-04  6.610734e-04  5.884565e-04
##  [6]  3.975650e-04 -4.936846e-06  1.232920e-04 -2.193250e-04  3.481126e-04
## [11]  1.575449e-04  3.298857e-05 -6.527114e-05 -1.530200e-04  9.674756e-05
## [16] -3.491530e-04 -4.573570e-04  1.440058e-04 -2.908708e-04 -8.824144e-05
## [21] -1.013183e-04  8.467591e-05  9.237973e-05 -2.433860e-04 -1.940740e-04
## [26] -5.100576e-04 -4.990608e-04  5.070866e-05 -2.150773e-04 -5.879494e-05
## [31] -3.960252e-04 -5.151059e-04 -3.302439e-04 -9.153577e-05 -4.670202e-04
## [36] -3.258616e-04 -3.979762e-04 -7.787954e-04 -1.483051e-04 -7.569220e-04
## [41] -7.488070e-04 -2.664263e-04 -1.799767e-04 -6.873960e-04 -2.188135e-04
## [46] -2.628708e-04 -3.672512e-04 -5.308043e-04 -1.813372e-04 -4.919206e-05
## [51] -1.223366e-04  2.597756e-04  3.077268e-04 -2.112806e-04 -3.615264e-04
## [56] -2.944750e-04  3.497917e-04 -1.189354e-04 -1.945476e-04 -1.955105e-04
## [61] -2.436871e-04  2.869707e-04  3.477626e-04 -1.113783e-04  4.925925e-05
## [66]  6.679881e-04  1.830437e-04  5.520072e-04  7.366946e-04 -1.409567e-05
## [71]  2.894507e-05  7.049967e-04  7.626185e-04  8.285215e-04  1.556682e-04
## [76]  3.380637e-04  7.405970e-04  3.963426e-04  3.584316e-04  6.207608e-05
##
## iteration = 17
## Parameter:
##  [1] -0.198507609  0.000127104  0.198755715  0.397355297  0.595719130
##  [6]  0.793826683  0.991576702  1.189040997  1.385931840  1.582456264
## [11]  1.778192405  1.973232720  2.167489394  2.360880285  2.553395731
## [16]  2.744744940  2.935062980  3.124350581  3.312172916  3.498801067
## [21]  3.683983794  3.867734061  4.049901521  4.230378833  4.409242789
## [26]  4.586265004  4.761516272  4.934971215  5.106282924  5.275639595
## [31]  5.442783233  5.607792363  5.770617609  5.931143668  6.089169038
## [36]  6.244863390  6.398012929  6.548535465  6.696643268  6.841812227
## [41]  6.984366399  7.124218810  7.261136562  7.395024189  7.526151324
## [46]  7.654158433  7.779088488  7.900890175  8.019632600  8.135118868
## [51]  8.247306592  8.356288353  8.461859150  8.563928126  8.662644227
## [56]  8.757933973  8.849835323  8.937964172  9.022595410  9.103627009
## [61]  9.181000572  9.254815630  9.324828574  9.391001231  9.453540404
## [66]  9.512388713  9.567201902  9.618362796  9.665637134  9.708850231
## [71]  9.748338419  9.784052404  9.815723206  9.843470277  9.867126290
## [76]  9.887008103  9.902976550  9.914827097  9.922771879  9.926688628
## Function Value
## [1] -2499.575
## Gradient:
##  [1]  5.620214e-05  6.149941e-05  5.898236e-05  1.217435e-04  8.551965e-05
##  [6]  2.635983e-05 -8.406108e-05 -5.076372e-05 -1.432109e-04  7.752154e-06
## [11] -4.103928e-05 -6.898477e-05 -8.864378e-05 -1.056473e-04 -3.248496e-05
## [16] -1.451309e-04 -1.702282e-04 -8.433572e-06 -1.233919e-04 -6.792759e-05
## [21] -6.998696e-05 -1.708031e-05 -8.351579e-06 -8.822836e-05 -6.423019e-05
## [26] -1.388030e-04 -1.289981e-04  2.302586e-05 -4.103110e-05  1.043394e-05
## [31] -6.817551e-05 -8.903666e-05 -3.059498e-05  4.244173e-05 -4.713920e-05
## [36]  1.475250e-06 -7.301677e-06 -1.004215e-04  7.406159e-05 -8.409047e-05
## [41] -8.134562e-05  4.368837e-05  6.256854e-05 -7.719523e-05  4.181603e-05
## [46]  2.311448e-05 -1.162242e-05 -6.406497e-05  1.800420e-05  4.171900e-05
## [51]  1.205195e-05  1.065670e-04  1.168821e-04 -1.974681e-05 -5.820462e-05
## [56] -4.233710e-05  1.261427e-04 -1.266791e-06 -2.401420e-05 -3.009981e-05
## [61] -5.303659e-05  7.571242e-05  7.972561e-05 -5.409348e-05 -2.447289e-05
```

```
## [66]  1.268260e-04 -1.432892e-05  7.476803e-05  1.180534e-04 -8.560825e-05
## [71] -7.955136e-05  9.298605e-05  1.031197e-04  1.214814e-04 -5.366160e-05
## [76] -9.029322e-07  1.103263e-04  2.415414e-05  2.065122e-05 -5.396554e-05
##
## Relative gradient close to zero.
## Current iterate is probably solution.
##
## Post processing for method  nlm
## Successful convergence!
## Method:  nlminb
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
##   0:    -1399.7627: 0.0378826 0.265317 0.442313 0.884269 0.843264 0.660107 0.223223 0.421135 0.05517
##   1:    -2484.6770: -0.0215222 0.0241462 0.0650472 0.129799 0.149977 0.156656 0.139791 0.180612 0.16
##   2:    -2499.3695: -0.0213533 0.000943246 0.0230458 0.0460467 0.0672715 0.0879155 0.107587 0.129438
##   3:    -2499.5615: -0.0208614 0.000163679 0.0211562 0.0422901 0.0631139 0.0838188 0.104333 0.125173
##   4:    -2499.5695: -0.0207965 9.90501e-05 0.0209756 0.0419311 0.0626986 0.0833861 0.103948 0.124683
##   5:    -2499.5727: -0.0207423 0.000283300 0.0212487 0.0424703 0.0631478 0.0836303 0.103799 0.124587
##   6:    -2499.5747: -0.0206924 -3.12116e-07 0.0206936 0.0413706 0.0620472 0.0827038 0.103336 0.12390
##   7:    -2499.5747: -0.0206906 8.79463e-06 0.0207081 0.0413996 0.0620736 0.0827225 0.103338 0.123913
##   8:    -2499.5747: -0.0206906 8.79463e-06 0.0207081 0.0413996 0.0620736 0.0827225 0.103338 0.123913
## Post processing for method  nlminb
## Successful convergence!
## Method:  nvm
## parchanged =  FALSE
## Parameter scaling: [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [39] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## [77] 1 1 1 1
## nvm -- J C Nash 2009-2015 - an R implementation of Alg 21
## Problem of size n= 80
## Initial fn= -1399.763
## ig= 1   gnorm= 477.3902     1   1   -1399.763
## **ig= 2   gnorm= 113.3571     4   2   -1774.233
## ig= 3   gnorm= 14.82658     5   3   -2154.016
## ig= 4   gnorm= 14.754     6   4   -2169.486
## *ig= 5   gnorm= 14.57398     8   5   -2218.708
## *ig= 6   gnorm= 14.47004     10   6   -2257.172
## ig= 7   gnorm= 18.39418     11   7   -2261.356
## ig= 8   gnorm= 13.60833     12   8   -2341.328
## ig= 9   gnorm= 11.98564     13   9   -2395.325
## *ig= 10   gnorm= 5.124423     15   10   -2489.874
## ig= 11   gnorm= 3.559244     16   11   -2493.571
## ig= 12   gnorm= 0.7171531     17   12   -2499.357
## ig= 13   gnorm= 0.1668112     18   13   -2499.563
## ig= 14   gnorm= 0.01007066     19   14   -2499.575
## ig= 15   gnorm= 0.001835856     20   15   -2499.575
## ig= 16   gnorm= 8.800109e-05     21   16   -2499.575
## ig= 17   gnorm= 3.395708e-06     22   17   -2499.575
## ig= 18   gnorm= 7.651999e-08     23   18   -2499.575
## ig= 19   gnorm= 1.07111e-08     24   19   -2499.575
## *********No acceptable point
## Converged
```

```
## Seem to be done nvm
## Post processing for method  nvm
## Successful convergence!
## Above for n= 80
## opm: wrapper to call optimr to run multiple optimizers
## Method:  L-BFGS-B
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## final  value -3934.277448
## converged
## Post processing for method  L-BFGS-B
## Successful convergence!
## Method:  BFGS
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## initial  value -2237.154642
## iter  10 value -3934.235073
## final  value -3934.277448
## converged
## Post processing for method  BFGS
## Successful convergence!
## Method:  ncg
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## ncg -- J C Nash 2023 - bounds constraint version of new CG
## an R implementation of Alg 22 with Yuan/Dai modification
## stepredn = 0.2
## Initial function value= -2237.155
## Initial fn= -2237.155
## 1   0   1   -2237.155   last decrease= NA
## **5   1   2   -2907.599   last decrease= 670.4447
## 7   2   3   -3905.625   last decrease= 998.0256
## Yuan/Dai cycle reset
## 7   3   1   -3905.625   last decrease= NA
## 9   4   2   -3934.209   last decrease= 28.58366
## 11   5   3   -3934.277   last decrease= 0.06879172
## 13   6   4   -3934.277   last decrease= 3.888317e-05
## 15   7   5   -3934.277   last decrease= 2.000888e-09
## Very small gradient -- gradsqr = 6.09626569329027e-12
## ncg seems to have converged
## Post processing for method  ncg
## Successful convergence!
## Method:  spg
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## iter:  0  f-value:  -2237.155  pgrad:  148.8531
```

```
## Post processing for method  spg
## Successful convergence!
## Method:  ucminf
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## ucminf message: Stopped by zero step from line search
## Post processing for method  ucminf
## Successful convergence!
## Method:  nlm
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## iteration = 0
## Step:
##   [1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
##  [38] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
##  [75] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
## Parameter:
##   [1] 0.1585359152 0.4762453921 0.4380232764 0.1701498171 0.4801218351
##   [6] 0.3647829883 0.7646687499 0.9443224974 0.7350596376 0.2619442386
##  [11] 0.0002287955 0.5986175796 0.2814716145 0.0647890628 0.8922212212
##  [16] 0.0952049091 0.6640563395 0.6174164074 0.4844931453 0.6459475018
##  [21] 0.7642707105 0.6783028583 0.8785211504 0.4857212624 0.3298169356
##  [26] 0.5865077251 0.7045470248 0.1184838787 0.0158197992 0.7622758329
##  [31] 0.8002777502 0.2838014218 0.4090248127 0.3313423467 0.4931035615
##  [36] 0.4802794480 0.4947820590 0.4936057834 0.9300969839 0.2912841630
##  [41] 0.5297782614 0.4573613480 0.2097127903 0.6164930279 0.5734564762
##  [46] 0.3283607832 0.8535736294 0.8191099835 0.3021842327 0.0740116104
##  [51] 0.6063881246 0.1441132606 0.4222857542 0.5606543086 0.3289747741
##  [56] 0.6875254167 0.3105044353 0.6699969911 0.8799084169 0.0498549487
##  [61] 0.1270133492 0.0340392077 0.8018223653 0.8176605711 0.0601480748
##  [66] 0.4157611448 0.3224741688 0.2237361851 0.6135258093 0.0135858178
##  [71] 0.2777832630 0.0183129238 0.3623589480 0.3048335123 0.6231244437
##  [76] 0.6597636011 0.3536810330 0.0404737038 0.9938483685 0.0013091320
##  [81] 0.1616805047 0.1147503117 0.0698698778 0.8970386917 0.5911166149
##  [86] 0.6950782521 0.2318912477 0.9603010069 0.0008434469 0.7129498015
##  [91] 0.5858605648 0.3815900665 0.1967063388 0.4274506147 0.1706040257
##  [96] 0.9911640761 0.6013529962 0.2046215783 0.7430194425 0.9297521936
## Function Value
## [1] -2237.155
## Gradient:
##   [1]   27.8115394   74.0270131   64.9582595   20.2285487   65.3169023
##   [6]   44.3386731  103.4597783  128.4065500   92.9752534   16.5882519
##  [11]  -26.9256734   63.2340794   11.1470225  -25.3068757  100.5155028
##  [16]  -26.0648039   59.6387395   49.7314628   26.4573363   48.9677883
##  [21]   64.8198990   48.9751221   77.6553165   14.2322874  -12.3396894
##  [26]   25.2350071   41.3018727  -52.0108158  -70.1874831   43.6035083
##  [31]   47.3426419  -35.0371792  -17.6670845  -31.8032961   -8.7031721
##  [36]  -12.7020546  -12.4205413  -14.5413691   51.3925023  -49.7280335
##  [41]  -14.4810859  -27.5177402  -67.7563339   -6.2718657  -14.6531023
##  [46]  -54.3976190   25.5987024   18.6718861  -63.1801397 -100.1343973
```

```
##  [51]  -18.8822662  -92.1723673  -50.3753365  -30.2766639  -67.6502740
##  [56]  -13.2699999  -73.1612665  -18.5652090   12.8301857 -117.3400629
##  [61] -106.5119180 -122.1162020   -3.9429595   -2.5779818 -121.3469563
##  [66]  -67.1146093  -82.6194482  -98.9490366  -39.3382299 -133.5042400
##  [71]  -93.3688194 -134.6002558  -82.0345278  -91.8540956  -43.2447207
##  [76]  -38.3651135  -86.7039529 -136.1254994   11.3024389 -143.6260022
##  [81] -119.3739945 -127.3286535 -134.9567689   -7.0499208  -55.1807628
##  [86]  -39.5681569 -112.0509985    0.6750758 -148.8530833  -38.5939826
##  [91]  -58.7094416  -90.7793406 -119.8096991  -84.2312816 -124.4023962
##  [96]    2.8842105  -57.8708871 -119.6597000  -36.1005549   -7.1440100
##
## iteration = 1
## Step:
##   [1] -1.10141698 -2.93168270 -2.57253397 -0.80110873 -2.58673725 -1.75593902
##   [7] -4.09730488 -5.08526882 -3.68208753 -0.65694250  1.06633413 -2.50425147
##  [13] -0.44145416  1.00222509 -3.98070310  1.03224122 -2.36186567 -1.96950901
##  [19] -1.04778664 -1.93926530 -2.56705449 -1.93955574 -3.07537395 -0.56363953
##  [25]  0.48868720 -0.99937888 -1.63567299  2.05977794  2.77962627 -1.72682438
##  [31] -1.87490483  1.38757310  0.69966738  1.25950202  0.34467065  0.50303791
##  [37]  0.49188917  0.57588005 -2.03529095  1.96937320  0.57349267  1.08978168
##  [43]  2.68334576  0.24838393  0.58030501  2.15430222 -1.01378226 -0.73946041
##  [49]  2.50211531  3.96561023  0.74779207  3.65029093  1.99500826  1.19904300
##  [55]  2.67914549  0.52553018  2.89739664  0.73523569 -0.50811227  4.64700409
##  [61]  4.21817839  4.83615294  0.15615254  0.10209550  4.80568859  2.65793162
##  [67]  3.27196785  3.91866654  1.55790708  5.28715201  3.69767388  5.33055739
##  [73]  3.24880332  3.63768646  1.71261536  1.51936887  3.43372599  5.39096142
##  [79] -0.44760910  5.68800291  4.72755363  5.04258101  5.34467633  0.27919715
##  [85]  2.18531696  1.56701286  4.43754192 -0.02673495  5.89501036  1.52843275
##  [91]  2.32506280  3.59512307  4.74480879  3.33580109  4.92669281 -0.11422304
##  [97]  2.29185363  4.73886839  1.42968584  0.28292335
## Parameter:
##   [1] -0.94288106 -2.45543731 -2.13451069 -0.63095891 -2.10661542 -1.39115603
##   [7] -3.33263613 -4.14094632 -2.94702789 -0.39499826  1.06656293 -1.90563390
##  [13] -0.15998255  1.06701415 -3.08848188  1.12744613 -1.69780933 -1.35209260
##  [19] -0.56329349 -1.29331780 -1.80278378 -1.26125288 -2.19685280 -0.07791827
##  [25]  0.81850414 -0.41287116 -0.93112596  2.17826182  2.79544607 -0.96454855
##  [31] -1.07462708  1.67137452  1.10869219  1.59084437  0.83777421  0.98331735
##  [37]  0.98667123  1.06948584 -1.10519397  2.26065736  1.10327093  1.54714303
##  [43]  2.89305855  0.86487696  1.15376149  2.48266300 -0.16020864  0.07964957
##  [49]  2.80429954  4.03962184  1.35418019  3.79440419  2.41729401  1.75969731
##  [55]  3.00812026  1.21305559  3.20790108  1.40523268  0.37179615  4.69685904
##  [61]  4.34519174  4.87019214  0.95797490  0.91975607  4.86583667  3.07369277
##  [67]  3.59444202  4.14240272  2.17143289  5.30073782  3.97545714  5.34887031
##  [73]  3.61116227  3.94251997  2.33573980  2.17913247  3.78740702  5.43143512
##  [79]  0.54623927  5.68931204  4.88923414  5.15733132  5.41454621  1.17623584
##  [85]  2.77643358  2.26209111  4.66943317  0.93356606  5.89585381  2.24138255
##  [91]  2.91092336  3.97671313  4.94151513  3.76325171  5.09729683  0.87694104
##  [97]  2.89320662  4.94348997  2.17270528  1.21267554
## Function Value
## [1] -2498.356
## Gradient:
##   [1]  -5.3459150 -14.9294000 -13.3727486  -4.6318757 -14.0037539 -10.0605220
##   [7] -22.2727546 -27.6043320 -20.7740377  -5.6946854   2.7545252 -15.7473856
##  [13]  -5.5731764   1.4476358 -24.2502239   0.9359883 -16.6798434 -15.0228234
```

```
##  [19] -10.6755080 -15.5628014 -19.1124253 -16.2772152 -22.4248558 -10.0087088
##  [25]  -5.0246519 -12.9739177 -16.5889841   1.8468607   5.1373967 -18.1750221
##  [31] -19.3012060  -3.0673137  -6.9435151  -4.4652547  -9.4922886  -9.0544289
##  [37]  -9.4785721  -9.4172545 -23.0771012  -3.0576212 -10.5303203  -8.2659557
##  [43]  -0.5141176 -13.2670583 -11.9319756  -4.2716080 -20.7504131 -19.7049061
##  [49]  -3.5536495   3.5505098 -13.1706167   1.2715207  -7.4834541 -11.8584732
##  [55]  -4.6421251 -15.9204330  -4.1561506 -15.4699172 -22.1039443   3.8383441
##  [61]   1.3612744   4.2242279 -19.8757090 -20.4224871   3.2543429  -7.9408139
##  [67]  -5.0679367  -2.0208822 -14.2768321   4.4781157  -3.8343618   4.2687518
##  [73]  -6.5282489  -4.7381831 -14.7208515 -15.8819356  -6.3086364   3.4912392
##  [79] -26.3881416   4.6976967  -0.3322345   1.1433949   2.5654278 -23.3285679
##  [85] -13.7262184 -16.9718286  -2.4504144 -25.2634495   4.8036449 -17.4979371
##  [91] -13.5075101  -7.1011162  -1.2989634  -8.5129083  -0.4448530 -26.1307200
##  [97] -13.9007589  -1.4569513 -18.3093603 -24.1485300
##
## iteration = 2
## Step:
##   [1]  0.166860429  0.470061265  0.422559769  0.148901417  0.445526810
##   [6]  0.322967052  0.709079873  0.878614619  0.665307953  0.192333938
##  [11] -0.071883447  0.512429933  0.193922038 -0.025296019  0.785606660
##  [16] -0.005499921  0.550984772  0.500704474  0.365748898  0.521556178
##  [21]  0.635274699  0.548005193  0.743584395  0.354733182  0.199886710
##  [26]  0.452234598  0.568136399 -0.010202621 -0.111725357  0.624297665
##  [31]  0.661820577  0.152795095  0.276909496  0.200947421  0.361270011
##  [36]  0.349528755  0.364921273  0.365023390  0.797123819  0.168873218
##  [41]  0.406176625  0.336905597  0.094849680  0.498342518  0.458277242
##  [46]  0.219055990  0.739818251  0.708849665  0.202293941 -0.019450909
##  [51]  0.508880273  0.056039525  0.333501394  0.473035833  0.247605438
##  [56]  0.604436139  0.235774171  0.593676594  0.804230435 -0.010859611
##  [61]  0.068759642 -0.019806376  0.740478028  0.759199893  0.015251145
##  [66]  0.369167982  0.280123187  0.185550789  0.572734706 -0.016467055
##  [71]  0.246466720 -0.007487290  0.333549163  0.278243585  0.593549115
##  [76]  0.631088467  0.330640604  0.023013235  0.964598451 -0.013274026
##  [81]  0.145867060  0.100119573  0.055991590  0.871825628  0.570069213
##  [86]  0.672797025  0.216117167  0.934837780 -0.011349262  0.691190675
##  [91]  0.565917798  0.364548683  0.182155878  0.409505848  0.155663815
##  [96]  0.964484089  0.579542935  0.187838889  0.718450684  0.902296937
## Parameter:
##   [1] -0.77602063 -1.98537604 -1.71195092 -0.48205749 -1.66108861 -1.06818898
##   [7] -2.62355626 -3.26233170 -2.28171994 -0.20266432  0.99467948 -1.39320396
##  [13]  0.03393949  1.04171813 -2.30287522  1.12194621 -1.14682456 -0.85138813
##  [19] -0.19754460 -0.77176162 -1.16750908 -0.71324769 -1.45326841  0.27681492
##  [25]  1.01839085  0.03936344 -0.36298956  2.16805920  2.68372071 -0.34025088
##  [31] -0.41280650  1.82416962  1.38560168  1.79179179  1.19904423  1.33284611
##  [37]  1.35159250  1.43450923 -0.30807015  2.42953058  1.50944755  1.88404863
##  [43]  2.98790823  1.36321947  1.61203873  2.70171899  0.57960962  0.78849924
##  [49]  3.00659348  4.02017093  1.86306047  3.85044372  2.75079541  2.23273314
##  [55]  3.25572570  1.81749173  3.44367525  1.99890927  1.17602658  4.68599943
##  [61]  4.41395139  4.85038577  1.69845293  1.67895596  4.88108781  3.44286075
##  [67]  3.87456521  4.32795351  2.74416760  5.28427077  4.22192386  5.34138302
##  [73]  3.94471143  4.22076356  2.92928892  2.81022094  4.11804762  5.45444836
##  [79]  1.51083772  5.67603802  5.03510120  5.25745090  5.47053780  2.04806147
##  [85]  3.34650279  2.93488813  4.88555033  1.86840384  5.88450454  2.93257323
##  [91]  3.47684116  4.34126182  5.12367100  4.17275755  5.25296065  1.84142513
```

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##  [97]  3.47274956  5.13132886  2.89115597  2.11497248
## Function Value
## [1] -3052.553
## Gradient:
##   [1]  -4.8848068 -13.6340046 -12.2064027  -4.2152219 -12.7656101  -9.1530663
##   [7] -20.2936481 -25.1465080 -18.8874552  -5.0912542   2.6503615 -14.2231568
##  [13]  -4.9045566   1.5346205 -21.9087038   1.1238613 -14.9351624 -13.3900605
##  [19]  -9.3859316 -13.8145231 -17.0194198 -14.3904985 -19.9635241  -8.5792579
##  [25]  -3.9817920 -11.1962736 -14.4510267   2.4351239   5.4878293 -15.7577328
##  [31] -16.7386486  -1.8624740  -5.3533764  -3.0407679  -7.5831676  -7.1341361
##  [37]  -7.4721142  -7.3664351 -19.7919957  -1.4558301  -8.2290798  -6.1091965
##  [43]   1.0222611 -10.5746843  -9.3044268  -2.2569070 -17.2565312 -16.2494297
##  [49]  -1.4446805   5.0962197 -10.1248381   3.1157817  -4.8315260  -8.7792772
##  [55]  -2.1406034 -12.3942085  -1.6024043 -11.8890182 -17.9010737   5.8403575
##  [61]   3.6237121   6.2816798 -15.6883681 -16.1462376   5.5204563  -4.6641243
##  [67]  -2.0013784   0.8189772 -10.3388473   6.8255108  -0.7338677   6.6972691
##  [73]  -3.1361693  -1.4752465 -10.5679881 -11.6039988  -2.8361468   6.1371734
##  [79] -21.1315586   7.2809261   2.7056370   4.0692961   5.3811873 -18.2576048
##  [85]  -9.4763827 -12.4294463   0.8442816 -19.9816349   7.4892887 -12.8713879
##  [91]  -9.2193868  -3.3612839   1.9434723  -4.6418136   2.7286202 -20.7309063
##  [97]  -9.5603201   1.8052239 -13.5878227 -18.9221194
##
## iteration = 3
## Step:
##   [1]  0.85954122  2.41633066  2.16967153  0.75989942  2.28173636  1.64811753
##   [7]  3.62919400  4.49611391  3.39395584  0.95619326 -0.40851924  2.58912755
##  [13]  0.94499653 -0.18881150  3.97324932 -0.10118600  2.75232287  2.48605050
##  [19]  1.78406361  2.57660667  3.15236471  2.69433816  3.68998197  1.68050987
##  [25]  0.87384727  2.16052671  2.74548499 -0.23895249 -0.77185985  3.00135475
##  [31]  3.18324653  0.55494923  1.18168215  0.77985227  1.59283559  1.52119188
##  [37]  1.58899577  1.57814298  3.78810072  0.54640560  1.75462059  1.38671880
##  [43]  0.13058461  2.19345517  1.97591943  0.73447436  3.40021934  3.22921270
##  [49]  0.61294490 -0.53888869  2.16584360 -0.17357996  1.24184345  1.94834500
##  [55]  0.77860464  2.60245900  0.69651131  2.52603499  3.59795448 -0.60316853
##  [61] -0.20410359 -0.66914378  3.23027541  3.31718410 -0.51694527  1.29352468
##  [67]  0.82699673  0.33237355  2.31483605 -0.72217652  0.62221229 -0.69027024
##  [73]  1.05669915  0.76642149  2.38177727  2.56918706  1.01902452 -0.56772413
##  [79]  4.26825176 -0.76385425  0.05001246 -0.18887916 -0.41884089  3.77281474
##  [85]  2.21881342  2.74429926  0.39390835  4.08659621 -0.78012409  2.82980989
##  [91]  2.18401777  1.14712618  0.20812058  1.37609117  0.07059536  4.22868757
##  [97]  2.24945586  0.23555028  2.96369408  3.90909563
## Parameter:
##   [1] 0.08352059 0.43095462 0.45772061 0.27784193 0.62064776 0.57992855
##   [7] 1.00563774 1.23378221 1.11223590 0.75352894 0.58616025 1.19592359
##  [13] 0.97893602 0.85290664 1.67037409 1.02076021 1.60549831 1.63466237
##  [19] 1.58651901 1.80484505 1.98485563 1.98109047 2.23671356 1.95732479
##  [25] 1.89223812 2.19989015 2.38249543 1.92910671 1.91186086 2.66110387
##  [31] 2.77044003 2.37911884 2.56728383 2.57164406 2.79187981 2.85403799
##  [37] 2.94058827 3.01265220 3.48003058 2.97593618 3.26406814 3.27076743
##  [43] 3.11849284 3.55667464 3.58795816 3.43619335 3.97982895 4.01771194
##  [49] 3.61953838 3.48128224 4.02890406 3.67686375 3.99263886 4.18107814
##  [55] 4.03433034 4.41995073 4.14018656 4.52494427 4.77398106 4.08283089
##  [61] 4.20984780 4.18124199 4.92872835 4.99614006 4.36414255 4.73638543
##  [67] 4.70156194 4.66032705 5.05900365 4.56209425 4.84413615 4.65111278
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##  [73] 5.00141058 4.98718504 5.31106619 5.37940800 5.13707214 4.88672423
##  [79] 5.77908948 4.91218377 5.08511365 5.06857174 5.05169691 5.82087621
##  [85] 5.56531621 5.67918740 5.27945868 5.95500005 5.10438045 5.76238312
##  [91] 5.66085893 5.48838799 5.33179158 5.54884873 5.32355601 6.07011270
##  [97] 5.72220542 5.36687914 5.85485005 6.02406811
## Function Value
## [1] -3915.099
## Gradient:
##   [1]  0.88245664  2.18522503  1.86263322  0.49286123  1.77367428  1.11066216
##   [7]  2.81338488  3.51567646  2.44649093  0.17624475 -1.12286150  1.51917019
##  [13] -0.02921501 -1.11513639  2.58396363 -1.15347653  1.36907716  1.07672280
##  [19]  0.39450617  1.06545451  1.54446638  1.09423169  1.96176892  0.11947461
##  [25] -0.63368385  0.50554779  1.01359153 -1.70002819 -2.19972490  1.18946883
##  [31]  1.33752027 -1.04937668 -0.49495813 -0.86917860 -0.14548935 -0.21970069
##  [37] -0.16653593 -0.18301264  1.80882080 -1.12060219 -0.02925629 -0.36066015
##  [43] -1.49388871  0.37077977  0.17691554 -0.94042974  1.47264296  1.32649919
##  [49] -1.02542431 -2.05476976  0.39807842 -1.70550918 -0.41823421  0.22856935
##  [55] -0.81874555  0.83849227 -0.87231554  0.79163996  1.77329796 -2.00599197
##  [61] -1.63154865 -2.04072145  1.49062941  1.58010910 -1.87022925 -0.22300177
##  [67] -0.63363074 -1.07067599  0.72884914 -2.00605910 -0.78545318 -1.96734637
##  [73] -0.38835156 -0.65115903  0.80683529  0.97602213 -0.42308397 -1.85615807
##  [79]  2.51112457 -2.03433038 -1.30102710 -1.52183454 -1.73776238  2.03847259
##  [85]  0.62640116  1.09470871 -1.03378719  2.29661843 -2.10352225  1.15228519
##  [91]  0.56444633 -0.37578656 -1.22841751 -0.17956286 -1.36641366  2.38130895
##  [97]  0.58751539 -1.23648097  1.22233967  2.07253897
##
## iteration = 4
## Step:
##   [1] -0.0851358386 -0.2292189134 -0.2024247444 -0.0654641328 -0.2065799032
##   [6] -0.1434215567 -0.3283395920 -0.4078244795 -0.3008523916 -0.0674269379
##  [11]  0.0641807930 -0.2178907847 -0.0598992044  0.0496481170 -0.3430263174
##  [16]  0.0451933540 -0.2235948754 -0.1966667573 -0.1285558756 -0.2022758238
##  [21] -0.2555787585 -0.2110180286 -0.3044682125 -0.1131289453 -0.0359394061
##  [26] -0.1573015993 -0.2122191721  0.0712367514  0.1222490354 -0.2350322147
##  [31] -0.2518724460 -0.0023635254 -0.0615467626 -0.0232125850 -0.1000912186
##  [36] -0.0931269804 -0.0994352179 -0.0983341719 -0.3078994327 -0.0006305214
##  [41] -0.1153612422 -0.0806706620  0.0382290730 -0.1575856300 -0.1372096923
##  [46] -0.0197915203 -0.2729056681 -0.2571412807 -0.0096139997  0.0990604046
##  [51] -0.1579068598  0.0634380620 -0.0712331010 -0.1386753276 -0.0282269858
##  [56] -0.2016574392 -0.0214490480 -0.1954699103 -0.2977229464  0.0999596064
##  [61]  0.0614937658  0.1050794174 -0.2651205134 -0.2738705344  0.0891173212
##  [66] -0.0830935639 -0.0393494512  0.0070989948 -0.1813020970  0.1062798163
##  [71] -0.0215188157  0.1027131522 -0.0630641517 -0.0356019470 -0.1888169187
##  [76] -0.2066631695 -0.0597878179  0.0905929941 -0.3680064939  0.1090905974
##  [81]  0.0319198032  0.0546929492  0.0767374619 -0.3204233562 -0.1728165594
##  [86] -0.2224637991  0.0005596826 -0.3494507628  0.1121391278 -0.2300234165
##  [91] -0.1686567050 -0.0702330697  0.0189413612 -0.0916103562  0.0324370766
##  [96] -0.3615680370 -0.1736869575  0.0174449195 -0.2410567351 -0.3305704291
## Parameter:
##   [1] -0.00161525  0.20173571  0.25529586  0.21237779  0.41406786  0.43650699
##   [7]  0.67729815  0.82595773  0.81138351  0.68610200  0.65034104  0.97803281
##  [13]  0.91903681  0.90255475  1.32734778  1.06595356  1.38190343  1.43799562
##  [19]  1.45796314  1.60256923  1.72927687  1.77007244  1.93224535  1.84419584
##  [25]  1.85629871  2.04258856  2.17027626  2.00034346  2.03410989  2.42607166
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##  [31]  2.51856758  2.37675532  2.50573707  2.54843148  2.69178859  2.76091101
##  [37]  2.84115305  2.91431803  3.17213114  2.97530566  3.14870690  3.19009677
##  [43]  3.15672191  3.39908901  3.45074847  3.41640183  3.70692329  3.76057066
##  [49]  3.60992438  3.58034265  3.87099720  3.74030181  3.92140576  4.04240282
##  [55]  4.00610336  4.21829329  4.11873751  4.32947436  4.47625812  4.18279050
##  [61]  4.27134157  4.28632141  4.66360783  4.72226952  4.45325987  4.65329187
##  [67]  4.66221248  4.66742605  4.87770155  4.66837407  4.82261734  4.75382593
##  [73]  4.93834642  4.95158310  5.12224927  5.17274483  5.07728432  4.97731723
##  [79]  5.41108299  5.02127437  5.11703346  5.12326469  5.12843437  5.50045285
##  [85]  5.39249965  5.45672360  5.28001837  5.60554929  5.21651958  5.53235970
##  [91]  5.49220222  5.41815492  5.35073294  5.45723837  5.35599308  5.70854466
##  [97]  5.54851846  5.38432406  5.61379331  5.69349768
## Function Value
## [1] -3929.904
## Gradient:
##   [1]  0.45811272  1.07753957  0.89715510  0.20179347  0.81325920  0.46784542
##   [7]  1.28932055  1.61960940  1.07913193 -0.05156893 -0.70317591  0.58743108
##  [13] -0.18606808 -0.73123554  1.08183066 -0.76846691  0.46648404  0.31532234
##  [19] -0.02683852  0.29872194  0.53085972  0.30646226  0.73279113 -0.17477888
##  [25] -0.54489177  0.01791386  0.27048232 -1.06369432 -1.30712115  0.36546607
##  [31]  0.44179939 -0.72996533 -0.45209139 -0.63170128 -0.27017053 -0.30149316
##  [37] -0.26966750 -0.27177944  0.71634269 -0.71960714 -0.17399910 -0.32922874
##  [43] -0.87946517  0.04742637 -0.03969681 -0.58151646  0.61654509  0.55441923
##  [49] -0.59379751 -1.09045672  0.12824806 -0.89836225 -0.25445581  0.07373102
##  [55] -0.43276144  0.39346354 -0.43971935  0.39008126  0.88417328 -0.96718603
##  [61] -0.77236045 -0.96469642  0.78400397  0.83715873 -0.85347348 -0.03273254
##  [67] -0.22645734 -0.43364291  0.46082688 -0.87931841 -0.27115540 -0.84774144
##  [73] -0.06483457 -0.19009975  0.53225512  0.61970174 -0.06542052 -0.76770972
##  [79]  1.38762549 -0.84868315 -0.48457473 -0.59170788 -0.69754425  1.16306245
##  [85]  0.46745042  0.69883040 -0.34925726  1.29245606 -0.87492902  0.72973869
##  [91]  0.44042043 -0.02244676 -0.44255793  0.07361493 -0.51246128  1.33263544
##  [97]  0.44763568 -0.45208275  0.75821749  1.17620440
##
## iteration = 5
## Step:
##   [1] -0.0782009247 -0.2056478267 -0.1799166206 -0.0554544877 -0.1804594900
##   [6] -0.1223212815 -0.2867752901 -0.3568017203 -0.2597566222 -0.0494171155
##  [11]  0.0695577235 -0.1825171757 -0.0399505987  0.0591310842 -0.2922666303
##  [16]  0.0566509873 -0.1837570671 -0.1590059767 -0.0974026169 -0.1630816633
##  [21] -0.2105371893 -0.1703216861 -0.2539966833 -0.0823466301 -0.0131390027
##  [26] -0.1220485087 -0.1714232818  0.0826362357  0.1282270004 -0.1923666438
##  [31] -0.2076785320  0.0158238679 -0.0375664013 -0.0035005007 -0.0727815215
##  [36] -0.0668873007 -0.0729270511 -0.0723507001 -0.2607724287  0.0142917051
##  [41] -0.0891649702 -0.0586216774  0.0474421386 -0.1287582961 -0.1110808202
##  [46] -0.0063924681 -0.2340591844 -0.2206331907  0.0006006806  0.0973065042
##  [51] -0.1338983720  0.0639091184 -0.0575728459 -0.1187473473 -0.0203781922
##  [56] -0.1766270957 -0.0157096534 -0.1725098358 -0.2649793681  0.0909086082
##  [61]  0.0556536976  0.0940703066 -0.2385917065 -0.2471001009  0.0777707969
##  [66] -0.0773493086 -0.0387430089  0.0023221059 -0.1672206123  0.0901705420
##  [71] -0.0249387805  0.0860701786 -0.0629621023 -0.0386289824 -0.1763244775
##  [76] -0.1926190475 -0.0611901091  0.0734149214 -0.3381428415  0.0895254889
##  [81]  0.0201176781  0.0404343394  0.0601815786 -0.2960081103 -0.1636159749
##  [86] -0.2081773496 -0.0081869659 -0.3221517010  0.0918049001 -0.2151009490
##  [91] -0.1600762751 -0.0718230792  0.0081648672 -0.0909328373  0.0204291707
```

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##  [96] -0.3328302348 -0.1642286127  0.0072750659 -0.2244768927 -0.3046886676
## Parameter:
##   [1] -0.07981617 -0.00391212  0.07537924  0.15692330  0.23360837  0.31418571
##   [7]  0.39052286  0.46915601  0.55162688  0.63668488  0.71989876  0.79551563
##  [13]  0.87908622  0.96168584  1.03508115  1.12260455  1.19814637  1.27898964
##  [19]  1.36056052  1.43948756  1.51873968  1.59975076  1.67824867  1.76184921
##  [25]  1.84315971  1.92054005  1.99885297  2.08297969  2.16233689  2.23370501
##  [31]  2.31088905  2.39257919  2.46817067  2.54493098  2.61900707  2.69402371
##  [37]  2.76822600  2.84196733  2.91135871  2.98959736  3.05954193  3.13147509
##  [43]  3.20416405  3.27033071  3.33966765  3.41000936  3.47286410  3.53993747
##  [49]  3.61052506  3.67764915  3.73709883  3.80421093  3.86383291  3.92365547
##  [55]  3.98572516  4.04166619  4.10302786  4.15696452  4.21127875  4.27369911
##  [61]  4.32699526  4.38039172  4.42501613  4.47516942  4.53103067  4.57594256
##  [67]  4.62346948  4.66974816  4.71048094  4.75854461  4.79767856  4.83989611
##  [73]  4.87538432  4.91295412  4.94592479  4.98012579  5.01609422  5.05073215
##  [79]  5.07294015  5.11079986  5.13715113  5.16369903  5.18861595  5.20444474
##  [85]  5.22888368  5.24854625  5.27183140  5.28339759  5.30832448  5.31725875
##  [91]  5.33212595  5.34633184  5.35889781  5.36630553  5.37642225  5.37571443
##  [97]  5.38428985  5.39159913  5.38931642  5.38880901
## Function Value
## [1] -3933.703
## Gradient:
##   [1]  0.028376817 -0.021913366 -0.053333373 -0.072093380 -0.117707717
##   [6] -0.141301228 -0.188077932 -0.221506647 -0.232884761 -0.229064311
##  [11] -0.234617831 -0.281485255 -0.282903201 -0.288485879 -0.343996455
##  [16] -0.319311353 -0.359802411 -0.369052994 -0.372436275 -0.388623930
##  [21] -0.400959901 -0.401341586 -0.413460471 -0.394766372 -0.386336421
##  [26] -0.397194892 -0.400134500 -0.367843788 -0.359178932 -0.391968394
##  [31] -0.389182859 -0.358011853 -0.357434459 -0.346845634 -0.347607544
##  [36] -0.339416332 -0.331944741 -0.323118419 -0.334504102 -0.292462186
##  [41] -0.292401718 -0.276923486 -0.252795257 -0.260471133 -0.245845597
##  [46] -0.220887539 -0.232817476 -0.216316985 -0.175216321 -0.148298685
##  [51] -0.158934222 -0.121574302 -0.120561227 -0.112954135 -0.087263807
##  [56] -0.090076342 -0.056954108 -0.059392311 -0.053834761  0.002862825
##  [61]  0.014773118  0.033375052  0.009303311  0.022312883  0.073472689
##  [66]  0.070039300  0.087663671  0.104885204  0.097820850  0.138284009
##  [71]  0.135730794  0.157160814  0.147951316  0.157251406  0.147886225
##  [76]  0.152386917  0.173795064  0.194898453  0.153894707  0.207299919
##  [81]  0.203818176  0.208713059  0.211838016  0.171673396  0.186865089
##  [86]  0.182851534  0.206463667  0.172243876  0.219964734  0.186090526
##  [91]  0.192813695  0.203409144  0.212443556  0.200329033  0.210916125
##  [96]  0.168803671  0.186041132  0.203854305  0.175857876  0.165392660
##
## iteration = 6
## Step:
##   [1]  0.0055417802  0.0187070193  0.0178332212  0.0078901094  0.0206513895
##   [6]  0.0166496448  0.0328743645  0.0403740985  0.0325010770  0.0141934373
##  [11]  0.0041821920  0.0279973623  0.0157736216  0.0074864254  0.0402572857
##  [16]  0.0090936087  0.0316370167  0.0299370429  0.0248050183  0.0312203795
##  [21]  0.0358995328  0.0324961381  0.0403024813  0.0247344150  0.0184584698
##  [26]  0.0283980598  0.0328572160  0.0095947725  0.0053542317  0.0345245366
##  [31]  0.0357990296  0.0152284024  0.0198905477  0.0165685500  0.0226615240
##  [36]  0.0218771440  0.0221554912  0.0218050796  0.0386499772  0.0131665818
##  [41]  0.0221835061  0.0189637364  0.0088490047  0.0244816571  0.0224099069
```

```
##  [46]  0.0123789894  0.0326414947  0.0308593065  0.0100718732  0.0006494259
##  [51]  0.0211628882  0.0025560517  0.0130889615  0.0181283388  0.0086092975
##  [56]  0.0223077268  0.0070658065  0.0207949891  0.0286244161 -0.0044711011
##  [61] -0.0018585862 -0.0059009012  0.0239405825  0.0241893246 -0.0059974680
##  [66]  0.0076197627  0.0035962316 -0.0006246784  0.0143875026 -0.0095219416
##  [71]  0.0005894110 -0.0098650227  0.0034550702  0.0010020021  0.0133445235
##  [76]  0.0146040671  0.0023763937 -0.0101149961  0.0272358503 -0.0119613364
##  [81] -0.0057811238 -0.0077153899 -0.0095287968  0.0229871777  0.0109215186
##  [86]  0.0149657218 -0.0033020053  0.0253078775 -0.0124802887  0.0155014498
##  [91]  0.0104745908  0.0024099974 -0.0048761073  0.0042122873 -0.0058568438
##  [96]  0.0264690320  0.0111675173 -0.0044103609  0.0168100877  0.0241870767
## Parameter:
##   [1] -0.07427439  0.01479490  0.09321246  0.16481341  0.25425975  0.33083536
##   [7]  0.42339722  0.50953011  0.58412796  0.65087832  0.72408095  0.82351299
##  [13]  0.89485984  0.96917226  1.07533843  1.13169816  1.22978338  1.30892668
##  [19]  1.38536554  1.47070794  1.55463922  1.63224689  1.71855115  1.78658363
##  [25]  1.86161818  1.94893811  2.03171019  2.09257447  2.16769112  2.26822955
##  [31]  2.34668808  2.40780759  2.48806122  2.56149953  2.64166860  2.71590085
##  [37]  2.79038149  2.86377241  2.95000869  3.00276394  3.08172544  3.15043883
##  [43]  3.21301306  3.29481237  3.36207755  3.42238835  3.50550560  3.57079678
##  [49]  3.62059693  3.67829858  3.75826172  3.80676698  3.87692187  3.94178381
##  [55]  3.99433446  4.06397392  4.11009366  4.17775951  4.23990317  4.26922801
##  [61]  4.32513668  4.37449082  4.44895671  4.49935875  4.52503320  4.58356232
##  [67]  4.62706571  4.66912348  4.72486844  4.74902267  4.79826797  4.83003109
##  [73]  4.87883939  4.91395612  4.95926932  4.99472985  5.01847061  5.04061715
##  [79]  5.10017600  5.09883852  5.13137001  5.15598364  5.17908715  5.22743192
##  [85]  5.23980520  5.26351197  5.26852940  5.30870547  5.29584419  5.33276020
##  [91]  5.34260054  5.34874184  5.35402170  5.37051782  5.37056541  5.40218346
##  [97]  5.39545736  5.38718877  5.40612651  5.41299609
## Function Value
## [1] -3933.946
## Gradient:
##   [1]  0.060975112  0.081917642  0.043828234 -0.031920779 -0.008503246
##   [6] -0.056082095 -0.014567254 -0.008097427 -0.064858970 -0.164303106
##  [11] -0.227058368 -0.143377837 -0.214239448 -0.267398252 -0.142594575
##  [16] -0.292399748 -0.209284654 -0.229448902 -0.262760945 -0.244758685
##  [21] -0.232506036 -0.253119152 -0.223217338 -0.292190443 -0.319825261
##  [26] -0.276725580 -0.256091017 -0.354047833 -0.370039503 -0.242037449
##  [31] -0.233247357 -0.317275352 -0.291829541 -0.300659508 -0.268562714
##  [36] -0.265654699 -0.257540289 -0.251532900 -0.170273405 -0.270472032
##  [41] -0.221150422 -0.224298731 -0.257040268 -0.178668375 -0.176227541
##  [46] -0.207601094 -0.107711057 -0.101705359 -0.176546579 -0.202477281
##  [51] -0.099804766 -0.166234438 -0.107271962 -0.072187042 -0.099797562
##  [56] -0.027036462 -0.078937652 -0.005592933  0.043027544 -0.084320090
##  [61] -0.058268438 -0.062456523  0.078741972  0.092804280 -0.023878214
##  [66]  0.047964572  0.042967947  0.036480952  0.112458640  0.019960391
##  [71]  0.073262151  0.036408842  0.100863613  0.096291854  0.155177551
##  [76]  0.166432913  0.119756714  0.071318197  0.237399258  0.073064535
##  [81]  0.103687349  0.097648947  0.090508389  0.230606771  0.178651150
##  [86]  0.196916009  0.118989280  0.243409266  0.081277377  0.202575932
##  [91]  0.181287016  0.147023775  0.115526333  0.153760488  0.108376915
##  [96]  0.245591167  0.177843909  0.109152544  0.198887933  0.229340404
##
## iteration = 7
```

```
## Step:
##   [1] -0.0019959016  0.0039604840  0.0067355979  0.0074149715  0.0129230692
##   [6]  0.0146763757  0.0206673395  0.0245409441  0.0248107816  0.0226142691
##  [11]  0.0221437021  0.0289792798  0.0279054595  0.0276255540  0.0361948748
##  [16]  0.0307704667  0.0369102065  0.0376711707  0.0375449657  0.0397997687
##  [21]  0.0415288216  0.0413308644  0.0433843563  0.0401918091  0.0389113527
##  [26]  0.0410875221  0.0419725240  0.0367492426  0.0356671839  0.0418564768
##  [31]  0.0418883958  0.0370526555  0.0376463767  0.0364911738  0.0373551076
##  [36]  0.0366838501  0.0361904024  0.0355086361  0.0384712932  0.0321393565
##  [41]  0.0332446141  0.0316639016  0.0285721116  0.0310709950  0.0296850557
##  [46]  0.0265251307  0.0298974742  0.0283760152  0.0226274963  0.0193581001
##  [51]  0.0226387844  0.0174484031  0.0186159527  0.0186048573  0.0154209487
##  [56]  0.0172630121  0.0128009021  0.0146102570  0.0150751401  0.0065980614
##  [61]  0.0059338764  0.0039541549  0.0094031525  0.0083769432  0.0006900664
##  [66]  0.0025578814  0.0006552898 -0.0012362312  0.0010936455 -0.0049850725
##  [71] -0.0035902944 -0.0065419350 -0.0042215524 -0.0052450741 -0.0030230662
##  [76] -0.0032253702 -0.0063804051 -0.0095401507 -0.0018288317 -0.0107338125
##  [81] -0.0097121048 -0.0103127458 -0.0107491799 -0.0036544204 -0.0062699120
##  [86] -0.0054478837 -0.0094797247 -0.0033373838 -0.0116126831 -0.0055733939
##  [91] -0.0066906233 -0.0084799022 -0.0100502648 -0.0079862204 -0.0100022812
##  [96] -0.0027829661 -0.0059554950 -0.0092101714 -0.0044431420 -0.0027221768
## Parameter:
##   [1] -0.07627030  0.01875538  0.09994806  0.17222839  0.26718282  0.34551173
##   [7]  0.44406456  0.53407105  0.60893874  0.67349259  0.74622466  0.85249227
##  [13]  0.92276530  0.99679782  1.11153331  1.16246862  1.26669359  1.34659785
##  [19]  1.42291050  1.51050771  1.59616804  1.67357776  1.76193550  1.82677544
##  [25]  1.90052953  1.99002563  2.07368271  2.12932371  2.20335831  2.31008603
##  [31]  2.38857647  2.44486024  2.52570759  2.59799070  2.67902370  2.75258470
##  [37]  2.82657190  2.89928104  2.98847998  3.03490330  3.11497005  3.18210273
##  [43]  3.24158517  3.32588336  3.39176261  3.44891348  3.53540307  3.59917279
##  [49]  3.64322443  3.69765668  3.78090050  3.82421539  3.89553783  3.96038866
##  [55]  4.00975541  4.08123693  4.12289457  4.19236977  4.25497831  4.27582607
##  [61]  4.33107055  4.37844497  4.45835986  4.50773569  4.52572326  4.58612020
##  [67]  4.62772100  4.66788725  4.72596209  4.74403759  4.79467767  4.82348915
##  [73]  4.87461784  4.90871104  4.95624625  4.99150448  5.01209020  5.03107700
##  [79]  5.09834717  5.08810471  5.12165791  5.14567089  5.16833797  5.22377750
##  [85]  5.23353529  5.25806409  5.25904967  5.30536808  5.28423151  5.32718681
##  [91]  5.33590991  5.34026194  5.34397144  5.36253160  5.36056313  5.39940049
##  [97]  5.38950187  5.37797859  5.40168337  5.41027391
## Function Value
## [1] -3934.196
## Gradient:
##   [1]  0.051790739  0.103459349  0.078769024  0.005005422  0.056689193
##   [6]  0.016942686  0.089357010  0.115202426  0.058201998 -0.054895910
##  [11] -0.121888264 -0.002650758 -0.080976772 -0.137247168  0.032642363
##  [16] -0.148209110 -0.033164906 -0.050639789 -0.086056410 -0.057161964
##  [21] -0.036841720 -0.059839490 -0.020032516 -0.107615674 -0.143409845
##  [26] -0.089608437 -0.065255234 -0.192693654 -0.215573163 -0.054828455
##  [31] -0.046764751 -0.157971145 -0.130105148 -0.145983732 -0.109897387
##  [36] -0.111309013 -0.106501778 -0.104782516 -0.007935855 -0.143097722
##  [41] -0.088220043 -0.100389440 -0.150334535 -0.058695898 -0.064095447
##  [46] -0.112929665  0.005086590  0.002609424 -0.103695906 -0.147563774
##  [51] -0.027142715 -0.121898104 -0.056626625 -0.021620567 -0.066555484
##  [56]  0.016257360 -0.059884817  0.023370203  0.074595135 -0.098805990
```

```
##  [61] -0.076265302 -0.091100721  0.079899650  0.088525412 -0.069843278
##  [66]  0.012326243 -0.002855521 -0.019460103  0.069380930 -0.056016100
##  [71]  0.005069118 -0.047649126  0.029640898  0.019707089  0.090892753
##  [76]  0.101245485  0.037586750 -0.027862730  0.180371675 -0.032255849
##  [81]  0.004099625 -0.005033252 -0.014372108  0.164514974  0.098485100
##  [86]  0.121369576  0.021633254  0.179608175 -0.027450246  0.126824644
##  [91]  0.099551681  0.055634984  0.015669570  0.065213445  0.008929598
##  [96]  0.185501696  0.100538038  0.014171561  0.129885164  0.169724177
##
## iteration = 8
## Step:
##   [1] -0.0065865411 -0.0114504067 -0.0079279634  0.0009716511 -0.0040048561
##   [6]  0.0010748362 -0.0062710581 -0.0085449316 -0.0017560636  0.0111638286
##  [11]  0.0189738524  0.0062203917  0.0152479601  0.0218250765  0.0034238045
##  [16]  0.0237014588  0.0112946370  0.0134813138  0.0176075100  0.0146001231
##  [21]  0.0124964675  0.0151240007  0.0107669726  0.0204205003  0.0243292070
##  [26]  0.0183332625  0.0155598266  0.0295154444  0.0319382562  0.0141037799
##  [31]  0.0130929627  0.0252108111  0.0219678410  0.0235517007  0.0193960293
##  [36]  0.0193694112  0.0186434279  0.0182455489  0.0073242666  0.0219802948
##  [41]  0.0156484531  0.0167104934  0.0219403616  0.0115467969  0.0118532614
##  [46]  0.0169419412  0.0036041215  0.0035340542  0.0148944087  0.0193693648
##  [51]  0.0057296385  0.0158475673  0.0083149698  0.0041287354  0.0087632941
##  [56] -0.0007047529  0.0073656954 -0.0021609761 -0.0081712352  0.0105935514
##  [61]  0.0077518566  0.0090760433 -0.0100874275 -0.0113450116  0.0058122902
##  [66] -0.0035645100 -0.0021782509 -0.0006183426 -0.0106824202  0.0029113769
##  [71] -0.0040500651  0.0015851764 -0.0070948337 -0.0061236642 -0.0140959659
##  [76] -0.0153605578 -0.0084649853 -0.0013563285 -0.0244393793 -0.0010722754
##  [81] -0.0051629979 -0.0041909940 -0.0031544997 -0.0228692133 -0.0155633704
##  [86] -0.0180906290 -0.0070884156 -0.0245316970 -0.0016882487 -0.0187149800
##  [91] -0.0157024234 -0.0108578901 -0.0064350675 -0.0118677821 -0.0055968950
##  [96] -0.0250175634 -0.0155897681 -0.0060150430 -0.0187346732 -0.0230936556
## Parameter:
##   [1] -0.082856837  0.007304977  0.092020097  0.173200037  0.263177968
##   [6]  0.346586570  0.437793502  0.525526120  0.607182678  0.684656419
##  [11]  0.765198508  0.858712665  0.938013257  1.018622892  1.114957111
##  [16]  1.186170083  1.277988226  1.360079168  1.440518013  1.525107836
##  [21]  1.608664506  1.688701759  1.772702477  1.847195935  1.924858739
##  [26]  2.008358891  2.089242540  2.158839154  2.235296565  2.324189807
##  [31]  2.401669437  2.470071055  2.547675436  2.621542400  2.698419734
##  [36]  2.771954115  2.845215324  2.917526593  2.995804251  3.056883596
##  [41]  3.130618504  3.198813222  3.263525529  3.337430162  3.403615869
##  [46]  3.465855422  3.539007192  3.602706848  3.658118840  3.717026044
##  [51]  3.786630140  3.840062955  3.903852795  3.964517399  4.018518704
##  [56]  4.080532179  4.130260262  4.190208790  4.246807071  4.286419619
##  [61]  4.338822411  4.387521013  4.448272435  4.496390678  4.531535555
##  [66]  4.582555694  4.625542746  4.667268903  4.715279670  4.746948972
##  [71]  4.790627607  4.825074331  4.867523006  4.902587379  4.942150286
##  [76]  4.976143926  5.003625219  5.029720672  5.073907786  5.087032433
##  [81]  5.116494908  5.141479897  5.165183474  5.200908286  5.217971915
##  [86]  5.239973457  5.251961256  5.280836384  5.282543259  5.308471831
##  [91]  5.320207491  5.329404049  5.337536368  5.350663819  5.354966234
##  [96]  5.374382929  5.373912101  5.371963552  5.382948693  5.387180254
## Function Value
## [1] -3934.271
```

```
## Gradient:
##   [1]  0.0158731886  0.0402293246  0.0345133949  0.0094017634  0.0331250784
##   [6]  0.0209353532  0.0523029331  0.0650966310  0.0450701135  0.0027945286
##  [11] -0.0215728095  0.0267798974 -0.0021634318 -0.0225593727  0.0453229330
##  [16] -0.0239744778  0.0221973773  0.0164404761  0.0034692623  0.0154473483
##  [21]  0.0238613629  0.0150969273  0.0305947303 -0.0039266308 -0.0183490988
##  [26]  0.0021719111  0.0110626839 -0.0394508877 -0.0490398030  0.0131944787
##  [31]  0.0156553193 -0.0286488455 -0.0186564299 -0.0257276087 -0.0124903219
##  [36] -0.0139262592 -0.0129776061 -0.0132792890  0.0234694462 -0.0305611068
##  [41] -0.0104009173 -0.0164423448 -0.0372244076 -0.0026531018 -0.0060261837
##  [46] -0.0264135246  0.0183357226  0.0158512251 -0.0273462421 -0.0461015181
##  [51] -0.0005566726 -0.0390080267 -0.0148635518 -0.0024967575 -0.0213633212
##  [56]  0.0096658621 -0.0214264198  0.0097303771  0.0282689483 -0.0410405893
##  [61] -0.0336886025 -0.0407009469  0.0250174039  0.0272099604 -0.0359345714
##  [66] -0.0050346074 -0.0120653326 -0.0195632656  0.0142043532 -0.0356598263
##  [71] -0.0125286495 -0.0336570716 -0.0038133512 -0.0079168042  0.0197112344
##  [76]  0.0235214367 -0.0016471287 -0.0274419393  0.0537487317 -0.0294934185
##  [81] -0.0153560839 -0.0187617631 -0.0220233030  0.0483636709  0.0229890358
##  [86]  0.0322155677 -0.0065044669  0.0554215528 -0.0252866977  0.0352020798
##  [91]  0.0247587791  0.0077643667 -0.0076225071  0.0120757468 -0.0094518156
##  [96]  0.0600187154  0.0271981644 -0.0062360220  0.0393082736  0.0551128412
##
## iteration = 9
## Step:
##   [1] -2.783482e-03 -6.517097e-03 -5.410487e-03 -1.187261e-03 -4.869388e-03
##   [6] -2.762393e-03 -7.714204e-03 -9.693504e-03 -6.403718e-03  4.569230e-04
##  [11]  4.417668e-03 -3.373520e-03  1.320591e-03  4.631054e-03 -6.327433e-03
##  [16]  4.874132e-03 -2.591389e-03 -1.673714e-03  3.973321e-04 -1.573518e-03
##  [21] -2.981420e-03 -1.629452e-03 -4.216088e-03  1.264810e-03  3.491381e-03
##  [26]  7.072940e-05 -1.476002e-03  6.573517e-03  8.020178e-03 -2.128607e-03
##  [31] -2.623337e-03  4.430117e-03  2.710527e-03  3.755000e-03  1.522404e-03
##  [36]  1.663377e-03  1.419493e-03  1.378175e-03 -4.655902e-03  3.972950e-03
##  [41]  6.116919e-04  1.487742e-03  4.750888e-03 -9.246738e-04 -4.677344e-04
##  [46]  2.738236e-03 -4.583060e-03 -4.281307e-03  2.590363e-03  5.518594e-03
##  [51] -1.932564e-03  4.198160e-03  2.214029e-04 -1.846405e-03  1.134754e-03
##  [56] -3.946917e-03  1.010160e-03 -4.093191e-03 -7.164840e-03  3.953567e-03
##  [61]  2.692910e-03  3.773441e-03 -6.888769e-03 -7.291997e-03  2.855796e-03
##  [66] -2.188658e-03 -1.095270e-03  8.045124e-05 -5.407313e-03  2.624915e-03
##  [71] -1.128531e-03  2.286007e-03 -2.523050e-03 -1.836655e-03 -6.276089e-03
##  [76] -6.871863e-03 -2.791022e-03  1.395649e-03 -1.170324e-02  1.768523e-03
##  [81] -4.894358e-04  1.046245e-04  6.920610e-04 -1.061484e-02 -6.454188e-03
##  [86] -7.897856e-03 -1.598121e-03 -1.156775e-02  1.509348e-03 -8.230609e-03
##  [91] -6.509724e-03 -3.735765e-03 -1.218055e-03 -4.362488e-03 -8.366501e-04
##  [96] -1.201596e-02 -6.675525e-03 -1.242833e-03 -8.572490e-03 -1.110552e-02
## Parameter:
##   [1] -0.0856403187  0.0007878802  0.0866096098  0.1720127763  0.2583085794
##   [6]  0.3438241773  0.4300792973  0.5158326157  0.6007789600  0.6851133423
##  [11]  0.7696161757  0.8553391458  0.9393338473  1.0232539462  1.1086296779
##  [16]  1.1910442155  1.2753968369  1.3584054541  1.4409153450  1.5235343177
##  [21]  1.6056830860  1.6870723076  1.7684863885  1.8484607456  1.9283501199
##  [26]  2.0084296206  2.0877665384  2.1654126708  2.2433167435  2.3220611995
##  [31]  2.3990460996  2.4745011717  2.5503859624  2.6252974002  2.6999421375
##  [36]  2.7736174916  2.8466348170  2.9189047678  2.9911483492  3.0608565454
##  [41]  3.1312301956  3.2003009641  3.2682764173  3.3365054877  3.4031481349
```

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##  [46]  3.4685936577  3.5344241327  3.5984255411  3.6607092030  3.7225446376
##  [51]  3.7846975762  3.8442611153  3.9040741978  3.9626709941  4.0196534572
##  [56]  4.0765852622  4.1312704225  4.1861155990  4.2396422306  4.2903731856
##  [61]  4.3415153207  4.3912944546  4.4413836662  4.4890986816  4.5343913507
##  [66]  4.5803670359  4.6244474763  4.6673493542  4.7098723570  4.7495738866
##  [71]  4.7894990762  4.8273603384  4.8649999568  4.9007507238  4.9358741972
##  [76]  4.9692720636  5.0008341965  5.0311163211  5.0622045473  5.0888009557
##  [81]  5.1160054723  5.1415845215  5.1658755352  5.1902934492  5.2115177277
##  [86]  5.2320756015  5.2503631343  5.2692686314  5.2840526072  5.3002412213
##  [91]  5.3136977667  5.3256682842  5.3363183132  5.3463013310  5.3541295842
##  [96]  5.3623669721  5.3672365761  5.3707207188  5.3743762022  5.3760747338
## Function Value
## [1] -3934.277
## Gradient:
##    [1]  2.653815e-04  4.235356e-03  4.853601e-03  3.279605e-03  6.882345e-03
##    [6]  6.534916e-03  1.075929e-02  1.281413e-02  1.113287e-02  6.913056e-03
##   [11]  4.587725e-03  1.008988e-02  7.239916e-03  5.298140e-03  1.284237e-02
##   [16]  5.574423e-03  1.069559e-02  1.017959e-02  8.816211e-03  1.008169e-02
##   [21]  1.088963e-02  9.755910e-03  1.113151e-02  7.033997e-03  5.065448e-03
##   [26]  6.859104e-03  7.371108e-03  1.470909e-03  4.111124e-05  6.411250e-03
##   [31]  6.318390e-03  1.140804e-03  1.817842e-03  6.993717e-04  1.797410e-03
##   [36]  1.332151e-03  1.128486e-03  7.951266e-04  4.419722e-03 -1.767138e-03
##   [41]  2.824311e-05 -9.770487e-04 -3.538384e-03 -1.026836e-04 -7.466026e-04
##   [46] -3.224226e-03  1.302895e-03  6.872113e-04 -4.365869e-03 -6.750301e-03
##   [51] -2.145839e-03 -6.535783e-03 -4.141428e-03 -2.982451e-03 -5.177057e-03
##   [56] -2.000425e-03 -5.512926e-03 -2.332563e-03 -5.563855e-04 -8.275393e-03
##   [61] -7.689303e-03 -8.534694e-03 -1.492041e-03 -1.330807e-03 -8.250104e-03
##   [66] -5.018401e-03 -5.824858e-03 -6.645355e-03 -2.998542e-03 -8.328425e-03
##   [71] -5.744310e-03 -7.836885e-03 -4.367941e-03 -4.501641e-03 -1.213258e-03
##   [76] -5.253856e-04 -2.999899e-03 -5.520123e-03  3.489479e-03 -5.216689e-03
##   [81] -3.406530e-03 -3.390014e-03 -3.265704e-03  4.822959e-03  2.552875e-03
##   [86]  3.925563e-03  1.026378e-04  7.077673e-03 -1.324780e-03  5.467182e-03
##   [91]  4.606727e-03  3.001450e-03  1.583360e-03  3.980147e-03  1.980872e-03
##   [96]  9.770939e-03  6.481410e-03  3.078308e-03  8.173242e-03  1.000518e-02
##
## iteration = 10
## Step:
##    [1] -3.109553e-04 -1.044902e-03 -9.902172e-04 -4.254959e-04 -1.130217e-03
##    [6] -8.949089e-04 -1.789704e-03 -2.192686e-03 -1.731021e-03 -6.826457e-04
##   [11] -9.883363e-05 -1.409671e-03 -7.032450e-04 -2.180259e-04 -2.032137e-03
##   [16] -2.671844e-04 -1.509256e-03 -1.394359e-03 -1.087071e-03 -1.426352e-03
##   [21] -1.668187e-03 -1.456883e-03 -1.872267e-03 -9.785511e-04 -6.057724e-04
##   [26] -1.142349e-03 -1.372963e-03 -5.276568e-05  1.994873e-04 -1.421807e-03
##   [31] -1.482242e-03 -3.200217e-04 -5.720952e-04 -3.795958e-04 -7.162951e-04
##   [36] -6.697439e-04 -6.841578e-04 -6.648113e-04 -1.608689e-03 -1.832375e-04
##   [41] -6.903291e-04 -5.140044e-04  4.648248e-05 -8.364152e-04 -7.293092e-04
##   [46] -1.775501e-04 -1.322466e-03 -1.232763e-03 -7.862144e-05  4.376487e-04
##   [51] -7.242385e-04  3.022068e-04 -3.043198e-04 -6.045695e-04 -9.044820e-05
##   [56] -8.765052e-04 -4.268587e-05 -8.308413e-04 -1.287854e-03  5.469824e-04
##   [61]  3.814582e-04  5.857820e-04 -1.108152e-03 -1.144305e-03  5.246585e-04
##   [66] -2.590984e-04 -5.607045e-05  1.574670e-04 -7.059522e-04  6.093730e-04
##   [71]  1.924953e-05  5.789013e-04 -1.936282e-04 -8.377756e-05 -8.017870e-04
##   [76] -8.978700e-04 -2.372564e-04  4.382793e-04 -1.676117e-03  4.959352e-04
##   [81]  1.275730e-04  2.120183e-04  2.883033e-04 -1.557352e-03 -9.052500e-04
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##  [86] -1.152337e-03 -1.487563e-04 -1.767891e-03  3.314664e-04 -1.250258e-03
##  [91] -9.828747e-04 -5.437037e-04 -1.476425e-04 -6.683605e-04 -1.167600e-04
##  [96] -1.937615e-03 -1.089766e-03 -2.245559e-04 -1.418562e-03 -1.835364e-03
## Parameter:
##   [1] -0.0859512740 -0.0002570214  0.0856193926  0.1715872803  0.2571783622
##   [6]  0.3429292684  0.4282895932  0.5136399301  0.5990479387  0.6844306966
##  [11]  0.7695173421  0.8539294746  0.9386306022  1.0230359203  1.1065975414
##  [16]  1.1907770312  1.2738875812  1.3570110953  1.4398282743  1.5221079654
##  [21]  1.6040148990  1.6856154245  1.7666141214  1.8474821945  1.9277443476
##  [26]  2.0072872720  2.0863935755  2.1653599051  2.2435162307  2.3206393927
##  [31]  2.3975638578  2.4741811500  2.5498138672  2.6249178045  2.6992258424
##  [36]  2.7729477477  2.8459506592  2.9182399565  2.9895396604  3.0606733079
##  [41]  3.1305398665  3.1997869597  3.2683228998  3.3356690725  3.4024188257
##  [46]  3.4684161077  3.5331016670  3.5971927782  3.6606305815  3.7229822863
##  [51]  3.7839733376  3.8445633220  3.9037698780  3.9620664246  4.0195630090
##  [56]  4.0757087570  4.1312277367  4.1852847577  4.2383543766  4.2909201680
##  [61]  4.3418967789  4.3918802366  4.4402755142  4.4879543770  4.5349160091
##  [66]  4.5801079375  4.6243914058  4.6675068212  4.7091664047  4.7501832595
##  [71]  4.7895183257  4.8279392397  4.8648063286  4.9006669462  4.9350724102
##  [76]  4.9683741936  5.0005969402  5.0315546004  5.0605284299  5.0892968909
##  [81]  5.1161330453  5.1417965397  5.1661638385  5.1887360972  5.2106124777
##  [86]  5.2309232647  5.2502143779  5.2675007403  5.2843840736  5.2989909631
##  [91]  5.3127148920  5.3251245805  5.3361706707  5.3456329704  5.3540128242
##  [96]  5.3604293571  5.3661468097  5.3704961629  5.3729576397  5.3742393696
## Function Value
## [1] -3934.277
## Gradient:
##   [1] -1.547169e-03 -1.542170e-03 -5.302016e-04  1.108012e-03  9.038310e-04
##   [6]  1.945126e-03  1.308806e-03  1.219549e-03  2.171530e-03  3.826309e-03
##  [11]  4.806121e-03  3.138112e-03  4.269076e-03  5.084226e-03  2.672504e-03
##  [16]  5.233711e-03  3.558371e-03  3.746164e-03  4.148297e-03  3.602986e-03
##  [21]  3.136880e-03  3.232067e-03  2.369660e-03  3.269273e-03  3.416296e-03
##  [26]  2.297016e-03  1.586083e-03  3.035164e-03  3.050120e-03  5.068122e-04
##  [31]  1.282037e-04  1.422612e-03  7.518265e-04  7.427020e-04  2.371671e-05
##  [36] -1.406463e-04 -3.814194e-04 -5.661923e-04 -2.119439e-03 -3.868275e-04
##  [41] -1.356906e-03 -1.350141e-03 -7.761716e-04 -2.185300e-03 -2.201674e-03
##  [46] -1.594373e-03 -3.363180e-03 -3.450685e-03 -2.092246e-03 -1.591934e-03
##  [51] -3.380321e-03 -2.065234e-03 -2.993813e-03 -3.464843e-03 -2.787796e-03
##  [56] -3.928085e-03 -2.802742e-03 -3.952197e-03 -4.676187e-03 -2.225971e-03
##  [61] -2.529502e-03 -2.219133e-03 -4.515000e-03 -4.528286e-03 -2.196453e-03
##  [66] -3.273514e-03 -2.933338e-03 -2.549169e-03 -3.651759e-03 -1.686156e-03
##  [71] -2.340560e-03 -1.314715e-03 -2.091907e-03 -1.593487e-03 -2.250187e-03
##  [76] -2.070018e-03 -8.698702e-04  3.667375e-04 -2.291234e-03  1.031653e-03
##  [81]  8.260861e-04  1.330562e-03  1.898034e-03 -1.957846e-04  1.159089e-03
##  [86]  1.183994e-03  2.926365e-03  9.659689e-04  4.184451e-03  2.245788e-03
##  [91]  2.876322e-03  3.710424e-03  4.492604e-03  4.020758e-03  5.080569e-03
##  [96]  2.812686e-03  4.217236e-03  5.602661e-03  4.100771e-03  3.630902e-03
##
## iteration = 11
## Step:
##   [1]  4.554561e-05 -3.333164e-05 -7.972628e-05 -1.039273e-04 -1.700255e-04
##   [6] -1.994692e-04 -2.644256e-04 -3.052631e-04 -3.074597e-04 -2.832206e-04
##  [11] -2.741802e-04 -3.323665e-04 -3.177698e-04 -3.105840e-04 -3.843516e-04
##  [16] -3.293870e-04 -3.795503e-04 -3.797508e-04 -3.703847e-04 -3.817684e-04
```

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##  [21] -3.868357e-04 -3.722378e-04 -3.758978e-04 -3.296718e-04 -3.006177e-04
##  [26] -3.039036e-04 -2.952169e-04 -2.308676e-04 -2.070146e-04 -2.523759e-04
##  [31] -2.412841e-04 -1.848731e-04 -1.790778e-04 -1.591828e-04 -1.593074e-04
##  [36] -1.465205e-04 -1.361378e-04 -1.247972e-04 -1.464852e-04 -8.260520e-05
##  [41] -8.717532e-05 -6.831911e-05 -3.708090e-05 -5.860480e-05 -4.523491e-05
##  [46] -1.603841e-05 -4.680103e-05 -3.150120e-05  2.368875e-05  5.474454e-05
##  [51]  2.359216e-05  6.819546e-05  5.302877e-05  4.743649e-05  7.038644e-05
##  [56]  4.718674e-05  8.152918e-05  5.863033e-05  4.927816e-05  1.227589e-04
##  [61]  1.231929e-04  1.322428e-04  7.163502e-05  7.154811e-05  1.338266e-04
##  [66]  1.080161e-04  1.154682e-04  1.218879e-04  8.912691e-05  1.326735e-04
##  [71]  1.064275e-04  1.175601e-04  7.852843e-05  6.880893e-05  2.950462e-05
##  [76]  1.374304e-05  2.655634e-05  3.908887e-05 -4.784284e-05  1.795436e-05
##  [81] -7.663263e-06 -2.088925e-05 -3.797846e-05 -1.246347e-04 -1.206356e-04
##  [86] -1.452659e-04 -1.241150e-04 -1.945383e-04 -1.320462e-04 -1.999070e-04
##  [91] -2.014365e-04 -1.952515e-04 -1.910877e-04 -2.208840e-04 -2.142086e-04
##  [96] -2.913975e-04 -2.711481e-04 -2.483495e-04 -2.984008e-04 -3.183885e-04
## Parameter:
##   [1] -0.085905728 -0.000290353  0.085539666  0.171483353  0.257008337
##   [6]  0.342729799  0.428025168  0.513334667  0.598740479  0.684147476
##  [11]  0.769243162  0.853597108  0.938312832  1.022725336  1.106213190
##  [16]  1.190447644  1.273508031  1.356631345  1.439457890  1.521726197
##  [21]  1.603628063  1.685243187  1.766238224  1.847152523  1.927443730
##  [26]  2.006983368  2.086098359  2.165129038  2.243309216  2.320387017
##  [31]  2.397322574  2.473996277  2.549634789  2.624758622  2.699066535
##  [36]  2.772801227  2.845814521  2.918115159  2.989393175  3.060590703
##  [41]  3.130452691  3.199718641  3.268285819  3.335610468  3.402373591
##  [46]  3.468400069  3.533054866  3.597161277  3.660654270  3.723037031
##  [51]  3.783996930  3.844631517  3.903822907  3.962113861  4.019633395
##  [56]  4.075755944  4.131309266  4.185343388  4.238403655  4.291042927
##  [61]  4.342019972  4.392012479  4.440347149  4.488025925  4.535049836
##  [66]  4.580215954  4.624506874  4.667628709  4.709255532  4.750315933
##  [71]  4.789624753  4.828056800  4.864884857  4.900735755  4.935101915
##  [76]  4.968387937  5.000623497  5.031593689  5.060480587  5.089314845
##  [81]  5.116125382  5.141775651  5.166125860  5.188611462  5.210491842
##  [86]  5.230777999  5.250090263  5.267306202  5.284252027  5.298791056
##  [91]  5.312513455  5.324929329  5.335979583  5.345412086  5.353798616
##  [96]  5.360137960  5.365875662  5.370247813  5.372659239  5.373920981
## Function Value
## [1] -3934.277
## Gradient:
##   [1] -1.311494e-03 -1.726487e-03 -9.548904e-04  5.655936e-04  1.177797e-05
##   [6]  9.059449e-04 -7.423144e-05 -3.743180e-04  5.801299e-04  2.383158e-03
##  [11]  3.426748e-03  1.450268e-03  2.674889e-03  3.542254e-03  7.345180e-04
##  [16]  3.611246e-03  1.669448e-03  1.866613e-03  2.330496e-03  1.731633e-03
##  [21]  1.246371e-03  1.430627e-03  5.557644e-04  1.718329e-03  2.032852e-03
##  [26]  9.017616e-04  2.448135e-04  2.055360e-03  2.207458e-03 -5.818499e-04
##  [31] -8.945466e-04  7.161121e-04  8.133846e-05  1.859702e-04 -5.302181e-04
##  [36] -6.205873e-04 -8.008703e-04 -9.200456e-04 -2.590570e-03 -5.021270e-04
##  [41] -1.495121e-03 -1.381810e-03 -6.328973e-04 -2.158984e-03 -2.099383e-03
##  [46] -1.328626e-03 -3.265663e-03 -3.266676e-03 -1.601142e-03 -9.271953e-04
##  [51] -2.885968e-03 -1.322189e-03 -2.332593e-03 -2.832410e-03 -2.026201e-03
##  [56] -3.292505e-03 -1.974818e-03 -3.248443e-03 -4.021593e-03 -1.162282e-03
##  [61] -1.460598e-03 -1.097170e-03 -3.725099e-03 -3.735579e-03 -1.055871e-03
##  [66] -2.272130e-03 -1.887004e-03 -1.463431e-03 -2.743170e-03 -5.324948e-04
```

```
##  [71] -1.327693e-03 -2.357247e-04 -1.224105e-03 -7.746192e-04 -1.643804e-03
##  [76] -1.545855e-03 -2.698514e-04  1.041098e-03 -2.092659e-03  1.599190e-03
##  [81]  1.256953e-03  1.693301e-03  2.171253e-03 -3.969444e-04  9.848291e-04
##  [86]  8.780929e-04  2.741854e-03  3.961301e-04  3.964226e-03  1.653947e-03
##  [91]  2.279492e-03  3.150991e-03  3.959113e-03  3.325095e-03  4.424169e-03
##  [96]  1.731352e-03  3.249531e-03  4.762300e-03  2.984437e-03  2.404465e-03
##
## iteration = 12
## Step:
##   [1]  1.536286e-04  1.441171e-04  4.160176e-05 -1.185430e-04 -1.067092e-04
##   [6] -2.102539e-04 -1.569193e-04 -1.534207e-04 -2.455431e-04 -4.021580e-04
##  [11] -4.957250e-04 -3.428346e-04 -4.502241e-04 -5.281232e-04 -3.055705e-04
##  [16] -5.455391e-04 -3.909362e-04 -4.094151e-04 -4.473769e-04 -3.967821e-04
##  [21] -3.530748e-04 -3.610075e-04 -2.791573e-04 -3.607244e-04 -3.720147e-04
##  [26] -2.654393e-04 -1.966694e-04 -3.288098e-04 -3.278728e-04 -8.925855e-05
##  [31] -5.192690e-05 -1.699814e-04 -1.051942e-04 -1.022436e-04 -3.341593e-05
##  [36] -1.630881e-05  7.877085e-06  2.681738e-05  1.733062e-04  1.422247e-05
##  [41]  1.066960e-04  1.081240e-04  5.656243e-05  1.893175e-04  1.924086e-04
##  [46]  1.374573e-04  3.036814e-04  3.137790e-04  1.896321e-04  1.451027e-04
##  [51]  3.131474e-04  1.920421e-04  2.794505e-04  3.240571e-04  2.617436e-04
##  [56]  3.685522e-04  2.644965e-04  3.721897e-04  4.405561e-04  2.137264e-04
##  [61]  2.428420e-04  2.140905e-04  4.275181e-04  4.287612e-04  2.120259e-04
##  [66]  3.123698e-04  2.805188e-04  2.443346e-04  3.463533e-04  1.626439e-04
##  [71]  2.224160e-04  1.252526e-04  1.954247e-04  1.465487e-04  2.051366e-04
##  [76]  1.860644e-04  7.233184e-05 -4.492710e-05  2.002020e-04 -1.112038e-04
##  [81] -9.442119e-05 -1.443038e-04 -2.006131e-04 -9.450097e-06 -1.389241e-04
##  [86] -1.440883e-04 -3.088607e-04 -1.287691e-04 -4.305018e-04 -2.522042e-04
##  [91] -3.128872e-04 -3.922395e-04 -4.668166e-04 -4.248875e-04 -5.257357e-04
##  [96] -3.168027e-04 -4.491985e-04 -5.794802e-04 -4.409789e-04 -3.981034e-04
## Parameter:
##   [1] -0.0857520999 -0.0001462358  0.0855812681  0.1713648100  0.2569016275
##   [6]  0.3425195453  0.4278682484  0.5131812463  0.5984949359  0.6837453179
##  [11]  0.7687474369  0.8532542735  0.9378626084  1.0221972131  1.1059076192
##  [16]  1.1899021051  1.2731170948  1.3562219294  1.4390105127  1.5213294149
##  [21]  1.6032749885  1.6848821791  1.7659590662  1.8467917982  1.9270717152
##  [26]  2.0067179291  2.0859016893  2.1648002277  2.2429813433  2.3202977582
##  [31]  2.3972706468  2.4738262955  2.5495295952  2.6246563781  2.6990331191
##  [36]  2.7727849183  2.8458223985  2.9181419767  2.9895664814  3.0606049252
##  [41]  3.1305593873  3.1998267646  3.2683423813  3.3357997852  3.4025659994
##  [46]  3.4685375266  3.5333585474  3.5974750560  3.6608439024  3.7231821335
##  [51]  3.7843100772  3.8448235596  3.9041023573  3.9624379182  4.0198951391
##  [56]  4.0761244960  4.1315737624  4.1857155778  4.2388442109  4.2912566534
##  [61]  4.3422628137  4.3922265700  4.4407746673  4.4884546863  4.5352618617
##  [66]  4.5805283234  4.6247873928  4.6678730437  4.7096018849  4.7504785769
##  [71]  4.7898471692  4.8281820524  4.8650802817  4.9008823039  4.9353070514
##  [76]  4.9685740011  5.0006958284  5.0315487622  5.0606807891  5.0892036415
##  [81]  5.1160309608  5.1416313467  5.1659252470  5.1886020124  5.2103529180
##  [86]  5.2306339104  5.2497814022  5.2671774330  5.2838215255  5.2985388518
##  [91]  5.3122005683  5.3245370895  5.3355127663  5.3449871989  5.3532728800
##  [96]  5.3598211568  5.3654264631  5.3696683333  5.3722182600  5.3735228776
## Function Value
## [1] -3934.277
## Gradient:
##   [1] -4.701880e-04 -9.293013e-04 -7.159349e-04 -7.231362e-05 -5.521146e-04
```

```
##   [6] -2.221274e-04 -8.992342e-04 -1.172065e-03 -7.193614e-04  2.249328e-04
##  [11]  7.578803e-04 -3.670282e-04  2.693324e-04  7.108855e-04 -8.619034e-04
##  [16]  6.915947e-04 -3.922157e-04 -2.947575e-04 -3.889659e-05 -3.566924e-04
##  [21] -5.995332e-04 -4.589209e-04 -8.814348e-04 -1.706129e-04  8.029146e-05
##  [26] -4.631416e-04 -7.418709e-04  3.355813e-04  4.900421e-04 -9.830059e-04
##  [31] -1.092664e-03 -1.383261e-04 -4.185436e-04 -3.014610e-04 -6.410410e-04
##  [36] -6.408644e-04 -6.914789e-04 -7.099831e-04 -1.574453e-03 -3.695326e-04
##  [41] -8.549139e-04 -7.373231e-04 -2.770123e-04 -1.072348e-03 -9.987733e-04
##  [46] -5.347465e-04 -1.555219e-03 -1.502621e-03 -5.254074e-04 -9.922989e-05
##  [51] -1.130031e-03 -2.368428e-04 -7.644836e-04 -1.017840e-03 -5.559840e-04
##  [56] -1.231026e-03 -4.876407e-04 -1.164222e-03 -1.557281e-03  5.030221e-05
##  [61] -8.406134e-05  1.236549e-04 -1.320433e-03 -1.319852e-03  1.660965e-04
##  [66] -4.901716e-04 -2.756561e-04 -4.626063e-05 -7.555713e-04  4.459398e-04
##  [71] -1.173766e-05  5.502072e-04 -4.265320e-05  1.442905e-04 -3.929545e-04
##  [76] -3.922377e-04  2.633221e-04  9.344289e-04 -8.352246e-04  1.143365e-03
##  [81]  9.025981e-04  1.071623e-03  1.246509e-03 -2.565111e-04  4.172994e-04
##  [86]  2.898399e-04  1.250041e-03 -9.257725e-05  1.813950e-03  4.960701e-04
##  [91]  7.920942e-04  1.230391e-03  1.631167e-03  1.233425e-03  1.778618e-03
##  [96]  2.442736e-04  1.032928e-03  1.827233e-03  8.164624e-04  4.741523e-04
##
## iteration = 13
## Parameter:
##   [1] -8.568002e-02 -2.036771e-05  8.567138e-02  1.713621e-01  2.569579e-01
##   [6]  3.425265e-01  4.279623e-01  5.133094e-01  5.985611e-01  6.836844e-01
##  [11]  7.686150e-01  8.532733e-01  9.377962e-01  1.022072e+00  1.105994e+00
##  [16]  1.189780e+00  1.273141e+00  1.356234e+00  1.438990e+00  1.521353e+00
##  [21]  1.603333e+00  1.684925e+00  1.766062e+00  1.846803e+00  1.927054e+00
##  [26]  2.006777e+00  2.086003e+00  2.164761e+00  2.242925e+00  2.320442e+00
##  [31]  2.397433e+00  2.473864e+00  2.549608e+00  2.624722e+00  2.699147e+00
##  [36]  2.772902e+00  2.845948e+00  2.918272e+00  2.989816e+00  3.060695e+00
##  [41]  3.130717e+00  3.199971e+00  3.268427e+00  3.335993e+00  3.402751e+00
##  [46]  3.468662e+00  3.533622e+00  3.597734e+00  3.660974e+00  3.723257e+00
##  [51]  3.784525e+00  3.844920e+00  3.904271e+00  3.962641e+00  4.020037e+00
##  [56]  4.076358e+00  4.131708e+00  4.185942e+00  4.239125e+00  4.291322e+00
##  [61]  4.342348e+00  4.392284e+00  4.441026e+00  4.488706e+00  4.535314e+00
##  [66]  4.580670e+00  4.624900e+00  4.667955e+00  4.709779e+00  4.750494e+00
##  [71]  4.789923e+00  4.828181e+00  4.865157e+00  4.900931e+00  4.935426e+00
##  [76]  4.968691e+00  5.000723e+00  5.031483e+00  5.060852e+00  5.089107e+00
##  [81]  5.115964e+00  5.141539e+00  5.165806e+00  5.188681e+00  5.210338e+00
##  [86]  5.230634e+00  5.249649e+00  5.267224e+00  5.283610e+00  5.298502e+00
##  [91]  5.312122e+00  5.324398e+00  5.335319e+00  5.344845e+00  5.353054e+00
##  [96]  5.359807e+00  5.365304e+00  5.369438e+00  5.372122e+00  5.373472e+00
## Function Value
## [1] -3934.277
## Gradient:
##   [1] -6.442649e-05 -2.331271e-04 -2.243854e-04 -1.007770e-04 -2.610417e-04
##   [6] -2.105065e-04 -4.125552e-04 -5.036938e-04 -4.002477e-04 -1.649162e-04
##  [11] -3.391813e-05 -3.277216e-04 -1.701619e-04 -6.261976e-05 -4.703906e-04
##  [16] -7.699504e-05 -3.572036e-04 -3.337908e-04 -2.673428e-04 -3.458030e-04
##  [21] -4.024257e-04 -3.573987e-04 -4.524005e-04 -2.540244e-04 -1.723845e-04
##  [26] -2.946503e-04 -3.484066e-04 -5.493580e-05 -1.165076e-06 -3.674534e-04
##  [31] -3.840201e-04 -1.263866e-04 -1.854042e-04 -1.450242e-04 -2.231686e-04
##  [36] -2.153840e-04 -2.210217e-04 -2.188405e-04 -4.317941e-04 -1.135693e-04
##  [41] -2.276540e-04 -1.884326e-04 -6.296319e-05 -2.604969e-04 -2.358912e-04
```

```
##  [46] -1.111899e-04 -3.658217e-04 -3.430265e-04 -8.091379e-05  3.868929e-05
##  [51] -2.173509e-04  1.689774e-05 -1.142036e-04 -1.763232e-04 -5.551873e-05
##  [56] -2.252183e-04 -3.156219e-05 -2.004190e-04 -2.941799e-04  1.261373e-04
##  [61]  9.876461e-05  1.539738e-04 -2.157091e-04 -2.133008e-04  1.718089e-04
##  [66]  8.046727e-06  6.543191e-05  1.254016e-04 -5.549169e-05  2.517253e-04
##  [71]  1.322766e-04  2.700104e-04  1.092218e-04  1.459004e-04 -2.635935e-06
##  [76] -1.150293e-05  1.494651e-04  3.136448e-04 -1.470108e-04  3.521759e-04
##  [81]  2.821402e-04  3.126121e-04  3.403248e-04 -6.291570e-05  9.309372e-05
##  [86]  4.787688e-05  2.824115e-04 -7.066290e-05  4.084136e-04  6.256499e-05
##  [91]  1.301375e-04  2.356745e-04  3.306245e-04  2.191438e-04  3.465573e-04
##  [96] -5.817189e-05  1.341709e-04  3.297172e-04  6.313587e-05 -3.004965e-05
##
## Relative gradient close to zero.
## Current iterate is probably solution.
##
## Post processing for method  nlm
## Successful convergence!
## Method:  nlminb
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##   0:    -2237.1546: 0.158536 0.476245 0.438023 0.170150 0.480122 0.364783 0.764669 0.944322 0.735060
##   1:    -3901.3693: 0.00203807 0.0596892 0.0724977 0.0563221 0.112578 0.115286 0.182492 0.221768 0.2
##   2:    -3933.7435: -0.0146860 0.00377473 0.0193849 0.0331444 0.0514716 0.0663849 0.0853499 0.102504
##   3:    -3934.2222: -0.0151670 0.000947161 0.0163475 0.0312812 0.0473512 0.0625574 0.0787645 0.09450
##   4:    -3934.2650: -0.0152307 0.000396807 0.0157268 0.0308600 0.0464619 0.0616969 0.0773396 0.09277
##   5:    -3934.2740: -0.0152479 0.000190527 0.0154876 0.0306891 0.0461091 0.0613487 0.0767737 0.09208
##   6:    -3934.2768: -0.0152545 7.03780e-05 0.0153449 0.0305828 0.0458936 0.0611325 0.0764274 0.09167
##   7:    -3934.2772: -0.0152546 4.19982e-05 0.0153097 0.0305548 0.0458384 0.0610757 0.0763386 0.09156
##   8:    -3934.2773: -0.0152185 0.000124680 0.0153761 0.0305641 0.0458912 0.0610974 0.0764186 0.09166
##   9:    -3934.2774: -0.0152840 -0.000102592 0.0151596 0.0304703 0.0456439 0.0609020 0.0760296 0.0911
##  10:    -3934.2774: -0.0152690 -5.16544e-05 0.0152078 0.0304908 0.0456984 0.0609447 0.0761154 0.0912
##  11:    -3934.2774: -0.0152589 -1.72806e-05 0.0152405 0.0305047 0.0457355 0.0609739 0.0761738 0.0913
##  12:    -3934.2774: -0.0152567 -9.50755e-06 0.0152479 0.0305080 0.0457440 0.0609806 0.0761872 0.0913
##  13:    -3934.2774: -0.0152563 -3.22534e-06 0.0152552 0.0305131 0.0457547 0.0609909 0.0762039 0.0914
##  14:    -3934.2774: -0.0152529 4.02044e-06 0.0152608 0.0305136 0.0457588 0.0609923 0.0762104 0.09140
##  15:    -3934.2774: -0.0152529 4.02044e-06 0.0152608 0.0305136 0.0457588 0.0609923 0.0762104 0.09140
## Post processing for method  nlminb
## Successful convergence!
## Method:  nvm
## parchanged =  FALSE
## Parameter scaling:  [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [75] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
## nvm -- J C Nash 2009-2015 - an R implementation of Alg 21
## Problem of size n= 100
## Initial fn= -2237.155
## ig= 1   gnorm= 710.8478     1   1   -2237.155
## **ig= 2   gnorm= 129.1917     4   2   -2491.574
## ig= 3   gnorm= 18.97807     5   3   -3435.185
## ig= 4   gnorm= 18.87864     6   4   -3465.305
## *ig= 5   gnorm= 18.24925     8   5   -3611.518
## *ig= 6   gnorm= 17.98799     10   6   -3667.053
```

```
## ig= 7   gnorm= 21.21118     11   7   -3717.593
## ig= 8   gnorm= 13.22653     12   8   -3839.63
## ig= 9   gnorm= 1.618186     13   9   -3933.267
## ig= 10   gnorm= 0.3185998     14   10   -3934.235
## ig= 11   gnorm= 0.1079007     15   11   -3934.273
## ig= 12   gnorm= 0.004133489     16   12   -3934.277
## ig= 13   gnorm= 0.001136791     17   13   -3934.277
## ig= 14   gnorm= 1.155324e-05     18   14   -3934.277
## ig= 15   gnorm= 3.614907e-06     19   15   -3934.277
## ig= 16   gnorm= 2.943182e-08     20   16   -3934.277
## *ig= 17   gnorm= 2.270494e-08     22   17   -3934.277
## ********ig= 18   gnorm= 2.270489e-08     31   18   -3934.277
## **********No acceptable point
## Converged
## Seem to be done nvm
## Post processing for method  nvm
## Successful convergence!
## Above for n= 100
```

The timings for these matrices of order 20 to 100 are likely too short to be very reliable in detail, but do show that the RQ problem using the scaling transformation and with an analytic gradient can be solved very quickly, especially by the limited memory methods such as L-BFGS-B and ncg. Below we use the latter to show the times over different matrix sizes.

```
ctable<-matrix(NA, nrow=10, ncol=2)
nmax<-5
for (ni in 1:nmax){
  n<-50*ni
  x<-runif(n) # generate a vector
  AA<-molerfast(n) # define matrix
  tcgu<-microbenchmark(arcgu<-optimr(x, fn=nobj, gr=ngrobj, method="ncg",
          AA=-AA), times=mbt)
  ctable[[ni,1]]<-n
  ctable[[ni,2]]<-mean(tcgu$time)*0.001
}
cgtime<-data.frame(n=ctable[,1], tcgmin=ctable[,2])
print(round(cgtime,0))
```

```
##       n tcgmin
## 1   50    511
## 2  100   1449
## 3  150   4257
## 4  200   3125
## 5  250   4861
## 6   NA     NA
## 7   NA     NA
## 8   NA     NA
## 9   NA     NA
## 10  NA     NA
```

A specialized minimizer - Geradin's method

For comparison, let us try the Geradin routine (Appendix 1) as implemented in R by one of us (JN).

```
cat("Test geradin with explicit matrix multiplication\n")
```

```
## Test geradin with explicit matrix multiplication
```

```
n<-10
AA<-molermat(n)
BB=diag(rep(1,n))
x<-runif(n)
tg<-microbenchmark(ag<-geradin(x, ax, bx, AA=AA, BB=BB,
   control=list(trace=FALSE)), times=mbt)
cat("Minimal eigensolution\n")
```

```
## Minimal eigensolution
```

```
print(ag)
```

```
## $x
##  [1] 386618.971 193310.315  96657.231  48332.971  24175.300  12105.330
##  [7]   6088.052   3114.812   1698.986   1132.655
##
## $RQ
## [1] 8.582807e-06
##
## $ipr
## [1] 44
##
## $msg
## [1] "Small gradient -- done"
```

```
cat("Geradin time=",msect(tg$time),"sd=",msecr(tg$time),"\n")
```

```
## Geradin time= 3009 sd= 11169
```

```
tgn<-microbenchmark(agn<-geradin(x, ax, bx, AA=-AA, BB=BB,
   control=list(trace=FALSE)), times=mbt)
cat("Maximal eigensolution (negative matrix)\n")
```

```
## Maximal eigensolution (negative matrix)
```

```
print(agn)
```

```
## $x
##  [1] -228931868277    7727064063  244106832738  472244009160  684423411715
##  [6]  873479917761 1033015295553 1157643756754 1243146919891 1286626829004
##
## $RQ
## [1] -31.58981
##
## $ipr
## [1] 35
##
## $msg
## [1] "Small gradient -- done"
```

```
cat("Geradin time=",msect(tgn$time),"sd=",msecr(tgn$time),"\n")
```

```
## Geradin time= 466 sd= 15
```

Let us time this routine with different matrix vector approaches.

```
naximp<-function(x, A=1){ # implicit moler A*x
   n<-length(x)
   y<-rep(0,n)
   for (i in 1:n){
      tt<-0.
      for (j in 1:n) {
          if (i == j) tt<-tt+i*x[i]
          else tt<-tt+(min(i,j) - 2)*x[j]
      }
      y[i]<- -tt # include negative sign
   }
   y
}

dyn.load("moler.so")
cat("Is the mat multiply loaded? ",is.loaded("moler"),"\n")
```

```
## Is the mat multiply loaded?  TRUE
```

```
naxftn<-function(x, A) { # ignore second argument
   n<-length(x) # could speed up by having this passed
   vout<-rep(0,n) # purely for storage
   res<-(-1)*(.Fortran("moler", n=as.integer(n), x=as.double(x), vout=as.double(vout)))$vout
}

require(microbenchmark)
nmax<-10
gtable<-matrix(NA, nrow=nmax, ncol=6) # to hold results
# loop over sizes
for (ni in 1:nmax){
  n<-50*ni
  x<-runif(n) # generate a vector
  gtable[[ni, 1]]<-n
  AA<-molermat(n)
  BB<-diag(rep(1,n))
  tgax<-microbenchmark(ogax<-geradin(x, ax, bx, AA=-AA, BB=BB, control=list(trace=FALSE)), times=mbt)
  gtable[[ni, 2]]<-msect(tgax$time)
  tgaximp<-microbenchmark(ogaximp<-geradin(x, naximp, ident, AA=1, BB=1, control=list(trace=FALSE)), ti
  gtable[[ni, 3]]<-msect(tgaximp$time)
  tgaxftn<-microbenchmark(ogaxftn<-geradin(x, naxftn, ident, AA=1, BB=1, control=list(trace=FALSE)), ti
  gtable[[ni, 4]]<-msect(tgaxftn$time)
}

gtym<-data.frame(n=gtable[,1], ax=gtable[,2], aximp=gtable[,3], axftn=gtable[,4])
print(gtym)
```

```
##      n    ax   aximp axftn
## 1   50  2677   27695   955
## 2  100  1451   90932  1166
## 3  150  2415  243310  2418
## 4  200  3760  432462  3754
## 5  250  4493  739758  5920
## 6  300  5488  975320  7571
## 7  350  7087 1354281 10484
```

```
## 8  400  9801 1772764 13675
## 9  450 17108 2319410 17501
## 10 500 19143 2824641 20550
```

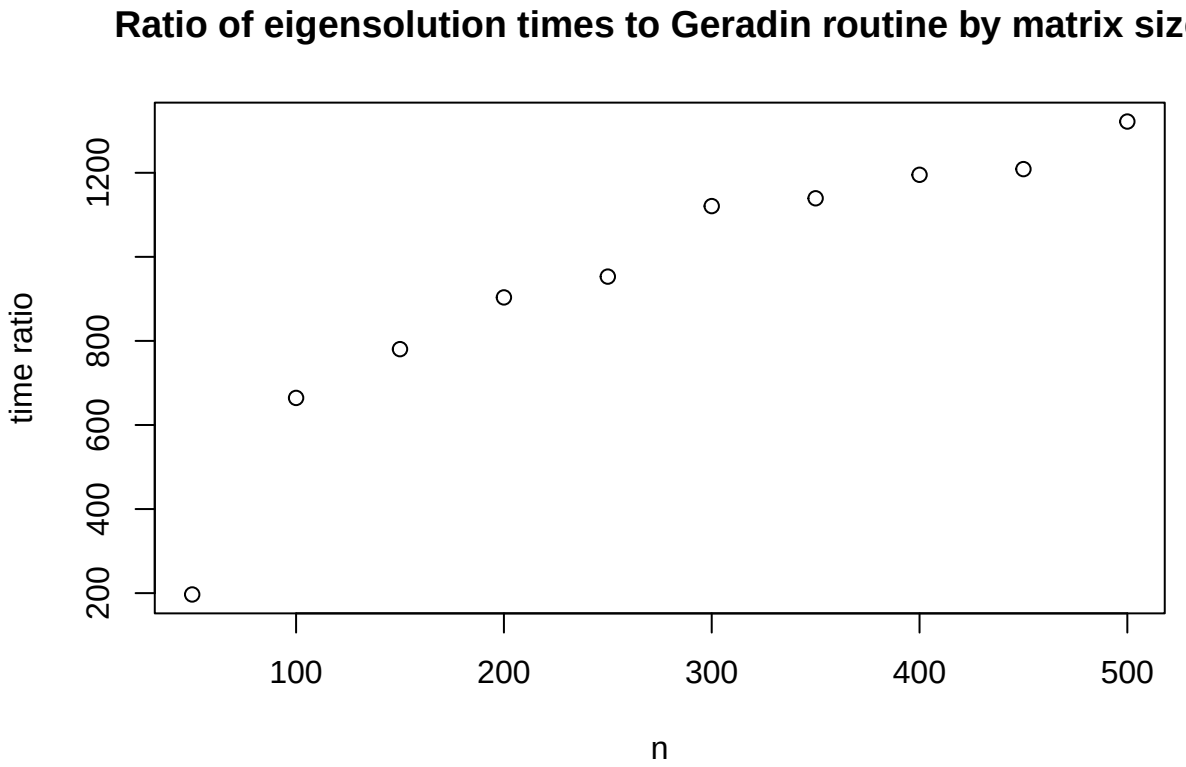
Let us check that the solution for `n = 100` by Geradin is consistent with the answer via `eigen()`.

```
n<-100
x<-runif(n)
evalmax<-emax$evalmax
evecmac<-emax$evecmax
ogaxftn<-geradin(x, naxftn, ident, AA=1, BB=1, control=list(trace=FALSE))
gvec<-ogaxftn$x
gval<- -ogaxftn$RQ
gvec<-sign(gvec[[1]])*gvec/sqrt(as.numeric(crossprod(gvec)))
diff<-gvec-evecmax
cat("Geradin eigenvalue - eigen result: ",gval-evalmax,"   max(abs(vector diff))=",
      max(abs(diff)), "\n")
```

```
## Geradin eigenvalue - eigen result:  -6.494235e-06    max(abs(vector diff))= 7.995635e-06
```

Perspective

We can compare the different approaches by looking at the ratio of the best solution time for each method (compiled or interpreted, with best choice of function) to the time for the Geradin approach for the different matrix sizes. In this we will ignore the fact that some approaches do not build the matrix.



To check the value of the Geradin approach, let us use a much larger problem, with `n=2000`.

```
## Times in seconds
## Build = 86951  eigen(): 836969   Rcgminu: 316848  Geradin: 313233
## Ratios: build= 0.2775921 eigen= 2.672033   Rcgminu= 1.011541
```

Conclusions}

The Rayleigh Quotient minimization approach to eigensolutions has an intuitive appeal and seemingly offers an interesting optimization test problem, especially if we can make it computationally efficient. To improve time efficiency, we can apply the R byte code compiler, use a Fortran (or other compiled language) subroutine, and choose how we set up our objective functions and gradients. To improve memory use, we can consider using a matrix implicitly.

From the tests in this vignette, here is what we may say about these attempts, which we caution are based on a relatively small sample of tests:

- The R byte code compiler offers a useful gain in speed when our code has statements that access array elements rather than uses them in vectorized form.}
- The `crossprod()` function is very efficient.
- Fortran is not very difficult to use for small subroutines that compute a function such as the implicit matrix-vector product, and it allows efficient computations for such operations.
- The `eigen()` routine is a highly effective tool for computing all eigensolutions, even of a large matrix. It is only worth computing a single solution when the matrix is very large, in which case a specialized method such as that of Geradin makes sense and offers significant savings, especially when combined with the Fortran implicit matrix-product routine.

Acknowledgements

This vignette originated due to a problem suggested by Gabor Grothendieck. Ravi Varadhan has provided inciteful comments and some vectorized functions which greatly altered some of the observations.

Appendix 1: Geradin routine

```
ax<-function(x, AA){
   u<-as.numeric(AA%*%x)
}
bx<-function(x, BB){
   v<-as.numeric(BB%*%x)
}
geradin<-function(x, ax, bx, AA, BB, control=list(trace=TRUE, maxit=1000)){
# Geradin minimize Rayleigh Quotient, Nash CMN Alg 25
# print(control)
  trace<-control$trace
  n<-length(x)
  tol<-n*n*.Machine$double.eps^2
  offset<-1e+5 # equality check offset
  if (trace) cat("geradin.R, using tol=",tol,"\n")
  ipr<-0 # counter for matrix mults
  pa<-.Machine$double.xmax
  R<-pa
  msg<-"no msg"
# step 1 -- main loop
  keepgoing<-TRUE
  while (keepgoing) {
    avec<-ax(x, AA); bvec<-bx(x, BB); ipr<-ipr+1
    xax<-as.numeric(crossprod(x, avec));
    xbx<-as.numeric(crossprod(x, bvec));
    if (xbx <= tol) {
       keepgoing<-FALSE # not really needed
       msg<-"avoid division by 0 as xbx too small"
```

```
   break
}
p0<-xax/xbx
if (p0>pa) {
   keepgoing<-FALSE # not really needed
   msg<-"Rayleigh Quotient increased in step"
   break
}
pa<-p0
g<-2*(avec-p0*bvec)/xbx
gg<-as.numeric(crossprod(g)) # step 6
if (trace) cat("Before loop: RQ=",p0," after ",ipr," products, gg=",gg,"\n")
if (gg<tol) { # step 7
   keepgoing<-FALSE # not really needed
   msg<-"Small gradient -- done"
   break
}
t<- -g # step 8
for (itn in 1:n) { # major loop step 9
   y<-ax(t, AA); z<-bx(t, BB); ipr<-ipr+1 # step 10
   tat<-as.numeric(crossprod(t, y)) # step 11
   xat<-as.numeric(crossprod(x, y))
   xbt<-as.numeric(crossprod(x, z))
   tbt<-as.numeric(crossprod(t, z))
   u<-tat*xbt-xat*tbt
   v<-tat*xbx-xax*tbt
   w<-xat*xbx-xax*xbt
   d<-v*v-4*u*w
   if (d<0) stop("Geradin: imaginary roots not possible") # step 13
   d<-sqrt(d) # step 14
   if (v>0) k<--2*w/(v+d) else k<-0.5*(d-v)/u
   xlast<-x # NOT as in CNM -- can be avoided with loop
   avec<-avec+k*y; bvec<-bvec+k*z # step 15, update
   x<-x+k*t
   xax<-xax+as.numeric(crossprod(x,avec))
   xbx<-xbx+as.numeric(crossprod(x,bvec))
   if (xbx<tol) stop("Geradin: xbx has become too small")
   chcount<-n - length(which((xlast+offset)==(x+offset)))
   if (trace) cat("Number of changed components = ",chcount,"\n")
   pn<-xax/xbx # step 17 different order
   if (chcount==0) {
     keepgoing<-FALSE # not really needed
     msg<-"Unchanged parameters -- done"
     break
   }
   if (pn >= p0) {
     if (trace) cat("RQ not reduced, restart\n")
     break # out of itn loop, not while loop (TEST!)
   }
   p0<-pn # step 19
   g<-2*(avec-pn*bvec)/xbx
   gg<-as.numeric(crossprod(g))
   if (trace) cat("Itn", itn," RQ=",p0," after ",ipr," products, gg=",gg,"\n")
```

```
      if (gg<tol){ # step 20
        if (trace) cat("Small gradient in iteration, restart\n")
        break # out of itn loop, not while loop (TEST!)
      }
      xbt<-as.numeric(crossprod(x,z)) # step 21
      w<-y-pn*z # step 22
      tabt<-as.numeric(crossprod(t,w))
      beta<-as.numeric(crossprod(g,(w-xbt*g)))
      beta<-beta/tabt # step 23
      t<-beta*t-g
    } # end loop on itn -- step 24
  } # end main loop -- step 25
  ans<-list(x=x, RQ=p0, ipr=ipr, msg=msg) # step 26
}
```

References

Nash, John C. 1979. *Compact Numerical Methods for Computers: Linear Algebra and Function Minimisation.* Bristol: Adam Hilger.