

# NOTES ON QUANTUM K THEORY OF FLAG MANIFOLDS

LEONARDO CONSTANTIN MIHALCEA

## CONTENTS

1. Preliminaries	2
1.1. Notation	2
1.2. Goal	4
1.3. The Schubert package	5
2. K theory	6
2.1. Generalities	6
2.2. K theory of flag manifolds	10
2.3. The K-theoretic Schubert package	13
3. Definition of quantum K theory and some properties	14
3.1. The moduli space	14
3.2. Definition of quantum K theory	14
3.3. Preview	17
4. Two point KGW invariants via curve neighborhoods	19
4.1. Definition and first properties	20
4.2. Calculation of curve neighborhoods	23
4.3. A K theoretic divisor axiom	26
5. The ‘quantum=classical’ statement and applications	27
5.1. The statement	27
5.2. A Pieri/Chevalley rule	29
6. Presentations of the quantum K ring of flag manifolds	31
6.1. A presentation for the QK ring of Grassmannians	31
6.2. The Coulomb branch presentation	33
7. Quantum K theory of Grassmannians from a Yang-Baxter algebra	37
7.1. Quantum K theory as a Yang-Baxter module	37
7.2. Graphical calculus	40
7.3. Bethe vectors	42
7.4. Example: $n = 2$	43
8. The (quantum K) Schubert package	48
8.1. The Schubert package	48
8.2. Some questions	49

---

*Date:* August 10, 2024.

These notes grew from my lectures on quantum K theory of flag manifolds at the summer school ‘Representation theory and flag or quiver varieties’ (June 2022, Paris, France), at the Schubert Calculus summer school at University of Illinois at Urbana-Champaign (June 2023), and at the Mathematical Society of Japan Summer Institute MSJ-SI 2023 ‘Elliptic Integrable Systems, Representation Theory and Hypergeometric Functions’ (Tokyo, July 2023). The author is grateful to the organizers of all these schools (Nicolas Perrin, Olivier Schiffmann, Michela Varagnolo, Eric Vasserot; Andrew Hardt, Reuven Hodges, Elizabeth Kelley, Avery St. Dizier, and Alexander Yong; Hitoshi Konno, Masatoshi Noumi, Jun’ichi Shiraishi, Kouichi Takemura, and Kohei Motegi) for the hospitality and support, and for the opportunity to lecture on this beautiful topic. Special thanks are due to Hitoshi Konno, Kohei Motegi, and the Math Society of Japan for their encouragement in publishing these notes. I would also like to thank the participants to all these schools for their questions, which helped me in various choices of the material. Finally, I am indebted to an anonymous referee for careful reading and for valuable suggestions, which helped me improve the exposition. The author was supported in part by the NSF grant DMS-2152294 and a Simons Collaboration Grant.

## 1. PRELIMINARIES

**1.1. Notation.** Throughout the notes,  $G := \mathrm{GL}_n$  denotes the complex general linear group,  $B, B^-$  a pair of opposite Borel subgroups, e.g. the upper/lower triangular matrices. The associated Weyl group is  $W = S_n$ , the symmetric group in  $n$  letters. It is generated by simple reflections  $s_i = (i, i+1)$  where  $1 \leq i \leq n-1$ . Denote by  $\ell : W \rightarrow \mathbb{N}$  the length function, defined to be the minimal number  $k = \ell(w)$  so that  $w = s_{i_1} \cdots s_{i_k}$ . Denote also by  $w_0$  the longest element; in the Lie type A (which is our case here),  $w_0$  is the permutation defined by  $w_0(i) = n+1-i$ , for  $1 \leq i \leq n$ . There is a partial order on  $W$ , called the **Bruhat order**, defined by the condition that  $u < v$  if there is a chain  $u = u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_k = v$  such that  $u_{j+1} = u_j \cdot (i_j, i_{j+1})$  and  $\ell(u_{j+1}) > \ell(u_j)$ .

A sequence  $I = (1 \leq i_1 < \cdots < i_k \leq n-1)$  determines a collection of simple roots  $s_i$  ( $i \in I$ ), and a parabolic subgroup  $P \subset G$  so that  $s_i \notin P$ . The **partial flag manifold**  $\mathrm{Fl}(i_1, i_2, \dots, i_k; n)$  is the homogeneous space

$$G/P = \{F_{i_1} \subset F_{i_2} \subset \cdots \subset F_{i_k} \subset \mathbb{C}^n : \dim F_{i_s} = i_s\}.$$

Of particular interest will be the **Grassmannians**

$$\mathrm{Gr}(k; n) = \{V \subset \mathbb{C}^n : \dim V = k\}$$

and the **complete flag manifolds**

$$\mathrm{Fl}(n) = \{F_1 \subset F_2 \subset \cdots \subset \mathbb{C}^n : \dim F_i = i\}.$$

The first example corresponds to  $I = \{k\}$  and the second to  $I = \{1, 2, \dots, n-1\}$ .

These are homogeneous spaces for  $G$ .<sup>1</sup> For a sequence  $I = (1 \leq i_1 < \dots < i_k \leq n-1)$  of simple roots, define  $W_P$  to be the subgroup generated by the simple reflections  $s_i = (i, i+1)$  where  $i \notin I$ , and set  $W^P := W/W_P$ , the set of minimal length representatives. One can check that

$$W^P = \{w \in W : w \text{ has descents at most in positions } i_1, \dots, i_s\}.$$

Recall that a permutation  $w \in W = S_n$  has a descent at position  $i$  if  $w(i) > w(i+1)$ .

If  $P$  is a maximal parabolic, i.e.  $G/P = \text{Gr}(k; n)$  is a Grassmann manifold, then  $W^P$  is in bijection with the set of **partitions**  $\lambda = (\lambda_1, \dots, \lambda_k)$  with  $0 \leq \lambda_k \leq \dots \leq \lambda_1 \leq n-k$ . The weight of such a partition is  $|\lambda| = \lambda_1 + \dots + \lambda_k$ , and the bijection satisfies  $\ell(w) = |\lambda|$  if  $w$  corresponds to  $\lambda$ .

Fix  $\{e_1, \dots, e_n\}$  the standard basis of  $\mathbb{C}^n$ . The maximal torus of diagonal matrices  $T \subset G$  acts on  $G/P$  and the  $T$ -fixed points are coordinate flags  $\{\mathbf{e}_w : w \in W^P\}$  where

$$\mathbf{e}_w := \langle e_{w(1)}, \dots, e_{w(i_1)} \rangle \subset \langle e_{w(1)}, \dots, e_{w(i_2)} \rangle \subset \dots \subset \langle e_{i_1}, \dots, e_{w(i_k)} \rangle \subset \mathbb{C}^n,$$

and  $e_i$  are the standard basis vectors. To each  $w \in W^P$  there are two **Schubert varieties**:

$$X_w = \overline{B \cdot \mathbf{e}_w}; \quad X^w = \overline{B^- \cdot \mathbf{e}_w},$$

where  $B, B^- \subset \text{GL}_n$  are opposite Borel subgroups. The orbits

$$X_w^\circ = B \cdot \mathbf{e}_w; \quad X^{w, \circ} = B^- \cdot \mathbf{e}_w$$

are called **Schubert cells**. With these conventions,  $X_w^\circ \simeq \mathbb{A}^{\ell(w)}$  and  $X^{w, \circ} \simeq \mathbb{A}^{\dim G/P - \ell(w)}$ . In particular,

$$\dim X_w = \text{codim } X^w = \ell(w); \quad X_w \cap X^w = \{\mathbf{e}_w\}; \quad X^w = w_0 X_{w_0 w w_P}$$

where  $w_P \in W_P$  is the longest element. With these definitions, the **Bruhat order** on  $W^P$  is defined by

$$v < w \text{ in } W^P \iff X_v \subset X_w \iff X^v \supset X^w.$$

(This coincides with the order induced from  $W$  using the inclusion  $W^P \subset W$ .) The partial flag manifolds have a stratification by Schubert cells:

$$G/P = \bigsqcup_{w \in W^P} X_w^\circ = \bigsqcup_{w \in W^P} X^{w, \circ}.$$

For further use we also recall that a partial flag variety  $\text{Fl}(i_1, \dots, i_k; n)$  is equipped with a tautological sequence of vector bundles:

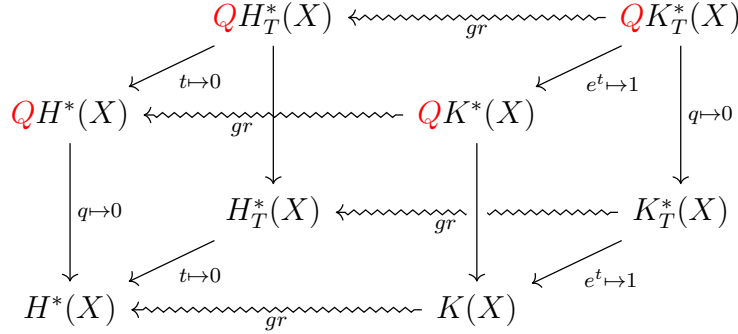
$$0 \rightarrow \mathcal{S}_1 \hookrightarrow \mathcal{S}_2 \hookrightarrow \dots \hookrightarrow \mathcal{S}_k \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathcal{Q}_{n-i_1} \twoheadrightarrow \mathcal{Q}_{n-i_2} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{Q}_{n-i_s} \twoheadrightarrow 0,$$

where  $\text{rank}(\mathcal{S}_j) = i_j$ .

---

<sup>1</sup>We work with  $\text{GL}_n$  for concreteness; often, the reader may replace  $G$  by any complex semisimple Lie group,  $\text{Fl}(n)$  by any  $G/B$ , and Grassmannians by any cominuscule Grassmannian.

**1.2. Goal.** The goal of these lectures is to introduce the quantum K theory ring, some of its basic properties, and computational techniques, with an emphasis on the case of Grassmannians. Before we proceed, we represent schematically the relationships between various (classical and quantum) intersection rings which are available in the literature.



The decorations on the arrows indicate the operations relating the rings on each arrow. For example, all vertical arrows are decorated with  $q \mapsto 0$ , meaning that the classical rings are obtained from the quantum ones upon dividing by the ideal of quantum parameters  $q$ . The decoration ‘ $gr$ ’ means ‘take associated graded’.

Until the section 7, we will not discuss much about the **equivariant** version of all these rings, but most techniques discussed below extend to this situation. We will attempt to point out if any changes are needed to make statements in that generality.

To illustrate some of the features of the quantum K multiplication, we give a (way too early) example<sup>2</sup> in  $QK(\text{Gr}(3,6))$ , where  $\mathcal{O}^\lambda$  denotes the structure sheaf of a Schubert variety. Here,  $\deg q = 6$ :

$$\begin{aligned}
 \mathcal{O}^{(3,2,1)} \star \mathcal{O}^{(2,1)} &= q\mathcal{O}^{(3)} + 2q\mathcal{O}^{(2,1)} - 2q\mathcal{O}^{(2,2)} \\
 (1.1) \quad &- 2q\mathcal{O}^{(3,1)} + q\mathcal{O}^{(3,2)} + q\mathcal{O}^{(1,1,1)} - 2q\mathcal{O}^{(2,1,1)} + q\mathcal{O}^{(2,2,1)} + q\mathcal{O}^{(3,1,1)} \\
 &- q\mathcal{O}^{(3,2,1)} + \mathcal{O}^{(3,3,3)}.
 \end{aligned}$$

To perform the calculation of this product we used the ‘quantum=classical’ statement, discussed later in these notes. One may observe that the sum of the coefficients is

$$q + 2q - 2q - 2q + q + q - 2q + q + q - q + 1 = 1.$$

One can also note a positivity statement: the sign of the coefficient of  $q^d \mathcal{O}^\nu$  is given by the parity of

$$|(3,2,1)| + |(2,1)| - d \cdot \deg q - |\nu| = 6 + 3 - 6d - |\nu|.$$

Is there a way to generalize, and explain, such properties ?

<sup>2</sup>The example was obtained utilizing A. Buch’s **Equivariant Schubert Calculator** available at <https://sites.math.rutgers.edu/~asbuch/equivcalc/>

**1.3. The Schubert package.** While these notes focus on the quantum K theory of Grassmanians, one may ask more generally for some common themes in the study of any (quantum, generalized) cohomology theory  $\mathfrak{C}(G/P)$  of a flag manifold  $G/P$ . The **Schubert package** below is an attempt to find such a list of objects and questions. As one moves along in the notes, it could be instructive for the reader to identify which (quantum) cohomology theory, and which portion of the Schubert package, is discussed at that point.

Any cohomology theory  $\mathfrak{C}(G/P)$  is equipped with an ‘intersection pairing’  $((\cdot, \cdot))$ , which is a symmetric, bilinear, non-degenerate pairing. For ‘classical’ theories (such as cohomology, or K-theory), this is the usual intersection pairing, given by

$$((a, b)) = \int_{G/P} a \cdot b \quad .$$

For ‘quantum’ theories, this pairing is part of the definition of the theory. Given the pair  $(\mathfrak{C}(G/P), ((\cdot, \cdot)))$  a *Schubert package* consists of :

- (Basis) Make a choice of a ‘Schubert basis’  $\{\mathfrak{s}_w\}_{w \in W^P}$  of  $\mathfrak{C}(G/P)$ ;
- (Duality) Identify the dual basis of the Schubert basis with respect to the intersection pairing;
- (Presentation + Giambelli problem) Find a presentation by generators and relations of the ring  $\mathfrak{C}(G/P)$  and identify polynomials representing the Schubert classes  $\mathfrak{s}_w$ ;
- (Positivity) Find and prove a positivity property for the structure constants  $c_{u,v}^w$  arising from the multiplication of two Schubert classes:

$$\mathfrak{s}_u \star \mathfrak{s}_v = \sum c_{u,v}^w \mathfrak{s}_w;$$

- (Schubert multiplication) Find appropriately positive formulae to multiply in  $\mathfrak{C}(G/P)$ . There are three types of formulae:
  - a Chevalley formula, i.e., a multiplication by any class associated to a line bundle;
  - a Pieri formula: a multiplication by any (ring) generator;
  - a Littlewood-Richardson (LR) rule, i.e., a manifestly positive formula to multiply any two Schubert classes.

To illustrate, consider the Grassmannian  $\text{Gr}(k; n)$ , the ordinary cohomology  $H^*(\text{Gr}(k; n))$ , equipped with the usual intersection pairing  $\langle \cdot, \cdot \rangle$ . Then a Schubert package is given by (see, e.g., [Ful97]):

- The basis obtained by taking the fundamental classes of Schubert varieties  $\{[X^\lambda]\}$ ;
- The Schubert classes are self-dual:  $\langle [X^\lambda], [X^\mu] \rangle = \delta_{\lambda, \mu^\vee}$ . (This uses that  $[X_\lambda] = [X^{\lambda^\vee}]$  and  $\dim(X^\lambda \cap X^\mu) = 0$  iff  $\lambda = \mu$ .)
- One may take a presentation of  $H^*(\text{Gr}(k; n))$  with Schur polynomials  $s_\lambda(x_1, \dots, x_k)$  representing the Schubert classes  $[X^\lambda]$ ;

- Transversality implies that the Schubert structure constants  $c'_{\lambda,\mu}$  from the multiplication  $[X^\lambda] \cdot [X^\mu] = \sum c'_{\lambda,\mu} [X^\nu]$  count points in the intersection of general translates of  $X^\lambda$ ,  $X^\mu$  and  $X^{\nu^\vee}$ , thus  $c'_{\lambda,\mu} \geq 0$ ;
- Finally, the classical Pieri-Chevalley rule, and, more generally, any of the known (positive) Littlewood-Richardson rules, give positive combinatorial algorithms to multiply any two Schubert classes.

In the case of the *equivariant* (potentially quantum) cohomology theories, one can enlarge the Schubert package by including the ‘fixed point basis’ (i.e., the classes associated to the fixed points). This basis is independent of choices, and one of the basic questions is to study how it interacts with the chosen Schubert basis, e.g., find transition matrices between the two bases. Proofs of several LR rules took full advantage of this equivariant structure. For example, see e.g., [KT03].

We note that, given a cohomology theory, there may be several choices of Schubert bases with good properties. For example, in the case of the cohomology ring  $H^*(\mathrm{Gr}(k;n))$  from before, instead of choosing the fundamental classes of Schubert varieties  $[X^\lambda]$  one may take the basis given by the Chern-Schwartz-MacPherson classes of Schubert cells,  $\{\mathrm{csm}(X^{\lambda,\circ})\} \subset H^*(\mathrm{Gr}(k;n))$ . The Schubert package arising from that choice gives the ‘Cotangent Schubert Calculus’, which includes a positivity property for the Schubert structure constants [KZJ21, AMSS24b, SSW23].

Finally, we comment that there is no *a priori* reason why, given a (quantum) cohomology theory  $\mathfrak{C}(G/P)$ , there exists a Schubert basis satisfying a positivity property. History suggests that this is the case for any classical or quantum generalized cohomology theory. (Many of these statements, especially in the quantum case, are still conjectural.) As we noticed above in relation to the ‘Cotangent Schubert Calculus’, such a basis, if exists, it may not be unique.

## 2. K THEORY

In this section we collect few facts we need about the K theory of an algebraic variety, then specialize to the case of flag manifolds. More details, including proofs, for this material included in this section, can be found e.g. in Brion’s ‘Lectures on flag manifolds’ [Bri05], and Chriss and Ginzburg’s ‘Representation theory and complex algebraic geometry’ [CG09].

**2.1. Generalities.** Let  $X$  be any algebraic variety. The (Grothendieck) K theory ring, denoted by  $K(X)$ , is defined as the ring generated by symbols  $[E]$  for (algebraic) vector bundles  $E \rightarrow X$ , modulo the relations  $[E_2] = [E_1] + [E_3]$  for any short exact sequence of vector bundles  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ . The addition and multiplication are given by the direct sum and the tensor product:

$$[E_1] + [E_2] = [E_1 \oplus E_2]; \quad [E_1] \cdot [E_2] = [E_1 \otimes E_2].$$

Then  $K(X)$  becomes a commutative ring with identity the (class of the) trivial, rank 1 vector bundle, which we often denote by  $\mathcal{O}$ . If  $X$  is complete (e.g., projective), this ring

is equipped with an **intersection pairing**:

$$\langle [E], [F] \rangle = \int_X [E] \cdot [F] = \chi(X, E \otimes F),$$

where  $\chi$  denotes the sheaf Euler characteristic. If  $X$  is further assumed to be a (quasi-projective) **manifold**, then one can construct *finite* resolutions of coherent sheaves:

**Theorem 2.1** (Resolutions of coherent sheaves). *Let  $X$  be a smooth, quasi-projective variety and let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Then  $\mathcal{F}$  has a finite resolution by locally free sheaves (i.e., vector bundles) on  $X$ :*

$$0 \rightarrow E_n \rightarrow E_{n-1} \rightarrow \dots \rightarrow E_1 \rightarrow \mathcal{F} \rightarrow 0.$$

Furthermore, one may assume that  $n \leq \dim X$ .

A proof can be found on [CG09, Prop. 5.1.29]. If  $X$  is smooth, the theorem allows us to define the class of any coherent sheaf as an element in K theory:

$$[\mathcal{F}] = \sum_{i=1}^n (-1)^{i-1} [E_i] \in K(X).$$

In the literature, the Grothendieck ring of vector bundles is sometimes denoted by  $K^\circ(X)$ . One may also define the Grothendieck *group* of coherent sheaves, denoted by  $K_\circ(X)$ , consisting of classes  $[\mathcal{F}]$  of coherent sheaves on  $X$  modulo the usual additive relations for short exact sequences. The addition is given by direct sum. Regarding a vector bundle as a locally free sheaf, then taking tensor products gives  $K_\circ(X)$  a structure of  $K^\circ(X)$ -module. (Note the similarity to the homology, realized as a module over cohomology.) If  $X$  is smooth, Theorem 2.1 implies that  $K^\circ(X) \simeq K_\circ(X)$ .

The most interesting coherent sheaves are the structure sheaves of subvarieties of  $X$ . If the subvarieties in question have nice singularities, then the product of classes becomes especially nice. For the following see [Bri05, Lemmas 4.1.1 and 4.1.2], who further refers to a lemma of Fulton and Pragacz.

**Lemma 2.2.** *Let  $Y, Z$  be equidimensional Cohen-Macaulay subvarieties of a nonsingular variety  $X$ . Assume that the intersection  $Y \cap Z$  is proper, i.e., it has the expected dimension  $\dim Y + \dim Z - \dim X$ . Then each component of the scheme theoretic intersection  $Y \cap Z$  has the expected dimension and  $Y \cap Z$  is Cohen-Macaulay. Furthermore,*

$$[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y \cap Z}] \in K(X).$$

Some examples of Cohen-Macaulay varieties are:

- Any smooth variety is Cohen-Macaulay.
- Any Schubert variety in a (generalized) flag manifold  $G/P$  has rational singularities (cf. Definition 2.11 below), therefore it is Cohen-Macaulay. (See, e.g., [Bri05, §2.2].)

- More generally, we have a Kleiman's transversality statement: if  $Y \subset X$  is Cohen-Macaulay, then for general  $g_1, \dots, g_k \in G$ ,  $Y \cap g_1 X^{w_1} \cap g_2 X^{w_2} \cap \dots \cap g_k X^{w_k}$  is either empty or purely-dimensional, of expected dimension, and Cohen-Macaulay. See [BCMP13], where we adapt Kleiman's argument from [Kle74].
- As a consequence of the previous point, any non-empty **Richardson variety**

$$R_u^v := X_u \cap X^v$$

must be Cohen-Macaulay.

- (To be defined later.) The moduli space of stable maps  $\overline{\mathcal{M}}_{0,n}(G/P, d)$  is Cohen-Macaulay, because it is locally a smooth variety modulo a finite group.
- Smooth pull-backs preserve the Cohen-Macaulay property.

**2.1.1. Functoriality.** For any morphism  $f : X \rightarrow Y$ , there is a **pull-back ring homomorphism**  $f^* : K^\circ(Y) \rightarrow K^\circ(X)$  given by  $[E] \mapsto [f^*E]$ . If  $f$  is flat and  $Z \subset X$  is a subvariety, then  $f^*[\mathcal{O}_Z] = [\mathcal{O}_{f^{-1}(Z)}]$ . For a proper morphism  $f : X \rightarrow Y$ , the **push-forward**  $f_* : K_\circ(X) \rightarrow K_\circ(Y)$  is defined by

$$f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}].$$

This sum is finite, as the higher direct images vanish beyond the dimension of  $X$ . The push-forward and pull-back satisfy the usual projection formula:

$$f_*(f^*[E] \otimes [\mathcal{F}]) = [E] \otimes f_*[\mathcal{F}] \in K(Y).$$

**2.1.2. The topological filtration and the Chern character.** For simplicity, assume that  $X$  is smooth, so we identify  $K_\circ(X) \simeq K^\circ(X) = K(X)$ . One important difference between K theory and (co)homology theory is that the K theory is not graded. However, one can define a **topological filtration** by defining  $\mathcal{K}^i(X)$  to be the subgroup of  $K_\circ(X)$  generated by sheaves  $[\mathcal{F}] \in K_\circ(X)$  which have support in codimension  $\geq i$ . Then

$$K^\circ(X) = \mathcal{K}^0(X) \supset \mathcal{K}^1(X) \supset \dots$$

is a decreasing filtration, and  $K(X)$  becomes a **filtered ring**, in the sense that  $\mathcal{K}^i(X) \cdot \mathcal{K}^j(X) \subset \mathcal{K}^{i+j}(X)$ .

Let  $A_*(X)$  denote the Chow group, generated by classes  $[Z]$  of irreducible subvarieties  $Z \subset X$  modulo rational equivalence, see [Ful84]. Let also

$$Gr(K(X)) = \bigoplus \mathcal{K}_i(X) / \mathcal{K}_{i+1}(X)$$

be the associated graded to the topological filtration. The class of a structure sheaf passes through rational equivalence, and one obtains a ring homomorphism

$$\Psi : A_*(X) \rightarrow Gr(K(X)); \quad [Z] \mapsto [\mathcal{O}_Z].$$

After tensoring with  $\mathbb{Q}$ , this is an isomorphism; cf. [Ful84, Ex. 15.1.5 and 15.2.16].

Furthermore, there is always a **Chern character**  $ch : K(X) \rightarrow A_*(X)_{\mathbb{Q}}$  defined by sending the class of a line bundle  $[L]$  to

$$ch[L] = e^{c_1(L)} = 1 + c_1(L) + c_1(L)^2/2! + \dots$$

For a general vector bundle  $E \rightarrow X$  one uses the splitting principle to define  $ch(E)$ . If  $X$  is smooth, it is shown e.g. in [Ful84, Ex. 15.2.16] that if  $Z \subset X$  is closed and irreducible, then

$$ch(Z) = [Z] + l.o.t.$$

where l.o.t. are terms in homological degree strictly less than  $\dim Z$ . In other words  $ch([\mathcal{O}_Z]) \in \oplus_{j \leq i} A_j(X)$ , where subscripts denote dimension. The Chern character is a **ring isomorphism**, if one works over  $\mathbb{Q}$ .

**2.1.3. The Hirzebruch  $\lambda_y$  class.** For a rank  $e$  vector bundle  $E \rightarrow X$ , the **Hirzebruch  $\lambda_y$  class** of  $E$  is defined by

$$\lambda_y(E) = 1 + y[E] + y^2[\wedge^2 E] + \dots + y^e[\wedge^e E] \in K(X)[y].$$

This class is multiplicative: if  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  is a short exact sequence then

$$\lambda_y(E_1) \cdot \lambda_y(E_3) = \lambda_y(E_2).$$

The class  $\lambda_{-1}(E^*)$  is sometimes called the K-theoretic Chern class of  $E$ , denoted by  $cK(E)$ . This is justified by the observation that if  $L$  is a line bundle with first Chern class  $c_1(L)$ , then

$$ch(\lambda_{-1}(L^*)) = 1 - e^{-c_1(L)} = c_1(L) + h.o.t.,$$

with h.o.t. are **cohomological** higher order terms. Furthermore, the identity

$$(1 - e^x)(1 - e^y) = (1 - e^x) + (1 - e^y) - (1 - e^{x+y})$$

implies that if  $L'$  is another line bundle, then

$$cK(L \oplus L') = cK(L) + cK(L') - cK(L \otimes L'),$$

recovering the formal group law for K theory.

Finally, note that the class  $\lambda_{-1}(E)$  appears geometrically as an Euler class, in the following sense: if  $E \rightarrow X$  is a vector bundle with a general section  $s : X \rightarrow E$ , then the zero locus of  $s$  has class

$$[\mathcal{O}_{Z(s)}] = \lambda_{-1}(E^*) \in K(X).$$

(This follows from the Koszul resolution of  $Z(s)$ .)

Virtually everything above extends to the equivariant case, provided one replaces all objects (bundles, sheaves, maps, varieties, etc.) with their equivariant versions. Informally, if  $X$  admits a  $G$ -action, an equivariant vector bundle  $\pi : E \rightarrow X$  is a vector bundle such that its total space admits a  $G$ -action,  $\pi$  is  $G$ -equivariant, and that the restriction to fibers induces a linear map. On flag manifolds  $G/P$ , many such vector

bundles are induced by representations: if  $V$  is a representation of the parabolic group  $P \subset G$ , define

$$\mathcal{V} = G \times^P V = \{[g, v] : [g, v] \equiv [gp, p^{-1}v]\}$$

This is equipped with a projection map  $\pi : \mathcal{V} \rightarrow G/P, [g, v] \mapsto gP$ , giving it a structure of a  $G$ -equivariant bundle. We refer again to Chriss and Ginzburg's book [CG09] for an introduction in equivariant K theory, and to the recent book by Anderson and Fulton [AF24] for more on equivariant intersection theory.

**2.2. K theory of flag manifolds.** For now we let  $X = G/P$  to be any flag manifold. For any Schubert variety  $\Omega \subset X$  with Schubert cell  $\Omega^\circ$ , define the **boundary** of  $\Omega$  to be  $\partial\Omega = \Omega \setminus \Omega^\circ$ . It is known that this is a Cohen-Macaulay Weil divisor in  $\Omega$ . The (Grothendieck) classes of the structure sheaves of  $X_w$ , or those  $X^w$  (for  $w \in W^P$ ) are denoted by  $\mathcal{O}_w, \mathcal{O}^w$  respectively. If  $G/P$  is a Grassmannian, and  $w$  corresponds to a partition  $\lambda$ , then we often use the notation  $\mathcal{O}_\lambda = \mathcal{O}_w$  and  $\mathcal{O}^\lambda = \mathcal{O}^w$ . The ideal sheaf of the boundary  $\partial X^w$  fits into an exact sequence:

$$0 \rightarrow \mathcal{I}_{\partial X^w} \rightarrow \mathcal{O}_{X^w} \rightarrow \mathcal{O}_{\partial X^w} \rightarrow 0.$$

We denote the K-theory classes of  $\mathcal{I}_{\partial X^w}$  and  $\mathcal{I}_{\partial X_w}$  by  $\mathcal{I}_w, \mathcal{I}^w$  respectively. Note that

$$\mathcal{O}_w = \mathcal{O}^{w^\vee} \text{ and } \mathcal{O}^w = \mathcal{O}_{w^\vee}$$

where  $w^\vee = w_0 w w_P$  is the minimal length representative for  $w_0 w$  in  $W^P$ , with  $w_P$  the longest element in  $W_P$ .<sup>3</sup> If  $G/P = \text{Gr}(k; n)$  is a Grassmannian, and  $w$  corresponds to a partition  $\lambda$ , then  $w^\vee$  corresponds to  $\lambda^\vee$ , the complement of  $\lambda$  in the  $k \times (n - k)$  rectangle.

**Theorem 2.3.** *Let  $X = G/P$  be a partial flag manifold. Then the following hold:*

(a) *The Grothendieck classes  $\{\mathcal{O}^w\}_{w \in W^P}$ , respectively  $\{\mathcal{O}_w\}_{w \in W^P}$ , form a  $\mathbb{Z}$ -basis of  $K(X)$ , i.e.,*

$$K(X) = \bigoplus_{w \in W^P} \mathbb{Z} \mathcal{O}^w = \bigoplus_{w \in W^P} \mathbb{Z} \mathcal{O}_w.$$

(b) *The dual of the Schubert classes are the (opposite) boundary classes, i.e., for any  $v, w \in W^P$ ,*

$$\langle \mathcal{O}_v, \mathcal{I}^w \rangle = \langle \mathcal{O}^v, \mathcal{I}_w \rangle = \delta_{v, w}.$$

(c) *Let  $P \subset Q$  be two parabolic subgroups and  $\pi : G/P \rightarrow G/Q$  the projection. Then for any  $v \in W^P$  and  $w \in W^Q$ ,*

$$\pi_* \mathcal{O}_v = \mathcal{O}_{vW_Q}; \quad \pi^* \mathcal{O}^v = \mathcal{O}^v.$$

For proofs of parts (a), (b) we refer to [Bri05, Thm. 3.4.1]. The pull back statement in (c) follows because  $\pi$  is flat, and the push-forward from the Frobenius split properties of Schubert varieties, see e.g. [BK05, Thm. 3.3.4(a)].

<sup>3</sup>We will not use this, but we note that if one works with equivariant K-theory classes,  $\mathcal{O}^w = w_0^L \cdot \mathcal{O}_{w^\vee}$ , where  $w_0^L$  denotes the left Weyl group multiplication; non-equivariantly,  $w_0^L = id$ .

To emphasize that we utilize a Poincaré dual class rather than its precise formula, we will use the notation

$$(\mathcal{O}_w)^\vee = \mathcal{I}^w; \quad (\mathcal{O}^w)^\vee = \mathcal{I}_w.$$

Note that, unlike for cohomology, the dual of a Schubert class is no longer a Schubert class. This can also be seen concretely by calculating the pairing  $\langle \mathcal{O}_u, \mathcal{O}^v \rangle$ :

$$\langle \mathcal{O}_u, \mathcal{O}^v \rangle = \chi(\mathcal{O}_{X_u \cap X^v}) = \begin{cases} 1 & v \leq u; \\ 0 & \text{else.} \end{cases}$$

(One uses Lemma 2.2, that the Richardson variety  $X_u \cap X^v$  is nonempty iff  $v \leq u$ , and that  $H^i(\mathcal{O}_{X_u \cap X^v}) = 0$  for  $i > 0$  since the Richardson variety is rational.)

**Remark 2.4.** *All statements in Theorem 2.3 extend to the equivariant setting, without any changes.*

**Remark 2.5.** *This theorem implies a recursive formula to generate any Schubert class from the class of a point. Let  $\text{Fl}(\hat{i}, n)$  denote the partial flag manifold parametrizing  $F_1 \subset \dots \subset \hat{F}_i \subset \dots \subset \mathbb{C}^n$ , and let  $p_i : \text{Fl}(n) \rightarrow \text{Fl}(\hat{i}, n)$  be the natural projection. Then  $\partial_i = p_i^*(p_i)_*$  is an endomorphism of  $K(\text{Fl}(n))$  called the **Demazure operator**. One can show that*

$$\partial_i(\mathcal{O}^w) = \begin{cases} \mathcal{O}^{ws_i} & \text{if } ws_i < w; \\ \mathcal{O}^w & \text{otherwise.} \end{cases}$$

*We leave this as an exercise, together with the fact that the Demazure operators satisfy  $\partial_i^2 = \partial_i$ , and the usual commutation and braid relations. (Hint: use that  $p_i^*(\mathcal{O}^w) = \mathcal{O}_{p_i^{-1}(X^w)}$  and  $(p_i)_*\mathcal{O}^w = \mathcal{O}_{p_i(X^w)}$  by Theorem 2.3(c), and that push forwards and pull-backs of Schubert varieties are Schubert varieties.)*

The previous remark helps relate Schubert classes to **Grothendieck polynomials**; these satisfy similar recursive relations which use the algebraic versions of the Demazure operators. See, e.g., [Las90, FL94, Len00, Buc05].

**Remark 2.6.** *As another exercise, one can use part (c) in the above theorem to show that for  $w \in W^P$ ,*

$$\pi_* \mathcal{I}_w = \begin{cases} \mathcal{I}_w & \text{if } w \in W^Q; \\ 0 & \text{otherwise.} \end{cases}$$

*(Hint: Use duality, and the projection formula.)*

**Remark 2.7.** *(See, e.g., [BCMP18a, §4.1]) Assume that  $G/P = \text{Gr}(k; n)$  is a Grassmann manifold (or, more generally, a minuscule Grassmannian) and let  $\iota : \text{Gr}(k; n) \rightarrow \mathbb{P} = \mathbb{P}(\wedge^k \mathbb{C}^n)$  be the Plücker embedding. Then the boundary of any Schubert variety  $X^\lambda$  is also a Cartier divisor, corresponding to the restriction of the line bundle  $\mathcal{O}_{\mathbb{P}}(1)$  to  $X^\lambda$ . More precisely, for any partition  $\lambda \subset k \times (n - k)$ ,*

$$\mathcal{I}^\lambda = \mathcal{O}^\lambda \cdot \mathcal{O}_{\text{Gr}(k; n)}(-1).$$

Using these formulae and the Möbius inversion one can write the ideal sheaf basis in terms of the Schubert classes and viceversa. Below we record two important situations (cf. [Bri05, Prop. 4.3.2], and [BCMP18a, Lemma 3.5]).

**Proposition 2.8.** (a) Let  $X = \text{Fl}(n)$ . Then

$$\mathcal{I}_w = \sum_{v \leq w} (-1)^{\ell(w) - \ell(v)} \mathcal{O}_v; \quad \mathcal{O}_w = \sum_{v \leq w} \mathcal{I}_v.$$

(b) Let  $X = \text{Gr}(k; n)$ . Then for any partition  $\lambda \subset k \times (n - k)$ ,

$$\mathcal{I}^\lambda = \sum_{\lambda \subset \mu} (-1)^{|\mu/\lambda|} \mathcal{O}^\mu; \quad \mathcal{O}^\lambda = \sum_{\lambda \subset \mu} \mathcal{I}^\mu,$$

where the sums are over partitions  $\mu \supset \lambda$  such that  $\mu/\lambda$  is a **rook strip**, i.e., the (possibly disconnected) skew shape  $\mu/\lambda$  does not have two boxes in the same row or column.

**Example 2.9.** In  $K(\text{Gr}(2, 4))$  we have

$$\mathcal{I}^{(1)} = \mathcal{O}^{(1)} - \mathcal{O}^{(2)} - \mathcal{O}^{(1,1)} + \mathcal{O}^{(2,1)}.$$

**2.2.1. Positivity.** Any (equivariant, quantum, K) cohomology theory of a flag manifold is expected to satisfy a positivity property. For K theory, this was discovered by Buch for Grassmannians [Buc02], then proved by Brion for any flag manifold  $G/P$  [Bri02]. An equivariant version was proved by Graham [Gra01] in equivariant cohomology, and by Anderson, Griffeth and Miller [AGM11] in equivariant K theory. (For quantum versions see [Mih06, BCMP22].)

**Theorem 2.10** (K-theoretic positivity). *Consider the Schubert expansion in  $K(G/P)$ :*

$$\mathcal{O}^u \cdot \mathcal{O}^v = \sum c_{u,v}^w \mathcal{O}^w.$$

Then  $(-1)^{\ell(u) + \ell(v) - \ell(w)} c_{u,v}^w \geq 0$ .

The proof relies on a more general result proved by Brion, stated next, which relies on the Kawamata-Viehweg vanishing theorem.

**Definition 2.11.** A variety  $X$  has **rational singularities** if has a proper resolution of singularities  $\pi : X' \rightarrow X$  such that (as sheaves)  $\pi_* \mathcal{O}_{X'} = \mathcal{O}_X$  and  $R^i \pi_* \mathcal{O}_{X'} = 0$  for  $i > 0$ .

A variety with rational singularities must be normal and Cohen-Macaulay. Brion [Bri02] proved the following general positivity statement.

**Theorem 2.12.** Let  $X = G/P$  and  $Y \subset X$  be a subvariety with rational singularities. Consider the expansion

$$[\mathcal{O}_Y] = \sum a_w \mathcal{O}_w.$$

Then  $(-1)^{\ell(w) - \dim Y} a_w \geq 0$ .

*Proof.* We give the proof in the case  $Y$  is smooth and  $X = \mathbb{P}^n$ . Then

$$a_w = \chi(Y \cdot (\mathcal{O}^w)^\vee) = \chi([\mathcal{O}_Y] \cdot \mathcal{O}_{\mathbb{P}^i} \cdot \mathcal{O}(-1)).$$

If nonempty, the general intersection is a (possibly disconnected) union of smooth varieties. The Kodaira vanishing applied to each component of this intersection implies that

$$\chi([\mathcal{O}_Y] \cdot \mathcal{O}_{\mathbb{P}^i} \cdot \mathcal{O}(-1)) = (-1)^{\dim Y - n - i} H^{\dim Y - n - i}(Y \cap \mathbb{P}^i; \mathcal{O}(-1))$$

proving the claim.  $\square$

I encourage the reader to go back to the example from (1.1) and compare positivity above to the one in the quantum K case.

Given all this, the proof of Theorem 2.10 follows from applying Theorem 2.12 to the case  $Y = X^u \cap X_{v^\vee}$  (a Richardson variety). Indeed, one has that the coefficients  $c_{u,v}^w$  may be calculated by

$$c_{u,v}^w = \chi(\mathcal{O}^u \cdot \mathcal{O}_{v^\vee} \cdot \mathcal{I}_w) = \chi([\mathcal{O}_{X^u \cap X_{v^\vee}}] \cdot \mathcal{I}_w).$$

The last quantity is the coefficient of  $\mathcal{O}^w$  in the expansion of  $[\mathcal{O}_{X^u \cap X_{v^\vee}}]$  into Schubert classes.

**2.2.2. Presentations.** We give presentations of the K theory rings in two extremal cases:  $X = \text{Gr}(k; n)$  and  $X = \text{Fl}(n)$ . Proofs are left as exercises, see also [GMS<sup>+</sup>23], or the classical [Las90]. We will revisit these in section 6 below.

**Proposition 2.13.** *Let  $X = \text{Gr}(k; n)$  equipped with the tautological sequence  $0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^n \rightarrow \mathcal{Q} \rightarrow 0$ . Then*

$$\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n)$$

*and a formal version of this leads to the full ideal of relations in  $K_T(\text{Gr}(k; n))$ . In other words,  $K(X)$  is generated by the exterior powers  $\wedge^i \mathcal{S}, \wedge^j \mathcal{Q}$ , modulo the Whitney relations. If one turns these powers into abstract variables, then one obtains a presentation of  $K(X)$ .*

**Proposition 2.14.** *Let  $X = \text{Fl}(n)$  equipped with the tautological sequence  $0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots \subset \mathcal{S}_{n-1} \subset \mathbb{C}^n$ . Then*

$$\lambda_y(\mathcal{S}_1) \cdot \lambda_y(\mathcal{S}_2/\mathcal{S}_1) \cdot \dots \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{n-1}) = \lambda_y(\mathbb{C}^n).$$

*A formal version of these equations leads to the full ideal of relations in  $K_T(\text{Fl}(n))$ .*

**2.3. The K-theoretic Schubert package.** For the convenience of the reader, we revisit the Schubert package we considered so far for the K-theory of partial flag manifolds:

- The Schubert basis consists of the classes  $\mathcal{O}^w$  (or  $\mathcal{O}_w$ ) of the structure sheaves of Schubert varieties.
- The dual basis consists of ideal sheaves  $\mathcal{I}_w$  (or  $\mathcal{I}^w$ ).
- There are presentations of the K-theory ring, such that Grothendieck polynomials represent Schubert classes. (For space reasons, we did not explore this in any depth.)

- The Schubert structure constants have are sign-alternating; cf. Theorem 2.10;
- Finally, there are various Pieri-Chevalley formulae for the K-theory of partial flags, and few Littlewood-Richardson rules, for  $k$ -step flag manifolds,  $k \leq 3$ ; see, e.g., [FL94, Len00, Buc02, CTY14, KZJ17] and references therein.

### 3. DEFINITION OF QUANTUM K THEORY AND SOME PROPERTIES

In this section we recall basic definitions of the K-theoretic Gromov-Witten invariants, and then define the (small) quantum K ring, following Givental and Lee [Giv00, Lee04]. Once the definition is given, we state a K-theoretic version of the ‘quantum=classical’ statement, followed by some consequences.

**3.1. The moduli space.** Let  $X$  be a projective manifold - very soon  $X$  will be equal to a flag manifold  $G/P$ . For an effective degree  $d \in H_2(X; \mathbb{Z})$ , denote by  $\overline{\mathcal{M}}_{0,n}(X, d)$  the Kontsevich moduli space of (genus 0,  $n$  pointed) stable maps of degree  $d$ ; cf. [FP97]. This is a projective scheme, with points given by (equivalence classes of) stable maps:

$$f : (C, p_1, \dots, p_n) \rightarrow X; \quad f_*[C] = d.$$

Here  $C$  is a tree of  $\mathbb{P}^1$ 's, and  $f$  satisfies a **stability condition**: if  $C'$  is a component such that  $f(C')$  is a constant, then  $C'$  must have at least three marked points, where a marked point is either a node or a marking  $p_i$ . There is a natural equivalence relation on this data ensuring that there are finitely many automorphisms. The moduli space comes equipped with evaluation maps  $\text{ev}_i : \overline{\mathcal{M}}_{0,n}(X, d) \rightarrow X$ , sending  $f \mapsto f(p_i)$ . If  $n \geq 3$  and  $d = 0$ , then  $\overline{\mathcal{M}}_{0,n}(X, 0) = X \times \overline{\mathcal{M}}_{0,n}$ , the product of  $X$  with the Mumford moduli space of stable curves. The evaluation maps are all equal to the projection to  $X$ .

More generally, for a sequence of effective degrees  $d_1, \dots, d_r \in H_2(X)$ , we can consider the fibre product

$$\overline{\mathcal{M}}_{0,n_1+\dots+n_r}(X, (d_1, \dots, d_r)) := \overline{\mathcal{M}}_{0,n_1+1}(X, d_1) \times_X \dots \times_X \overline{\mathcal{M}}_{0,n_r+1}(X, d_r)$$

This may be identified with a boundary component inside  $\overline{\mathcal{M}}_{0,n_1+\dots+n_r}(X, (d_1 + \dots + d_r))$ . We list some important properties of the Kontsevich moduli space.

**Theorem 3.1.** *Let  $X = G/P$  be a flag manifold. Then the following hold:*

- $\overline{\mathcal{M}}_{0,n}(X, d)$  has finite quotient singularities, hence rational singularities - this follows from construction, see e.g. [FP97];
- $\overline{\mathcal{M}}_{0,n}(G/P, d)$  is a connected, thus irreducible variety [Tho98, KP01];
- $\overline{\mathcal{M}}_{0,n}(X, d)$  is a rational variety [KP01].

**3.2. Definition of quantum K theory.** We follow the references [Giv00, Lee04] by Givental and Lee. From now on we will take  $X = G/P$  to be any partial flag manifold. This results in fewer technicalities, such as the replacement of the ‘virtual fundamental sheaves’ of Kontsevich moduli spaces by their structure sheaves. For the general construction, consult [Lee04].

Let  $a_1, \dots, a_n \in K(X)$  and let  $d \in H_2(X)$  be an effective degree. The **K-theoretic Gromov-Witten invariants** (KGW) invariant is defined by:

$$(3.1) \quad \langle a_1, \dots, a_n \rangle_d = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \text{ev}_1^*(a_1) \cdot \dots \cdot \text{ev}_n^*(a_n).$$

In general the moduli space is not smooth, but since  $X$  is, one may write each of the classes  $a_i$  as a finite alternating sum of classes of vector bundles. Then (3.1) may be written as a finite alternating sum of sheaf Euler characteristics of vector bundles. In the latter case, the product  $\cdot$  is the tensor product  $\otimes$ .

**Example 3.2.** Consider  $X = G/P$  a partial flag manifold. Then

$$\langle 1, \dots, 1 \rangle_d = 1$$

for any degree  $d$ . Indeed, from Theorem 3.1 we deduce that  $H^i(\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(X,d)) = 0$  for  $i > 0$ , hence  $\chi(\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(X,d)) = 1$ .

More generally, as explained in [Giv00, Cor. 1], if  $\pi : \overline{\mathcal{M}}_{0,n+1}(X,d) \rightarrow \overline{\mathcal{M}}_{0,n}(X,d)$  denotes the map forgetting the  $(n+1)$ -st marking, then  $\pi_*[\mathcal{O}_{\overline{\mathcal{M}}_{0,n+1}(X,d)}] = [\mathcal{O}_{\overline{\mathcal{M}}_{0,n}(X,d)}]$  since all the fibers are rational curves. This implies that

$$(3.2) \quad \langle a_1, \dots, a_n, 1 \rangle_d = \langle a_1, \dots, a_n \rangle_d,$$

which is the simplest case of the **string equation**; cf. [Lee04, §4.4].

Recall that  $H_2(X)$  has a basis of fundamental classes of irreducible curves, say  $[C_1], \dots, [C_r]$ . Consider the sequence of Novikov variables  $q = (q_1, \dots, q_r)$ . For  $d = d_1[C_1] + \dots + d_r[C_r]$ , set  $q^d = q_1^{d_1} \cdot \dots \cdot q_r^{d_r}$ . Define the  $\mathbb{Z}[[q]]$ -module

$$\text{QK}(X) = K(X) \otimes \mathbb{Z}[[q]].$$

Fix an additive basis  $\mathcal{O}^0 = 1, \dots, \mathcal{O}^n$  of  $K(X)$ , and denote by  $\mathcal{O}^{i,\vee}$  the dual basis with respect to the intersection pairing. (In our case  $X = G/P$  is a flag variety, and one may take Schubert classes  $\{\mathcal{O}^w\}_{w \in W^P}$ , with the dual basis given by the boundary classes  $\mathcal{I}_w$ .)

**Definition 3.3.** The (small) **QK pairing** is defined by

$$((a, b)) = \langle a, b \rangle + \sum_{d>0} \langle a, b \rangle_d q^d.$$

Here  $q$  stands for the sequence of Novikov variables indexed by a basis of  $H_2(X)$ , and  $q^d = q_1^{d_1} \cdot \dots \cdot q_r^{d_r}$ . The QK pairing is a symmetric, bilinear, nondegenerate pairing with values in the formal power series  $\mathbb{Z}[[q]]$ .

The (small) quantum K product on  $\text{QK}(X)$  is the unique product  $\star$  which satisfies:

$$((a \star b, c)) = \sum_{d \geq 0} \langle a, b, c \rangle_d q^d.$$

As a consequence, the pairing satisfies the **Frobenius property**  $((a \star b, c)) = ((a, b \star c))$ .

**Example 3.4.** *It follows from Example 3.2 that if  $X = \text{Gr}(k; n)$  then*

$$((1, 1)) = 1 + q + q^2 + \dots = \frac{1}{1 - q}.$$

*More generally, if  $X = G/P$ , then*

$$((1, 1)) = \frac{1}{\prod_{i=1}^{\text{rk}(H_2(G/P))} (1 - q_i)}.$$

*As a fun exercise, one can use the string equation (3.2) to check that  $a \star 1 = a$ .*

**Theorem 3.5** (Givental, Lee). *The product  $\star$  equips  $\text{QK}(X)$  with a structure of a commutative, associative ring with identity  $1 = [\mathcal{O}_X]$ .*

From definition it follows that  $K(X) \simeq \text{QK}(X)/\langle q \rangle$ . Since  $K(X)$  is filtered algebra, it induces a filtration on  $\text{QK}(X)$ , with  $\deg q_i = \int_X c_1(T_X) \cap [C_i]$ . The associated graded algebra is equal to

$$\text{Gr } \text{QK}(X) = \text{QH}^*(X),$$

the **quantum cohomology** of  $X$ .

Next we unravel the definition of the  $\text{QK}$  product and we discuss two equivalent formulations of the definition.

**Lemma 3.6.** *Consider the product*

$$\mathcal{O}^u \star \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w.$$

*Then we have the following equivalent formulae for the structure constants  $N_{u,v}^{w,d}$ :*

(a) *(A recursive formula)*

$$N_{u,v}^{w,d} = \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^w)^\vee \rangle_d - \sum_{d' > 0, \kappa} N_{u,v}^{\kappa, d-d'} \langle \mathcal{O}^\kappa, (\mathcal{O}^w)^\vee \rangle_{d'}.$$

(b) *(Inclusion-Exclusion)*

$$\begin{aligned} N_{u,v}^{w,d} = & \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^w)^\vee \rangle_d \\ & + \sum (-1)^s \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^{\kappa_0})^\vee \rangle_{d_0} \cdot \langle \mathcal{O}^{\kappa_0}, (\mathcal{O}^{\kappa_1})^\vee \rangle_{d_1} \cdot \dots \cdot \langle \mathcal{O}^{\kappa_s}, (\mathcal{O}^k)^\vee \rangle_{d_s}; \end{aligned}$$

*here the sum is over effective degrees  $d_0, \dots, d_s$  such that  $d_0 + \dots + d_s = d$  and  $d_p > 0$  if  $p > 0$ .*

(c) *Let  $\mathcal{D} \subset \overline{\mathcal{M}}_{0,3}(X, d)$  be the boundary divisors consisting of maps with reducible domain where markings 1, 2 are on the first component, and marking 3 on the last. Then*

$$N_{u,v}^{w,d} = \chi(\mathcal{O}_{\overline{\mathcal{M}}_{0,3}(X,d)}(-\mathcal{D}) \cdot \text{ev}_1^*(\mathcal{O}^u) \cdot \text{ev}_2^*(\mathcal{O}^v) \cdot \text{ev}_3^*((\mathcal{O}^w)^\vee)).$$

Note that, unlike in quantum cohomology, both 2 and 3-point KGW invariants are needed to calculate a single structure constants. However, the proof of the associativity is essentially the same as in the cohomological case: it is obtained from equalities obtained by pulling back points in  $\mathbb{P}^1 \simeq \mathcal{M}_{0,4}$ . The pull-backs are simple normal crossing

boundary divisors in  $\overline{\mathcal{M}}_{0,4}(X, d)$ ; while in cohomology the class of such a reducible divisor  $D = \bigcup D_i$  is the sum of its components  $[D_i]$ , in K-theory one uses a Koszul sequence to get an alternating sum

$$[\mathcal{O}_D] = \sum (-1)^{k-1} [\mathcal{O}_{D_{i_1} \cap \dots \cap D_{i_k}}].$$

This explains the shape of the formula in part (c).

The formulae in the lemma suggest that in general the QK multiplication may not be finite. Indeed, Example 3.2 shows that the KGW invariants are in general nonzero for any degree  $d$ . In fact, the QK multiplication of *Schubert classes is finite* for flag manifolds [BCMP13, Kat18, ACT22]. Nevertheless, it is convenient to work with the power series version, as certain inverses of commonly encountered elements naturally live there. For example, in  $\mathrm{QK}(\mathbb{P}^1)$ ,  $\mathcal{O}(1) \star \mathcal{O}(-1) = 1 - q$ , showing that the *quantum* inverse of  $\mathcal{O}(-1)$  is

$$\mathcal{O}(-1)^{-1} = \frac{\mathcal{O}(1)}{1 - q}.$$

For Grassmannians, we will explain this as an application of the ‘quantum=classical’ statement.

Informally, many calculations of KGW invariants can be traced to two geometric facts:

- (Transversality) If  $\Omega_1, \dots, \Omega_n$  satisfy a K-theoretic transversality property, then

$$[\mathcal{O}_{\Omega_1}] \cdot \dots \cdot [\mathcal{O}_{\Omega_n}] = [\mathcal{O}_{\Omega_1 \cap \dots \cap \Omega_n}];$$

- (Rational connectedness + mild singularities) If  $X$  is a rational/unirational/rationally connected projective variety which has rational singularities, then  $\chi(\mathcal{O}_X) = 1$ .

The following result, due to Graber, Harris, and Starr [GHS03], provides one of the main tools for proving that a variety is rationally connected.

**Theorem 3.7.** *Let  $f : X \rightarrow Y$  be any dominant morphism of complete irreducible complex varieties. If  $Y$  and the general fiber of  $f$  are rationally connected, then  $X$  is rationally connected.*

**3.3. Preview.** In this section we give a preview of some of the main theorems we will talk about in these notes. We are biased towards applications in (quantum K) Schubert Calculus, and obviously our list only scratches the surface of what has been done.

The next theorem, proved in [BM15, BCLM20] may be used to calculate the 2-point KGW invariants, and the quantum K pairing, for any flag manifold  $G/P$ .

**Theorem 3.8.** (a) *Let  $X = G/P$  and let  $u, v \in W^P$ . For each effective degree  $d$  there is a combinatorially defined element  $u(d) \in W^P$  with the property that the 2-point KGW invariants  $\langle \mathcal{O}_u, \mathcal{I}^v \rangle_d$  are given by*

$$\langle \mathcal{O}_u, \mathcal{I}^v \rangle_d = \delta_{u(d), v}.$$

*The Schubert variety  $X_{u(d)}$  is called the **curve neighborhood** of  $X_u$ .*

(b) The QK metric may be calculated by

$$((\mathcal{O}_u, \mathcal{O}^v)) = \frac{q^{d_{\min}(u,v)}}{\prod (1 - q_i)}$$

where  $q^{d_{\min}(u,v)}$  is the minimum power of  $q$  in the **quantum cohomology** product  $[X_u] \star [X^v]$ .

One can say a little more for Grassmannians, cf. [BM11, Sum24]:

**Proposition 3.9.** *The dual basis of the Schubert classes is given by the quantized ideal sheaves*

$$\mathcal{I}_q^\lambda := \mathcal{O}^\lambda \star \mathcal{O}_{\text{Gr}(k;n)}(-1).$$

That is,  $((\mathcal{O}_\lambda, \mathcal{I}_q^\mu)) = \delta_{\lambda,\mu}$ .

The following theorem, proved in [ACT22, Kat18], shows that the quantum K Schubert Calculus is finite; see also [BCMP13, BCMP16] for special cases.

**Theorem 3.10** (Finiteness). *Let  $X = G/P$ . Then the quantum K product of Schubert classes is finite, i.e., for any  $u, v \in W^P$ ,  $\mathcal{O}^u \star \mathcal{O}^v \in K(X) \otimes \mathbb{Z}[q]$ .*

Finally, the next result gives a combinatorial recipe to calculate any 3-point KGW invariants, in the case of Grassmannians <sup>4</sup> [BM11, CP11b].

**Theorem 3.11** (‘Quantum = classical’). *Assume  $X = \text{Gr}(k;n)$  is a Grassmannian. Consider the incidence diagram*

$$\begin{array}{ccc} Z_d := \text{Fl}(k-d, k, k+d; n) & \xrightarrow{p_d} & X := \text{Gr}(k; n) \\ \downarrow q_d & & \\ Y_d := \text{Fl}(k-d, k+d; n) & & \end{array}$$

Here, if  $d \geq k$  then we set  $Y_d := \text{Fl}(k+d; n)$  and if  $k+d \geq n$  then we set  $Y_d := \text{Gr}(k-d; n)$ . In particular, if  $d \geq \min\{k, n-k\}$ , then  $Y_d$  is a single point. Then for any  $a, b, c \in K(X)$  and any effective degree  $d$

$$\langle a, b, c \rangle_d = \int_{Y_d} (q_d)_* p_d^*(a) \cdot (q_d)_* p_d^*(b) \cdot (q_d)_* p_d^*(c).$$

The ‘quantum = classical’ theorem is at the heart of many applications, including:

- explicit combinatorial **Pieri/Chevalley formulae** for any (co)minuscule Grassmannians  $X$ ; see section 5.2;
- **Presentations** of  $\text{QK}(\text{Gr}(k;n))$  by generators and relations which quantize the Whitney presentation; see section 6;
- Geometric interpretations of certain monodromy operators from integrable systems; cf. [GKM25], see section 7.

---

<sup>4</sup>A similar statement extends to the case of cominuscule Grassmannians.

- An extension of **Seidel representation** and combinatorics of quantum shapes, generalizing Postnikov's cylinder; cf. e.g., [LLSY25, BCP23].

The ‘quantum = classical’ is also at the heart of a positivity statement proved in [BCMP22] for Grassmannians. Here we state the general positivity conjecture which is expected to hold for any  $G/P$ :

**Conjecture 1** (Positivity). *Let  $X = G/P$  and consider*

$$\mathcal{O}^u \star \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w.$$

*Then  $(-1)^{\ell(w) + \deg(q^d) - \ell(u) - \ell(v)} N_{u,v}^{w,d} \geq 0$ .*

For  $X = \text{Gr}(k; n)$ , see (1.1) above for an example illustrating this positivity. Other special cases of this conjecture were proved in [Xu24] for the incidence varieties  $\text{Fl}(1, n-1; n)$ , and in [BPX24] for the submaximal isotropic Grassmannian  $\text{IG}(2, 2n)$ .

**3.3.1. Notes.** As the reader undoubtedly observed, we are not saying much about the quantum K ring of (partial) flag manifolds, beyond Grassmannians. The ‘quantum=affine’ statement conjectured in [LLMS18], and recently proved by Syu Kato [Kat18], gives a ring isomorphism between a localization of  $\text{QK}(\text{Fl}(n))$  (and more generally  $\text{QK}(G/B)$ ) and the K-theory of ‘semi-infinite flag manifolds’. Under this isomorphism, multiplications by (antidominant) line bundles in the QK ring correspond to certain line bundle multiplications on the semi-infinite side. In [LNS24], Lenart, Naito and Sagaki used this to build a combinatorial model to multiply by line bundles in  $\text{QK}(\text{Fl}(n))$ . In addition, this leads to presentations of  $\text{QK}(\text{Fl}(n))$ , and to proofs that the (double) quantum Grothendieck polynomials represent Schubert classes in the quantum K ring; see [MNS25a, MNS25b].

Another direction we barely cover is the relation to integrable systems. In the last section of these notes we discuss some of the results from [GKM25], stemming off Gorbounov and Korff's earlier paper [GK17], where the quantum K theory of Grassmannians is related to Bethe Ansatz equations from a certain integrable system. In a different direction, the ideal of relations in the quantum K theory ring of the complete flag manifold  $G/B$  is expected to be generated by the integrals of motion of the finite-difference Toda lattice. In some cases this is known, see e.g. (a combination of the results in) [GL03, ACT22, MNS25a, IIM20], and also the related paper [IMT15]. A full account for partial flag manifolds is provided in [AHKM<sup>+</sup>25]. Closely related to this is an area with a high level of activity, that of quantum K theory of cotangent bundles of flag manifolds, or of Nakajima quiver varieties; cf. [Oko17, KPSZ21, OS22].

#### 4. TWO POINT KGW INVARIANTS VIA CURVE NEIGHBORHOODS

Unless otherwise stated, throughout this section  $X = G/P$  is a partial flag manifold. To perform the calculations required in Lemma 3.6, we need formulae for the two-point KGW invariants of the form  $\langle \mathcal{O}^u, (\mathcal{O}^v)^\vee \rangle_d$ . Their calculation relies on the notion of **curve**

**neighborhoods**, defined and studied in a series of papers [BCMP13, BM15, BCMP18b]; an earlier version also appeared in papers by Chaput, Manivel, and Perrin; see, e.g., [CMP08]. We present next the basic facts.

#### 4.1. Definition and first properties.

**Definition 4.1.** *Let  $\Omega_1, \dots, \Omega_n \subset X$  be closed subvarieties and fix an effective degree  $d \in H_2(X)$ .*

(a) *The (n-point) **Gromov-Witten variety** is the intersection*

$$\mathrm{GW}_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_1^{-1}(\Omega_1) \cap \dots \cap \mathrm{ev}_n^{-1}(\Omega_n) \subset \overline{\mathcal{M}}_{0,n+a}(X, d).$$

*If  $\Omega_2 = \dots = \Omega_n = X$  we will simply use the notation  $\mathrm{GW}_d(\Omega_1) := \mathrm{GW}_d(\Omega_1, X, \dots, X)$ .*

(b) *The (n-point) **curve neighborhood** of  $\Omega_1, \dots, \Omega_n$  is defined as the image of the corresponding Gromov-Witten variety:*

$$\Gamma_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)).$$

*As before,  $\Gamma_d(\Omega) := \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega))$ .*

*All these may be extended to the case when one has a sequence of degrees  $d_1, \dots, d_k$ , by replacing the moduli space with an appropriate stratum in the boundary.*

**Example 4.2.** (a) *If  $d = 0$ , then  $\Gamma_0(\Omega_1, \Omega_2) = \Omega_1 \cap \Omega_2$ .*

(b) *Take  $X = \mathbb{P}^n$  and  $d > 0$ . Then  $\Gamma_d(pt) = \mathbb{P}^n$  and*

$$\Gamma_d(pt, pt) = \begin{cases} \text{line} & d = 1 \\ \mathbb{P}^n & d \geq 2. \end{cases}$$

We next recall the notion of cohomological triviality.

**Definition 4.3.** *Let  $f : X \rightarrow Y$  be a morphism of algebraic varieties. We say that  $f$  is **cohomologically trivial** if  $f_*\mathcal{O}_X = \mathcal{O}_Y$  and  $R^i f_*\mathcal{O}_X = 0$  for  $i > 0$ .*

Most non-trivial examples of cohomologically trivial maps arise from special cases of a theorem of Kollár [Kol86]:

**Theorem 4.4.** *Let  $f : X \rightarrow Y$  be a surjective morphism of projective varieties with rational singularities. If the general fibers of  $f$  are rationally connected, then  $f$  is cohomologically trivial.*

The next result was proved in [BCMP13]; initial versions of (c) can be traced to work by Chaput and Perrin [CP11a].

**Theorem 4.5.** *Let  $\Omega_1, \dots, \Omega_n$  be general translates of Schubert varieties in  $X$ . Then the following hold:*

(a) *The GW variety  $\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)$  is either empty, or locally irreducible of expected dimension, and with rational singularities. Furthermore,*

$$\langle [\mathcal{O}_{\Omega_1}], \dots, [\mathcal{O}_{\Omega_n}] \rangle_d = \chi([\mathcal{O}_{\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)}]).$$

(b) *The non-empty Gromov-Witten varieties  $\text{GW}_d(\Omega_1, \Omega_2)$  are irreducible and rationally connected. In particular, the 2-point curve neighborhood  $\Gamma_d(\Omega_1, \Omega_2)$  is also irreducible and rationally connected.*

(c) *If  $\Omega$  is any Schubert variety, then  $\Gamma_d(\Omega)$  is again a Schubert variety and the evaluation map  $\text{ev}_i : \text{GW}(\Omega) \rightarrow \Gamma_d(\Omega)$  is cohomologically trivial.*

*Idea of proof.* Part (a) follows from a K-theoretic Kleiman-Bertini type statement, due to Sierra [Sie09]. For (b) we may assume  $\Omega_1 = X_u, \Omega_2 = X^v$ . The evaluation map  $\text{ev}_1 : \overline{\mathcal{M}}_{0,3}(X, d) \rightarrow X$  is a  $G$ -equivariant locally trivial fibration in Zariski topology. Its fibre  $F$  is irreducible and unirational. By base-change,  $\text{ev}_1^{-1}(X_u) \rightarrow X_u$  is also locally trivial, showing  $\text{GW}(X_u)$  is irreducible and rationally connected. The image  $\Gamma_d(X_u) = \text{ev}_2(\text{GW}(X_u))$  is irreducible and  $B$ -stable, thus a  $B$ -stable Schubert variety. Then  $\text{GW}_d(X_u)$  has an open dense set which is a locally trivial fibration over the cell  $\Gamma_d(X_u)^\circ$ . The intersection  $\text{ev}_1^{-1}(X_u) \cap \text{ev}_2^{-1}(X^v)$  is locally irreducible and it has an open dense set which is a locally trivial fibration over  $\Gamma_d(X_u)^\circ \cap X^v$ . If non-empty, the latter is irreducible and rational. Since all these varieties have rational singularities, and the (general) fibers of these maps are unirational, the statement follows from Theorem 4.4 and Theorem 3.7.  $\square$

Part (c) of the theorem implies that for any  $u \in W^P$  and  $d$  an effective degree one may define the elements  $u(d), u(-d) \in W^P$  by the requirement that

$$X_{u(d)} = \Gamma_d(X_u); \quad X^{u(-d)} = \Gamma_d(X^u).$$

Using these elements one can immediately calculate any 2-point GW invariant.

**Corollary 4.6.** *Let  $X = G/P$  and let  $u, v \in W^P$  be two Weyl group elements and  $d$  an effective degree. Then*

$$\langle \mathcal{O}^u, (\mathcal{O}^v)^\vee \rangle_d = \delta_{u(-d), v},$$

(the Kronecker delta symbol).

*Proof.* From definition,

$$\begin{aligned} \langle \mathcal{O}^u, (\mathcal{O}^v)^\vee \rangle_d &= \chi(\overline{\mathcal{M}}_{0,3}(X, d); \text{ev}_1^*(\mathcal{O}^u) \cdot \text{ev}_2^*((\mathcal{O}^v)^\vee)) \\ &= \chi(G/P; (\text{ev}_2)_*(\text{ev}_1^*(\mathcal{O}^u) \cdot \text{ev}_2^*((\mathcal{O}^v)^\vee))) \\ &= \chi(G/P; [\mathcal{O}_{\Gamma_d(X_u)} \cdot (\mathcal{O}^v)^\vee]) \\ &= \chi(G/P; \mathcal{O}^{u(-d)} \cdot (\mathcal{O}^v)^\vee) \\ &= \delta_{u(-d), v}. \end{aligned}$$

Here the third equality follows from the projection formula, and the last from duality.  $\square$

**Definition 4.7.** *For  $u, v \in W^P$ , define  $d_{\min}(u, v)$  the minimum degree  $d$  for which  $q^d$  appears in the quantum cohomology product  $[X_u] \star [X^v]$ .*

This minimum degree is obviously well defined if  $\text{Pic}(G/P) \simeq \mathbb{Z}$  (i.e., when  $P$  is a maximal parabolic), and more generally it is well defined by results of Postnikov [Pos05] and Fulton-Woodward [FW04]. One can prove that this is the same as the minimum degree  $d$  of  $q$  such that  $GW_d(X_u, X^v) \neq \emptyset$ . A combinatorial calculation of  $d_{\min}(u, v) = (d_1, \dots, d_k)$  may be found in [BCLM20]: the  $j$ th component  $d_j$  of  $d_{\min}(u, v)$  is obtained by calculating the quantum product of  $[X_{uW_{P_j}}], [X^{vW_{P_j}}]$  of the classes given by the images of the Schubert varieties  $X_u, X^v$  in the quantum cohomology of the Grassmannian  $\text{Gr}(j; n) = G/P_j$ .

**Example 4.8.** We calculate  $d_{\min}(u, v)$  for  $u = \text{id}$  and  $v = s_1$  as elements in  $W = S_3$ . Note that  $X_u$  is the  $B$ -stable Schubert point in  $\text{Fl}(3)$ , and  $X^v$  is the Schubert divisor pulled back from  $\text{Gr}(1, 3)$ . Let

$$\text{pr}_1 : \text{Fl}(3) \rightarrow \text{Gr}(1; 3), \quad \text{pr}_2 : \text{Fl}(3) \rightarrow \text{Gr}(2; 3)$$

be the projections. The images of the relevant Schubert varieties are:

$$\text{pr}_1(X_u) = X_\emptyset; \quad \text{pr}_1(X^v) = X^\square; \quad \text{pr}_2(X_u) = X_\emptyset; \quad \text{pr}_2(X^v) = \text{Gr}(2; 3).$$

We denote by  $q_1$  and  $q_2$  respectively the quantum parameters in  $\text{QH}^*(\text{Gr}(1; 3))$  and  $\text{QH}^*(\text{Gr}(2; 3))$ . In  $\text{QH}^*(\text{Gr}(1; 3))$ , the minimum quantum degree in the product  $[X_\emptyset] \star [X^\square]$  is  $q_1$ , and in  $\text{QH}^*(\text{Gr}(2; 3))$ , the minimum quantum degree in the product  $[X_\emptyset] \star [\text{Gr}(2; 3)] = [X_\emptyset] \star 1$  is  $q_2^0 = 1$ . It follows that the minimum power in the quantum  $K$ -theoretic multiplication  $\mathcal{O}_u \star \mathcal{O}^v$  is  $q_1^1 q_2^0 = q_1$ , i.e.,  $d_{\min}(u, v) = (1, 0)$ .

Using the minimum degree  $d_{\min}(u, v)$  one can calculate the QK pairing between any two Schubert classes:

$$(4.1) \quad ((\mathcal{O}_u, \mathcal{O}^v)) = \sum_{d \geq d_{\min}(u, v)} \langle \mathcal{O}_u, \mathcal{O}^v \rangle_d q^d = \frac{q^{d_{\min}(u, v)}}{\prod (1 - q_i)}.$$

(This generalizes equivariantly, with the same formula.) Note that in the non-equivariant case this formula may also be written more symmetrically as

$$((\mathcal{O}^u, \mathcal{O}^v)) = \frac{q^{d'_{\min}(u, v)}}{\prod (1 - q_i)},$$

where  $d'_{\min}(u, v)$  is the minimum degree of  $q$  which appears in the quantum cohomology product  $[X^u] \star [X^v]$ . This uses the fact that  $X_u = w_0.X^{v^\vee}$ , thus non-equivariantly  $\mathcal{O}_u = \mathcal{O}^{u^\vee}$ .

**Example 4.9.** Assume that  $X = \mathbb{P}^2$ . In this case  $K(X)$  has a basis  $1 = \mathcal{O}^0, \mathcal{O}^1, \mathcal{O}^2$ , where  $\mathcal{O}^i$  is the  $K$ -theoretic class representing the hyperplane of (complex) codimension  $i$ . With respect to this basis, the Poincaré metric  $g_{ij} = \int_X \mathcal{O}^i \cdot \mathcal{O}^j$  is given by the matrix

$$(g_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The QK metric is obtained by adding  $\frac{q}{1-q}$ :

$$((\mathcal{O}^i, \mathcal{O}^j)) = (g_{i,j}) + \frac{q}{1-q} Id.$$

**Corollary 4.10** ([BCLM20]). *Consider the polynomial version of the quantum K ring, namely  $\text{QK}^{\text{poly}}(X) = K(X) \otimes \mathbb{Z}[q]$ . Consider the specialization  $q_i \mapsto 1$  for all  $i$  of the usual pairing  $\chi : \text{QK}^{\text{poly}}(X) \rightarrow \text{QK}(pt) = \mathbb{Z}[q]$ . Then this is a **ring** homomorphism.*

*Proof.* Write  $\mathcal{O}^u \star \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w$ . By the Frobenius property of the QK pairing,

$$\sum N_{u,v}^{w,d} q^d \frac{1}{\prod (1 - q_i)} = ((\mathcal{O}^u \star \mathcal{O}^v, 1)) = ((\mathcal{O}^u, \mathcal{O}^v)) = \frac{q^{d'_{\min}(u,v)}}{\prod (1 - q_i)}.$$

It follows that  $\sum N_{u,v}^{w,d} = 1$ . Then the statement follows from the fact that  $\chi(\mathcal{O}^u) = 1$  for any  $u$ .  $\square$

Note that quantum K theory is the first time when  $\chi$  is a ring homomorphism. This property fails for any quotient such as the K-theory specialization, the quantum cohomology specialization etc.

**Example 4.11.** Take  $a = b = [\mathcal{O}_{pt}]$  in  $K(\mathbb{P}^1)$ . Then

$$\chi(a \cdot b) = 0 \neq \chi(a) \cdot \chi(b) = 1 \cdot 1 = 1.$$

Since  $[\mathcal{O}_{pt}] \star [\mathcal{O}_{pt}] = q$  in  $\text{QK}(\mathbb{P}^1)$ , we have

$$\chi([\mathcal{O}_{pt}] \star [\mathcal{O}_{pt}]) = 1 = \chi([\mathcal{O}_{pt}]) \cdot \chi([\mathcal{O}_{pt}]).$$

There is a more general, and rather surprising statement, due to Kato [Kat19]:

**Theorem 4.12.** *Let  $\pi : G/P \rightarrow G/Q$  be the natural projection for  $P \subset Q$ . Consider the  $\mathbb{Z}[q]$ -module projection  $\pi_* : \text{QK}(G/P) \rightarrow \text{QK}(G/Q)$  defined by extending the usual projection  $\pi_* : K(G/P) \rightarrow K(G/Q)$  and specializing  $q_i \mapsto 1$  for all  $i$  such that  $s_i \in W_Q \setminus W_P$ . Then this is a ring homomorphism.*

More refined applications require more refined knowledge of the Weyl group elements giving curve neighborhoods.

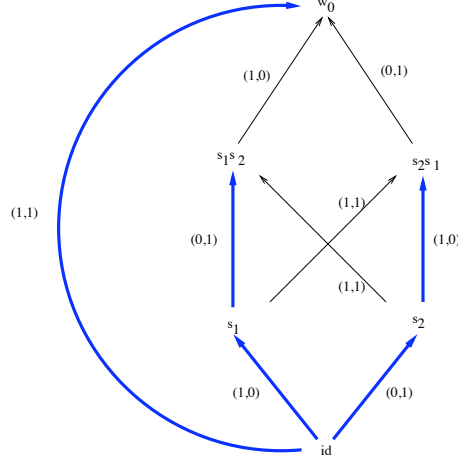
**4.2. Calculation of curve neighborhoods.** The goal is to give an algorithm to calculate the elements  $u(d)$  and  $u(-d)$ . To start,

$$X^{u(-d)W_P} = \Gamma_d(X^{uW_P}) = \Gamma_d(w_0 X_{w_0 u W_P}) = \Gamma_d(w_0 X_{u W_P}) = w_0 X_{u(d)W_P}.$$

This reduces the calculation of  $u(-d)$  to that of  $u(d)$ . For ‘small’ degrees  $d$ , a practical method to do this calculation is based on the **moment graph** of  $G/P$ .

The moment graph of  $G/P$  has vertices corresponding to  $u \in W^P$  and edges  $u \xrightarrow{d(i,j)} v$  if  $\ell(v) > \ell(u)$  and  $u \cdot (i, j) = v$  for  $i < j$ . The edge has (multi)degree  $\varepsilon_i - \varepsilon_j = (0^{i-1}, 1^{j-i}, 0^{n-j})$  modulo  $\Delta_P$  (the simple roots which are already in  $P$ ). Then  $\Gamma_d(X_u)$  is the (unique!) maximal element in the Bruhat order obtained from tracing a path from  $u$  of degree  $\leq d$ .

**Example 4.13.** Below is the moment graph for  $\text{Fl}(3)$ . In blue we drew the paths giving  $\Gamma_{(1,0)}(pt) = X_{s_1}, \Gamma_{(0,1)}(pt) = X_{s_2}, \Gamma_{(1,1)}(pt) = X_{s_1 s_2 s_1}$ .



4.2.1. *Curve neighborhoods of Grassmannians.* We now turn to the calculation of curve neighborhoods for Grassmannians. In this case, (or more generally in **cominuscule Grassmannians**) a formula follows from results in [BCMP13], and a procedure is explicitly reviewed in [BCMP18a]. Recall that in this case the Schubert classes are indexed by Young diagrams  $\lambda$  included in the  $k \times (n - k)$  rectangle, and the curve neighborhoods have particularly nice combinatorial descriptions:

- $\lambda(d)$  is obtained from  $\lambda$  by adding  $d$  rim hooks of maximal length;
- $\lambda(-d)$  is obtained from  $\lambda$  by removing  $d$  rim hooks of maximal length.

**Example 4.14.**

$\alpha_5$	$\alpha_6$	$\alpha_7$	$\alpha_8$	$\alpha_9$	$\alpha_{10}$	$\alpha_{11}$
$\alpha_4$	$\alpha_5$	$\alpha_6$	$\alpha_7$	$\alpha_8$	$\alpha_9$	$\alpha_{10}$
$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$	$\alpha_7$	$\alpha_8$	$\alpha_9$
$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$	$\alpha_7$	$\alpha_8$
$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$	$\alpha_7$


On the left:  $\emptyset(1), \emptyset(2), \dots$ ; on the right:  $\lambda(2)$ , for  $\lambda = (3, 2, 1)$ .

The key geometric fact which explains this formula for Grassmannians is the following:

**Corollary 4.15** ([BCMP13]). *Let  $X$  be a (cominuscule) Grassmannian. Then*

$$\Gamma_d(X_u) = \Gamma_1(\Gamma_1(\dots(\Gamma_1(X_u)))).$$

*In other words, if one point may be joined to  $X_u$  using a rational curve of degree  $d$ , then it may also be joined by a sequence of  $d$  lines.*

The fact that  $\Gamma_d$  is obtained by successive applications of  $\Gamma_1$ 's is special for (cominuscule) Grassmannians. It fails for example for submaximal isotropic Grassmanian

IG(2, 7), or for adjoint varieties. The corollary implies the following important simplification of the formulae from Lemma 3.6 for the QK product of Schubert classes in  $\text{QK}(\text{Gr}(k; n))$ .

**Corollary 4.16.** *Consider the QK product  $\mathcal{O}^\lambda \star \mathcal{O}^\mu = \sum N_{\lambda, \mu}^{\nu, d} q^d \mathcal{O}^\nu$  in  $\text{QK}(\text{Gr}(k; n))$ . Then*

$$N_{\lambda, \mu}^{\nu, d} = \langle \mathcal{O}^\lambda, \mathcal{O}^\mu, (\mathcal{O}^\nu)^\vee \rangle_d - \sum_{\eta} \langle \mathcal{O}^\lambda, \mathcal{O}^\mu, (\mathcal{O}^\eta)^\vee \rangle_{d-1} \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\nu)^\vee \rangle_1.$$

*Proof.* We need to show that for  $\lambda, \mu$  fixed and fixed  $d - d_0 := d_1 + \dots + d_r \geq 2$ , then

$$\sum_{d_1 + \dots + d_r = d - d_0} (-1)^r \langle \mathcal{O}^\lambda, (\mathcal{O}^{\kappa_1})^\vee \rangle_{d_1} \cdot \dots \cdot \langle \mathcal{O}^{\kappa_r}, (\mathcal{O}^\nu)^\vee \rangle_{d_r} = 0.$$

From Corollary 4.6 it follows that this equals to

$$\begin{aligned} \sum_{d_1 + \dots + d_r = d - d_0} (-1)^r \delta_{\lambda(-d_1), \kappa_1} \cdot \dots \cdot \delta_{\kappa(-d_r), \mu} &= \sum (-1)^r \delta_{\lambda(-d_1 - d_2 - \dots - d_r), \mu} \\ &= \sum_{r=1}^{d-d_0} (-1)^r \binom{d - d_0 + r - 1 - r}{r - 1} \\ &= (1 - 1)^{d-d_0-1} = 0. \end{aligned}$$

□

This formula may be interpreted as

$$\begin{aligned} N_{\lambda, \mu}^{\nu, d} &= \langle (\text{ev}_3)_* [\text{GW}_d(g_1 X^\lambda, g_2 X^\mu)] - (\text{ev}_3)_* [\text{GW}_{d-1,1}(g_1 X^\lambda, g_2 X^\mu)], (\mathcal{O}_\nu)^\vee \rangle \\ &= \langle [\mathcal{O}_{\Gamma_d(\lambda, \mu)}] - [\mathcal{O}_{\Gamma_{d-1,1}(\lambda, \mu)}], (\mathcal{O}_\nu)^\vee \rangle, \end{aligned}$$

where  $g_1, g_2$  are general in  $G$ . In fact, the second equality is slightly incorrect: while we can prove that  $(\text{ev}_3)_* [\text{GW}_d(g_1 X^\lambda, g_2 X^\mu)] = [\mathcal{O}_{\Gamma_d(\lambda, \mu)}]$ , we do *not know* whether  $(\text{ev}_3)_* [\text{GW}_{d-1,1}(g_1 X^\lambda, g_2 X^\mu)] = [\mathcal{O}_{\Gamma_{d-1,1}(\lambda, \mu)}]$ . This is true in many cases, and analyzing this carefully lies at the heart of the proof of positivity for  $\text{QK}(\text{Gr}(k; n))$  from [BCMP22].

**4.2.2. Curve neighborhoods for arbitrary flag manifolds.** For a general combinatorial procedure, we need two ingredients. The **Demazure product**  $\cdot$  of two Weyl group elements is defined as follows. If  $u \in W$  and  $s_i \in W$  is a simple reflection,

$$u \cdot s_i = \begin{cases} us_i & \ell(us_i) > \ell(u) \\ u & \ell(us_i) < \ell(u). \end{cases}$$

If  $v = s_{i_1} \dots s_{i_k}$  is a reduced decomposition, then  $u \cdot v = (((u \cdot s_{i_1}) \cdot s_{i_2}) \dots) \cdot s_{i_k}$ . This equips  $(W, \cdot)$  with a structure of an associative monoid. Let also  $z_d \in W$  be the unique element defined by

$$X_{u(d)} = \Gamma_d(pt) \subset \text{Fl}(n).$$

The following combinatorial algorithm to calculate  $u(d)$  for any flag manifold has been proved in [BM15].

**Theorem 4.17.** *The following hold:*

- (a) *In  $\mathrm{Fl}(n)$ ,  $\Gamma_d(X_u) = X_{u \cdot z_d}$ .*
- (b) *Take  $\alpha > 0$  be the largest positive root such that  $d - \alpha^\vee \geq 0$  in  $H_2(\mathrm{Fl}(n))$ . Then*

$$z_d = z_{d-\alpha^\vee} \cdot s_\alpha = s_\alpha \cdot z_{d-\alpha^\vee}.$$

- (c) *Same procedure applies to any  $G/P$ : take  $\alpha \in R^+ \setminus R_P^+$  maximal such that  $d - \alpha^\vee \geq 0$  in  $H_2(G/P)$ . Then*

$$z_d W_P = s_\alpha \cdot z_{d-\alpha^\vee} W_P.$$

**4.3. A K theoretic divisor axiom.** We end this section with an expression for the ‘Chevalley’ KGW invariants of any partial flag manifold, inspired by the behavior of curve neighborhoods. This was still conjectural at the time these lectures were given, but a proof was recently announced in [LNS<sup>+</sup>25].

Let  $X = G/P$  be any partial flag manifold  $\mathrm{Fl}(i_1, \dots, i_k; n)$ , fix a simple reflection  $s_i := s_{i_p} \in W^P$ , and an effective degree  $d = (\dots, d_i, \dots) \in H_2(X)$ . If the component  $d_i > 0$ , then for any subvarieties  $\Omega_1, \Omega_2 \subset X$ , the curve neighborhood

$$\Gamma_d(\Omega_1, \Omega_2, X^{s_i}) = \Gamma_d(\Omega_1, \Omega_2).$$

Motivated by this and some computational evidence, Buch and I made the following (unpublished) conjecture in 2011:

**Conjecture 2.** *Let  $u, v \in W^P$  and let  $d \in H_2(G/P)$  be an effective degree as above, with  $d_i > 0$ . Then for a general  $g \in G$ , the evaluation map*

$$\mathrm{ev}_3 : \mathrm{GW}_d(X_u, X^v, g \cdot X^{s_i}) \rightarrow \Gamma_d(X_u, X^v, g \cdot X^{s_i}) = \Gamma_d(X_u, X^v)$$

*is cohomologically trivial.*

It is not difficult to prove the conjecture if the degree component  $d_i = 1$ . The conjecture has important consequences for calculations of the (equivariant) ‘Chevalley coefficients’ in the quantum K theory of partial flag manifolds, leading to expressions which may be thought as replacements for the ‘divisor axiom’ in quantum K theory.

**Corollary 4.18.** *The following hold:*

$$\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}^v \rangle_d = \begin{cases} \langle \mathcal{O}^u, \mathcal{O}^v \rangle_d & \text{if } d_i > 0; \\ \langle \mathcal{O}^{s_i}, \mathcal{O}^{u(-d)}, \mathcal{O}^v \rangle_0 & \text{if } d_i = 0. \end{cases}$$

*Equivalently, if  $\mathcal{S}_i$  is the tautological subbundle of rank  $i$  on  $X$ ,*

$$\langle \det \mathcal{S}_i, \mathcal{O}^u, \mathcal{O}^v \rangle_d = \begin{cases} 0 & \text{if } d_i > 0; \\ \langle \mathcal{O}^{s_i}, \mathcal{O}^{u(-d)}, \mathcal{O}^v \rangle_0 & \text{if } d_i = 0. \end{cases}$$

The conjecture was proved for (cominuscule) Grassmannians [BM11] and for incidence flag manifolds  $\mathrm{Fl}(1, n-1; n)$  by Weihong Xu [Xu24]. A proof of Corollary 4.18 for any homogeneous space  $G/P$ , which relies on the quantum K Chevalley formula, was recently given by Lenart, Maeno, Naito, Sagaki and Xu in [LNS<sup>+</sup>25]. A different proof of the

corollary, for Grassmannians, using Quot schemes, was recently given by Sinha and Zhang in [SZ24].

## 5. THE ‘QUANTUM=CLASSICAL’ STATEMENT AND APPLICATIONS

The ‘quantum=classical’ statement is at the heart of most calculations in the quantum K theory of Grassmannians. In this section we give the statement and sketch its proof, following [BM11]. We then indicate how to use this statement to perform Chevalley and Pieri multiplications in the quantum K ring.

**5.1. The statement.** We start with Buch’s notion of **kernel** and **span** of a rational curve [Buc03]; see also [CMP08] for a generalization to cominuscle Grassmannians.

**Definition 5.1.** *Let  $f : \mathbb{P}^1 \rightarrow \mathrm{Gr}(k; n)$  be a morphism of degree  $d$ . The **kernel** and **span** of  $f$  are the linear subspaces of  $\mathbb{C}^n$  defined by*

$$\ker(f) = \bigcap_{x \in \mathbb{P}^1} f(x); \quad \mathrm{span}(f) = \mathrm{span}\{f(x) : x \in \mathbb{P}^1\}$$

The following key statement was proved by Buch [Buc03], and Buch, Kresch, and Tamvakis [BKT03].

**Proposition 5.2.** *(a) If  $f : \mathbb{P}^1 \rightarrow \mathrm{Gr}(k; n)$  is of degree  $d$ , then  $\dim \ker(f) \geq k - d$  and  $\dim \mathrm{span} f \leq k + d$ . Furthermore, for a general map  $f$ , equalities occur.*

*(b) Let  $U, V, W \subset \mathrm{Gr}(d, 2d)$  be three general spaces. Then there exists a unique morphism  $f : \mathbb{P}^1 \rightarrow \mathrm{Gr}(d, 2d)$  of degree  $d$  such that  $f(0) = U, f(1) = V, f(\infty) = W$ .*

*Proof.* Let  $S$  be the tautological bundle on  $\mathrm{Gr}(k; n)$ . Then  $f^*(S) \subset \mathbb{C}^n$ , thus  $f^*S = \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(-a_i)$  where  $a_i \geq 0$  and  $\sum a_i = d$ . A map  $f : \mathbb{P}^1 \rightarrow \mathrm{Gr}(k; n)$  is then given by

$$\sum_{j=0}^{a_i} \alpha_j u^{-j} v^{j-a_i} \mapsto \sum_{j=0}^{a_i} \alpha_j \otimes v_j^{(i)}.$$

We have

$$\sum_{i=1}^k (1 + a_i) = k + d$$

$v_j^{(i)}$ ’s, showing that the span is at most of dimension  $k + d$ . But at least  $k - d$  of  $a_i$ ’s equal to 0, giving that (for these  $a_i$ ’s)  $v_0^{(i)}$  are in the kernel; there are at least  $k - d$  of these.

Regarding part (b), observe that  $\mathbb{C}^{2d} = U \oplus W$ . Take a basis  $v_1, \dots, v_d$  of  $V$  and project to  $U, W$ :  $v_i = u_i + w_i$ . Define  $f[s : t] = [su_1 + tw_1 : \dots : su_d + tw_d]$ .  $\square$

For small degrees  $d$ , the proposition allows one to replace a rational curve, i.e., a map  $f : \mathbb{P}^1 \rightarrow \mathrm{Gr}(k; n)$ , by its kernel - span pair. This idea was used by Buch, Kresch and Tamvakis [BKT03] to prove the cohomological version of the ‘quantum=classical’ statement. In that case, a GW invariant *counts* rational curves subject to incidence

conditions, and the statement shows that this count is equal to the count of pairs of kernels - spans, subject to appropriately modified incidence conditions. If one works in K-theory, the count is replaced by the ‘K-theoretic count’, i.e., by a sheaf Euler characteristic. Still, the same statement holds, without any restrictions to degree. We formalize all of this next, following [BM11]. We also mention that all statements work equivariantly.

Consider the ‘kernel-span incidence’:

$$\begin{array}{ccc} Z_d := \mathrm{Fl}(k-d, k, k+d; n) & \xrightarrow{p_d} & X := \mathrm{Gr}(k; n) \\ \downarrow q_d & & \\ Y_d := \mathrm{Fl}(k-d, k+d; n) & & \end{array}$$

Here, if  $d \geq k$  then we set  $Y_d := \mathrm{Fl}(k+d; n)$  and if  $k+d \geq n$  then we set  $Y_d := \mathrm{Gr}(k-d; n)$ . In particular, if  $d \geq \min\{k, n-k\}$ , then  $Y_d$  is a single point.

**Theorem 5.3** (Quantum = classical). *Let  $a, b, c \in K(\mathrm{Gr}(k; n))$  and  $d \geq 0$  a degree. Then the following equality holds in  $K(pt)$ :*

$$\langle a, b, c \rangle_d = \int_{Y_d} (q_d)_*(p_d^*a) \cdot (q_d)_*(p_d^*b) \cdot (q_d)_*(p_d^*c).$$

*Idea of proof.* The proof of this is based on the ‘quantum = classical’ diagram which we explain below. Define  $M_d := \overline{\mathcal{M}}_{0,3}(X, d)$ ,

$$\mathrm{Bl}_d := \{((K, S), f) \in Y_d \times M_d, K \subset \ker(f), \mathrm{span}(f) \subset S\}$$

$$Z_d^{(3)} := \{K \subset V_1, V_2, V_3 \subset S : (K, V_i, S) \in Z_d, i = 1, 2, 3\}$$

There is the following commutative diagram from [BM11]:

$$(5.1) \quad \begin{array}{ccccc} \mathrm{Bl}_d & \xrightarrow{\pi} & M_d & & \\ \downarrow \phi & & \downarrow \mathrm{ev}_i & & \\ Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p_d} & X \\ & & \downarrow q_d & & \\ & & Y_d & & \end{array}$$

By Proposition 5.2, the map  $\pi : \mathrm{Bl}_d \rightarrow M_d$  is birational, and if  $d \leq \min\{k, n-k\}$  then  $\phi : \mathrm{Bl}_d \rightarrow Z_d^{(3)}$  is also birational. A diagram chase proves the theorem in this case. The key point for general  $d$  is that the general fibre of  $\phi$  is rationally connected, thus  $\phi$  is cohomologically trivial by Theorem 4.4. This is proved in type A in [BM11] by an argument using local coordinates, and in other cominuscule types in [CP11b]. This implies that  $\phi_*[\mathcal{O}_{\mathrm{Bl}_d}] = [\mathcal{O}_{Z_d^{(3)}}]$ , and again a diagram chase proves the theorem in this case.  $\square$

In practice, it is often difficult to calculate triple K-theoretic intersections in two-step flag manifolds. Fortunately, there is a version of the ‘quantum=classical’ which goes from a Grassmannian to another Grassmannian. It requires some restrictions which in many interesting situations are satisfied. Form the following incidence diagram:

$$(5.2) \quad \begin{array}{ccccc} Z_d := \mathrm{Fl}(k-d, k, k+d; n) & \xrightarrow{p'_d} & \mathrm{Fl}(k-d, k; n) & \xrightarrow{p''_d} & X := \mathrm{Gr}(k; n) \\ & q_d \downarrow & & q'_d \downarrow & \\ Y_d := \mathrm{Fl}(k-d, k+d; n) & \xrightarrow{pr} & \mathrm{Gr}(k-d; n) & & \end{array}$$

Here all maps are the natural projections. As before, denote by  $p_d : \mathrm{Fl}(k-d, k, k+d; n) \rightarrow \mathrm{Gr}(k; n)$  the composition  $p_d := p''_d \circ p'_d$ .

**Corollary 5.4.** *Let  $a, b, c \in K(\mathrm{Gr}(k; n))$  and  $d \geq 0$  a degree. Assume that  $(q_d)_*(p_d^*(a)) = pr^*(a')$  for some  $a' \in K(\mathrm{Gr}(k-d; n))$ . Then*

$$\langle a, b, c \rangle_d = \int_{\mathrm{Gr}(k-d; n)} a' \cdot (q'_d)_*(p''_d^*(b)) \cdot (q'_d)_*(p''_d^*(c)).$$

A similar statement holds, relating to the  $\mathrm{QK}(\mathrm{Gr}(k+d; n))$ .

The proof is again a standard diagram chase, see [BM11].

**5.2. A Pieri/Chevalley rule.** We sketch next the main calculations leading to formula to multiply by a Schubert divisor, and more generally by Pieri classes.

To start, using Theorem 2.3, one proves that  $(q'_d)_*(p''_d)^*(\mathcal{O}^\lambda) = \mathcal{O}^{\bar{\lambda}_d}$ , where  $\bar{\lambda}_d$  is the result of removing the top  $d$  rows of  $\lambda$ . Similarly, if one uses  $\mathrm{Gr}(k+d; n)$  instead of  $\mathrm{Gr}(k-d; n)$ , one needs to remove the leftmost  $d$  columns. Therefore one has explicit calculations of the coefficients in the products

$$\mathcal{O}^i \star \mathcal{O}^\lambda = \sum N_{i, \lambda}^{\mu, d} q^d \mathcal{O}^\mu$$

in terms of the classical coefficients for  $\mathcal{O}^i \cdot \mathcal{O}^\lambda$ , found by Lenart [Len00]. We illustrate the calculation for the **QK Chevalley formula** of  $\mathrm{Gr}(k; n)$ , following mainly [BM11], see also [BCMP18a]. Recall that if  $\lambda \subset \mu$  are two partitions, the skew shape  $\mu/\lambda$  is a **rook strip** if the skew shape  $\mu/\lambda$  has no two boxes on the same row, and on the same column.

**Theorem 5.5** (The QK Chevalley formula). *The following holds in  $\mathrm{QK}(\mathrm{Gr}(k; n))$ :*

$$\mathcal{O}^1 \star \mathcal{O}^\lambda = \sum_{\mu} (-1)^{|\mu/\lambda|} \mathcal{O}^\mu + \sum_{\nu} (-1)^{\nu/\lambda(-1)} \mathcal{O}^\nu,$$

where the first sum is over those  $\mu$  such that  $\mu/\lambda$  is a non-empty rook strip; the second sum is empty unless  $\lambda_1 = n - k$ ,  $\ell(\lambda) = k$ , in which case the sum is over  $\nu$  such that  $\nu = \mu(-1)$  and  $\mu/\lambda$  is a rook strip.

**Example 5.6.** In  $\text{QK}(\text{Gr}(3, 7))$  we consider the multiplication  $\mathcal{O}^1 \star \mathcal{O}^{(4,3,1)}$ . In this case

$$(4, 3, 1)(-1) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} (-1) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

and

$$\mathcal{O}^\square \star \mathcal{O}^{\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}} = q\mathcal{O}^{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}} - q\mathcal{O}^{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}} - q\mathcal{O}^{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}} + q\mathcal{O}^{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}} + \mathcal{O}^{\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}} + \mathcal{O}^{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}} - \mathcal{O}^{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}}$$

*Idea of proof for Theorem 5.5.* The classical part follows from Lenart's Pieri rule [Len00]. For the quantum part, note that by Corollary 4.16 we have

$$N_{\lambda, (1)}^{\mu, d} = \langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\mu)^\vee \rangle_d - \sum_{\eta} \langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\eta)^\vee \rangle_{d-1} \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\mu)^\vee \rangle_1.$$

It is an instructive exercise using the ‘quantum=classical’ statement to show that

$$\langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\mu)^\vee \rangle_d = \langle \mathcal{O}^\lambda, (\mathcal{O}^\mu)^\vee \rangle_d$$

whenever  $d \geq 1$ . In particular, if  $d \geq 2$ , the right hand side contains the same terms occurring in  $1 \star \mathcal{O}^\lambda$ , therefore it must vanish. Thus only the terms involving  $q^1$  may appear. In this case, the right hand side is equal to

$$\delta_{\lambda(-1), \nu} - \sum_{\eta} \langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\eta)^\vee \rangle_0 \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\mu)^\vee \rangle_1 = \delta_{\lambda(-1), \nu} - \sum_{\eta} N_{\lambda, (1)}^{\eta, 0} \cdot \delta_{\eta(-1), \nu}$$

A combinatorial exercise based on the classical Pieri rule [Len00] shows that the latter expression is the one claimed.  $\square$

We now give the general **Pieri formula**. Recall that the **outer rim** of a partition  $\lambda$  consists of the set of boxes which do not have any box strictly South-East. One obtains the following formula:

**Theorem 5.7** (Pieri rule). *The constants  $N_{i, \lambda}^{\mu, d} = 0$  for  $d \geq 2$ . Furthermore,  $N_{i, \lambda}^{\mu, 1}$  is nonzero only if  $\ell(\lambda) = k$ , and  $\mu$  can be obtained from  $\lambda$  by removing a subset of the boxes in the outer rim of  $\lambda$ , with at least one box removed from each row. When these conditions hold, we have*

$$N_{i, \lambda}^{\mu, 1} = (-1)^e \binom{r}{e}$$

where  $e = |\mu| + n - i - |\lambda|$  and  $r$  is the number of rows of  $\mu$  that contain at least one box from the outer rim of  $\lambda$ , excluding the bottom row of this rim.

**Example 5.8.** On  $X = \text{Gr}(3, 6)$  we have  $N_{2, (3, 2, 1)}^{(2, 1), 1} = -2$ , with  $e = 1$  and  $r = 2$ .



## 6. PRESENTATIONS OF THE QUANTUM K RING OF FLAG MANIFOLDS

Presentations of the (equivariant) quantum K rings of the complete flag manifold  $\mathrm{Fl}(n)$  have been obtained by Lenart, Naito and Sagaki [LNS24] (non-equivariantly) and extended to the equivariant setting by Maeno, Naito and Sagaki [MNS25a]. The source of these presentations is given by the theory of quantum Grothendieck polynomials pioneered by Lenart and Maeno [LM06], see also [MNS25b].

In what follows we focus on the Grassmannian case, where the ‘quantum Whitney’ presentations also arise from physics considerations. Notably, the ideal of relations is generated by the partial derivatives of a superpotential from supersymmetric gauge theory, and also follows from the Bethe Ansatz equations associated to an integrable system.

Since these lectures were given, the physics inspired approach was generalized to give presentations for the quantum K theory of any partial flag manifold. While we will barely discuss these below, we mention the continuation work [GMS<sup>+</sup>24, GMS<sup>+</sup>23], where conjectural presentations were given, and the work by Huq-Kuruvilla [HK24a] where these have been proved. He used an approach based on a K-theoretic version of the abelian-nonabelian correspondence, combined with Iritani, Milanov and Tonita’s work [IMT15] relating  $K$ -theoretic  $J$ -functions and relations in quantum K theory. A different proof of relations for quantum K theory of partial flag manifolds, based on Kato’s pushforward in quantum K theory, can be found in [AHKM<sup>+</sup>25].

**6.1. A presentation for the QK ring of Grassmannians.** Let  $0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^n \rightarrow \mathcal{Q} \rightarrow 0$  be the tautological sequence, where  $\mathrm{rk}(\mathcal{S}) = k$ . An influential result by Witten [Wit95] proves that  $(\mathrm{QH}^*(\mathrm{Gr}(k; n)), \star)$ , the quantum cohomology ring of the Grassmannian, is determined by the ‘quantum Whitney relations’:

$$(6.1) \quad c(\mathcal{S}) \star c(\mathcal{Q}) = c(\mathbb{C}^n) + (-1)^k q,$$

where  $c(E) = 1 + c_1(E) + \dots + c_e(E)$  is the total Chern class of the rank  $e$  bundle  $E$ . This equation leads to a presentation of  $\mathrm{QH}^*(\mathrm{Gr}(k; n))$  by generators and relations:

$$(6.2) \quad \mathrm{QH}^*(\mathrm{Gr}(k; n)) = \frac{\mathbb{Z}[q][e_1(x), \dots, e_k(x); e_1(\tilde{x}), \dots, e_{n-k}(\tilde{x})]}{\left\langle \left( \sum_{i=0}^k e_i(x) \right) \left( \sum_{j=0}^{n-k} e_j(\tilde{x}) \right) = 1 + (-1)^k q \right\rangle}.$$

The idea of proof is explained in [FP97] (and it is originally due to Siebert-Tian [ST97]) and it goes as follows.

**Proposition 6.1.** *Consider a graded ring  $R := \mathbb{Z}[q][e_1, \dots, e_k, e_1(\tilde{x}), \dots, e_{n-k}(\tilde{x})] / \langle P_1, \dots, P_n \rangle$  where  $P_i$ ’s are polynomials in  $e_i$ ’s,  $\tilde{e}_j$ ’s, and  $q$ . Assume that:*

- *The specializations  $P_i|_{q=0}$  generate the ideal of relations for  $H^*(X)$ ;*
- *Each  $P_i = 0$  in  $\mathrm{QH}^*(X)$ .*

*Then  $R \simeq \mathrm{QH}^*(X)$ .*

The goal is to extend this setup to quantum K theory. We already observed that the (quantum) K theory is not a graded ring, therefore Proposition 6.1 does not apply in this case. We will need the following Nakayama-type result from commutative algebra, proved in [GMS<sup>+</sup>25], see also [GMSZ22, Appendix]:

**Proposition 6.2.** *Let  $R$  be a Noetherian integral domain, and let  $I \subset R$  be an ideal. Assume that  $R$  is complete in the  $I$ -adic topology. Let  $M, N$  be finitely generated  $R$ -modules.*

*Assume that the  $R$ -module  $N$ , and the  $R/I$ -module  $N/IN$ , are both free modules of the same rank  $p < \infty$ , and that we are given an  $R$ -module homomorphism  $f : M \rightarrow N$  such that the induced  $R/I$ -module map  $\bar{f} : M/IM \rightarrow N/IN$  is an isomorphism of  $R/I$ -modules.*

*Then  $f$  is an isomorphism.*

The strategy is to apply this result in the case

$$R = \mathbb{Z}[[q]], \quad I = \langle q \rangle, \quad N = \mathrm{QK}(X)$$

and  $M$  the claimed presentation. We notice the hypothesis that  $R$  is complete with respect to  $I$  - this is another place where power series in  $q$  is necessary. Another hypothesis is that  $M$  is finitely generated as an  $R$ -module. This turns out to be automatic in the cases of our interest, when  $M$  is a quotient of a polynomial ring which is  $I$ -adically complete; we refer to [GMS<sup>+</sup>25] for details.

To see how this leads to a presentation, we observe that the short exact sequence  $0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^n \rightarrow \mathcal{Q} \rightarrow 0$  implies the ‘K-theoretic Whitney relations’

$$\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n).$$

Introduce formal variables  $X = (X_1, \dots, X_k); \tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_{n-k})$  such that

$$\lambda_y(\mathcal{S}) = \prod_{i=1}^k (1 + yX_i) = \sum y^i e_i(X); \quad \lambda_y(\mathcal{Q}) = \prod_{j=1}^{n-k} (1 + y\tilde{X}_j) = \sum y^j e_j(\tilde{X}).$$

We already observed in Proposition 2.13 (and it is not difficult to show) that one may formalize the Whitney relations into a presentation for the K-theory ring  $\mathrm{K}(\mathrm{Gr}(k; n))$ :

$$\mathrm{K}(\mathrm{Gr}(k; n)) \simeq \frac{\mathbb{Z}[e_1(X), \dots, e_k(X), e_1(\tilde{X}), \dots, e_{n-k}(\tilde{X})]}{\langle \sum_{i+j=s} e_i(X) e_j(\tilde{X}) = \binom{n}{s}; 1 \leq s \leq n \rangle}.$$

(A similar presentation holds equivariantly, after replacing  $\lambda_y(\mathbb{C}^n) = \sum \binom{n}{s} y^s$  by its equivariant version  $\lambda_y(\mathbb{C}^n) = \sum e_s(T_1, \dots, T_n) y^s$ , where  $\mathbb{C}^n = \bigoplus \mathbb{C}_{T_i}$  as a  $T$ -module.) By Proposition 6.2, we need to quantum deform the Whitney relations. This has been done in [GMSZ22]:

**Theorem 6.3** (Quantum K Whitney relations). *The following equality holds in  $\mathrm{QK}(X)$ :*

$$(6.3) \quad \lambda_y(\mathcal{S}) \star \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n) - \frac{q}{1-q} y^{n-k} (\lambda_y(\mathcal{S}) - 1) \star \det \mathcal{Q}.$$

At the heart of the proof of this theorem lies the ‘quantum=classical’ statement, applied now to the  $\lambda_y$  classes of the tautological bundles. For example, one uses the variant of the ‘quantum=classical’ from Corollary 5.4 to prove the following:

**Proposition 6.4.** *Fix arbitrary  $b, c \in K_T(\mathrm{Gr}(k; n))$  and any degree  $d \geq 0$ . Then the equivariant KGW invariant  $\langle \lambda_y(\mathcal{S}), b, c \rangle_d$  satisfies:*

$$\langle \lambda_y(\mathcal{S}), b, c \rangle_d = \int_{\mathrm{Gr}(k-d; n)} \lambda_y(\mathcal{S}_{k-d}) \cdot q_* p^*(b) \cdot q_* p^*(c).$$

Taking in particular  $y = 0$ , gives that the 2-point KGW invariant  $\langle b, c \rangle_d$  satisfies:

$$\langle b, c \rangle_d = \int_{\mathrm{Gr}(k-d; n)} q_* p^*(b) \cdot q_* p^*(c).$$

The quantum K Whitney relations formalize into the following presentation of the ring  $\mathrm{QK}(\mathrm{Gr}(k; n))$ .

**Corollary 6.5.** *Let  $X = (X_1, \dots, X_k)$  and  $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_{n-k})$ . The quantum K theory ring  $\mathrm{QK}(\mathrm{Gr}(k; n))$  has a presentation with generators and relations*

$$\frac{\mathbb{Z}[[q]][e_1(X), \dots, e_k(X), e_1(\tilde{X}), \dots, e_{n-k}(\tilde{X})]}{\langle \prod_{i=1}^k (1 + yX_i) \prod_{j=1}^{n-k} (1 + y\tilde{X}_j) = (1 + y)^n - \frac{q}{1-q} y^{n-k} \tilde{X}_1 \cdots \tilde{X}_{n-k} (\prod_{i=1}^k (1 + yX_i) - 1) \rangle}$$

While in cohomology Chern classes of a vector bundle and its dual differ by a sign, in K theory the relation is more subtle. The previous presentation implies that the dual bundles  $\mathcal{S}^\vee, \mathcal{Q}^\vee$  and their exterior powers may be generated in the (quantum) K ring as polynomials in  $\wedge^i \mathcal{S}, \wedge^j \mathcal{Q}$ . For example, in  $\mathrm{K}(\mathrm{Gr}(k; n))$ ,  $\det \mathcal{S}^\vee = \det \mathcal{Q}$ , and, more generally,

$$\wedge^i(\mathcal{S}) \cdot \det(\mathcal{S}^\vee) = \wedge^{k-i}(\mathcal{S}^\vee).$$

(For the proof, take Chern character.) The quantum analogue of this is the following.

**Theorem 6.6** ([GMSZ22]). *Let  $i > 0$ . Then the following holds in  $\mathrm{QK}(\mathrm{Gr}(k; n))$ :*

$$\wedge^i(\mathcal{S}) \star \det(\mathcal{Q}) = (1 - q) \wedge^{k-i}(\mathcal{S}^\vee) \cdot \det(\mathbb{C}^n).$$

(Here we included  $\det \mathbb{C}^n$ , because that is how this statement generalizes to the equivariant setting; non-equivariantly,  $\det \mathbb{C}^n = 1$ .)

**6.2. The Coulomb branch presentation.** The study of presentations generated by bundles (rather than Schubert classes) was actually inspired by results in physics [JM20, JMNT20, JM19, UY20], see also the recent [DN23, §5]. Informally, the (Schur) bundles correspond to ‘Wilson line operators’ and in physics the quantum K ring arises as

$$\mathrm{QK}(X) = \text{algebra of Wilson operators} / \text{relations}.$$

Such a description holds more generally when  $X$  may be realized as a GIT quotient of the form  $V//G$  where  $V$ . In the physics literature (cf. e.g. [MP95, CK16]), one considers the ‘twisted superpotential’

$$(6.4) \quad \begin{aligned} \mathcal{W} = & \frac{k}{2} \sum_{a=1}^k (\ln X_a)^2 - \frac{1}{2} \left( \sum_{a=1}^k \ln X_a \right)^2 \\ & + \ln((-1)^{k-1} q) \sum_{a=1}^k \ln X_a + n \sum_{a=1}^k \text{Li}_2(X_a). \end{aligned}$$

Here

$$\text{Li}_2(z) = \int_1^{1-z} \frac{\ln(t)}{1-t} dt$$

is the **dilogarithm**, and the only thing we need is that it satisfies

$$(6.5) \quad y \frac{\partial}{\partial y} \text{Li}_2(y) = -\ln(1-y),$$

The variables  $X_i$  are interpreted as the exponentials of the Chern roots  $X_i = e^{x_i}$ . In this context, the exterior powers  $\wedge^i \mathcal{S}, \wedge^j \mathcal{Q}$  are the aforementioned Wilson line operators considered in the physics literature. The Coulomb branch (or vacuum) equations for  $\mathcal{W}$  are

$$(6.6) \quad \exp \left( \frac{\partial \mathcal{W}}{\partial \ln X_i} \right) = 1, \quad 1 \leq i \leq k.$$

This implies that

$$(6.7) \quad (-1)^{k-1} q (X_a)^k = \left( \prod_{b=1}^k X_b \right) (1 - X_a)^n.$$

These equations turn out to be the **Bethe Ansatz** equations associated to an integrable system studied by Gorbounov and Korff [GK17]. (See also section 7 below.) There is also an equivariant version of these identities, explained next. Let  $T_1, \dots, T_n \in K_T(pt)$  denote the equivariant parameters, i.e., the weights in the decomposition of the  $T$ -module

$$\mathbb{C}^n = \bigoplus \mathbb{C}_{T_i}.$$

In physical theories these appear as exponentials of “twisted masses.” Concretely, in cases with twisted masses, the superpotential (6.4) for  $\text{Gr}(k; n)$  generalizes to (see

[UY20])<sup>5</sup>:

$$\begin{aligned} \mathcal{W} = & \frac{k}{2} \sum_{a=1}^k (\ln X_a)^2 - \frac{1}{2} \left( \sum_{a=1}^k \ln X_a \right)^2 \\ & + \ln((-1)^{k-1} q) \sum_{a=1}^k \ln X_a + \sum_{i=1}^n \sum_{a=1}^k \text{Li}_2(X_a T_i^{-1}). \end{aligned}$$

Simplifying

$$(6.8) \quad \exp \left( \frac{\partial \mathcal{W}}{\partial \ln X_a} \right) = 1$$

for each  $1 \leq a \leq k$ , we find

$$(6.9) \quad (-1)^{k-1} q (X_a)^k \prod_{j=1}^n T_j = \left( \prod_{b=1}^k X_b \right) \cdot \prod_{i=1}^n (T_i - X_a).$$

(Compare to equations (7.6) below.) The equations (6.7) are not  $S_k \times S_{n-k}$  symmetric, thus they cannot represent relations in the quantum K ring of  $\text{Gr}(k; n)$ ; one needs to symmetrize them. To (eventually) relate to the quantum cohomology presentation, we also work with the ‘shifted Wilson line operators’, or, equivalently, with variables

$$z_i = 1 - X_i, \quad (1 \leq i \leq k).$$

The Coulomb branch equations show that  $z_i$  are the roots of a ‘characteristic polynomial’:

$$(6.10) \quad f(\xi, z, q) = \xi^n + \sum_{i=0}^{n-1} (-1)^{n-i} \xi^i g_{n-i}(z, \lambda, q),$$

where  $g_j(z, \lambda, q)$  is symmetric in  $z_i$ ’s. (See example below.) This means that  $f(\xi, z_i, q) = 0$  for  $1 \leq i \leq k$ . The polynomial  $f(\xi, z, q)$  has degree  $n$ , and, aside from the roots  $z_1, \dots, z_k$ , denote by  $\hat{z}_1, \dots, \hat{z}_{n-k}$  the other roots. The Vieta relations are polynomials which are symmetric separately in  $z_1, \dots, z_k$  and  $\hat{z}_1, \dots, \hat{z}_{n-k}$ . Denote by  $I$  the ideal generated by these polynomials. This gives another presentation of the quantum K ring, equal to the one proved in [GK17], in the context of integrable systems. This version was proved in [GMSZ22].

**Theorem 6.7** (‘Coulomb branch’ presentation). *The Vieta relations applied to the characteristic polynomial  $f(\xi, z_i, q)$  generate an ideal  $I$  such that*

$$\mathbb{C}[[q]][z_1, \dots, z_k; \hat{z}_1, \dots, \hat{z}_{n-k}] / I$$

*is isomorphic to  $\text{QK}(\text{Gr}(k; n))$ .*

---

<sup>5</sup>The superpotential  $\mathcal{W}$  also depends on certain variables called the Chern-Simons (CS) levels. Here we already specialized these levels to recover Givental and Lee’s quantum K theory ring. Very recently, we related the more general CS levels to a twisted version of quantum K theory, see [HKMSZ25].

**Example 6.8.** *The Coulomb branch relations for  $\mathrm{Gr}(2; 5)$  are*

$$\sum_{i+j=\ell} e_i(z) e_j(\hat{z}) = g_\ell(z, q) \quad ,$$

for  $1 \leq \ell \leq 5$ , where the polynomials  $g_\ell(z, \lambda, q)$  are given by

$$g_1 = z_1 z_2; g_2 = g_3 = 0; g_4 = g_5 = -q.$$

In fact, one may eliminate the variables  $\hat{z}$  by solving for  $e_i(\hat{z})$  in terms of  $e_i(z)$  to obtain:

$$\begin{aligned} e_1(\hat{z}) &= -G_1(z); \\ e_2(\hat{z}) &= G_2(z); \\ e_3(\hat{z}) &= -G_3(z). \end{aligned}$$

Here  $G_i(z)$  are the Grothendieck polynomials, given by

$$\begin{aligned} G_1(z) &= z_1 + z_2 - z_1 z_2; \\ G_2(z) &= z_1^2 + z_1 z_2 + z_2^2 - z_1^2 z_2 - z_1 z_2^2; \\ G_3(z) &= z_1^3 + z_1^2 z_2 + z_1 z_2^2 + z_2^3 - z_1^3 z_2 - z_1^2 z_2^2 - z_1 z_2^3. \end{aligned}$$

This gives the presentation

$$\mathrm{QK}(\mathrm{Gr}(2; 5)) = \frac{\mathbb{C}[[q]][z_1, z_2]}{\langle -e_1(z)G_3(z) + e_2(z)G_2(z) = -q, \quad e_2(z)G_3(z) = q \rangle}$$

To obtain the quantum cohomology presentation, one needs to take the associated graded of this presentation. Equivalently, one needs to take the leading terms of each of the generators. Denote by  $h_i(z)$  the complete homogeneous symmetric function in  $z = (z_1, z_2)$ . Since  $\deg q = 5$ , the leading terms are:

$$-e_1(z)h_3(z) + e_2(z)h_2 = 0; \quad e_2h_3(z) = q.$$

It is an exercise to check that the presentation given by these relations generate a ring isomorphic to the Witten presentation of  $\mathrm{QH}^*(\mathrm{Gr}(2, 5))$  from (6.2).

**6.2.1. A Whitney presentation for the quantum  $K$  ring of partial flag manifolds.** We briefly discuss below an analogue of the Whitney presentation above for any partial flag manifold  $X = \mathrm{Fl}(i_1, \dots, i_k; n)$ , as stated in [GMS<sup>+</sup>23], and following the physics considerations from [GMS<sup>+</sup>24]. As mentioned in the introduction to this section, proofs of this presentation have been obtained recently in [HK24a], after a Toda-type presentation for the full flag manifold was proved in [MNS25a]. It was proved in [AHKM<sup>+</sup>25] that the ‘Whitney’ presentation implies the ‘Toda’ presentation proved in [MNS25a], in the sense that the latter may be obtained by eliminating some of the generators from the former.

Consider the tautological sequence  $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \subset \mathcal{S}_k \subset \mathcal{S}_{k+1} = \mathbb{C}^n$  on  $X$ , where  $\mathcal{S}_j$  has rank  $r_j$ .

**Theorem 6.9.** *For  $j = 1, \dots, k$ , the following relations hold in  $\mathrm{QK}(X)$ :*

$$\lambda_y(\mathcal{S}_j) \star \lambda_y(\mathcal{S}_{j+1}/\mathcal{S}_j) = \lambda_y(\mathcal{S}_{j+1}) - y^{r_{j+1}-r_j} \frac{q_j}{1-q_j} \det(\mathcal{S}_{j+1}/\mathcal{S}_j) \star (\lambda_y(\mathcal{S}_j) - \lambda_y(\mathcal{S}_{j-1})).$$

*The same relations hold equivariantly.*

Note that this generalizes the presentation from Theorem 6.3 above. We also note that there is also an analogue of the twisted superpotential  $\mathcal{W}$  from (6.4) in this case, giving a Jacobi presentation as in (6.8); again, the resulting equations are the Bethe Ansatz equations for a certain integrable system. We refer to [GMS<sup>+</sup>24] for details.

## 7. QUANTUM K THEORY OF GRASSMANNIANS FROM A YANG-BAXTER ALGEBRA

Quantum cohomology theories of flag manifolds have long been related to lattice models in mathematical physics. In these notes I will mention one relatively recent, and explicit, connection, due to Gorbounov and Korff [GK17]. They realized the sum of quantum K theory rings of Grassmannian as a module over a Yang-Baxter algebra in such a way that the ‘monodromy matrix operator’ from a 5-vertex lattice model is directly related to the (geometrically defined) quantum K multiplication. This result, along with a full geometric interpretation of the results from [GK17] may be found in [GKM25]. We present below an brief outline of some of the salient points.

**7.1. Quantum K theory as a Yang-Baxter module.** An important feature of integrable systems is that one considers the sum of the quantum K rings of *all* Grassmannians. To be precise, let  $T \subset G = \mathrm{GL}_n$  be the maximal torus, and denote by  $R(T) = K_T(pt)$  the representation ring of  $T$ . This is a Laurent polynomial ring  $\mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$  in a basis of characters of  $T$ . Recall that  $K_T(X)$  denotes the  $T$ -equivariant  $K$ -theory of a  $T$ -variety  $X$ . It is defined in analogy to the ordinary  $K$ -theory, except that all objects (vector bundles, coherent sheaves, morphisms etc) are appropriately equivariant. The (equivariant!) structure morphism  $X \rightarrow pt$  turns  $K_T(X)$  into a  $K_T(pt)$ -algebra, by pullback.

Fix  $\mathbb{C}^2$  with standard basis  $e_1, e_2$ . For an indeterminate  $y$  (the spectral parameter), the  $R$ -matrix is a certain function  $\mathbf{R}(y)$  with values in  $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ . The  $R$ -matrix solves the (quantum) Yang-Baxter equation:

$$\mathbf{R}_{12}(y_1/y_2) \mathbf{R}_{13}(y_1) \mathbf{R}_{23}(y_2) = \mathbf{R}_{23}(y_2) \mathbf{R}_{13}(y_1) \mathbf{R}_{12}(y_1/y_2);$$

this is regarded as composition of endomorphisms of  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ , where  $\mathbf{R}_{ij}$  acts on the  $i$  and  $j$  factors and identity on the others. The Yang-Baxter algebra  $\mathbf{YB}$  from [GK17] has generators  $t_{ij}(y)$ , thought as matrix entries in the *monodromy matrix*

$$\mathbf{T}(y) = \begin{pmatrix} t_{00}(y) & t_{01}(y) \\ t_{10}(y) & t_{11}(y) \end{pmatrix}$$

subject to relations of the form  $\mathbf{RTT} = \mathbf{TTR}$ . For the purpose of these notes, we omit the precise form of the  $R$  matrix and the relations, and we refer to [GKM25] for precise

definitions in the notation conventions used here. The geometric conventions we use below (and in [GKM25]) are slightly different from those in [GK17]; to relate the two one needs to make changes of variables:

$$-y = 1 - z, \quad T_i = 1 - y_i, \quad \beta = -1.$$

(In addition, in [GKM25], the variable  $\varepsilon_i$  is denote by  $T_i$  here, and we use opposite notation for Schubert classes.) Consider the vector space:

$$\mathbf{V}_n = \mathbb{C}^2[T_1^{\pm 1}] \otimes \cdots \otimes \mathbb{C}^2[T_n^{\pm 1}].$$

We may identify  $\mathbf{V}_n$  with the sum of the equivariant K theory modules:

$$\Phi : \mathbf{V}_n := \bigoplus_{k=0}^n V_{k,n} \rightarrow \bigoplus_{k=0}^n K_T(\mathrm{Gr}(k; n)); \quad v_\lambda \rightarrow \mathcal{O}^\lambda;$$

here  $V_{k,n}$  is spanned by vectors  $v_\lambda := e_{i_1} \otimes \cdots \otimes e_{i_n}$  with  $k$   $e_2$ 's, obtained from the usual identification of the Young diagram  $\lambda$  in the  $k \times (n - k)$  rectangle with its path. Equivalently,  $\Phi$  is an isomorphism of  $K_T(pt)$ -modules.

Each vector space  $\mathbb{C}^2[T_i^{\pm 1}]$  may be given a **YB**-module structure, referred in the literature as an *evaluation module*. The tensor product therefore admits a **YB**-module structure, giving  $\mathbf{V}_n$  a structure of a highest weight module. This means that for

$$(7.1) \quad v_o = e_1 \otimes \cdots \otimes e_1 \in K_T(\mathrm{Gr}(0; n))$$

we have

$$(7.2) \quad t_{00}(y).v_o = \prod_{k=1}^n (1 + y/T_k)v_o, \quad t_{01}(y).v_o = 0, \quad t_{11}(y).v_o = v_o.$$

Furthermore, each  $V_{k,n}$  is a weight space preserved by the diagonal operators, i.e.,

$$t_{ii} : V_{k;n} \rightarrow V_{k;n}, \quad i = 0, 1.$$

The off-diagonal operators relate the weight spaces:

$$t_{10} : V_{k;n} \rightarrow V_{k+1;n}; \quad t_{01} : V_{k+1;n} \rightarrow V_{k;n}.$$

It is natural to ask for geometric interpretation of the actions of the  $R$ -matrix, and of the monodromy matrix  $\mathbf{T}(y)$ , on the module  $\mathbf{V}_n$ . It turns out that the components of the  $R$ -matrix correspond to the the *left* Weyl group action on  $\mathrm{QK}_T(\mathrm{Gr}(k; n))$ , defined in [Knu03, MNS22]. This is the action induced by left multiplication by  $\mathrm{GL}_n(\mathbb{C})$  on  $\mathrm{Gr}(k; n)$ , therefore defining an action on any *equivariant* (quantum) cohomology theory of  $\mathrm{Gr}(k; n)$ . In various forms, this relation was observed in many other contexts, see for example the identification of the  $R$ -matrix with the left Weyl group action on cohomological stable envelopes from [MNS22]. Slightly more precisely, a twisted version  $\check{\mathbf{R}}$  of the  $R$ -matrix operator from [GK17], is given by the left Weyl group multiplication:

$$\Phi(\check{\mathbf{R}}_{i,i+1}(T_i/T_{i+1})v_\lambda) = s_i^L.\mathcal{O}^\lambda,$$

where  $s_i^L$  is the operator induced by multiplication by the simple reflection  $s_i = (i, i+1)$ .

To give the action of the monodromy matrix  $\mathbf{T}(y)$ , let  $0 \leq k \leq n-1$ , and consider the incidence diagram:

$$(7.3) \quad \begin{array}{ccc} \mathrm{Fl}(k, k+1; n) & \xrightarrow{p_2} & \mathrm{Gr}(k+1; n) \\ p_1 \downarrow & & \\ \mathrm{Gr}(k; n) & & \end{array}$$

**Theorem 7.1.** *The following hold:*

- (1) *The quantum trace operator  $t(y) := t_{00}(y) + qt_{11}(y)$  restricted to  $V_{k,n} \otimes \mathbb{C}[q]$  satisfies:*

$$\Phi(t(y).v_\lambda) = \lambda_y(\mathcal{Q}_{n-k}^\vee) \star \mathcal{O}^\lambda.$$

- (2) *The off-diagonal operators are given by convolutions. More precisely, let*

$$\tau_{10}(y) : K_T(\mathrm{Gr}(k; n))[y] \rightarrow K_T(\mathrm{Gr}(k+1; n))[y]$$

*and*

$$\tau_{01}(y) : K_T(\mathrm{Gr}(k+1; n))[y] \rightarrow K_T(\mathrm{Gr}(k; n))[y]$$

*be the convolution operators defined by*

$$\tau_{10}(\kappa) = \lambda_y(\mathcal{Q}_{n-k-1}^\vee) \cdot (p_2)_*(p_1)^*(\kappa) - (p_2)_*p_1^*(\lambda_y(\mathcal{Q}_{n-k}^\vee) \cdot \kappa),$$

*and by*

$$\tau_{01}(\kappa) = (p_1)_*(p_2)^*(\lambda_y(\mathcal{Q}_{n-k-1}^\vee) \cdot \kappa).$$

*Then  $\Phi(t_{10}(y).v_\lambda) = \tau_{10}(\mathcal{O}^\lambda)$  and  $\Phi(t_{01}(y).v_\mu) = \tau_{01}(\mathcal{O}^\mu)$ .*

This generalizes to K theory the results in [GKS20] in the situation of the (equivariant) quantum cohomology of the Grassmannian.

We comment next on some of the features of this theorem. The quantum parameter  $q$  has meaning only on geometry; on the integrable system side, the ‘ $q$ -part’ is external to the theory. The theorem states that  $t_{11}$  gives the coefficient of  $q$  in the quantum  $K$  multiplication by  $\lambda_y(\mathcal{Q}^\vee)$ .

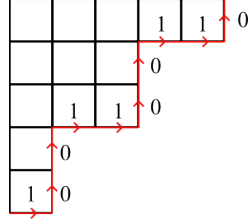
The two terms in  $\tau_{10}$  correspond to the two factors from Corollary 4.16 above. More precisely, the coefficient of  $q^d \mathcal{O}^\nu$  in the multiplication  $\lambda_y(\mathcal{Q}^\vee) \star \mathcal{O}^\mu$  is equal to 0 if  $d \geq 2$ , and, for  $d = 1$  is equal to:

$$(7.4) \quad \langle \lambda_y(\mathcal{Q}^\vee), \mathcal{O}^\mu, (\mathcal{O}^\nu)^\vee \rangle_1 - \sum_{\eta} \langle \lambda_y(\mathcal{Q}^\vee), \mathcal{O}^\mu, (\mathcal{O}^\eta)^\vee \rangle_0 \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\nu)^\vee \rangle_1.$$

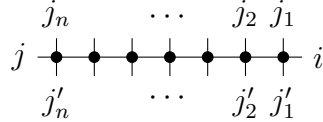
In other words,  $\tau_{10}$  is precisely the convolution arising in the ‘quantum=classical’ phenomenon, and one can use this statement to calculate the values of the diagonal operators  $t_{ii}$ , for  $i = 0, 1$ . In practice, it is non-trivial to find cancellation free formulae for the multiplication  $\lambda_y(\mathcal{Q}^\vee) \star \mathcal{O}^\mu$ . Thankfully, the *graphical calculus* associated to a 5-vertex model, developed in [GK17], provides such formulae, for all operators  $t_{ij}$ . We will give below the relevant definitions and examples.

The idea of proof of Theorem 7.1 exploits the fact that both sets of operators  $t_{ij}$  and  $\tau_{ij}$  commute with the left Weyl group action. This implies that it suffices to check equality on the class of the Schubert point class, as this class generates the full ring under the (left) nil-Hecke algebra action. The equalities  $t_{ij}(v_{(n-k)^k}) = \tau_{ij}(\mathcal{O}^{(n-k)^k})$  are verified using the ‘quantum = classical’ statement in geometry; for integrable systems, this follows from the graphical calculus.

**7.2. Graphical calculus.** We explain next the main formulae behind the graphical calculus, using the conventions from [GKM25]. For each partition  $\lambda$  in the  $k \times (n-k)$  rectangle we associate the 01 word  $J_\lambda = j_1 j_2 \dots j_n$  with 0’s in positions  $\lambda_i + k - i + 1$ , for  $1 \leq i \leq k$ . Graphically,  $J_\lambda$  is obtained by tracing the outline of the Young diagram of  $\lambda$ , starting from the SW corner, and placing 0’s for the vertical steps, and 1’s for the horizontal steps. To illustrate, in the next example,  $\lambda = (5, 3, 3, 1, 1)$  and its associated 01-word  $I = 1001100110$  for  $\text{Gr}(5; 10)$ . Consider now two 01-words  $J = j_1 \dots j_n, J' =$



$j'_1 \dots j'_n$  and one of the matrix entries  $t_{ij}$ . We consider configurations of the form:



where each bullet at a single vertex is a placeholder for one of the following five possible types of vertices,

(7.5)

$$(I) \quad 0 \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} 0 \quad (II) \quad 0 \begin{array}{c} 1 \\ \vdots \\ 0 \end{array} 1 \quad (III) \quad 1 \begin{array}{c} 0 \\ \vdots \\ 1 \end{array} 0 \quad (IV) \quad 0 \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} 0 \quad (V) \quad 1 \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} 1 \quad .$$

Each vertex of type (I) in column  $i$  (labelled from left to right) contributes a factor  $1 + y/T_{n+1-i}$ , each vertex of type (II) a factor  $-y/T_{n+1-i}$  and vertices (III-V) each contribute a factor 1. Then the total weight of the configuration is the product of individual weights. If there is no possible row configuration consisting only of these types of vertices, then the total weight is zero. If  $J = J_\lambda$  and  $J' = J_\mu$  for  $\lambda, \mu$  in the appropriate rectangles, then the coefficient of  $v_\mu$  in  $t_{ij}(v_\lambda)$  is equal to the sum of the weights of the pictures above.

**Example 7.2.** Take  $k = 2$ ,  $n = 7$  and  $J = 1101011$ ,  $J' = 1001101$ . Then  $J$  corresponds to  $\lambda = (3, 2) \subset (5, 5)$ , and  $J'$  to  $\mu = (3, 3, 1) \subset (4, 4, 4)$ . We calculate the coefficient of  $v_\mu$  in  $t_{10}(v_\lambda)$ . One vertex configuration is:

$$\begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}$$

which produces the term  $(1 + y/T_7)(1 + y/T_4)(-y/T_2)(-y/T_6)$  as there is a vertex of type (I) in columns 4 and 7 and a vertex of type (II) in columns 2 and 6 (numbered from right to left). There are many other possible configurations and summing up all their contributions then gives the matrix element.

**Example 7.3.** Consider  $\text{Gr}(2, 4)$  and the partitions  $\lambda = \mu = (1)$ . The associated word is 0101. There is a single configuration with the labels  $i = j = 0$  and the top and bottom labels given by 0101:

$$\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & \text{---} \text{---} \text{---} \text{---} & 0 \\ 1 & 0 & 1 & 0 \end{array}$$

The coefficient of  $v_{(1)}$  in  $t_{00}v_{(1)}$  is equal to the weight  $(1 + y/T_2)(1 + y/T_4)$  of this configuration.

We now calculate the coefficient of  $v_\emptyset$  in  $t_{11}v_{(2,1)}$ . Again, there is a single configuration:

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & \text{---} \text{---} \text{---} \text{---} & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

which has weight  $(-y/T_1) \cdot (1 + y/T_3)$ .

One can use the graphical calculus to calculate  $t(y)(v_\lambda) = (t_{00} + qt_{11})(v_\lambda)$ . Theorem 7.1 states that under the isomorphism  $\Phi$ , this is equal to the quantum K multiplication  $\lambda_y(\mathcal{Q}_2^\vee) \star \mathcal{O}^\lambda$ . We leave it as an exercise in the graphical calculus to check the following:

$$\begin{aligned} \lambda_y(\mathcal{Q}_2^\vee) \star \mathcal{O}^\emptyset &= \frac{(T_3 + y)(T_4 + y)}{T_3 T_4} \mathcal{O}^\emptyset - \frac{(T_4 + y)y}{T_3 T_4} \mathcal{O}^{(1)} - \frac{y}{T_4} \mathcal{O}^{(2)} \\ \lambda_y(\mathcal{Q}_2^\vee) \star \mathcal{O}^{(1)} &= \frac{(T_4 + y)(T_2 + y)}{T_2 T_4} \mathcal{O}^{(1)} - \frac{(T_4 + y)y}{T_4 T_2} \mathcal{O}^{(1,1)} - \frac{y(T_2 + y)}{T_4 T_2} \mathcal{O}^{(2)} + \frac{y^2}{T_4 T_2} \mathcal{O}^{(2,1)} \\ \lambda_y(\mathcal{Q}_2^\vee) \star \mathcal{O}^{(1,1)} &= -\frac{yq}{T_1} \mathcal{O}^\emptyset + \frac{(T_4 + y)(T_1 + y)}{T_4 T_1} \mathcal{O}^{(1,1)} - \frac{y(T_1 + y)}{T_4 T_1} \mathcal{O}^{(2,1)} \\ \lambda_y(\mathcal{Q}_2^\vee) \star \mathcal{O}^{(2)} &= \frac{(T_2 + y)(T_3 + y)}{T_2 T_3} \mathcal{O}^{(2)} - \frac{y(T_3 + y)}{T_2 T_3} \mathcal{O}^{(2,1)} - \frac{y}{T_3} \mathcal{O}^{(2,2)} \\ \lambda_y(\mathcal{Q}_2^\vee) \star \mathcal{O}^{(2,1)} &= -\frac{yq(T_3 + y)}{T_3 T_1} \mathcal{O}^\emptyset + \frac{y^2 q}{T_3 T_1} \mathcal{O}^{(1)} + \frac{(T_1 + y)(T_3 + y)}{T_1 T_3} \mathcal{O}^{(2,1)} - \frac{y(T_1 + y)}{T_3 T_1} \mathcal{O}^{(2,2)} \\ \lambda_y(\mathcal{Q}_2^\vee) \star \mathcal{O}^{(2,2)} &= -\frac{yq(T_2 + y)}{T_2 T_1} \mathcal{O}^{(1)} - \frac{yq}{T_2} \mathcal{O}^{(1,1)} + \frac{(T_1 + y)(T_2 + y)}{T_1 T_2} \mathcal{O}^{(2,2)} \end{aligned}$$

**Example 7.4.** We now illustrate the calculation of the off-diagonal operator  $t_{01}(v_{(2,2)}) \in K_T(\text{Gr}(1, 4))$  for  $v_{(2,2)} \in K_T(\text{Gr}(2, 4))$ , using both the graphical calculus, and the ‘quantum=classical’ statement.

First, specializing  $q = 0$  in the previous example one obtains

$$\lambda_y(\mathcal{Q}_2^\vee) \cdot \mathcal{O}^{(2,2)} = \frac{(T_1 + y)(T_2 + y)}{T_1 T_2} \mathcal{O}^{(2,2)}$$

One may check that  $(p_1)_*(p_2)^*(\mathcal{O}^{(2,2)}) = \mathcal{O}^{(2)}$ , giving

$$\tau_{01}(\mathcal{O}^{(2,2)}) = \frac{(T_1 + y)(T_2 + y)}{T_1 T_2} \mathcal{O}^{(2)}.$$

One may check the result using graphical calculus; there is a single configuration, of weight  $\frac{(T_1+y)(T_2+y)}{T_1 T_2}$ :

$$\begin{array}{ccccccc} & & 0 & 0 & 1 & 1 & \\ & & & & | & | & \\ 1 & \frown & & & | & | & 0 \\ & & 1 & 0 & 1 & 1 & \end{array}$$

**7.3. Bethe vectors.** The geometry - integrable systems dictionary developed in Theorem 7.1 has a number of consequences. In these notes we only mention one of them, in relation to *Bethe vectors*. To start, observe that the operator of multiplication  $\lambda_y(\mathcal{Q}_{n-k}^\vee) \star$  is diagonalizable with distinct eigenvalues. In the classical case ( $q = 0$ ) its eigenvectors are the classes  $\mathbf{e}_\lambda \in K_T(\text{Gr}(k; n))$  of torus fixed points, for  $\lambda$  varying in the  $k \times (n - k)$  rectangle. A Bethe vector  $\mathbf{b}_\lambda$  is any eigenvector of  $\lambda_y(\mathcal{Q}_{n-k}^\vee) \star$ . We will choose a normalization of the eigenvectors to coincide with that from [GKS20], and which satisfies

$$\Phi(\mathbf{b}_\lambda) \bmod q = \mathbf{e}_\lambda.$$

**Corollary 7.5.** (a) The set  $\{\mathbf{e}_\lambda^q := \Phi(\mathbf{b}_\lambda)\}$  of Bethe vectors diagonalize the operators  $\lambda_y(\mathcal{Q}_{n-k}^\vee) \star$ , and are orthogonal:

$$\mathbf{e}_\lambda^q \star \mathbf{e}_\mu^q = 0, \quad \forall \lambda \neq \mu.$$

In particular,  $\text{QK}_T(\text{Gr}(k; n))$  is a semisimple ring.

(b) The elements  $\mathbf{e}_\lambda^q \in \text{QK}_T(\text{Gr}(k; n))$  may be calculated from the convolution operators and the roots of the Bethe Ansatz equations:

$$\mathbf{e}_\lambda^q = \tau_{10}(-x_1^\lambda) \star \dots \star \tau_{10}(-x_k^\lambda) \mathcal{O}^\emptyset$$

where  $x^\lambda = (x_1^\lambda, \dots, x_k^\lambda)$ , with  $\lambda \subset k \times (n - k)$ , are the distinct solutions of the Bethe Ansatz equations for  $\text{Gr}(k; n)$  from (6.9), i.e., for  $1 \leq a \leq k$ ,

$$(7.6) \quad \prod_{i=1}^n (1 - x_a/T_i) \prod_{i \neq a}^k (x_i/x_a) + (-1)^k q = 0,$$

and  $\mathcal{O}^0 = \Phi(v_o)$  is the identity in  $K_T(\text{Gr}(0, n))$ .<sup>6</sup>

It is interesting to note that in the integrable systems context, an equation which determines the quantum K ring relations, analogous to the quantum K Whitney relations, from the previous section is a functional equation calculating, in geometric terms, the product  $\lambda_y(\mathcal{Q}_{n-k}^\vee) \star \lambda_{1/y}(\mathcal{S}_k)$ . We provide below a table summarizing the dictionary between the geometric and integrable systems perspectives.

Quantum Schubert Calculus	Quantum Integrable System
left Weyl group action	<b>R</b> matrix (solution of the quantum Yang-Baxter equation)
multiplication/convolution operators	entries of the monodromy matrix $\mathbf{T}(y) = \begin{pmatrix} t_{00}(y) & t_{01}(y) \\ t_{10}(y) & t_{11}(y) \end{pmatrix}$
quantum multiplication by $\lambda_y(\mathcal{Q}_{n-k}^\vee)$ (push-pull) convolution operators	quantum trace: $t(y) = t_{00}(y) + qt_{11}(y)$ off diagonal operators: $t_{01}(y), t_{10}(y)$
Schubert classes $\mathcal{O}^\lambda$ quantization of fixed point basis $\mathbf{e}_\lambda^q$	spin basis $v_\lambda$ Bethe vectors $\mathbf{b}_\lambda$

There are several other applications of these results, such as explicit combinatorial formulae for an action of the extended affine Weyl group (generalizing, for instance, the Seidel representation in quantum K theory [BCP23, LLSY25]); we refer the reader to [GKM25]. One application we briefly mention, and which is also discussed in the extended example below, is that of **quantum localization**.

We already mentioned that the elements  $\mathbf{e}_\lambda^q$  quantize the fixed points  $\mathbf{e}_\lambda \in K_T(\text{Gr}(k; n))$ . Using these one can define a morphism

$$\iota_\lambda^q : \text{QK}_T(\text{Gr}(k; n)) \rightarrow K_T^q(pt), \quad \kappa \mapsto ((\kappa, \mathbf{e}_\lambda^q));$$

here  $K_T^q(pt)$  is an appropriate extension of  $K_T(pt)$  ensuring the map is well defined. The map  $\iota_\lambda^{q=0}$  is the usual localization map, and it is a ring homomorphism. It was proved in [GKM25] that the same properties hold for arbitrary  $q$ , thus motivating the terminology of quantum localization.

**7.4. Example:  $n = 2$ .** In this section we illustrate the calculations of the monodromy matrix  $\mathbf{T}(y)$  in the case  $n = 2$ , i.e., on the the Yang-Baxter module is

$$\mathbf{V}_2 = K_T(\text{Gr}(0, 2)) \oplus K_T(\text{Gr}(1, 2)) \oplus K_T(\text{Gr}(2, 2)).$$

We also work out the algorithm calculating the Bethe vectors on  $\text{QK}_T(\text{Gr}(1, 2)) \simeq \text{QK}_T(\mathbb{P}^1)$ .

---

<sup>6</sup>Here the solutions  $x^\lambda$  of the Bethe Ansatz equations are labelled by partitions  $\lambda$  in the  $k \times (n - k)$  rectangle as follows: we attach the partition  $\lambda$  to the solution  $x^\lambda$  if at  $q = 0$  it specializes to  $x^\lambda|_{q=0} = \{T_{\lambda_k+1}, \dots, T_{\lambda_1+k}\}$ ; cf. [GK17, Lemma 4.6], see also [GKM25, Cor. 1.2].

The Grassmannians  $\text{Gr}(0; 2)$  and  $\text{Gr}(2; 2)$  have a single Schubert variety:  $X_0^\emptyset = 0$ , respectively  $X_2^\emptyset = \langle e_1, e_2 \rangle$ . The Schubert varieties on  $\text{Gr}(1, 2)$  are

$$X_1^\square = \langle e_2 \rangle; \quad X_1^\emptyset = \text{Gr}(1, 2) = \mathbb{P}^1.$$

Note that the quotient bundle on  $\text{Gr}(0, 2)$  is  $\mathcal{Q}_2 = \mathbb{C}^2$ , while the quotient bundle  $\mathcal{Q}_0$  on  $\text{Gr}(2, 2)$  is trivial of rank 0. The multiplications by the classes  $\lambda_y(\mathcal{Q}^\vee)$  are given by:

$$\lambda_y(\mathcal{Q}_2^\vee) \cdot \mathcal{O}_0^\emptyset = (1 + y/T_1)(1 + y/T_2)\mathcal{O}_0^\emptyset; \quad \lambda_y(\mathcal{Q}_0^\vee) = 1.$$

The Yang-Baxter module has the Schubert/spin basis

$$v_{11} := \mathcal{O}_0^\emptyset, \quad v_{12} := \mathcal{O}_1^\emptyset, \quad v_{21} := \mathcal{O}_1^\square, \quad v_{22} := \mathcal{O}_2^\emptyset.$$

Here the lower indices indicate the label  $k$  from  $\text{Gr}(k; n)$ , so  $\mathcal{O}_k^\lambda \in K_T(\text{Gr}(k; n))$ . We describe next the monodromy matrix  $\mathbf{T}(y) = \begin{pmatrix} t_{00}(y) & t_{01}(y) \\ t_{10}(y) & t_{11}(y) \end{pmatrix}$  using Theorem 7.1; we will identify  $t_{ij}$  with the convolution operators  $\tau_{ij}$ . Since there are no quantum corrections in  $\text{QK}_T(\text{Gr}(0, 2))$  and  $\text{QK}_T(\text{Gr}(2, 2))$  it follows that

$$t_{11}(y)v_{11} = t_{11}(y)v_{22} = 0$$

and that

$$t_{00}(y)v_{11} = \lambda_y(\mathcal{Q}_2^\vee) \cdot v_{11} = (1 + y/T_1)(1 + y/T_2)v_{11}; \quad t_{00}(y)v_{22} = \lambda_y(\mathcal{Q}_0^\vee) \cdot v_{22} = v_{22}.$$

One calculates that

$$\lambda_y(\mathcal{Q}_1^\vee) = (1 + y/T_2)\mathcal{O}_1^\emptyset - (y/T_2)\mathcal{O}_1^\square.$$

Using for example the ‘quantum=classical’ statement one calculates that

$$\lambda_y(\mathcal{Q}_1^\vee) \star \mathcal{O}_1^\square = -(yq/T_1)\mathcal{O}_1^\emptyset + (1 + y/T_1)\mathcal{O}_1^\square.$$

Therefore it follows from Theorem 7.1 that

$$t_{00}(y)v_{12} = (1 + y/T_2)v_{12} - (y/T_2)v_{21}; \quad t_{00}(y)v_{21} = (1 + y/T_1)v_{21},$$

and

$$t_{11}(y)v_{12} = 0; \quad t_{11}(y)v_{21} = -y/T_1 v_{12}.$$

We now calculate the values of the off-diagonal operators on the spin basis. We start with  $t_{01}(y)$ :

$$t_{01}(y)v_{11} = 0;$$

$$t_{01}(y)v_{12} = (p_1)_* p_2^*(\lambda_y(\mathcal{Q}_1^\vee) \cdot \mathcal{O}_1^\emptyset) = \mathcal{O}_0^\emptyset = v_{11};$$

$$t_{01}(y)v_{21} = (p_1)_* p_2^*(\lambda_y(\mathcal{Q}_1^\vee) \cdot \mathcal{O}_1^\square) = (p_1)_* p_2^*((1 + y/T_1)\mathcal{O}_1^\square) = (1 + y/T_1)v_{11};$$

$$t_{01}(y)v_{22} = (p_1)_* p_2^*(\lambda_y(\mathcal{Q}_0^\vee) \cdot \mathcal{O}_2^\emptyset) = \mathcal{O}_1^\emptyset = v_{12}.$$

We continue with  $t_{10}(y)$ . We have

$$\begin{aligned}
t_{10}(y)v_{11} &= \lambda_y(\mathcal{Q}_1^\vee) \cdot (p_2)_*(p_1)^*(\mathcal{O}_0^\emptyset) - (p_2)_*p_1^*(\lambda_y(\mathcal{Q}_2^\vee) \cdot \mathcal{O}_0^\emptyset) \\
&= \lambda_y(\mathcal{Q}_1^\vee) - (1 + y/T_1)(1 + y/T_2)\mathcal{O}_1^\emptyset \\
&= (1 + y/T_2)\mathcal{O}_1^\emptyset - (y/T_2)\mathcal{O}_1^\square - (1 + y/T_1)(1 + y/T_2)\mathcal{O}_1^\emptyset \\
&= -y/T_1(1 + y/T_2)\mathcal{O}_1^\emptyset - (y/T_2)\mathcal{O}_1^\square \\
&= -y/T_1(1 + y/T_2)v_{12} - (y/T_2)v_{21}.
\end{aligned}$$

$$\begin{aligned}
t_{10}(y)v_{12} &= \lambda_y(\mathcal{Q}_0^\vee) \cdot (p_2)_*(p_1)^*(\mathcal{O}_1^\emptyset) - (p_2)_*p_1^*(\lambda_y(\mathcal{Q}_1^\vee) \cdot \mathcal{O}_1^\emptyset) \\
&= \mathcal{O}_2^\emptyset - (p_2)_*p_1^*((1 + y/T_2)\mathcal{O}_1^\emptyset - (y/T_2)\mathcal{O}_1^\square) \\
&= \mathcal{O}_2^\emptyset - \mathcal{O}_2^\emptyset \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
t_{10}(y)v_{21} &= \lambda_y(\mathcal{Q}_0^\vee) \cdot (p_2)_*(p_1)^*(\mathcal{O}_1^\square) - (p_2)_*p_1^*(\lambda_y(\mathcal{Q}_1^\vee) \cdot \mathcal{O}_1^\square) \\
&= \mathcal{O}_2^\emptyset - (p_2)_*p_1^*((1 + y/T_1)\mathcal{O}_1^\square) \\
&= \mathcal{O}_2^\emptyset - (1 + y/T_1)\mathcal{O}_2^\emptyset \\
&= -(y/T_1)v_{22}.
\end{aligned}$$

and, finally,  $t_{10}(y)v_{22} = 0$ .

We now proceed to determine the Bethe basis for  $\mathrm{QK}_T(\mathrm{Gr}(1, 2))$ . For  $\mathrm{Gr}(1, 2)$  there is a single Bethe Ansatz equation:

$$(1 - x/T_1)(1 - x/T_2) - q = 0,$$

with roots

$$x_\pm := \frac{T_1}{2} + \frac{T_2}{2} \pm \frac{\sqrt{4qT_1T_2 + (T_1 - T_2)^2}}{2}$$

At  $q = 0$ , the root  $x_+ = T_1$  corresponds to  $\lambda = \emptyset$  and the root  $x_- = T_2$  to  $\lambda = (1)$ . Then the Bethe vectors are

$$\mathbf{e}_\emptyset^q = \Phi(\mathbf{b}_{12}) = t_{10}(-x_+)v_{11} = \frac{x_+}{T_1}(1 - \frac{x_+}{T_2})\mathcal{O}_1^\emptyset + \frac{x_+}{T_2}\mathcal{O}_1^{(1)}$$

and

$$\mathbf{e}_{(1)}^q = \Phi(\mathbf{b}_{21}) = t_{10}(-x_-)v_{11} = \frac{x_-}{T_1}(1 - \frac{x_-}{T_2})\mathcal{O}_1^\emptyset + \frac{x_-}{T_2}\mathcal{O}_1^{(1)}$$

Alternatively, one can check directly that if  $q = 0$ , the Bethe vectors are precisely the classes of the corresponding torus fixed points in  $\mathrm{Gr}(1, 2)$ . Using that in the quantum ring  $\mathrm{QK}_T(\mathrm{Gr}(1, 2))$ ,

$$(7.7) \quad \mathcal{O}_1^{(1)} \star \mathcal{O}_1^{(1)} = q(T_2/T_1)\mathcal{O}_1^\emptyset - (T_2/T_1)\mathcal{O}_1^{(1)} + \mathcal{O}_1^{(1)}$$

(for example by the ‘quantum=classical’) an algebra calculation gives that

$$t_{10}(-x_+) \star t_{10}(-x_-) = 0,$$

verifying the orthogonality property of the idempotents.

In passing, note that if one ‘quantum integrates’ (i.e., take  $q \mapsto 1$ , and push-forward to a point, thanks to Kato’s functoriality) then from the Bethe Ansatz equations one obtains

$$\int_{\mathrm{QK}_T(\mathbb{P}^1)} \mathbf{e}_\emptyset^q = \frac{x_+}{T_1} \left(1 - \frac{x_+}{T_2}\right) + \frac{x_+}{T_2} = 1 - q \mapsto 0.$$

Similarly,

$$\int_{\mathrm{QK}_T(\mathbb{P}^1)} \mathbf{e}_{(1)}^q = 0.$$

We may also calculate the quantum localizations:

$$(\langle \mathcal{O}_1^\emptyset, \mathbf{e}_\emptyset^q \rangle) = \frac{\frac{x_+}{T_1} \left(1 - \frac{x_+}{T_2}\right)}{1 - q} + \frac{\frac{x_+}{T_2}}{1 - q} = \frac{1 - q}{1 - q} = 1;$$

$$(\langle \mathcal{O}_1^\emptyset, \mathbf{e}_\square^q \rangle) = \frac{\frac{x_-}{T_1} \left(1 - \frac{x_-}{T_2}\right)}{1 - q} + \frac{\frac{x_-}{T_2}}{1 - q} = \frac{1 - q}{1 - q} = 1;$$

Note that these coincide with the classical localizations. Furthermore, in this case the Bethe Ansatz equations are equivalent to these localizations being 1.

We now turn to the quantum localization of  $\mathcal{O}_1^\square$ . We have:

$$\begin{aligned} (\langle \mathcal{O}_1^\square, \mathbf{e}_\emptyset^q \rangle) &= \frac{\frac{x_+}{T_1} \left(1 - \frac{x_+}{T_2}\right)}{1 - q} + \frac{x_+}{T_2} (\langle \mathcal{O}_1^\square, \mathcal{O}_1^\square \rangle) \\ &= \frac{\frac{x_+}{T_1} \left(1 - \frac{x_+}{T_2}\right)}{1 - q} + \frac{x_+}{T_2} \cdot \frac{qT_2/T_1 - T_2/T_1 + 1}{1 - q} \\ &= \frac{1 - q - x_+/T_2 \cdot T_2/T_1(1 - q)}{1 - q} = 1 - x_+/T_1; \end{aligned}$$

$$\begin{aligned} (\langle \mathcal{O}_1^\square, \mathbf{e}_\square^q \rangle) &= \frac{\frac{x_-}{T_1} \left(1 - \frac{x_-}{T_2}\right)}{1 - q} + \frac{x_-}{T_2} (\langle \mathcal{O}_1^\square, \mathcal{O}_1^\square \rangle) \\ &= \frac{\frac{x_-}{T_1} \left(1 - \frac{x_-}{T_2}\right)}{1 - q} + \frac{x_-}{T_2} \cdot \frac{qT_2/T_1 - T_2/T_1 + 1}{1 - q} \\ &= \frac{1 - q - x_-/T_2 \cdot T_2/T_1(1 - q)}{1 - q} = 1 - x_-/T_1; \end{aligned}$$

We now calculate the quantum Euler pairing  $((\mathbf{e}_\emptyset^q, \mathbf{e}_\emptyset^q))$ . From the expansion into Schubert classes we obtain

$$\begin{aligned} ((\mathbf{e}_\emptyset^q, \mathbf{e}_\emptyset^q)) &= ((\frac{x_+}{T_1}(1 - \frac{x_+}{T_2})\mathcal{O}_1^\emptyset + \frac{x_+}{T_2}\mathcal{O}_1^{(1)}, \mathbf{e}_\emptyset^q)) \\ &= \frac{x_+}{T_1}(1 - \frac{x_+}{T_2}) + \frac{x_+}{T_2}((\mathcal{O}_1^{(1)}, \mathbf{e}_\emptyset^q)) \\ &= \frac{x_+}{T_1}(1 - \frac{x_+}{T_2}) + \frac{x_+}{T_2}(1 - \frac{x_+}{T_1}) \\ &= \frac{x_+}{T_1} + \frac{x_+}{T_2} - \frac{2(x_+)^2}{T_1 T_2} \\ &= 1 - q - \frac{(x_+)^2}{T_1 T_2}. \end{aligned}$$

From the  $W$ -equivariance of the quantum pairing we obtain

$$((\mathbf{e}_\square^q, \mathbf{e}_\square^q)) = s^L.((\mathbf{e}_\emptyset^q, \mathbf{e}_\emptyset^q)) = 1 - q - \frac{(x_-)^2}{T_1 T_2}.$$

Note that if  $q = 0$ ,  $x_+ = T_1$  and  $x_- = T_2$ , giving that  $((\mathcal{O}_1^\square, \mathbf{e}_\emptyset)) = 0$ ,  $((\mathcal{O}_1^\square, \mathbf{e}_\square)) = 1 - T_2/T_1$ , and  $((\mathbf{e}_\emptyset, \mathbf{e}_\emptyset)) = 1 - T_1/T_2$ , consistent with the classical case.

We already recorded that the ‘quantum localization map’

$$\mathrm{QK}_T(\mathrm{Gr}(k; n)) \rightarrow \bigoplus_{\lambda \subset (n-k)^k} \mathrm{K}_T^q(pt); \quad \kappa \mapsto ((\kappa, \mathbf{e}_\lambda^q))$$

is a **ring** homomorphism. In particular, we have a ‘quantum Atiyah-Bott’ theorem: for any class  $\kappa \in \mathrm{QK}_T(\mathrm{Gr}(k; n))$ , the quantum character defined by

$$\mathrm{qch}(\kappa) := ((\kappa, 1))$$

satisfies

$$\mathrm{qch}(\kappa) = \sum_{\lambda} \frac{((\kappa, \mathbf{e}_\lambda^q))}{((\mathbf{e}_\lambda^q, \mathbf{e}_\lambda^q))}.$$

We now illustrate the calculation of the quantum character of  $(\det \mathcal{Q}_1)^{\star d}$ . To start, we have

$$\lambda_y \mathcal{Q}_1 = (1 + yT_2)\mathcal{O}_1^\emptyset + yT_1\mathcal{O}_1^\square,$$

thus

$$\det \mathcal{Q}_1 = T_2\mathcal{O}_1^\emptyset + T_1\mathcal{O}_1^\square.$$

Then

$$\begin{aligned} ((\det \mathcal{Q}_1, \mathbf{e}_\emptyset^q)) &= T_2 + T_1(1 - \frac{x_+}{T_1}) = T_1 + T_2 - x_+; \\ ((\det \mathcal{Q}_1, \mathbf{e}_\square^q)) &= T_1 + T_2 - x_-. \end{aligned}$$

Then the quantum character is

$$\mathrm{qch}(\det \mathcal{Q}) = \frac{T_1 + T_2 - x_+}{1 - q - \frac{(x_+)^2}{T_1 T_2}} + \frac{T_1 + T_2 - x_-}{1 - q - \frac{(x_-)^2}{T_1 T_2}}$$

We illustrate this in the non-equivariant case  $T_1 = T_2 = 1$ :

$$\begin{aligned} \frac{2 - x_+}{1 - q - (x_+)^2} + \frac{2 - x_-}{1 - q - (x_-)^2} &= \frac{2}{1 - q - (x_+)^2} + \frac{2}{1 - q - (x_-)^2} - \frac{x_+}{1 - q - (x_+)^2} - \frac{x_-}{1 - q - (x_-)^2} \\ &= \frac{2}{1 - q} - \frac{x_+}{1 - q - (x_+)^2} - \frac{x_-}{1 - q - (x_-)^2} \\ &= \frac{2}{1 - q} \end{aligned}$$

Here the first equality follows from the quantum Atiyah-Bott applied to  $((1, 1))$ , and the second equality because

$$\frac{x_+}{1 - q - (x_+)^2} + \frac{x_-}{1 - q - (x_-)^2} = 0$$

using the Vieta relations of the Bethe Ansatz equations. One may check that in the equivariant case one obtains  $\frac{T_1 + T_2}{1 - q}$ .

## 8. THE (QUANTUM K) SCHUBERT PACKAGE

To end these notes, we revisit the Schubert package for the quantum K theory rings of flag manifolds  $G/P$  built above, and in the process indicate some of the (many!) things left to do. The subject is moving very fast, and the author apologizes if some references are omitted, because of exposition constraints, or simply because his own ignorance.

### 8.1. The Schubert package.

- (Schubert basis) As for  $K(G/P)$ , the basis of choice for  $QK(G/P)$  consists of the classes  $\mathcal{O}^w$  (or  $\mathcal{O}_w$ );
- (Dual basis) This has been worked out only for (cominuscule) Grassmannians; see, e.g., [Sum24], cf. Proposition 3.9.
- (Presentations and polynomial representatives) The *undeformed* (Grassmannian) Grothendieck polynomials represent quantum Schubert classes, in the presentations from section 6; this is proved in [GK17]. For  $QK_T(\text{Fl}(n))$ , the **(double) quantum Grothendieck polynomials** represent Schubert classes in the ‘Toda presentation’ from [MNS25a, MNS25b]; see also [IIM20, AHKM<sup>+</sup>25]. Based on the history of this problem, the author expects that the K-theoretic versions of the (factorial)  $P$  and  $Q$ -Schur functions, defined by Ikeda and Naruse [IN13], will represent Schubert classes in the equivariant quantum K theory of the maximal orthogonal, respectively the Lagrangian Grassmannian. This would generalize the quantum cohomological statement from [IMN16]. For further related work, see also [IKNY24, BPX24].
- (Positivity) Aside from the positivity statements proved in [BCMP22] for the quantum K theory of (minuscule) Grassmannians, in [Xu24] for the incidence varieties  $\text{Fl}(1, n - 1; n)$ , and in [BPX24] for the submaximal isotropic Grassmannian  $\text{IG}(2, 2n)$ , the general positivity statement from Conjecture 1 is wide open.

- (Schubert multiplication) Pieri-Chevalley rules for the quantum K theory of (cominuscule) Grassmannians have been obtained in [BM11, BCMP18a, BCP23]. Based on Kato's ‘quantum=affine’ statement and the ring homomorphism

$$\mathrm{QK}(G/B) \rightarrow \mathrm{QK}(G/P)$$

from Theorem 4.12, Chevalley rules for the quantum K theory of complete flag manifolds  $G/B$ , and also for some  $G/P$  with  $P$  maximal parabolic, were obtained in [LNS24, KLNS21, BPX24]; see also [LNS<sup>+</sup>25].

**8.2. Some questions.** There are many more questions in this wide open area, and we mention few of them next.

There is a ‘twisted’ version of KGW invariants, where the (virtual) structure sheaf is twisted by certain classes [CG07, RZ23]. There are twisted quantum K rings [HK24b], which appear naturally in the study of a K-theoretic version of the abelian-nonabelian correspondence from [CFKS08], see [HKMSZ25]. It is natural to ask how these twisted rings are related to the ‘quasi-map quantum K theory’  $\mathrm{QK}_{T \times \mathbb{C}^*}(T^*(G/P))$  of cotangent bundles defined by Okounkov [Oko17], see for example [GY21].

It is expected that  $\mathrm{QK}_T(G/P)$  is a limit of the quasi-map ring  $\mathrm{QK}_{T \times \mathbb{C}^*}(T^*(G/P))$  (see, e.g., [KPSZ21, Kor21]), but a complete proof seems to still be missing. Furthermore, in the ‘classical’ situation, by the homotopy property,  $\mathrm{K}_{T \times \mathbb{C}^*}(T^*(G/P))$  is isomorphic to (a localization of)  $\mathrm{K}_T(G/P)[y]$ , with the caveat that the ‘good’ Schubert basis for  $\mathrm{K}_{T \times \mathbb{C}^*}(T^*(G/P))$ , given by the K-theoretic stable envelopes, corresponds to the basis given by the motivic (Segre) classes of Schubert cells in  $\mathrm{K}_T(G/P)[y]$ ; see, e.g., [AMSS24a, AMSS24b, KZJ21]. Is there an isomorphism between the *quantum* K rings  $\mathrm{QK}_{T \times \mathbb{C}^*}(T^*(G/P))$  and  $\mathrm{QK}_T(G/P)[y]$ , appropriately localized?

Another question is how to extend the dictionary mentioned in section 7 of these notes to the quasi-map ring  $\mathrm{QK}_{T \times \mathbb{C}^*}(T^*(G/P))$ . In the integrable system context, and for  $G = \mathrm{GL}_n$ , one expects to use an integrable system based on a 6-vertex model, i.e., on a graphical calculus with 6 vertices. Such models are available in the literature, and one may ask for analogues of Theorem 7.1 in this context.

Often, the approach to quantum K theory inspired from physics leads to alternative formulae for various objects. One such example are conjectural expressions from [GGM<sup>+</sup>25] (see Conjectures 1 and 2) for the quantum K pairing, very different from those given in the equation (4.1) above. For example, if  $a, b \in K(\mathrm{Gr}(k, n))$ , then Conjecture 1 from *loc. cit.* states that

$$((a, b)) = \frac{\oint_{\mathrm{Gr}(k, n)} (a \star b \star \det \mathcal{Q}^{\star k})}{(1 - q)^k},$$

where  $\oint_{\mathrm{Gr}(k, n)} \kappa$  denotes a certain ‘top form’ operator, obtained by collecting the coefficient of Schur function indexed by the full rectangle  $s_{(n-k)^k}(x_1, \dots, x_k)$  in the expansion of  $\kappa$ ; see [GGM<sup>+</sup>25] for details. We seek a natural mathematical context to explain, and prove, these conjectures.

Finally, another concrete question, which arose from experiments with quantum K multiplications is:

*Is there a maximum quantum degree in the multiplication  $\mathcal{O}^u \star \mathcal{O}^v$  ?*

It is known that the quantum degrees form (finite) integer intervals in the multiplication in  $\mathrm{QK}(\mathrm{Gr}(k;n))$ , and more generally in the quantum K ring of cominuscule Grassmannians [BCMP22]. Examples show that the quantum (multi) degrees appearing in  $\mathrm{QK}(\mathrm{Fl}(n))$  do not form convex sets. However, it is known that a unique minimum quantum degree exists [Pos05, BCLM20], and examples suggest that for any  $u, v \in W$ ,  $\mathcal{O}^u \star \mathcal{O}^v$  has a unique maximum degree.

## REFERENCES

- [ACT22] David Anderson, Linda Chen, and Hsian-Hua Tseng. On the finiteness of quantum K-theory of a homogeneous space. *Int. Math. Res. Not. IMRN*, (2):1313–1349, 2022.
- [AF24] David Anderson and William Fulton. *Equivariant cohomology in algebraic geometry*, volume 210 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2024.
- [AGM11] Dave Anderson, Stephen Griffeth, and Ezra Miller. Positivity and Kleiman transversality in equivariant  $K$ -theory of homogeneous spaces. *J. Eur. Math. Soc. (JEMS)*, 13(1):57–84, 2011.
- [AHKM<sup>+</sup>25] Kamyar Amini, Irit Huq-Kuruvilla, Leonardo C. Mihalcea, Daniel Orr, and Weihong Xu. Toda-type presentations for the quantum K theory of partial flag varieties. *submitted, preprint available at arXiv:2504.07412*, 2025.
- [AMSS24a] Paolo Aluffi, Leonardo C. Mihalcea, Jörg Schürmann, and Changjian Su. Motivic Chern classes of Schubert cells, Hecke algebras, and applications to Casselman’s problem. *Ann. Sci. Éc. Norm. Supér. (4)*, 57(1):87–141, 2024.
- [AMSS24b] Paolo Aluffi, Leonardo C. Mihalcea, Jörg C. Schürmann, and Changjian C. Su. From motivic Chern classes of Schubert cells to their Hirzebruch and CSM classes. In *A glimpse into geometric representation theory*, volume 804 of *Contemp. Math.*, pages 1–52. Amer. Math. Soc., [Providence], RI, [2024] ©2024.
- [BCLM20] Anders S. Buch, Sjuvon Chung, Changzheng Li, and Leonardo C. Mihalcea. Euler characteristics in the quantum  $K$ -theory of flag varieties. *Selecta Math. (N.S.)*, 26(2):Paper No. 29, 11, 2020.
- [BCMP13] Anders S. Buch, Pierre-Emmanuel Chaput, Leonardo C. Mihalcea, and Nicolas Perrin. Finiteness of cominuscule quantum  $K$ -theory. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(3):477–494 (2013), 2013.
- [BCMP16] Anders S. Buch, Pierre-Emmanuel Chaput, Leonardo C. Mihalcea, and Nicolas Perrin. Rational connectedness implies finiteness of quantum  $K$ -theory. *Asian J. Math.*, 20(1):117–122, 2016.
- [BCMP18a] Anders S. Buch, Pierre-Emmanuel Chaput, Leonardo C. Mihalcea, and Nicolas Perrin. A Chevalley formula for the equivariant quantum  $K$ -theory of cominuscule varieties. *Algebr. Geom.*, 5(5):568–595, 2018.
- [BCMP18b] Anders S. Buch, Pierre-Emmanuel Chaput, Leonardo C. Mihalcea, and Nicolas Perrin. Projected Gromov-Witten varieties in cominuscule spaces. *Proc. Amer. Math. Soc.*, 146(9):3647–3660, 2018.
- [BCMP22] Anders S. Buch, Pierre-Emmanuel Chaput, Leonardo C. Mihalcea, and Nicolas Perrin. Positivity in minuscule quantum K theory. *available on arXiv:arXiv:2205.08630*, 2022.

- [BCP23] Anders S. Buch, Pierre-Emmanuel Chaput, and Nicolas Perrin. Seidel and Pieri products in cominuscule quantum K-theory. *available on arXiv:2308.05307*, 2023.
- [BK05] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progr. Math.* Birkhäuser Boston, Inc., Boston, MA, 2005.
- [BKT03] A. S. Buch, A. Kresch, and H. Tamvakis. Gromov-Witten invariants on Grassmannians. *J. Amer. Math. Soc.*, 16(4):901–915(electronic), 2003.
- [BM11] Anders S. Buch and Leonardo C. Mihalcea. Quantum  $K$ -theory of Grassmannians. *Duke Math. J.*, 156(3):501–538, 2011.
- [BM15] Anders S. Buch and Leonardo C. Mihalcea. Curve neighborhoods of Schubert varieties. *J. Differential Geom.*, 99(2):255–283, 2015.
- [BPX24] V. Benedetti, N. Perrin, and W. Xu. Quantum  $K$ -theory of  $\mathrm{IG}(2, 2n)$ . *Int. Math. Res. Not. IMRN*, (22):14061–14093, 2024.
- [Bri02] Michel Brion. Positivity in the Grothendieck group of complex flag varieties. *J. Algebra*, 258(1):137–159, 2002. Special issue in celebration of Claudio Procesi’s 60th birthday.
- [Bri05] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.
- [Buc02] Anders Skovsted Buch. A Littlewood-Richardson rule for the  $K$ -theory of Grassmannians. *Acta Math.*, 189(1):37–78, 2002.
- [Buc03] Anders Skovsted Buch. Quantum cohomology of Grassmannians. *Compositio Math.*, 137(2):227–235, 2003.
- [Buc05] Anders Skovsted Buch. Combinatorial  $K$ -theory. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 87–103. Birkhäuser, Basel, 2005.
- [CFKS08] Ionuț Ciocan-Fontanine, Bumsig Kim, and Claude Sabbah. The abelian/nonabelian correspondence and Frobenius manifolds. *Invent. Math.*, 171(2):301–343, 2008.
- [CG07] Tom Coates and Alexander Givental. Quantum Riemann-Roch, Lefschetz and Serre. *Ann. of Math. (2)*, 165(1):15–53, 2007.
- [CG09] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Springer Science & Business Media, 2009.
- [CK16] Cyril Closset and Heeyeon Kim. Comments on twisted indices in 3d supersymmetric gauge theories. *JHEP*, 08:059, 2016, 1605.06531.
- [CMP08] P. E. Chaput, L. Manivel, and N. Perrin. Quantum cohomology of minuscule homogeneous spaces. *Transform. Groups*, 13(1):47–89, 2008.
- [CP11a] P. E. Chaput and N. Perrin. On the quantum cohomology of adjoint varieties. *Proc. Lond. Math. Soc. (3)*, 103(2):294–330, 2011.
- [CP11b] P.-E. Chaput and N. Perrin. Rationality of some Gromov-Witten varieties and application to quantum  $K$ -theory. *Commun. Contemp. Math.*, 13(1):67–90, 2011.
- [CTY14] Edward Clifford, Hugh Thomas, and Alexander Yong.  $K$ -theoretic Schubert calculus for  $\mathrm{OG}(n, 2n + 1)$  and jeu de taquin for shifted increasing tableaux. *J. Reine Angew. Math.*, 690:51–63, 2014.
- [DN23] Mykola Dedushenko and Nikita Nekrasov. Interfaces and quantum algebras, I: Stable envelopes. *J. Geom. Phys.*, 194:Paper No. 104991, 74, 2023.
- [FL94] William Fulton and Alain Lascoux. A Pieri formula in the Grothendieck ring of a flag bundle. *Duke Math. J.*, 76(3):711–729, 1994.
- [FP97] William Fulton and Rahul Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic Geometry — Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96, Providence, RI, 1997. Amer. Math. Soc.
- [Ful84] William Fulton. *Intersection theory*. Springer-Verlag, Berlin, 1984.

- [Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.
- [FW04] W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. of Alg. Geom.*, 13(4):641–661, 2004.
- [GGM<sup>+</sup>25] Wei Gu, Jirui Guo, Leonardo Mihalcea, Yaoxiong Wen, and Xiaohan Yan. A correspondence between the quantum K theory and quantum cohomology of Grassmannians. *J. Geom. Phys.*, 210:Paper No. 105437, 24, 2025.
- [GHS03] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67, 2003.
- [Giv00] Alexander Givental. On the WDVV equation in quantum  $K$ -theory. *Michigan Math. J.*, 48:295–304, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.
- [GK17] Vassily Gorbounov and Christian Korff. Quantum integrability and generalised quantum Schubert calculus. *Adv. Math.*, 313:282–356, 2017.
- [GKM25] V. Gorbounov, C. Korff, and L. C. Mihalcea. Quantum K-theory of Grassmannians from a Yang-Baxter algebra. *submitted, preprint available at arXiv:2503.08602*, 2025.
- [GKS20] Vassily Gennadievich Gorbunov, Christian Korff, and Catharina Stroppel. Yang–baxter algebras, convolution algebras, and grassmannians. *Russian Mathematical Surveys*, 75(5):791, 2020.
- [GL03] Alexander Givental and Yuan-Pin Lee. Quantum  $K$ -theory on flag manifolds, finite-difference Toda lattices and quantum groups. *Invent. Math.*, 151(1):193–219, 2003.
- [GMS<sup>+</sup>23] Wei Gu, Leonardo C. Mihalcea, Eric Sharpe, Weihong Xu, Hao Zhang, and Hao Zou. Quantum K Whitney relations for partial flag varieties. 2023.
- [GMS<sup>+</sup>24] Wei Gu, Leonardo Mihalcea, Eric Sharpe, Weihong Xu, Hao Zhang, and Hao Zou. Quantum K theory rings of partial flag manifolds. *J. Geom. Phys.*, 198:Paper No. 105127, 30, 2024.
- [GMS<sup>+</sup>25] Wei Gu, Leonardo C. Mihalcea, Eric Sharpe, Weihong Xu, Hao Zhang, and Hao Zou. A Nakayama result for the quantum K theory of homogeneous spaces. *to appear in Epjournal de Geometrie Algebrique, preprint available at arXiv:2507.15183*, 2025.
- [GMSZ22] Wei Gu, Leonardo C. Mihalcea, Eric Sharpe, and Hao Zou. Quantum K theory of Grassmannians, Wilson line operators, and Schur bundles. *to appear in Forum Math. Sigma, preprint available on arXiv:2208.01091*, 2022.
- [Gra01] William Graham. Positivity in equivariant Schubert calculus. *Duke Math. J.*, 109(3):599–614, 2001.
- [GY21] Alexander Givental and Xiaohan Yan. Quantum K-theory of grassmannians and non-abelian localization. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 17:Paper No. 018, 24, 2021.
- [HK24a] Irit Huq-Kuruvilla. Quantum K-rings of partial flag varieties, Coulomb branches, and the Bethe Ansatz. *arXiv:2409.15575*, 2024.
- [HK24b] Irit Huq-Kuruvilla. Relations in twisted quantum K-rings. *arXiv:2406.00916*, 2024.
- [HKMSZ25] Irit Huq-Kuruvilla, Leonardo C. Mihalcea, Eric Sharpe, and Hao Zhang. Quantum K-theory levels in physics and math. *arXiv:2507.00116*, 2025.
- [IIM20] Takeshi Ikeda, Shinsuke Iwao, and Toshiaki Maeno. Peterson isomorphism in  $K$ -theory and relativistic Toda lattice. *Int. Math. Res. Not. IMRN*, (19):6421–6462, 2020.
- [IKNY24] Takeshi Ikeda, Takafumi Kouno, Yusuke Nakayama, and Kohei Yamaguchi. Quantum K-theory of Lagrangian Grassmannian via parabolic Peterson isomorphism. *available on arXiv:2405.17854*, 2024.

- [IMN16] Takeshi Ikeda, Leonardo C. Mihalcea, and Hiroshi Naruse. Factorial  $P$ - and  $Q$ -Schur functions represent equivariant quantum Schubert classes. *Osaka J. Math.*, 53(3):591–619, 2016.
- [IMT15] Hiroshi Iritani, Todor Milanov, and Valentin Tonita. Reconstruction and convergence in quantum  $K$ -theory via difference equations. *Int. Math. Res. Not. IMRN*, (11):2887–2937, 2015.
- [IN13] Takeshi Ikeda and Hiroshi Naruse.  $K$ -theoretic analogues of factorial Schur  $P$ - and  $Q$ -functions. *Adv. Math.*, 243:22–66, 2013.
- [JM19] Hans Jockers and Peter Mayr. Quantum  $K$ -theory of Calabi-Yau manifolds. *JHEP*, 11:011, 2019, 1905.03548.
- [JM20] Hans Jockers and Peter Mayr. A 3d Gauge Theory/Quantum  $K$ -theory correspondence. *Adv. Theor. Math. Phys.*, 24(2):327 – 457, 2020, 1808.02040.
- [JMNT20] Hans Jockers, Peter Mayr, Urmi Ninad, and Alexander Tabler. Wilson loop algebras and quantum  $K$ -theory for grassmannians. *JHEP*, 10:036, 2020, 1911.13286.
- [Kat18] Syu Kato. Loop structure on equivariant  $K$ -theory of semi-infinite flag manifolds. *arXiv preprint arXiv:1805.01718*, 2018.
- [Kat19] Syu Kato. On quantum  $K$ -groups of partial flag manifolds. *available on arXiv:1906.09343*, 2019.
- [Kle74] Steven L. Kleiman. The transversality of a general translate. *Compositio Math.*, 28:287–297, 1974.
- [KLNS21] Takafumi Kouno, Cristian Lenart, Satoshi Naito, and Daisuke Sagaki. Quantum  $K$ -theory Chevalley formulas in the parabolic case (with an appendix by Takafumi Kouno, Cristian Lenart, Satoshi Naito, Daisuke Sagaki, Weihong Xu). *available on arXiv:2109.11596*, 2021.
- [Knu03] Allen Knutson. A Schubert calculus recurrence from the noncomplex  $W$ -action on  $G/B$ . *arXiv preprint math/0306304*, 2003.
- [Kol86] János Kollár. Higher direct images of dualizing sheaves. II. *Ann. of Math. (2)*, 124(1):171–202, 1986.
- [Kor21] Peter Koroteev. A-type quiver varieties and ADHM moduli spaces. *Comm. Math. Phys.*, 381(1):175–207, 2021.
- [KP01] B. Kim and R. Pandharipande. The connectedness of the moduli space of maps to homogeneous spaces. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 187–201. World Sci. Publ., River Edge, NJ, 2001.
- [KPSZ21] Peter Koroteev, Petr P. Pushkar, Andrey V. Smirnov, and Anton M. Zeitlin. Quantum  $K$ -theory of quiver varieties and many-body systems. *Selecta Math. (N.S.)*, 27(5):Paper No. 87, 40, 2021.
- [KT03] Allen Knutson and Terence Tao. Puzzles and (equivariant) cohomology of Grassmannians. *Duke Math. J.*, 119(2):221–260, 2003.
- [KZJ17] A. Knutson and P. Zinn-Justin. Schubert puzzles and integrability I: invariant trilinear forms. *available on arXiv:1706.10019*, 2017.
- [KZJ21] Allen Knutson and Paul Zinn-Justin. Schubert puzzles and integrability II: multiplying Segre classes. *available on arXiv:2102.00563*, 2021.
- [Las90] Alain Lascoux. Anneau de Grothendieck de la variété de drapeaux. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 1–34. Birkhäuser Boston, Boston, MA, 1990.
- [Lee04] Y.-P. Lee. Quantum  $K$ -theory. I. Foundations. *Duke Math. J.*, 121(3):389–424, 2004.
- [Len00] Cristian Lenart. Combinatorial aspects of the  $K$ -theory of Grassmannians. *Ann. Comb.*, 4(1):67–82, 2000.

- [LLMS18] Thomas Lam, Changzheng Li, Leonardo C. Mihalcea, and Mark Shimozono. A conjectural Peterson isomorphism in  $K$ -theory. *J. Algebra*, 513:326–343, 2018.
- [LLSY25] Changzheng Li, Zhaoyang Liu, Jiayu Song, and Mingzhi Yang. On Seidel representation in quantum  $K$ -theory of Grassmannians. *Sci. China Math.*, 68(7):1523–1548, 2025.
- [LM06] Cristian Lenart and Toshiaki Maeno. Quantum Grothendieck polynomials. available on arXiv:math/0608232v1 [math.CO], 2006.
- [LNS24] Cristian Lenart, Satoshi Naito, and Daisuke Sagaki. A general Chevalley formula for semi-infinite flag manifolds and quantum  $K$ -theory. *Selecta Math. (N.S.)*, 30(3):Paper No. 39, 44, 2024.
- [LNS<sup>+</sup>25] Cristian Lenart, Satoshi Naito, Daisuke Sagaki, Weihong Xu with an Appendix by Leonardo C. Mihalcea, and Weihong Xu. Quantum  $K$ -theoretic divisor axiom for flag manifolds. *preprint available at arXiv:2505.16150*, 2025.
- [Mih06] Leonardo Constantin Mihalcea. Positivity in equivariant quantum Schubert calculus. *Amer. J. Math.*, 128(3):787–803, 2006.
- [MNS22] Leonardo C. Mihalcea, Hiroshi Naruse, and Changjian Su. Left Demazure-Lusztig operators on equivariant (quantum) cohomology and  $K$ -theory. *Int. Math. Res. Not. IMRN*, (16):12096–12147, 2022.
- [MNS25a] Toshiaki Maeno, Satoshi Naito, and Daisuke Sagaki. A presentation of the torus-equivariant quantum  $K$ -theory ring of flag manifolds of type  $A$ , Part I: The defining ideal. *J. Lond. Math. Soc. (2)*, 111(3):Paper No. e70095, 43, 2025.
- [MNS25b] Toshiaki Maeno, Satoshi Naito, and Daisuke Sagaki. A presentation of the torus-equivariant quantum  $K$ -theory ring of flag manifolds of type  $A$ , Part II: quantum double Grothendieck polynomials. *Forum Math. Sigma*, 13:Paper No. e19, 26, 2025.
- [MP95] David R. Morrison and M. Ronen Plesser. Summing the instantons: quantum cohomology and mirror symmetry in toric varieties. *Nuclear Phys. B*, 440(1-2):279–354, 1995.
- [Oko17] Andrei Okounkov. Lectures on  $K$ -theoretic computations in enumerative geometry. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 251–380. Amer. Math. Soc., Providence, RI, 2017.
- [OS22] A. Okounkov and A. Smirnov. Quantum difference equation for Nakajima varieties. *Invent. Math.*, 229(3):1203–1299, 2022.
- [Pos05] Alexander Postnikov. Quantum Bruhat graph and Schubert polynomials. *Proc. Amer. Math. Soc.*, 133(3):699–709, 2005.
- [RZ23] Yongbin Ruan and Ming Zhang. Verlinde/Grassmannian correspondence and rank 2  $\delta$ -wall-crossing. *Peking Math. J.*, 6(1):217–306, 2023.
- [Sie09] Susan J. Sierra. A general homological Kleiman-Bertini theorem. *Algebra Number Theory*, 3(5):597–609, 2009.
- [SSW23] Jörg Schürmann, Connor Simpson, and Botong Wang. A new generic vanishing theorem on homogeneous varieties and the positivity conjecture for triple intersections of Schubert cells. *available on arXiv:2303.13833*, 2023.
- [ST97] Bernd Siebert and Gang Tian. On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator. *Asian J. Math.*, 1(4):679–695, 1997.
- [Sum24] Kevin Summers. A dual basis for the equivariant quantum  $K$ -theory of cominuscule varieties. *preprint arXiv:2407.02703*, 2024.
- [SZ24] Shubham Sinha and Ming Zhang. Quantum  $K$ -invariants via quot schemes I. *preprint arXiv:2406.12191*, 2024.
- [Tho98] Jesper Funch Thomsen. Irreducibility of  $\overline{M}_{0,n}(G/P, \beta)$ . *Internat. J. Math.*, 9(3):367–376, 1998.

- [UY20] Kazushi Ueda and Yutaka Yoshida. 3d  $\mathcal{N} = 2$  Chern-Simons-matter theory, Bethe ansatz, and quantum  $K$ -theory of Grassmannians. *JHEP*, 08:157, 2020, 1912.03792.
- [Wit95] Edward Witten. The Verlinde algebra and the cohomology of the Grassmannian. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 357–422. Int. Press, Cambridge, MA, 1995.
- [Xu24] Weihong Xu. Quantum  $K$ -theory of incidence varieties. *Eur. J. Math.*, 10(2):Paper No. 22, 47, 2024.

DEPARTMENT OF MATHEMATICS, VIRGINIA TECH UNIVERSITY, BLACKSBURG, VA 24061 USA  
*Email address:* lmihalce@vt.edu